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Log Hölder Continuity of the Integrated Density of States for Stochastic Jacobi Matrices

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Abstract. We consider the integrated density of states, $k(E)$, of a general operator on $\ell_2(\mathbb{Z}^\nu)$ of the form $h = h_0 + v$, where $(h_0 u)(n) = \sum_{|i|=1} u(n+i)$ and $(vu)(n) = v(n)u(n)$, where v is a general bounded ergodic stationary process on $\mathbb{Z}^\nu$. We show that $|k(E) - k(E')| \leqq C[-\log(|E-E'|]^{-1}$ when $|E-E'| \leqq \frac{1}{2}$, The key is a "Thouless formula for the strip."

1. Introduction

In this paper, we discuss general multidimensional stochastic Jacobi matrices. Explicitly, let (Ω, μ, Σ) be a probability measure space on which $\mathbb{Z}^\nu$ acts, that is, ν commuting measure preserving invertible transformations, $T_1, \ldots, T_\nu$ are given. If $n = (n_1, \ldots, n_\nu) \in \mathbb{Z}^\nu$, we let $T^n = T_1^{n_1} \ldots T_1^{n_\nu}$. We suppose that the action is ergodic. Fix a measurable real valued function f on Ω and let $v_\omega(n) \equiv f(T_\omega^n)$. On $\ell_2(\mathbb{Z}^\nu)$ let h_0 be the finite difference Laplacian given by

$$(h_0 u)(n) = \sum_{|\delta|=1} u(n+\delta), \tag{1.1}$$

where the sum is over the 2ν nearest neighbors of n. Let v_ω be the diagonal operator $(v_\omega u)(n) = v_\omega(n) u_\omega(n)$. We consider the operators

$$h_\omega = h_0 + v_\omega. \tag{1.2}$$

In the bulk of this paper, we assume that the function $f(\omega)$ is bounded. In fact our main theorem extends, with minor modifications of the proof, to the case that $\ln(|f|+1)$ is in L^1; these modifications are sketched in Sect. 3.

Examples of interest include the following cases: (a) The periodic case where Ω is finite and each T_i is periodic. (b) The almost periodic case where Ω is a compact metric space and the T's are isometries (see e.g. [3]). (c) The random case where the process $v_\omega(n)$ is a set of independent, identically distributed random variables.

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A basic quantity of interest is the integrated density of states, defined as follows (see Benderskii-Pastur [5]): Let g be a continuous function on $(-\infty,\infty)$ and let $g(h_\omega)(i,j)$ be the matrix elements of the operator $g(h_\omega)$ defined by the functional calculus. By the ergodic theorem, for a.e. ω,

$$\mathscr{L}(g)(\omega)=\lim_{R\to\infty}(2R+1)^{-\nu}\sum_{|n_i|\leqq R}g(h_\omega)(n,n)$$

exists and is independent of ω. Moreover, the a.e. constant value obeys

$$\mathscr{L}(g)=\mathsf{E}(g(h_\omega)(0,0)), \tag{1.3}$$

where E means expectation over the underlying probability measure on Ω and $M(i,j)$ denotes a matrix element of the operator M. In the periodic and almost periodic cases, one can replace a.e. ω by all ω (see [3]). It is not hard to see that $\mathscr{L}$ defines a positive linear functional; therefore there is a measure, dk, called *the density of states measure*, with

$$\mathscr{L}(g)=\int g(E)\mathrm{d}k(E). \tag{1.4}$$

The ergodic theorem implies that in the limit defining the density of states, we can replace $(2R+1)^{-1}\sum_{|n_1|\leqq R}$ by $R^{-1}\sum_{n_1=1}^{R}$ without changing dk.
The integrated density of states, $k(E)$, is the measure that dk assigns to the open interval $(-\infty,E)$. Our main concern here is the continuity of k in E; equivalently, the fact that dk has no pure points. This is somewhat connected with eigenvalues of h_ω:

Proposition 1.1. *$k(E)$ is continuous at $E=E_0$ if and only if the probability that E_0 is an eigenvalue of h_ω is* 0.

Proof. By taking limits in (1.3) (see e.g. [4]), one sees that

$$k(E_0+0)-k(E_0)=\mathsf{E}(\mathsf{P}_{\{E_0\}}(h_\omega)(0,0)), \tag{1.5}$$

where $\mathsf{P}_{\{E_0\}}$ is the projection onto the eigenspace of h_ω for eigenvalue E_0. From (*1*.5), we see that if $\mathsf{P}_{\{E_0\}}=0$ for a.e. ω, then k is continuous. Conversely, since P is positive, if k is continuous, then $\mathsf{P}_{\{E_0\}}(h_\omega)(0,0)=0$ for a.e. ω. By stationarity, $\mathsf{P}_{\{E_0\}}(h_\omega)(n,n)=0$ for all n and thus $\mathrm{Tr}(\mathsf{P}_{\{E_0\}}(h(\omega))=0$ a.e., which implies that $\mathsf{P}_{\{E_0\}}(h_\omega)=0$ a.e. □

This result shows that the continuity of $k(E)$ is not a simple question in dimension $\nu>1$, even in the periodic case. In that case, it is a theorem of Thomas [16] that h_ω has no eigenvalues, but the proof is quite subtle. Our continuity result below implies Thomas' result (in the $\mathbb{Z}^\nu$ case; he also has a Schrödinger operator result) since *in the periodic case*, all h_ω are unitarily equivalent, so E_0 is an eigenvalue of one h_ω if and only if it is an eigenvalue of all h_ω.

We make the following useful definition.

Definition. *A functional k on $(-\infty,\infty)$ is called log Hölder continuous if, for all R,* *there exists* C_R *with*

$$|k(E)-k(E')| \leqq C_R[\log(|E-E'|^{-1})]^{-1}$$

so long as $|E|<R$ *and* $|E-E'|<\frac{1}{2}$.
Our main result in this note is the following:

Theorem 1.2. *Consider an operator of the form (1.2) whose potential* v_ω *is generated by a bounded function f. Then the associated integrated density of states k(E) is log Hölder continuous.*

As already mentioned, with minor modifications, our proof extends to the case where $\ln(|f|+1)$ is in L^1. As we will see shortly, the theorem can be false if f is allowed to be infinite on a set of positive measure. Let us mention some examples which show that our theorem cannot be significantly strengthened:

Example 1. In [6], Craig constructed weakly almost periodic potentials (and Pöschel [11] allows limit periodic potentials in a similar construction) where k is not Hölder continuous of order θ for any $\theta \in (0,1)$. Indeed, for any δ there are examples where $k(E)$ obeys

$$\sup_{|E-E'|=\varepsilon} |k(E)-k(E')| \geqq C(\ln \varepsilon^{-1})^{-1}(\ln(\ln \varepsilon^{-1}))^{-1-\delta}$$

for ε small. This shows that our log Hölder continuity can't be much improved (if at all) and also that the Russman estimates required in [6, 11] cannot be much weakened. Of course, for special classes of potentials, one can hope to improve our results, and such results would be very interesting (see, e.g. Avron et al. [1]).

Example 2. One case where our theorem can be improved is in the random case where the distribution, $d\gamma$, of $v(0)$ has the form $g(\lambda)d\lambda$. In that case, Wegner [18] proved that k is Lipschitz. However, in the general random case, it is likely our result is close to optimal. There is much evidence ([12, 8, 15]) that if $d\gamma(\lambda)=\phi\delta(\lambda-\lambda_0)+(1-\phi)\delta(\lambda-\lambda_1)$, then for any θ, there are suitable $\phi, \lambda_0, \lambda_1$, so that k is only Hölder continuous of order θ.

Example 3. ([2]). In $\nu=1$, if v is random, with the values either 0 with probability $p<1$ or ∞ with probability, $1-p$, it is easy to compute k and see that dk has only pure points. This shows that some finiteness condition on v is required for continuity of k.

Theorem 1.2 has the following corollaries (using Proposition 1.1 and our discussion above of Thomas' theorem);

Corollary 1.3. *Under the hypotheses of Theorem 1.2, for any E, the probability that E is an eigenvalue of* h_ω *is 0.*

Corollary 1.4. *If V is a bounded periodic* $\mathbb{Z}^\nu$*-sequence, then* h_ω *has no eigenvalues.*

The special case of Theorem 1.2 where $\nu=1$ was proven by us in [7]; it is worth recalling that proof since it will motivate us here. We begin with a lemma whose proof we repeat because it is so basic and elementary:

Lemma 1.5.([7]). *Let* $d\mu$ *be a positive measure on* $(-\infty, \infty)$ *of compact support so that for any complex* E *with* $\operatorname{Im} E \neq 0$, *we have*

$$c(E) = \int \ln|E - E'| d\mu(E') \geqq 0. \tag{1.6}$$

Then $\mu(-\infty, E)$ *is log Hölder continuous in* E.

