LOGARITHMIC CONFORMAL FIELD THEORY: BEYOND AN INTRODUCTION

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ABSTRACT. This article aims to review a selection of central topics and examples in logarithmic conformal field theory. It begins with the remarkable observation of Cardy that the horizontal crossing probability of critical percolation may be computed analytically within the formalism of boundary conformal field theory. Cardy's derivation relies on certain implicit assumptions which are shown to lead inexhorably to indecomposable modules and logarithmic singularities in correlators. For this, a short introduction of the fusion algorithm of Nahm, Gaberdiel and Kausch is provided.

While the percolation logarithmic conformal field theory is still not completely understood, there are several examples for which the formalism familiar from rational conformal field theory, including bulk partition functions, correlation functions, modular transformations, fusion rules and the Verlinde formulae, has been successfully generalised. This is illustrated for three examples: The singlet model $\mathfrak{M}(1,2)$, related to the triplet model $\mathfrak{W}(1,2)$, symplectic fermions and the fermionic *bc* ghost system; the fractional level Wess-Zumino-Witten model based on $\widehat{\mathfrak{sl}}(2)$ at $k=-\frac{1}{2}$, related to the bosonic $\beta\gamma$ ghost system; and the Wess-Zumino-Witten model for the Lie supergroup $\mathsf{GL}(1|1)$, related to $\mathsf{SL}(2|1)$ at $k=-\frac{1}{2}$ and 1, the Bershadsky-Polyakov algebra $W_3^{(2)}$ and the Feigin-Semikhatov algebras $W_n^{(2)}$. These examples have been chosen because they represent the most accessible, and most useful, members of the three best-understood families of logarithmic conformal field theories: The logarithmic minimal models $\mathfrak{W}(q,p)$, the fractional level Wess-Zumino-Witten models, and the Wess-Zumino-Witten models on Lie supergroups (excluding $\mathsf{OSP}(1|2n)$).

In this review, the emphasis lies on the representation theory of the underlying chiral algebra and the modular data pertaining to the characters of the representations. Each of the archetypal logarithmic conformal field theories is studied here by first determining its irreducible spectrum, which turns out to be continuous, as well as a selection of natural reducible, but indecomposable, modules. This is followed by a detailed description of how to obtain character formulae for each irreducible, a derivation of the action of the modular group on the characters, and an application of the Verlinde formula to compute the Grothendieck fusion rules. In each case, the (genuine) fusion rules are known, so comparisons can be made and favourable conclusions drawn. In addition, each example admits an infinite set of simple currents, hence extended symmetry algebras may be constructed and a series of bulk modular invariants computed. The spectra of the extended theories is typically discrete and this is how the triplet model $\mathfrak{W}(1,2)$ arises, for example. Moreover, simple current technology admits a derivation of the extended algebra fusion rules from those of its continuous parent theory. Finally, each example is concluded by a brief description of the computation of some bulk correlators, a discussion of the structure of the bulk state space, and remarks concerning more advanced developments and generalisations.

The final part gives a very short account of the theory of staggered modules, the (simplest class of) representations that are responsible for the logarithmic singularities in correlators which distinguish logarithmic conformal field theory from its rational cousin. Staggered modules are discussed in a generality suitable to encompass all the examples met in this review and some of the very basic structure theory is proven. Then, the important quantities known as logarithmic couplings are reviewed for Virasoro staggered modules and their role as fundamentally important parameters, akin to the three-point constants of rational conformal field theory, is discussed. An appendix is also provided in order to introduce some of the necessary, but perhaps unfamiliar, language of homological algebra.

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CONTENTS

1. INTRODUCTION

Ever since the pioneering work of Belavin, Polyakov and Zamolodchikov [1], two-dimensional conformal field theory has been at the forefront of much of the progress in modern mathematical physics. Its application to the study of critical statistical models and string theory is well known, see [2–5] for example, but it also provides the basic inspiration for the mathematical theory of vertex operator algebras [6–8]. The simplest conformal field theories are constructed mathematically from irreducible representations of an infinite-dimensional symmetry algebra. However, recent attention to non-local observables for statistical models and string theories with fermionic degrees of freedom has led to the conclusion that the corresponding field-theoretic models require, in addition, certain reducible, but indecomposable, representations. Such models have come to be known as *logarithmic conformal field theories* because the type of indecomposability required leads to logarithmic singularities in correlation functions.

As a field of study, logarithmic conformal field theory dates back to the works of Rozansky and Saleur on the $\mathsf{U}(1|1)$ (or perhaps $\mathsf{GL}(1|1)$) Wess-Zumino-Witten model [9, 10] and that of Gurarie on a fermionic ghost system [11] related to the theory now known as symplectic fermions. Since then, things have progressed rather rapidly with many of the standard features of rational conformal field theory now understood in the logarithmic setting. In particular, there are two fine reviews of the subject [12, 13] which focus on, among other things, modular transformations and module structure, mostly for a family of theories related to symplectic fermions.

Both reviews are accounts of lectures given at a workshop in 2001. The present review aims to build upon the state of knowledge summarised there, introducing the reader to some of the recent advances that seem to be converging towards a more unified picture of logarithmic conformal field theory. