ACI JOURNAL TECHNICAL PAPER

Title no. 82-59

Log Double Power Law for Concrete Creep


by Zdeněk P. Bažant and Jenn-Chuan Chern

An improved law of creep of concrete at constant humidity and temperature is proposed. The well-known double power law gives too high a final slope of creep curves compared to available test data. This is remedied by a new formula which exhibits a continuous transition from a power curve to a straight line in the logarithm of creep duration. The straight line has the same slope for all ages at loading, and the higher the age at loading, the longer is the duration at which the transition occurs. The exponent of the initial power curve is higher than that used in the double power law and is much too high in comparison with the existing test results for very short load durations in the dynamic range. This penalty, however, is outweighed by better extrapolation to very long load durations. The new formula significantly restricts the occurrence of divergence of creep curves at various ages at loading but does not eliminate it completely unless closeness of data fit is sacrificed. The new formula also greatly reduces the occurrence of negative values at the end of calculated stress relaxation curves.

Extensive statistical analysis of most test data available in the literature reveals a relatively modest improvement in the overall coefficient of variation for the deviations of the formula from test data and a significant improvement for the deviations of the final slope from its measured value. The same improvements were achieved in an earlier study in which the transition from the power law to the logarithmic law was abrupt, with a discontinuity in curvature. The continuity of curvature in the present formulation is desirable for applications in data extrapolation.

Keywords: coefficient of variation; computer programs; **concretes; creep properties**; dynamic modulus of elasticity; **humidity**; loads (forces); **measurement**; strains, strength; stresses; stress relaxation; **temperature**.

Although existing test data on creep of concrete at constant temperature and constant specific water content can be accurately described by the double power law,[1-3] certain deviations seem to be systematic rather than random. In particular, the final slope of the curves of strain versus the logarithm of loading duration appears too steep for tests of long duration.

A preceding work[4] showed that final slopes that agree with test data can be attained by introducing at a certain time a transition from the double power law to a logarithmic law, in which the curves of strain versus log-time are straight lines of the same slope for all ages at loading with the load duration at the transition increasing with a higher age at loading. In that work[4] the transition from the power curve to the logarithmic curve was considered to occur suddenly, with a discontinuous change in curvature, but without a discontinuous change in slope. However, from the viewpoint of the physical mechanism of creep, there is no reason for a sudden transition from one creep law to another. Therefore, a creep law that is smooth and approaches asymptotically the double power law for very short load durations and the logarithmic law for very long load durations seems more realistic. Furthermore, a smooth creep law is preferable for the extrapolation of short-time creep data. Development of such a creep law is the objective of this work.

REVIEW OF DOUBLE POWER LAW

The basic creep of concrete, i.e., the creep at constant temperature and constant specific water content, may be relatively well-described by the double power law[1-3]

$$J(t,t') = \frac{1}{E_0} + \frac{\phi_1}{E_0}(t'^{-m} + \alpha)(t - t')^n \qquad (1)$$

in which $J(t,t')$ is the compliance function (or the creep function) = the strain at age t caused by a unit uniaxial constant stress acting since age t'; E_0 is the asymptotic modulus, such that $1/E_0$ represents the asymptotic value of the creep curve $J(t,t')$ versus log $(t - t')$ as log $(t - t') \to -\infty$ or $(t - t') \to 0$; n, m, α, and ϕ_1 are material parameters whose typical values are $n \simeq 1/8$, $m \simeq 1/3$, $\alpha \simeq 0.05$ if t and t' are in days, and $\phi_1 =$ 3 to 6; and $E_0 \simeq 1.5\ E_{28}$, where E_{28} is the conventional (static) elastic modulus at age 28 days. Since $(t - t')^n = \exp[n\ ln(t - t')]$, the plots of $J(t,t')$ versus log $(t - t')$ at constant t' have the shape of exponentials. When the double power law was originally proposed,[1] α was considered as 0. A nonzero value of α does not improve data fits but does agree with the idea that even an infinitely old concrete should exhibit some creep.

Received Feb. 3, 1984, and reviewed under Institute publication policies. Pertinent discussion will be published in the July-August 1986 ACI JOURNAL if received by Apr. 1, 1986.

Zdeněk P. Bažant, FACI, is a professor and director of the Center for Concrete and Geomaterials, Northwestern University, Evanston, Ill. Dr. Bažant is a registered structural engineer, serves as consultant to Argonne National Laboratory and several other firms, and is on editoral boards of five journals. He is a member of ACI-ASCE Committee 334, Concrete Shell Design and Construction, and also of ACI Committees 209, Creep and Shrinkage in Concrete, and 348, Structural Safety. Dr. Bažant serves as chairman of RILEM Committee TC69 on creep, ASCE-EMD Committee on Properties of Materials, and IA-SMiRT Division H. His works on concrete and geomaterials, inelastic behavior, fracture, and stability have been recognized by a RILEM medal, ASCE Huber Prize and T.Y. Lin Award, Guggenheim Fellowship, Ford Foundation Fellowship, and election as Fellow of American Academy of Mechanics.

Jenn-Chuan Chern is an associate professor of civil engineering, National Taiwan University, Taipai. He received his PhD in civil engineering from Northwestern University. Dr. Chern has several years research and engineering experience from the Argonne National Laboratory, Chinese Marine Corps, and Fu-Tai Engineering Co. in Taiwan. His research interests include concrete creep, moisture effects, cracking, and high temperature effects on concretes, on which he cowrote a state-of-the-art report for the Electric Power Research Institute. He has also developed two large computer codes now in use in some U.S. government laboratories.

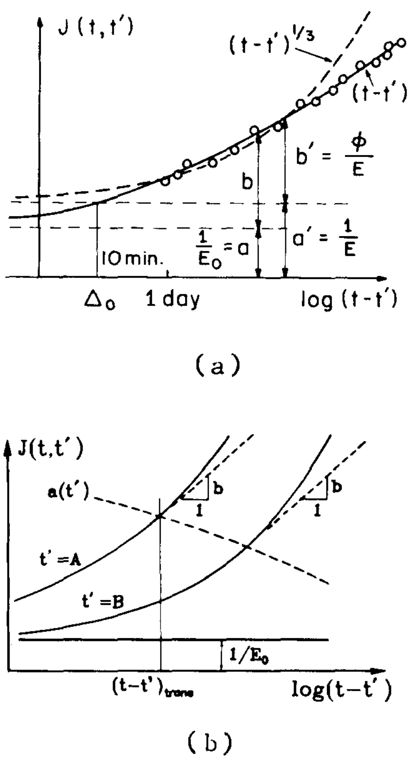

Fig. 1—(a) Creep curve in log-time scale (a = true elastic deformation, b = true creep, a' = conventional elastic deformation, and b' = conventional creep); and (b) transition from power law to the logarithmic law

Eq. (1) has a remarkably broad range of applicability. It works from a loading age of 1 day to many years and for load durations from 1 sec to several decades. It also yields reasonable compliance values for rapidly (dynamically) applied loads, and the dynamic modulus is acceptably estimated by Eq. (1) as $1/J(t' + \Delta, t')$ for $\Delta \simeq 10^{-7}$ day. The conventional (static) elastic modulus is obtained as the value of $1/J(t' + \Delta, t')$ for $\Delta \simeq 0.1$ day.[5] Parameters E_0, ϕ_1, m, and α are needed to describe the age-dependence of the elastic modulus, and thus only the one additional parameter n is needed to describe all creep.

Various other creep laws have been proposed in the past.[1-3,6-9] Extensive statistical studies of practically all test data documented in the literature revealed, however, that the double power law allows the smallest coefficient of variation for deviations from the available data. The power function of load duration was introduced by Straub and Shank.[10,11] Wittman et al.[7] presented supporting arguments based on the activation energy theory, and Cinlar, Bažant, and Osman[2] gave other supporting arguments based on a certain reasonable hypothesis for the stochastic nature of the creep process.

Some investigators believed the power function of $t - t'$ predicted too much creep for load durations beyond about one month. This conclusion, however, was reached by applying the power function to only that part of the creep strain that is in addition to the conventional short-time strain, which corresponds roughly to the load duration of 0.1 day. The horizontal asymptote $1/E_0$ is then too high, and to fit the data for medium load durations (up to about 30 days) a relatively high exponent n (about $n \simeq 1/3$) is needed. This causes the power curve to shoot above the data points for longer durations. However, if the power function is applied to the entire creep strain, including that which occurs for extremely short load durations in the dynamic range (about 0.001 sec), the power function becomes acceptable even for relatively long durations (several years). In this case, exponent n is then about $1/8$, which is small enough to prevent the power curve from overshooting the long-time data points [Fig. 1(a)]. In addition, the inclusion of all creep strain under the power law allows $1/E_0$ to be considered as age-independent, whereas in the earlier formulas it was necessary to use an age-dependent short-time modulus $E(t')$.

Although the double power law is formulated for basic creep, i.e., the creep of sealed specimens, it is not irrelevant for many structures exposed to weather. For walls over approximately 1 ft thick, only a small part of the initial water content is lost during a normal lifetime of 50 years, whereas for a 6-in. cylinder most of the initial water content is lost within 10 years, as diffusion calculations confirm. Consequently, the creep of such walls is closer to that of a sealed cylinder than to that of a 6-in. cylinder exposed to drying.[3]

LOG DOUBLE POWER LAW

With $(t - t')^n = e^{n\xi}$ if $\xi = \ln(t - t')$, the power curves plotted in log-time scale appear as exponential curves, the slope of which is steadily increasing. However, the prevalent trend of long-time creep measurements indicates that the slope becomes constant when a certain limiting slope b, common to the curves for all ages t' at loading, is approached. This was already noted in Reference 4. Thus it appears that for very long load durations the creep law should asymptotically approach the form

$$J(t,t') = a(t') + b \ln(t - t') \quad (2)$$

which is similar to the logarithmic law proposed by Hanson and Harboe.[14,15] At the same time, the creep law should asymptotically approach the double power law in Eq. (1) as $t - t' \to 0$ or $\log(t - t') \to -\infty$. The simplest smooth formula satisfying both asymptotic conditions is the log double power law

$$J(t,t') = \frac{1}{E_0} + \frac{\psi_0}{E_0} \ln [1 + \psi_1 (t'^{-m} + \alpha)(t - t')^n] \quad (3)$$

where ψ_0 and ψ_1 are constants. Indeed, denoting $x = \psi_1 (t'^{-m} + \alpha)(t - t')^n$ and realizing that $\ln(1 + x) \simeq x$ when x is small (approximately $x \leqslant 0.1$), Eq. (3) becomes nearly identical to Eq. (1) for small $(t - t')$ if we set

$$\phi_1 = \psi_0 \psi_1 \quad (4)$$

Furthermore, for large x, $\ln(1 + x) = \ln(1 + 1/x) + \ln x \simeq \ln x$; therefore, Eq. (3) becomes nearly identical to Eq. (2) for large $(t - t')$ if we set

$$b = n\psi_0/E_0;\ a(t') = E_0^{-1} \{1 + \psi_0 \ln[\psi_1(t'^{-m} + \alpha)]\} \quad (5)$$

The transition from Eq. (1) to (2) is gradual and centers on the time for which $x = 1$ or $\psi_1(t'^{-m} + \alpha)(t - t')^n = 1$, i.e., on the time

$$(t - t')_{\text{trans}} = [\psi_1(t'^{-m} + \alpha)]^{-1/n} \quad (6)$$

This transition time becomes longer as the age at loading t' increases [Fig. 1(b)]. This agrees with the fact that a constant slope b in the log-time scale appears to be reached fairly early if t' is small, and very late or hardly at all if t' is very large.

As with the double power law, Eq. (3) can in principle be applied to only basic creep, which represents a constitutive property of the material. It is often overlooked that the additional creep due to drying, as known, is not a constitutive property. Rather, it is an average property of the specimen as a whole, since the available test data represent the overall deformation of specimens in which drying caused highly nonuniform transient distributions of moisture content and stress and produced microcracking. Consequently, an empirical description of the mean creep of drying specimens requires much more complicated formulas.

The logarithmic law of Hanson and Harboe,[14,15] which has the form $J(t,t') = 1/E(t') + B \log(1 + t - t')$ where $t - t'$ is in days, is equivalent to the limiting case of Eq. (3) for very long $t - t'$, i.e., to $m \to \infty$, $\alpha\psi_1 = 1$, $n = 1$, and $E_0 \to E(t')$.

VERIFICATION BY TEST DATA FROM LITERATURE

Eq. (3) contains six material parameters (E_0, n, m, α, ψ_0, ψ_1) that must be determined from test data. This may be efficiently accomplished by computer optimization in which the sum of squared deviations Δ of Eq. (3) from the given data is minimized. A suitable library subroutine for this purpose is the Marquardt-Levenberg algorithm, which is often used for the fitting of creep data.[1] The data used in this optimization were drawn from a computerized data bank set up at Northwestern University,[2,16] which contains most of the test data available in the literature.

The raw measurements as found in the literature were replaced by visually hand-smoothed curves. These curves were characterized by data points placed on the curves at regularly spaced intervals in the log $(t - t')$ scale. This eliminates the bias due to crowding of data points in some portions of the log-time scale, as found in most reported data, and mitigates also to some extent the measurement error, which is not felt by real structures.

The solid lines in Fig. 2 represent the calculated optimal fits achieved for the test data from References 14, 15, and 17 through 23. These fits seem satisfactory and better than any others achieved previously. The corresponding values of the material parameters for each concrete, obtained by computer optimization, are also listed in Fig. 2.

The error of the log double power law may be characterized by the overall coefficient of variation ω of the deviations from test data

$$\bar{\omega} = \left(\frac{1}{N} \sum_{j=1}^{N} \omega_j^2\right)^{1/2}; \text{ with}$$
$$\omega_j = \frac{1}{\bar{J}_j}\left(\frac{1}{n - 1} \sum_{i=1}^{n} \Delta_{ij}^2\right)^{1/2};\ \bar{J}_j = \frac{1}{n} \sum_{i=1}^{n} \tilde{J}_{ij} \quad (7)$$

in which $\tilde{J}_{ij}$ $(i = 1, \ldots n)$ are the characteristic data points for the data series number j which are placed at regular spacings in log-time on the hand-smoothed measured curves; Δ_{ij} is the vertical deviations of Eq. (3) from these characteristic data points; $\bar{J}_j$ is the mean compliance value for data series number j; ω_j is the coefficient of variation of the deviations from test data for data series number j; and $j = 1, \ldots N$ are all the data series considered. The values of ω_j and $\bar{\omega}$ are summarized in Table 1 (Column LDPL). For comparison, Table 1 also lists the values of ω_j and $\bar{\omega}$ for the optimum fits attainable with the double power law (DPL), plotted as the dashed line curves in Fig. 2 and characterized by the material parameter values listed in Reference 1. For the double power law, $\bar{\omega} = 5.5$ percent (for $\alpha = 0$),[1] while with the log double power law $\bar{\omega} = 3.4$ percent is achieved (Table 1). The improvement is appreciable but not drastic, which is not surprising since the capability of the double power law to represent test data is already quite good.

Similar to a previous study[4] in which an abrupt transition from power curves to logarithmic curves was used, a more significant improvement is found in the representation of the final slopes of the measured long-term creep curves, which is important for the extrapolation of creep measurements. The final slopes were

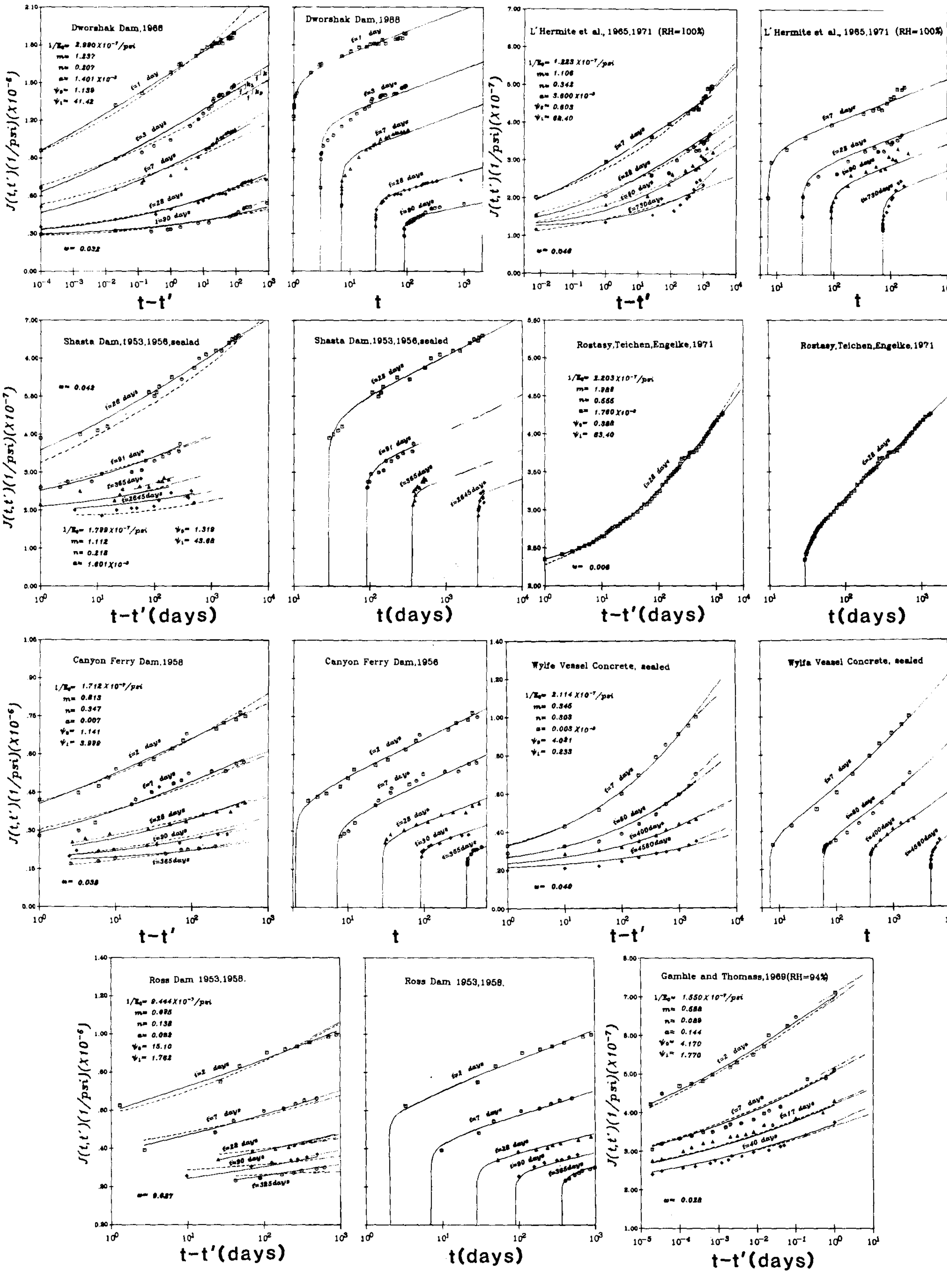

Fig. 2— Optimum fits of creep data from the literature[14,15,17-23]

Table 1 — Coefficients of variation for test data

j	Creep law / Data set	Optimum fits DPL[2]	Optimum fits DPL[1]	Optimum fits LDPL	Prediction formula DPL[2,]*	Prediction formula ACI*	Prediction formula CEB*	Prediction formula LDPL
1.	Canyon Ferry Dam[14,15]	4.60	5.58	3.80	39.6	47.3	18.7	15.8
2.	Ross Dam[14,15]	3.50	7.00	2.70	27.7	16.3	25.5	14.2
3.	Dworshak Dam[17]	5.46	5.63	3.20	21.2	35.4	46.0	11.8
4.	Rostasy et al[22]	1.00	1.20	0.60	5.1	12.2	7.9	8.0
5.	L'Hermite et al.[18]	4.90	6.28	4.60	25.2	52.3	19.8	16.9
6.	Shasta Dam[14,15]	4.10	5.37	4.20	16.6	27.5	20.3	23.9
7.	Wylfa Vessel[19-21]	4.14	4.15	4.00	21.0	35.4	46.0	9.82
8.	Gamble-Thomass[23]	2.82	6.20	2.80	—	—	—	—
9.	McDonald[32]	—	—	—	20.0	24.1	4.0	14.7
10.	Meyers-Maity[33]	—	—	—	14.2	15.4	27.5	14.1
11.	York et al.[34]	—	—	—	16.1	23.2	8.7	14.4
12.	Mossiossian-Gamble[35]	—	—	—	7.81	—	—	3.46
13.	Keeton[36]	—	—	—	26.9	36.4	52.4	16.1
14.	Ross[26] 93 percent	—	—	—	33.7	35.0	14.2	41.2
Number of data sets *N*		8	8	8	13.0	12.0	12.0	13.0
$\bar{\omega} = \left(\sum_i \omega_i^2/N\right)^{1/2}$		4.03	5.45	3.44	23.11	32.35	28.72	17.97

*Statistical values are provided by Reference 2.

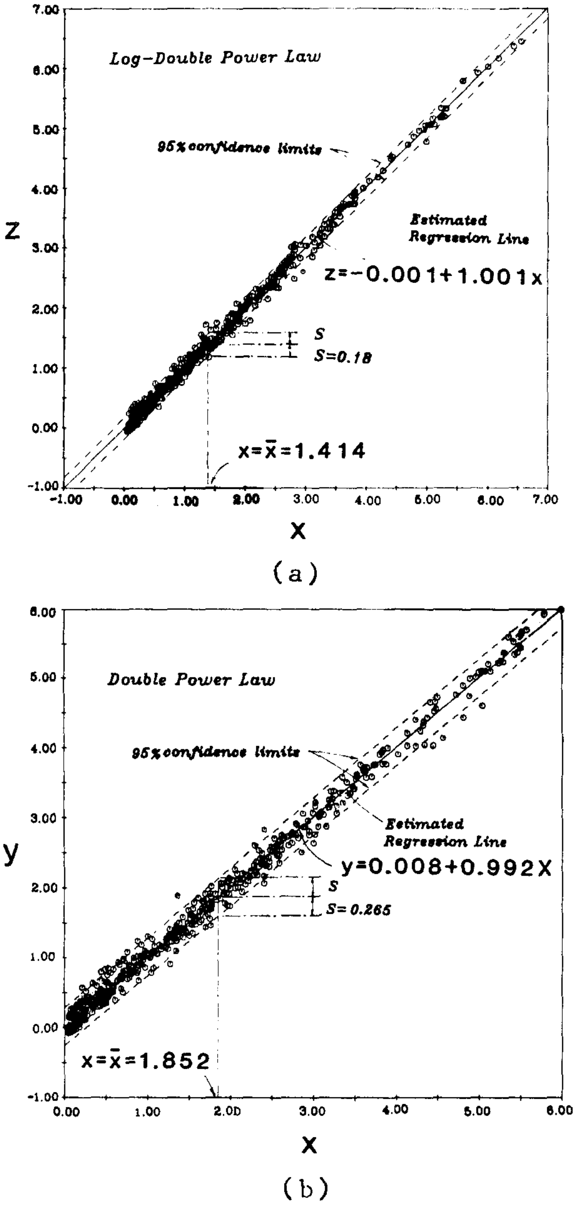

Fig. 3—Statistical regression analysis of test data for log double power law and double power law

determined graphically as illustrated in Fig. 2 and 3 by the dashed straight lines. The deviations of the slope $\partial J(t,t')/\partial \log(t - t')$ calculated from Eq. (3) for the last sampling time have been analyzed statistically, and their coefficients of variation (combined for all t'), along with the corresponding values for the double power law, are listed for each data series in Table 2. For the latter, the coefficient of variation for the final slope deviations is found to be $\bar{\omega}_f = 34$ percent while for our log double power law this is reduced to $\bar{\omega}_f = 19$ percent. The improvement here is indeed significant.

In the fits shown in Fig. 2, we may detect one undesirable feature: the curvature of the creep curves at very short load durations (less than 1 day) is often too high, as are the compliance values at the beginning of the creep curves, particularly for high t'. The problem is in the value of exponent n. The optimum values of n listed in the figures are mostly around ⅓, while fitting of the same test data with the double power law yields $n \approx 1/8$. The latter value gives just about the correct leftward extrapolation into the dynamic range, i.e., a correct dynamic modulus, whereas for $n \simeq 1/3$ the dynamic modulus is obtained too close to the static modulus of elasticity.

Compared to the previous study in which a sudden transition from the power curve to the logarithmic function was used, here we have the advantage of a smooth formula without a sudden change in curvature, but a disadvantage in the loss of applicability of the formula to very short loading durations, particularly the dynamic range.

If some material parameters are estimated, then the remaining parameters can be easily determined by linear regression. For example, we may define $x = \ln[1 + \psi_1 (t'^{-m} + \alpha)(t - t')^n]$ and choose the values of m, n, α, and ψ_1 based on experience (e.g., on the basis of the values listed in Fig. 2 and 4). Then, if we plot the measured values $y = J(t,t')$ versus x, straight-line regression gives us E_0^{-1} as the intercept and ψ_0/E_0 as the slope, since $y = E_0^{-1} + (\psi_0/E_0)x$. The errors in the intercept and slope may be characterized by correction coefficients c_0 and c_1, defined by the relation $y = (1 + c_0)E_0^{-1} + c_1(\psi/E_0)x$. Then, if we set $z = E_0 J(t,t') - 1$, the plot of z versus x must be a straight line, i.e., $z =$

Table 2 — Final slopes

Data set	t'	Optimum fits DPL,[1] Δ_D	Optimum fits LDPL, Δ_L	Prediction formulas DPL,[2] Δ_D	Prediction formulas LPDL, Δ_L
1	2	0.38	0.07	1.20	0.00
	7	0.17	0.20	0.85	0.03
	28	0.00	0.08	0.00	0.05
	90	0.40	0.20	0.10	0.25
	365	0.32	0.00	0.20	0.00
2	2	0.70	0.19	1.00	0.18
	7	0.00	0.00	0.50	0.20
	28	0.26	0.00	0.00	0.20
	90	0.50	0.05	0.50	0.20
	365	0.60	0.00	0.40	0.33
3	1	0.60	0.00	1.64	0.22
	3	0.05	0.00	1.00	0.00
	7	0.20	0.00	0.80	0.10
	28	0.00	0.00	0.00	0.23
	90	0.50	0.40	0.30	0.07
4	28	0.25	0.00	0.11	0.20
5	7	0.04	0.33	0.54	0.05
	28	0.17	0.17	0.50	0.23
	90	0.17	0.17	0.25	0.15
	730	0.60	0.50	0.60	0.70
6	28	0.39	0.16	0.35	0.10
	91	0.00	0.10	0.08	0.05
	365	0.00	0.20	0.10	0.75
	2645	0.50	0.20	0.10	0.50
7	7	0.39	0.39	0.20	0.14
	60	0.17	0.17	0.40	0.22
	400	0.05	0.05	0.30	0.60
	4560	0.29	0.29	0.15	0.00
8	2	0.20	0.00	—	—
	7	0.00	0.08	—	—
	17	0.00	0.00	—	—
	40	0.28	0.20	—	—
9	90	—	—	0.96	1.33
10	72	—	—	0.04	0.00
	28	—	—	0.52	0.00
11	90	—	—	0.41	0.90
12	4	—	—	0.05	0.00
13	8	—	—	0.16	0.40
14	8	—	—	1.38	1.72
	14	—	—	0.42	0.79
	28	—	—	1.30	2.00
	60	—	—	1.50	1.90
	90	—	—	1.50	1.45
N		32	32	39	39
ω_f		33.7	19.1	71.40 (58.00)	69.00 (38.80)

Δ_D and Δ_L are normalized errors of final slopes for curves of double power law k_D and log double power law k_L in comparisons with the final slope of test data k, defined as $|k_D/k - 1|$ and $|k_L/k - 1|$, respectively.

$$\bar{\omega}_f = \left(\frac{1}{n-1}\sum_{i=1}^{n}\Delta_{D_i}^2\right)^{1/2} \text{ or } \left(\frac{1}{n-1}\sum_{i=1}^{n}\Delta_{L_i}^2\right)^{1/2}.$$

$c_1x + c_0$. Thus, the sum ϕ of the squared vertical deviations of the data points from the straight regression line is a measure of the error. The initial estimates of the values of m, n, α, and ψ_1 may be improved by carrying out the regression for various estimates and then picking the one which gives the smallest ϕ.

If the formula [Eq. (3)] were perfect, then c_1 should be 1 and c_0 should be 0. Statistical regression analysis, shown in the plot in Fig. 3, yields for c_1 and c_0 values that differ only slightly from 1 and 0, respectively, if the values of the material parameters m, n, α, and ψ_1 previously obtained by nonlinear optimization are used. This is another confirmation of the optimization results.

Regression analysis has the advantage of providing statistics, such as the confidence limits, as a function of x; see the dashed curves in Fig. 3, which represent the 95 percent confidence limits. Although these curves look straight, due to the large size of the data set and a small coefficient of variation, they actually have the shape of hyperbolas and diverge as the distance from the centroid of the data set increases. In this manner, the confidence limits increasingly separate when the test data are extrapolated to longer times. A useful feature of the regresssion plot in Fig. 3 is that it permits combining the data from different laboratories for different concretes. This greatly broadens the statistical basis.

Alternatively, another linear regression is possible by choosing the values of E_0, ψ_0, m, and α in advance and defining $x = \log(t - t')$ and $y = \log [(\exp\{[E_0J(t,t') - 1]/\psi_0\} - 1)/(t'^{-m} + \alpha)]$. Here, straight-line regression of y versus x yields n as the slope and $\log \psi_1$ as the intercept. This type of regression plot, however, is usually quite scattered, since the value of y appears to be sensitive to small changes in the parameters.

For comparison, the same type of regression as in Fig. 3(a) is shown in 3(b) for the double power law. Here $y = E_0J(t,t') - 1$, and $x = \phi_1(t'^{-m} + \alpha)(t - t')^n$. Again, the log double power law yields narrower confidence limits than does the double power law (Fig. 3). (The definitions of x for Fig. 3(a) and (b) are different, and thus the scales are not the same.)

Another interesting plot is that of $J(t,t')$ versus $\log t$ (Fig. 2). This plot should approach a straight line asymptotically for very large t (different straight lines for different t'). The plots in Fig. 2 clearly confirm such an asymptotic trend, thus confirming the transition to a logarithmic law. However, such plots are not useful for small values of $(t - t')/t$ because they crowd the data points together.

DIVERGENCE OF CREEP CURVES

The double power law is known to exhibit a certain questionable property.[12,24,25] The creep curves for different ages t' at loading begin to diverge after a certain load duration $t - t' = \theta_D$ is exceeded. This property causes their shape to be nonmonotic when creep recovery curves are calculated from $J(t,t')$ on the basis of the principle of superposition.

Although this might seem puzzling, no fundamental (thermodynamic) reason exists that would prohibit divergence and nonmonotonic recovery in the case of an aging material.[12,24,25] In fact, both properties have

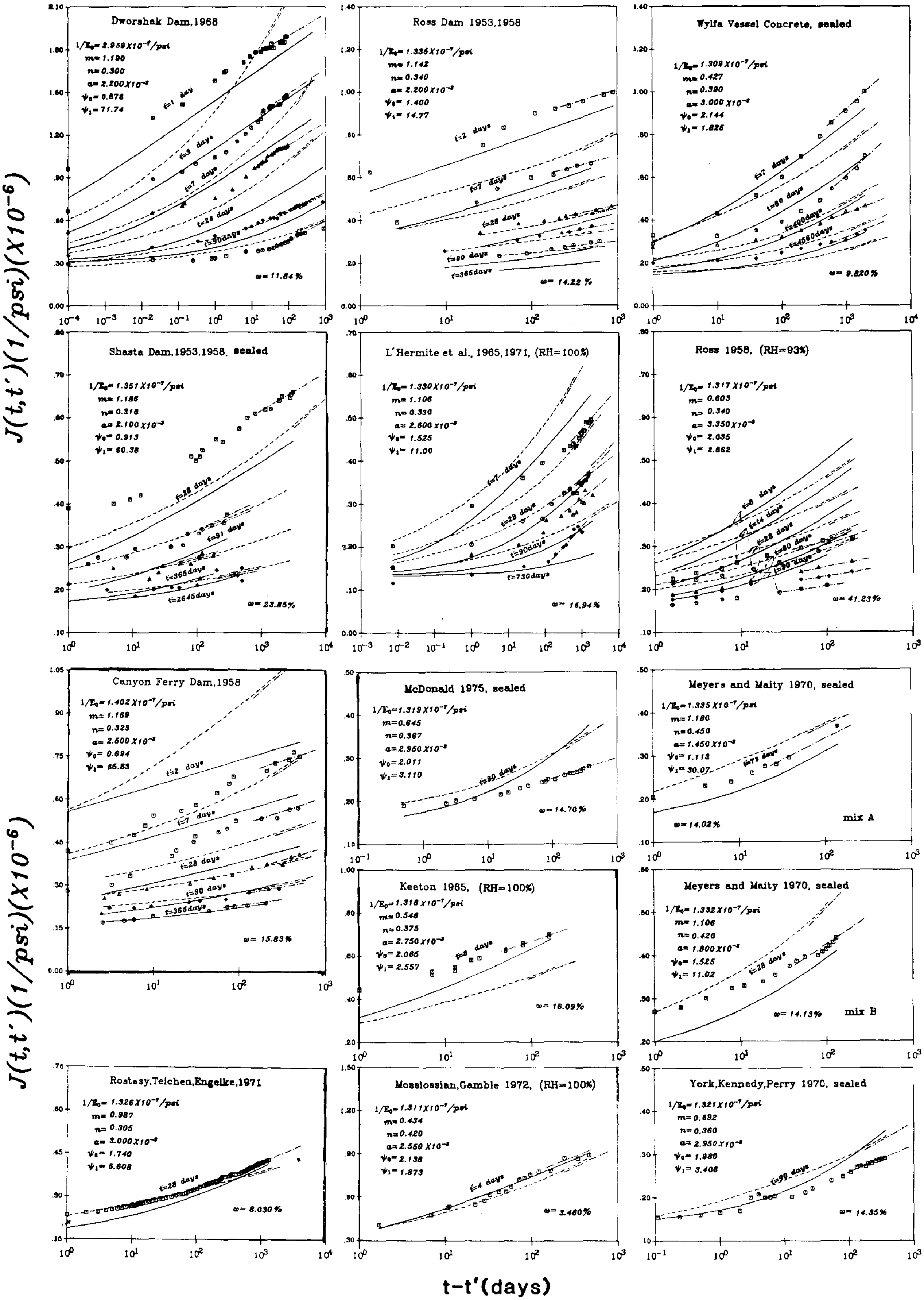

Fig. 4—Curves obtained when material parameters are predicted from strength and composition of concrete

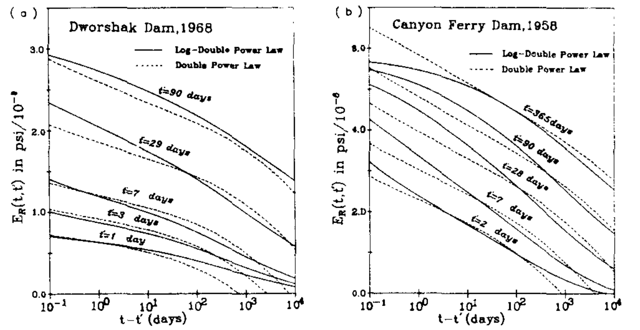

Fig. 5—Stress-relaxation curves for Dworshak Dam[17] and Canyon Ferry Dam[14,15] calculated from superposition principle after data smoothing with double power law and log double power law

been observed in some experiments. For example, the divergence can be visually recognized in the measured plots of $J(t,t')$ versus log t in Fig. 2. Nevertheless, there exists no reason why divergence should occur. Moreover, it might be that observed divergence is caused by some nonlinear phenomena,[24] which cannot be described by the compliance function. For these reasons, and because divergence causes some computational problems, it is probably preferable to avoid or limit its occurrence as far as test data permit.

Divergence occurs when the creep curve for a higher age at loading t' and the same t has a smaller slope $\partial J(t,t')/\partial t$, i.e., when $\partial^2 J(t,t')/\partial t\, \partial t'$ is negative, which is what we want to avoid. Differentiation of Eq. (3) yields

$$\frac{\partial J(t,t')}{\partial t} = \frac{n\psi_1 (t'^{-m} + \alpha)(t - t')^{n-1}}{1 + \psi_1 (t'^{-m} + \alpha)(t - t')^n} \quad (8)$$

and so divergence does not take place as long as

$$\frac{\partial^2 J(t,t')}{\partial t\, \partial t'} = \frac{n\psi_0 (f_1 - f_2 + f_3)}{E_0 (t - t')^n [1 + \psi_1 (t'^{-m} + \alpha)(t - t')^n]^2} \geqslant 0 \quad (9)$$

in which

$$f_1 = (1 - n)(t'^{-m} + \alpha),$$
$$f_2 = mt'^{-1-m}(t - t'),$$
$$f_3 = \psi_1 (t'^{-m} + \alpha)^2 (t - t')^n \quad (10)$$

The inequality of Eq. (9) implies that $f_1 + f_3 \geqslant f_2$. The duration $t - t'$ at which divergence begins is the solution of this inequality. An explicit solution, however, is impossible.

Evaluating the magnitudes of f_1, f_2, and f_3 for the various data sets and various t and t', f_1 is usually small as compared to f_2 and f_3. Thus the condition of nondivergence may be approximately stated as $f_3 \geqslant f_2$, and this inequality can be solved explicitly for $t - t'$, yielding

$$t - t' \leqslant \left[\frac{\psi_1}{m} (t'^{-m} + \alpha)^2\, t'^{m+1}\right]^{\frac{1}{1-n}} \quad (11)$$

Table 3 — Elasped time limits for nondivergence for creep test data of L'Hermite et al. (1965) fitted by double power law and log double power law (in days)

t'	DPL*	LDPL†	
		Exact, Eq. (9)	Approximate, Eq. (11)
7	33.8	450	430
28	152.0	500	491
90	558.0	1080	1010
730	6283.9	50000	49500

*m = 0.329, n = 0.084, and α = 0.198.
†m = 1.106, n = 0.342, ψ = 68.4, and α = 0.0036.

This is a sufficient condition but not a necessary one because f_1 was neglected. The exact limit on $t - t'$ is larger than Eq. (11) indicates, but usually it is only slightly larger (Table 3).

Divergence cannot be totally prevented and will always occur at sufficiently long load durations. For the double power law,[24] the nondivergence condition leads to $f_1 \geqslant f_2$ rather than $f_1 + f_3 \geqslant f_2$. Since always $f_3 > 0$, the range of nondivergence for our log double power law is always broader (actually much broader) than it is for the double power law. Moreover, after the adjacent creep curves start to diverge, they are found to increase their separation by only a little, much less than one finds for the double power law; see the plots of $J(t,t')$ versus log t in Fig. 2. Thus the recovery curves calculated from $J(t,t')$ on the basis of the superposition principle would never be too far from a monotonic curve.

STRESS RELAXATION PREDICTIONS

Stress relaxation at constant strain, unlike creep recovery, may be closely predicted from $J(t,t')$ on the basis of the principle of superposition. This is confirmed by relaxation measurements.[26] Although measurements for very long durations $t - t'$ of imposed strain are lacking, frequently it is found that the relaxation curves calculated from $J(t,t')$ cross into negative values of stress at very long $t - t'$ (over 10 years) if t' is small. This is usually found when the actual measured compliance values are used and when they are smoothed by the double power law.

From the thermodynamic viewpoint, no fundamental law exists that would prohibit such negative values when one deals with an aging material.[25] Nevertheless, in absence of experimental support, such behavior seems suspicious, especially since it is not clear whether it might be caused merely by some random scatter of compliance measurements or by an error in the formula for the compliance function (or by some nonlinear phenomena). Although not imperative, we prefer a compliance function that avoids or minimizes the occurrence of negative values of the associated relaxation function.

Fig. 5 shows examples of the relaxation curves, which were calculated on the basis of the superposition principle (using a highly accurate step-by-step algorithm[27,28]) from the compliance functions that give the optimum fits of the test data for Dworshak Dam[17] and for Canyon Ferry Dam.[14,15] While the relaxation curves do cross at long times into negative values for the double power law, this does not happen for the present log double power law.

EXTRAPOLATION OF TEST DATA

The more realistic representation of the slope of the long-time creep curves, as well as the smoothness of the log double power law, is particularly useful for extrapolating short-time measurements into long-time measurements. This will be demonstrated using the measurements of Rostasy et al.,[22] which represent one of the best controlled and least scattered creep measurements available (Fig. 6). Assume that only the measurements up to load-durations $t - t' = 6$ months are known. The optimum fit of these data is obtained by linear regression as described previously. This yields the solid curve plotted in Fig. 4. This curve is very close to the remaining measurements, which demonstrates the extrapolation capability of the log double power law.

For comparison, Fig. 6 also shows the extrapolation obtained when the data up to 6 months duration are fitted by the so-called Ross' hyperbola (dashed lines). This hyperbola represents a widely used method.[13,29] The extrapolation is very poor, which reinforces recent negative conclusions about this extrapolation approach.[13,29]

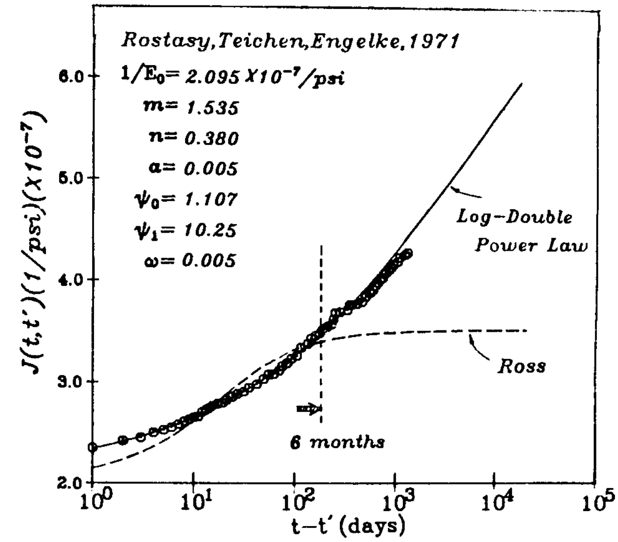

Fig. 6—Example of extrapolation based on data for load duration up to six months

CONCLUSIONS

1. The final slopes of long-time creep curves as given by the double power law are predominantly on the high side when compared to long-term measurements. This may be remedied by the log double power law, which exhibits a continuous transition from a power curve to a straight line in the logarithmic scale of load duration. The slope of this line is the same for all ages at loading, and the younger the concrete the earlier the transition occurs.

2. The log double power law appears suitable for extrapolating short-time creep measurements to long load durations.

3. The exponent of the initial power curve is higher than that for the double power law (about ⅓ instead of about ⅛). This has the disadvantage that the log double power law does not apply for very short load durations (under 0.1 day) and especially not for the dynamic range, whereas the double power law does apply for this range. This is the price paid for better long-term extrapolation.

4. The aforementioned disadvantage for very short load durations does not exist for the previously formulated double power logarithmic law, in which the transition from a power curve to a logarithmic curve is sudden, with a discontinuous change in curvature. The continuity of curvature in the present formulation is, however, advantageous for extrapolation of test data.

5. The log double power law (same as the previous double power logarithmic law) significantly restricts the occurrence of divergence of creep curves for various ages at loading. However, divergence cannot be eliminated completely if a good fit of test data should not be sacrificed.

6. The present formulation (similar to the previous double power logarithmic law) also greatly reduces the occurrence of negative stress values at the long-time tail of stress relaxation curves calculated on the basis of the principle of superposition.

7. Compared to the double power law, the present formulation achieves only a small improvement in the overall coefficient of variation for the deviations from the bulk of the test data reported in the literature. However, a significant improvement is achieved in the coefficient of variation for the deviations of the final slope of the measured creep curves from that predicted by the present formula. (These features are the same as for the previous double power logarithmic law.)

A further improvement of the concrete creep law is given in the paper "Triple Power Law for Concrete Creep," by Bažant and Chern, *Journal of Engineering Mechanics*, ASCE, V. 111, 1985, pp. 63-83.

ACKNOWLEDGMENT

Financial support by the U.S. National Science Foundation under Grant No. CEE-8303148 to Northwestern University is gratefully acknowledged.

REFERENCES

1. Bažant, Z. P., and Osman, E., "Double Power Law for Basic Creep of Concrete," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 9, No. 49, Jan.-Feb. 1976, pp. 3-11.

2. Bažant, Z. P., and Panula, L., "Practical Prediction of Time-Dependent Deformation of Concrete," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 11, No. 65, Sept.-Oct. 1978, pp. 307-328; V. 11, No. 66, Nov.-Dec. 1978, pp. 415-434; and V. 12, No. 69, May-June 1979, pp. 169-183.

3. Bažant, Zdeněk P., and Panula, Liisa, "Creep and Shrinkage Characterization for Analyzing Prestressed Concrete Structures," *Journal*, Prestressed Concrete Institute, V. 25, No. 3, May-June 1980, pp. 86-122.

4. Bažant, Zdeněk, and Chern, Jenn-Chuan, "Double Power Log-

arithmic Law for Concrete Creep," *Report* No. 83-11/679d, Center for Concrete and Geomaterials, Technological Institute, Northwestern University, Evanston, 1983. Also, *Cement and Concrete Research,* V. 14, No. 6, Nov. 1984, pp. 793-806.

5. Bažant, Zdeněk P., and Oh, Byung, "Strain-Rate Effect in Rapid Nonlinear Triaxial Deformation of Concrete," *Proceedings,* ASCE, V. 108, EM5, Oct. 1982, pp. 764-782.

6. Bažant, Z. P., "Theory of Creep and Shrinkage in Concrete Structures: A Précis of Recent Developments," *Mechanics Today,* V. 2, 1975, pp. 1-93.

7. Wittmann, F., and Setzer, M., "Vergleich einiger Kriechfunktionen mit Versuchsergebnissen," *Cement and Concrete Research,* V. 1, No. 6, Nov. 1971, pp. 679-690.

8. Bažant, Zdeněk, and Osman, El Mamoun, Reply to Rüsch, Jungwirth, and Hilsdorf's Second Discussion of the Paper "On the Choice of Creep Function for Standard Recommendations on Practical Analysis of Structures," *Cement and Concrete Research,* V. 7, No. 1, Jan. 1977, pp. 119-130.

9. Bažant, Z. P., and Thonguthai, W., "Optimization Check of Certain Recent Practical Formulations for Concrete Creep," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 9, No. 50, Mar.-Apr. 1976, pp. 91-98.

10. Straub, Lorenz G., "Plastic Flow in Concrete Arches," *Proceedings,* ASCE, V. 56, 1930, P. 49. Also, *Transactions,* ASCE, V. 95, 1931, pp. 613-678.

11. Shank, J. R., "The Plastic Flow of Concrete," *Bulletin* No. 91, Engineering Experiment Station, Ohio State University, Columbus, 1935, 62 pp.

12. Çinlar, Erhan; Bažant, Zdeněk P.; and Osman, El Mamoun, "Stochastic Process for Extrapolating Concrete Creep," *Proceedings,* ASCE, V. 103, EM6, Dec. 1977, pp. 1069-1088.

13. Bažant, Zdeněk P., and Chern, J. C., Reply to Hilsdorf and Muller's Discussion of "Comments on the Use of Ross' Hyperbola and Recent Comparisons of Various Practical Creep Prediction Models," *Cement and Concrete Research,* V. 13, No. 3, May 1983, pp. 444-448.

14. Hanson, J. A., "A Ten-Year Study of Creep Properties of Concrete," *Concrete Laboratory Report* No. SP-38, U.S. Bureau of Reclamation, Denver, July 1953, 34 pp.

15. Harboe, E. M., et al., "A Comparison of the Instantaneous and the Sustained Modulus of Elasticity of Concrete," *Concrete Laboratory Report* No. C-854, U.S. Bureau of Reclamation, Denver, Mar. 1958.

16. Bažant, Zdeněk, and Zebich, Steven, "Statistical Linear Regression Analysis of Prediction Models for Creep and Shrinkage," *Cement and Concrete Research,* V. 13, No. 6, Nov. 1983, pp. 869-876.

17. Pirtz, D., "Creep Characteristics of Mass Concrete for Dworshak Dam," *Report* No. 65-2, Structural Engineering Laboratory. University of California, Berkeley, Oct. 1968, 62 pp.

18. L'Hermite, R. G.; Mamillan, M.; and Lefèvre, C., "Nouveaux résultats de recherches sur la déformation et la rupture du béton," *Annales,* Institut Technique du Bâtiment et des Travaux Publics (Paris), V. 18, No. 207-208, 1965, p. 325. Also, *The Structure of Concrete and Its Behaviour Under Load,* Cement and Concrete Association, London, 1968, pp. 423-433.

19. Browne, R. D., and Blundell, R., "The Influence of Loading Age and Temperature on the Long Term Creep Behavior of Concrete in a Sealed, Moisture Stable State," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 2, No. 8, Mar.-Apr. 1969, pp. 133-144.

20. Browne, R. D., and Burrow, R. E. D., "Utilization of the Complex Multiphase Material Behavior in Engineering Design," *Proceedings,* Southampton Conference on Structure, Solid Mechanics and Engineering Design in Civil Engineering Materials, Wiley-Interscience, London, 1971, p. 1343.

21. Browne, R. D., and Bamforth, P. P., "The Long Term Creep of the Wylfa P.V. Concrete for Loading Ages up to 12½ Years," 3rd International Conference on Structural Mechanics in Reactor Technology, London, Sept. 1975, Paper H 1/8.

22. Rostasy, F. S.; Teichen, K.-Th.; and Engelke, H., "Beitrag zur Klärung der Zusammenhanges von Kriechen und Relaxation bei Normal-Beton," *Heft* 139, Amtliche Forschungs- und Materialprüfunsanstalt für das Bauwesen. Otto-Graf-Institut, Universität Stuttgart, Strassenbau- und Strassenverkehrstechnik, 1972.

23. Gamble, B. R., and Thomass, L. H., "The Creep of Maturing Concrete Subject to Time-Varying Stress," *Proceedings,* 2nd Australasian Conference on the Mechanics of Structures and Materials, Adelaide, Aug. 1969, 25 pp.

24. Bažant, Z. P., and Kim, S. S., "Can the Creep Curves for Different Loading Ages Diverge?" *Cement and Concrete Research,* V. 8, No. 5, Sept. 1978, pp. 601-602.

25. Bažant, Zdeněk P., "Thermodynamics of Solidifying or Melting Viscoelastic Material," *Proceedings,* ASCE, V. 105, EM6, Dec. 1979, pp. 933-952.

26. Ross, A. D., "Creep of Concrete Under Variable Stress," ACI Journal, *Proceedings* V. 54, No. 9, Mar. 1958, pp. 739-758.

27. "Time-Dependent Effects," *Finite Element Analysis of Reinforced Concrete,* American Society of Civil Engineers, New York, 1982, pp. 309-400.

28. Bažant, Z. P., "Input of Creep and Shrinkage Characteristics for a Structural Analysis Program," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 15, No. 88, July-Aug. 1982, pp. 283-290.

29. Bažant, Zdeněk P., and Chern, Jenn-Chuan, "Comment on the Use of Ross' Hyperpola and Recent Comparisons of Various Practical Creep Prediction Models," *Cement and Concrete Research,* V. 12, No. 4, July 1982, pp. 527-532.

30. ACI Committee 209, "Prediction of Creep, Shrinkage, and Temperature Effects in Concrete Structures" (ACI 209R-82), American Concrete Institute, Detroit, 1982, 108 pp. Also, *Designing for Creep and Shrinkage in Concrete Structures,* SP-76, American Concrete Institute, Detroit, 1982, pp. 193-300.

31. *CEB-FIP Model Code for Concrete Structures,* 3rd Edition, Comité Euro-International du Béton/Fédération Internationale de la Précontrainte, Paris, 1978, 348 pp.

32. McDonald, J. E., "Time Dependent Deformation of Concrete under Multiaxial Stress Conditions," *Technical Report* No. C-75-4, U.S. Army Engineer Waterways Experiment Station, Vicksburg, Oct. 1975, 110 pp.

33. Maity, K., and Meyers, B. L., "The Effect of Loading History on the Creep and Creep Recovery of Sealed and Unsealed Plain Concrete Specimens," *Report* No. 70-7, Department of Civil Engineering, University of Iowa, Iowa City, Sept. 1970.

34. York, Guy P.; Kennedy, Thomas W.; and Perry, Ervin S., "Experimental Investigation of Creep in Concrete Subjected to Multiaxial Compressive Stresses and Elevated Temperatures," *Research Report* No. 2864-2, University of Texas at Austin, June 1970, 428 pp.

35. Mossiossian, V., and Gamble, W. L., "Time-Dependent Behavior of Non-Composite and Composite Prestressed Concrete Structures under Field and Laboratory Conditions," Civil Engineering Studies, *Structural Research Series* No. 385, University of Illinois, Urbana, May 1972, 517 pp.

36. Keeton, John R., "Study of Creep in Concrete," *Technical Reports* No. R333-I, R333-II, and R333-III, U.S. Naval Civil Engineering Laboratory, Port Hueneme, 1965.

APPENDIX—PREDICTIONS OF CREEP FROM CONCRETE STRENGTH AND COMPOSITION

Demonstration of the capability to fit the bulk of the existing test data confirms the applicability of the type of the mathematical formula. However, it is another matter to predict the values of the material parameters in the formula from the given strength and composition of concrete. This problem is generally known for huge statistical scatter,[2,3] no matter which formula is used. Because the uncertainty of predicting material parameters in the creep law from the strength and composition is much larger than that of the creep law itself, it is not important to use a very accurate creep law if only the strength and composition are specified and no creep measurements per se are taken.

Therefore, the use of the present formulation can at best bring only a minor advantage over the double power law in the prediction problem. Nevertheless, more realistic final slopes are still an advantage, and thus the previously developed prediction formulas for the parameters of the double power law[2,3] have been modified, in a purely empirical manner, to yield prediction formulas for the material parameters in the present log double power law. Analysis of the test data previously used for the BP Model[2] indicated that the six parameters in Eq. (3) can be predicted by the following empirical formulas

$$E_0 = E_0' + 0.77\,(1 + 526e^{-2.91 f_c'})^{-1} \quad \text{(A1)}$$

with

$$E_0' = 0.0062\,(f_c' - 5) \text{ if } f_c' \geqslant 5 \text{ ksi, else } E_0' = 0 \quad \text{(A2)}$$

$$m = 0.4 + 0.79\,(1 + 1.16 \times 10^{-7} e^{2.66 f_c'})^{-1},$$
$$n = 2.5\, n_{DPL},\ \alpha = 0.5\, \alpha_{DPL} \quad \text{(A3)}$$

$$\psi_0 = 0.87 + 1.31\,(1 + 8330e^{-1.74 f_c'})^{-1},$$
$$\psi_1 = 1.5 \pm 42500\,(586 + e^{-1.57 f_c'})^{-1} \quad \text{(A4)}$$

in which f_c' is the standard cylindrical strength of concrete at age 28 days, which must be given in ksi (1 ksi = 1000 psi = 6.895 MPa); the subscript DPL refers to the values given by the BP Model formulas for the double power law[2]

$$\alpha = 0.0025 \text{ or } \alpha = \frac{1}{16\ \text{w/c}} \quad \text{(A5)}$$

$$n = 0.288 = 0.000325\,(f_c')^{3.4} \quad \text{(A6)}$$

$$\text{or for } x > 0,\ n = 0.3 + \frac{0.175\, x^6}{5130 + x^6} \quad \text{(A7a)}$$

$$\text{for } x \leqslant 0,\ n = 0.3 \quad \text{(A7b)}$$

with x being defined as in Eq. (18) of Reference 2, i.e.

$$x = \left[2.1\, \frac{a/c}{(s/c)^{1.4}} + 0.1\, \left(f_c'\right)^{1.5} \left(\frac{w}{c}\right)^{1/3} \left(\frac{a}{g}\right)^{2.2}\right] a_1 - 4 \quad \text{(A8)}$$

where c = cement content in kg/m³; w/c = water-cement ratio; a/c = aggregate-cement ratio; g/s = gravel-sand ratio; s/c = sand-cement ratio (all ratios by weight); f_c' is 28-day cylinder strength in ksi; and a_1 is a coefficient taken as 1.00 for ordinary cements of ASTM Types I and II, 0.93 for cements of Type III, and 1.05 for cements of Type IV.

The prediction results are illustrated in Fig. 4. The overall coefficient of variation for the deviations of the predicted compliance values from test data for all the data sets used is $\bar{\omega}$ = 18.0 percent, while for the BP Model $\bar{\omega}$ = 23.1 percent.[2] For the ACI Committee 209 recommendations,[30] the comparable value is 32.4 percent,[2] and for the CEB-FIP Model Code[31] 28.7 percent[2] (see Table 1). For the deviations of the final predicted slopes from the measured slopes, the combined coefficient of variation for all data sets is $\bar{\omega}_f$ = 69 percent, and 71 percent for the BP Model. Calculation of these last values included Ross's data,[26] which conspicuously differ from other data sets. If these data were omitted, $\bar{\omega}_f$ would be 38.8 percent for the log double power law and 58.0 percent for the double power law (see Table 2).

These prediction formulas must be considered rather crude and a more careful study of the prediction problem needs to be carried out in the future, particularly for the effect of concrete composition.