Proof. Let $E_0 \leqq E_1 \leqq E_0 + \frac{1}{2}$. Then

$$0 \leqq c(E_0 + i\varepsilon) = \int \ln|E_0 - E' + i\varepsilon| d\mu(E')$$

$$= \int_{E_0 \leqq E' \leqq E_1} \ln|E_0 - E' + i\varepsilon| d\mu(E') + \int_{\substack{E' \notin \{E_0, E_1\} \\ |E' - E_0| \leqq 1}} \ln|E_0 - E' + i\varepsilon| d\mu(E')$$

$$+ \int_{\substack{E' \notin \{E_0, E_1\} \\ |E' - E_0| \geqq 1}} \ln|E_0 - E' + i\varepsilon| d\mu(E').$$

Letting $\varepsilon \to 0$, and estimating $\ln|E_0 - E'| \geqq \ln|E_0 - E_1|$ in the first integral, we obtain

$$0 \leqq \ln|E_0 - E_1| \cdot \mu([E_0, E_1]) + \int_{|E' - E_0| \geqq 1} \ln|E_0 - E'| d\mu. \tag{1.7}$$

If μ is supported in $(-c, c)$, the last integral is dominated by $\ln(|c| + |E_0| + 1) \times \mu((-\infty, \infty)) \equiv d$. Thus

$$\mu([E_0, E_1]) \leqq d(\ln(E_1 - E_0)^{-1})^{-1},$$

which is the advertised result. □

In [7], we noticed that the Thouless formula says that the $c(E)$ associated to the one dimensional dk is the Lyaponov exponent, $\gamma(E)$, which is of necessity non-negative, so the lemma implies log Hölder continuity. In this paper, we will prove log Hölder continuity of k by proving that its Hilbert transform (given by (1.6)) is positive when $\operatorname{Im} E \neq 0$. Thus, we want to find an object $\gamma(E)$, so that $\gamma(E) \geqq 0$ and so that

$$\gamma(E) = \int \ln|E - E'| dk(E'). \tag{1.8}$$

It is not clear how to directly describe γ in the $\mathbb{Z}^\nu$ case. However, dk is a suitable limit of densities of states associated to strips, i.e. for fixed L, look at an operator on sites n with $-\infty < n_1 < \infty$ and $|n_2|, \ldots, |n_\nu| \leqq L$. This operator has $2(2L+1)^{\nu-1} \equiv 2\alpha$ Lyaponov exponents $\gamma_1 \geqq \gamma_2 \geqq \ldots \geqq \gamma_\alpha \geqq 0 \geqq \gamma_{\alpha+1} \geqq \cdots \geqq \gamma_{2\alpha}$ (as we shall see in Sect. 2), and we will prove (in Sect. 3) that

$$\gamma = \alpha^{-1} \sum_{1}^{\alpha} \gamma_j$$

(which is clearly positive) obeys (1.8), where dk is the density of states for the strip. Taking limits, we will see that the lemma is applicable and thereby prove our basic theorem.

In [7], we also proved log Hölder continuity of the density of states for suitable one dimensional stochastic Schrödinger operators by exploiting the Thouless formula for that case. Independently, Kotani [9] demonstrates a similar one dimensional result. We are currently studying the extension of our result here to ν-dimensional Schrödinger operators, which involves some subtle "renormalization" questions.

2. The Growth Index and Reduction to the Strip

As explained in the introduction, we will prove Theorem 1.2 by studying Jacobi matrices in strips, i.e. operators which are infinite in only one dimension. Accordingly, we begin by studying such operators. Since there is no loss in using greater generality, we do not restrict ourselves to operators which are finite difference in the transverse directions. Explicitly, suppose we have m sites in the transverse direction, so our underlying space is $\ell_2(-\infty,\infty;\mathbb{C}^m)$, i.e. a "wave function" is a sequence $u(n)$ of vectors in $\mathbb{C}^m$. Let $\langle\ ,\ \rangle$ denote the usual sesquilinear product on $\mathbb{C}^m$, so e.g.

$$\langle \bar{u},u\rangle = \sum_1^m u_j v_j$$

(note the $\bar{}$ on u). We begin by noting the analog of the constancy of Wronskian.

Lemma 2.1. *Let u,v solve the equation*

$$\varphi(n+1)+\varphi(n-1)+A(n)\varphi(n)=0, \tag{2.1}$$

where $A(n)$ is, for each n, a symmetric $m\times m$ matrix (perhaps complex and non-selfadjoint). Then

$$\langle \bar{u}(n+1),v(n)\rangle - \langle \bar{u}(n),v(n+1)\rangle \equiv W(n)$$

is constant.

Proof. We note that $W(n)-W(n-1)=\langle \bar{u}(n+1)+\bar{u}(n-1),v(n)\rangle - \langle \bar{u}(n), v(n+1)+v(n-1)\rangle = -\langle \overline{Au}(n),v(n)\rangle + \langle \bar{u}(n),Av(n)\rangle = 0$ by the complex symmetry $(A^t \equiv \bar{A}^* = A)$. □

This lemma can be rephrased in terms of "transfer matrices." Let $\Phi(n)$ be the $2m$ component vector $(u(n+1),\ u(n))$, so that (2.1) is equivalent to $\Phi(n+1)=Q(n+1)\Phi(n)$, where $Q(n)$ is the $2m\times 2m$ block matrix:

$$\left(\begin{array}{c|c} -A(n) & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) = Q(n)$$

Let Γ be the Block matrix $\left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$, so that $W(n)=\langle \Phi(n),\Gamma\Psi(n)\rangle$, where Ψ is built out of v and $\langle\ ,\ \rangle$ is now used for the obvious inner product on $\mathbb{C}^{2m}$. Then Lemma 2.1 is equivalent to

$$Q(n)^t \Gamma Q(n) = \Gamma, \tag{2.2}$$

that is, that $Q(n)$ is a symplectic transformation. Since products of such transformations obey $B^t \Gamma B = \Gamma$ also, the *transfer matrix* $T(n) = Q(n)\ldots Q(1)$ is symplectic.

Lemma 2.2. *The singular values* $s_1,\ldots,s_{2m}$ *of a symplectic transformation obey* $s_j = s_{2m-j+1}^{-1}$.

Proof. If B is symplectic, i.e. $B^t \Gamma B = \Gamma$, then

$$B^{-1} = \Gamma^{-1} B^t \Gamma = \Gamma^t B^t \Gamma, \tag{2.3}$$

since $\Gamma^{-1} = \Gamma^t = -\Gamma$. Taking transposes, we see that B^t is symplectic and so B^* is symplectic since $\bar{\Gamma} = \Gamma$. Thus B^*B is symplectic. Equation (2.3) shows that the square root of a positive symplectic transformation is symplectic. Thus $|B|$ is symplectic. But $|B|$ and $|B|^t$ have the same eigenvalues, so by (2.3) for $|B|$, $|B|^{-1}$ and $|B|$ have the same eigenvalues. Thus $s_1 \geqq \ldots \geqq s_{2m}$ is just a relabeling of $s_1^{-1}\ldots s_{2m}^{-1}$, i.e. $s_j = s_{2m-j+1}^{-1}$ $\square$

Now suppose that $A(1),\ldots,A(n),\ldots$ are such that for any k (with $\bigwedge^k$ = antisymmetric product)

$$\lim_{n\to\infty} \frac{1}{n} \ln \|\textstyle\bigwedge^k T(n)\| \tag{2.4}$$

exists. The Lyaponov exponents are then defined recursively by setting the limit (2.4) to be $\gamma_1 + \cdots + \gamma_k$. Lemma (2.2) immediately implies $\gamma_j + \gamma_{2m-j+1} = 0$, i.e.

Theorem 2.3. *The Lyaponov exponents* $\gamma_1,\ldots,\gamma_{2m}$ *associated to solving* (2.1) *with a symmetric* A, *obey*

$$\gamma_1 \geqq \gamma_2 \cdots \geqq \gamma_m \geqq 0 \geqq \gamma_{m+1} = -\gamma_m \geqq \cdots \geqq \gamma_{2m} = -\gamma_1. \tag{2.5}$$

We define the *growth index*, γ, by

$$\gamma = \frac{1}{m} \sum_{j=1}^{m} \gamma_j, \tag{2.6}$$

the average of the positive γ's. Equivalently

$$m\gamma = \lim_{n\to\infty} \frac{1}{n} \ln \|\textstyle\bigwedge^m T(n)\|. \tag{2.7}$$

We note that by (2.5), this limit is the largest among the limits (2.4).

Now, suppose that $A(n) = W(n) - E\mathbb{1}$, where $W(n)$ is real and symmetric but E might be complex. Consider the operators h_M on $\ell^2(-M,M;\mathbb{C}^m)$

$$(h_M u)(j) = u(j+1) + u(j-1) + W(j)u(j) \tag{2.8}$$

with the boundary conditions $u(\pm(M+1))=0$. If

$$\lim_{M\to\infty}\frac{1}{mM}\mathrm{Tr}(f(h_M))$$

exists for all $f\in C_0(\mathbb{R})$, we say the density of states exists; we define the measure $dk_{(m)}$ by setting the limit to $\int f(E)dk_{(m)}(E)$. In the next section we will prove that for bounded matrices W;

Theorem 2.4. *If the density of states exists, then so does* $\gamma(E)=\lim(1/mn)\ln\|\bigwedge{}^m(T_E(n))\|$ *whenever* $\mathrm{Im}\,E\neq 0$. *For such* E:

$$\gamma(E)=\int\ln|E-E'|dk_{(m)}(E).$$

Assuming this result for the moment, let us prove Theorem 1.2.

Proof of Theorem 1.2 Fix L and look at the operator on $\{\alpha|-\infty<\alpha_1<\infty;|\alpha_2|\leqq L,\ldots|\alpha_\nu|\leqq L\}$. It has the form (2.8) with $m=(2L+1)^{\nu-1}$. Moreover, by the ergodic theorem, for a.e. ω, the limit $dk^{\omega}_{(m)}$ exists (but it may be ω dependent if T_1 is not ergodic by itself). Thus, since $\gamma(E)$ is trivially non-negative: $\int\ln|E-E'|dk^{\omega}_{(m)}(E)\geqq 0$, if $\mathrm{Im}\,E\neq 0$. By the ergodic theorem again, $dk^{\omega}_{(m)}$ converges weakly a.e. ω as $L\to\infty$, to the density of states dk for the infinite $\mathbb{Z}^\nu$ problem (and now there is independence, since we assume ergodicity of $\{T_j\}_{j=1}^{\nu}$). Thus, since $\ln|E-E'|$ is a continuous function for $\mathrm{Im}\,E\neq 0$ and the $dk^{\omega}_{(m)}(E)$ have a common bounded support, we have $\int\ln|E-E'|dk(E)\geqq 0$. By Lemma 1.5, k is log-Hölder continuous. □

We end this section with some remarks about Lyaponov indices in the strip:

(1) By the argument in [7], the quantities $\gamma_1(E)+\cdots+\gamma_k(E)=\Gamma_k(E)$ (which exist for E fixed for a.e. ω by the subadditive ergodic theorem) are subharmonic. It would be interesting to know if they are harmonic for E non-real. For $k=m$, we of course prove harmonicity. If $\Gamma_k(E)=\int\ln|E-E'|d\mu_k(E')$ for some μ_k, what is the interpretation of μ_k?

(2) By the argument of Pastur [10], if $\gamma_m(E)>0$ for E in some subset A of $\mathbb{R}$ then there is no a.c. spectrum on A for the strip operator. For the one dimensional case, there is a converse due to Kotani [9] (see [13] for the discrete case). Does this extend? Specifically, is σ_{ac} the essential closure of $\{E|\gamma_m(E)=0,E\text{ real}\}$? Note from this point of view, it is γ_m and not the growth index γ which is most important.

3. The Thouless Formula for the Strip

Our goal in this section is to prove Theorem 2.4 and thereby complete the proof of Theorem 1.2. The proof is patterned after the one Thouless gave in case $m=1$ [17] but there is an extra complication: In the case $m=1$, $\mathrm{Im}\,E\neq 0$, one can show that each and every matrix element of $T_E(n)$ grows at the same exponential rate. For $m\neq 1$, it is no longer true that every matrix element of $\bigwedge^m T_E(n)$ grows at the same rate:

Example. Take $m=4$ and

$$A(n)=\begin{pmatrix} V(n) & 1 & 0 & 1 \\ 1 & V(n) & 1 & 0 \\ 0 & 1 & V(n) & 1 \\ 1 & 0 & 1 & V(n) \end{pmatrix},$$

where the $V(n)$ are independent identically distributed random variables. The $A(n)$ arise precisely from a Jacobi matrix in the strip with periodic boundary conditions on the strip edge. Since the diagonal matrix elements are all equal, by a standard plane wave analysis, the transfer matrix breaks up in a direct sum of 4 distinct 2×2 blocks with corresponding Lyaponov exponents, $\pm\gamma_1$, $\pm\gamma_2$, $\pm\gamma_3$, $\pm\gamma_4$. Here $\gamma_1>0$ and $\gamma_2=\gamma_3$ corresponding to the fact that the plane wave $(1, i, -1, -i)$ and $(1, -i, -1, i)$ yield equivalent transfer matrices. If we start out with initial data of the form $u(0)=(a,0,-a,0)$, $u(1)=(b,0,-b,0)$, then $u(n)=(c(n),0,-c(n),0)$ for all n. It follows that the space given by $(u(0),u(1))=(a_1,0,a_2,0;\ b_1,0,b_2,0)\equiv a_1\mathbf{e}_1+a_2\mathbf{e}_3+b_1\mathbf{d}_1+b_2\mathbf{d}_3$ has two eigenvectors for the transfer matrix with eigenvalues acting as $e^{\pm\gamma_2}$, and thus the rate of growth of a matrix element of $\bigwedge^4 T$ of the form $(\mathbf{e}_1\wedge\mathbf{e}_3\wedge\mathbf{d}_1\wedge\mathbf{d}_3,\ \bigwedge^4 T(n)(\mathbf{e}_1\wedge\mathbf{e}_3\wedge\mathbf{d}_1\wedge\mathbf{d}_3))$ is at most $e^{n\alpha}$, where $\alpha=\max(\gamma_i+\gamma_j)<\gamma=\gamma_1+\gamma_2+\gamma_3+\gamma_4$. While this phenomenon of slower growth of certain matrix elements is not typical, the fact that it occurs shows that the proof must have an extra element beyond that of Thouless.

Let $\mathbf{e}_1,\dots,\mathbf{e}_m,\mathbf{d}_1,\dots,\mathbf{d}_m$ denote the standard basis of the space on which $T(n)$ acts. Thus $T(n)(\sum\alpha_i\mathbf{e}_i+\beta_j\mathbf{d}_j)$ represents $(u(n+1),u(n))$ for the solution of (2.1) with $u(0)=\alpha$, $u(1)=\beta$. We begin by identifying the zeros of

$$(\mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_a}\wedge\mathbf{d}_{j_1}\cdots\wedge\mathbf{d}_{j_{m-a}},\ \textstyle\bigwedge^m(T_E(n))(\mathbf{e}_{l_1}\wedge\cdots\mathbf{e}_{l_b}\wedge\mathbf{d}_{k_1}\wedge\cdots\wedge\mathbf{d}_{k_{m-b}}). \tag{3.1}$$

If P, Q are the projections onto the span of $\mathbf{e}_{i_1},\dots,\mathbf{d}_{j_1},\dots$ and $\mathbf{e}_{\ell_1},\dots,\mathbf{d}_{k_1},\dots$, then the above quantity is precisely $\det(\mathrm{P}\bigwedge^m(T)\mathrm{Q})$, viewed as an operator from $\mathrm{Q}\mathbb{R}^{2m}$ to $\mathrm{P}\mathbb{R}^{2m}$. This is zero if and only if there exists $\Phi\in\mathrm{Ran\,Q}$ with $\mathrm{P}\bigwedge^m(T)\Phi=0$. We therefore see that

Proposition 3.1. *(3.1) is zero if and only if (2.1) has a solution obeying* $u_\ell(0)=0$ *if* $\ell\neq\ell_1,\dots,\ell_b$; $u_k(1)=0$ $k\neq k_1,\dots,k_{m-b}$; $u_i(n)=0$ *if* $i=i_1,\dots,i_a$; $u_j(n+1)=0$ *if* $j=j_1,\dots j_{n-a}$.

This proposition is particularly useful in case $a=b=0$. It shows that $P(E)\equiv(\mathbf{d}_1\wedge\cdots\wedge\mathbf{d}_m,\ \bigwedge^m(T_E(n))\mathbf{d}_1\wedge\cdots\mathbf{d}_m)=0$ exactly when E is an eigenvalue of the operator on $\ell_2(1,\dots,m)$ with boundary values $u(0)=u(n+1)=0$. Here $P(E)$ can be seen to be a monic polynomial of degree mn and thus the Thouless argument [17] (see also [4]) immediately implies that if $\mathrm{Im}\,E\neq 0$:

$$\lim_{n\to\infty}\frac{1}{mn}\ln|P(E)|=\int\ln|E-E'|\,dk_{(m)}(E'),$$

showing that $\gamma(E)\geqq\int\ln|E-E'|\,dk_{(m)}(E')$.

We remark that for other choices of a, b and the sequences $i_1,\dots i_a$, $j_1\dots j_{m-a}$, the boundary value problem for the strip is in general non-selfadjoint.

As the example above shows, we should not directly try to control all matrix elements of $\wedge^m(T)$. Rather, we will analyze enough linear combinations of matrix elements.

Proposition 3.2. *Suppose that for any choice of vectors* $\mathbf{d}_1^\flat, \ldots \mathbf{d}_m^\flat, \mathbf{d}_1^\#, \ldots \mathbf{d}_m^\#$ *in the span of the* $\mathbf{d}$'s

$$\overline{\lim_{n\to\infty}} \frac{1}{nm} \ln|((\mathbf{e}_1 + \mathbf{d}_1^\#) \wedge \cdots \wedge (\mathbf{e}_m + \mathbf{d}_m^\#),\ \wedge^m T_E(n)[(\mathbf{e}_1 + \mathbf{d}_1^\flat) \wedge \cdots \wedge (\mathbf{e}_m + \mathbf{d}_m^\flat)])|$$

$$\leqq \int \ln|E - E'|\mathrm{d}k_{(m)}(E'). \tag{3.2}$$

Then Theorem 2.4 holds.

Proof. It is easy to see that $\{(\mathbf{e}_1 + \mathbf{d}_1^\flat) \wedge \cdots \wedge (\mathbf{e}_m + \mathbf{d}_m^\flat)\}$ span $\wedge^m(\mathbb{R}^m)$. Moreover, for any finite spanning sets S_1 and S_2 in a vector space, we have for all A

$$\sup_{\varphi\in S_1, \varphi\in S_2} |(\varphi, A\psi)| \geqq c\|A\|$$

for some c. Thus, the hypothesis shows that $\gamma(E) \leqq \int \ln|E - E'|\mathrm{d}k$. We already noted the opposite inequality above. □

Given E fixed and $\mathbf{d}_1^\flat, \ldots, \mathbf{d}_n^\flat$ we can find $\tilde{W}(0)$, a non-necessarily symmetric matrix on $\mathbb{R}^n$, so that $\tilde{Q}(0) = \begin{pmatrix} E - \tilde{W}(0) & -1 \\ 1 & 0 \end{pmatrix}$ maps $\mathbf{d}_i$ to $\mathbf{d}_i^\flat + \mathbf{e}_i$ and similarly, we can find $\tilde{W}(n+1)$ so $\tilde{Q}(n+1)^t$ (note the transpose) maps $\mathbf{d}_i$ to $-(\mathbf{d}_i^\# + \mathbf{e}_i)$. Then

$$|((\mathbf{e}_1 + \mathbf{d}_1^\#) \wedge \cdots \wedge (\mathbf{e}_m + \mathbf{d}_m^\#), \wedge^m T_E(n)[(\mathbf{e}_1 + \mathbf{d}_1^\flat) \wedge \cdots \wedge (\mathbf{e}_m + \mathbf{d}_m^\flat)])|$$
$$= |(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_m, \wedge^m[Q(n+1)T_E(n)Q(0)]\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_m)|,$$

so, by the Thouless argument, the log is just

$$\sum_{j=1}^{n+2} \ln|E - \tilde{E}_j^n| \equiv (n+2) \int \ln|E - E'|\mathrm{d}k_{(m);n}(E'),$$

where $\tilde{E}_j^n$ are the eigenvalues of the operator on $\ell^2(-1, N+1; \mathbb{C}^m)$ with "potential" $\tilde{W}(0)$ and $\tilde{W}(n+1)$ at the ends, vanishing boundary conditions at -2 and $n+2$ and the usual potential at $0, 1, \ldots, n$. The required inequality (3.2), and thus Theorem 2.4, then follows from the following two lemmas.

Lemma 3.3. *As* $n \to \infty$, $\mathrm{d}\tilde{k}_{(m);n}$ *as a measure on* $\mathbb{C}$ *converges weakly to* $\mathrm{d}k_{(m)}$.

Lemma 3.4. *If* $\mathrm{d}\mu_n$ *and* $\mathrm{d}\mu$ *are measures on* $\mathbb{C}$ *supported in* $\{t \mid |t| \leqq R\}$ *for some* R, *and* $\mathrm{d}\mu_n \to \mathrm{d}\mu$ *weakly, then for any* E

$$\overline{\lim_{n\to\infty}} \int \ln|E - E'|\mathrm{d}\mu_n(E') \leqq \int \ln|E - E'|\mathrm{d}\mu(E').$$

Proof of Lemma 3.4. Let $f_\varepsilon(E') = \ln|E - E'|$ if $|E - E'| \geqq \varepsilon$ and $\ln|\varepsilon|$ if $|E' - E| \leqq \varepsilon$. By the weak convergence and the support hypothesis,

$$\int f_\varepsilon(E')\mathrm{d}\mu_n(E') \to \int f_\varepsilon(E')\mathrm{d}\mu(E').$$

But $\ln|E-E'| \leqq f_\varepsilon(E')$, so

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int \ln|E-E'|\mathrm{d}\mu(E') &\leqq \overline{\lim_{n\to\infty}} \int f_\varepsilon(E')\mathrm{d}\mu_n(E') \\ &= \int f_\varepsilon(E')\mathrm{d}\mu_n(E').\end{aligned}$$

Now take $\varepsilon \downarrow 0$ using monotone convergence to obtain the desired result. □

Proof of Lemma 3.3. Let $\tilde{E}_j$ be the eigenvalues of the problem defining $\mathrm{d}k$, and let E_j be the eigenvalues of the problem with the correct potential at 0 and $n+1$. We need only show that

$$\frac{1}{n+2}\left(\sum_j E_j^{\ell+m} - \sum_j \tilde{E}_j^\ell \bar{\tilde{E}}_j^m\right) \to 0, \tag{3.3}$$

or equivalently, (3.3) for $m=0$ and

$$\frac{1}{n+2}\left(\sum_j \tilde{E}_j^\ell[\bar{\tilde{E}}_j^m - \tilde{E}_j^m]\right) \to 0. \tag{3.4}$$

When $m=0$, (3.3) just says that $(1/(n+2))\mathrm{Tr}(H_n^\ell - \tilde{H}_n^\ell) \to 0$, which is immediate since diagonal matrix elements are equal for all sites at least a distance ℓ from the ends and matrix elements are bounded. To prove (3.4), we proceed as follows: Since $\tilde{E}$ is bounded and $|z^m - \bar{z}^m| \leqq |z-\bar{z}|m(|z|+|\bar{z}|)^{m-1}$, (3.4) follows from $\frac{1}{n+2}\sum_j |\mathrm{Im}\tilde{E}_j| \to 0$, which follows (by Schwarz) from

$$\frac{1}{n+2}\sum_j |\mathrm{Im}\tilde{E}_j|^2 \to 0. \tag{3.5}$$

There exists an orthonormal basis ("Schur basis"; see e.g. [14]), so that $\tilde{E}_j = (\varphi_j, \tilde{H}\varphi_j)$. Thus

$$\sum_j |\mathrm{Im}\tilde{E}_j|^2 \leqq \sum_j \|\mathrm{Im}\tilde{H}\varphi_j\|^2 = \mathrm{Tr}((\mathrm{Im}\tilde{H})^2)$$

is bounded, so (3.5) holds. □

We end this section with a series of remarks outlining the proof of Theorem 1.2, weakening the hypothesis that f be bounded to the requirement that f satisfy $\mathsf{E}(\ln(|f|+1)) < \infty$. Recall that the potential is defined by $v_\omega(n) \equiv f(T^n\omega)$.

(1) The condition is natural, since if $\mathrm{d}k_\varepsilon(E)$ denotes the density of states for the operator $\varepsilon h_0 + v_\omega$, then $\int \ln(|E'|+1)\mathrm{d}k_0(E') = \ (\ln(|f|+1))$. Furthermore, since $-2v + v_\omega \leqq h_0 + v_\omega \leqq 2v + v_\omega$, for any bounded monotone function g,

$$\int g(E'-2v)\mathrm{d}k_0(E') \leqq \int g(E')\mathrm{d}k(E') \leqq \int g(E'+2v)\mathrm{d}k_0(E').$$

It follows that

$$\int \ln(|E'|+1)\mathrm{d}k(E') < \infty. \tag{3.6}$$

(2) The proof of Lemma 1.5 must be modified to exclude the conditions on the boundedness of the support of $d\mu$. It suffices to bound the integral in (1.7) by $\int \ln(|E'|+1)d\mu(E') < \infty$. Hence the extended version of Theorem 1.2 will follow once the positivity of $\int \ln|E-E'|dk(E')$, $\operatorname{Im} E > 0$, is established.

(3) We may make approximations to the integral $\int \ln|E-E'|dk(E')$ by modifying $f(\omega)$ as follows: Let

$$f_M(\omega) = \begin{cases} f(\omega) & \text{if } |f(\omega)| \leqq M \\ M & \text{if } f(\omega) > M \\ -M & \text{if } f(\omega) < -M. \end{cases}$$

Theorem 1.2 implies that $\int \ln|E-E'|dk_M(E') \geqq 0$, so we need only show that $\lim\limits_{M\to\infty} \int \ln|E-E'|(dk(E')-dk_M(E'))=0$. The bound (3.6) allows for each fixed E a choice of cutoff R, uniformly in $M\to\infty$, such that $\left|\int\limits_{|E'|>R} \ln|E-E'|(dk(E')-dk_M(E'))\right| < \varepsilon$. Thus we only need address the weak convergence of the measure $\chi_{[-R,R]}(dk(E')-dk_M(E'))$.

(4) We now address the weak convergence of $(dk_M - dk)$ by first restricting the operator to a finite region $\Lambda \subset \mathbb{Z}^\nu$, and then taking limits. Denoting h^Λ and h_n^Λ the operators with potential v and v_n respectively, restricted to the region Λ, we have

$$\begin{aligned} \mathsf{E}\left(\frac{1}{|\Lambda|}\left|\operatorname{Tr}\left(\frac{1}{h_M^\Lambda+i}-\frac{1}{h^\Lambda+i}\right)\right|\right) &= \mathsf{E}\left(\frac{1}{|\Lambda|}\left|\operatorname{Tr}\left(\frac{1}{h_M^\Lambda+i}(v-v_M)\frac{1}{h^\Lambda+i}\right)\right|\right) \\ &\leqq \text{const}\, \mathsf{E}\left(\frac{1}{|\Lambda|}|\operatorname{Tr}\chi_\Lambda(v-v_M)|\right) \\ &\leqq \text{const}\left|\int (dk_{0M}(E')-dk_0(E'))\right|, \end{aligned}$$

which tends to zero as $M\to\infty$. This implies the weak convergence of $(dk_M - dk)$ to zero.

(5) For simplicity, the above procedure avoids proving the Thouless formula for the strip for unbounded potentials, but with straightforward estimates and the bound $(\ln(|f|+1))$ this can be done as well.

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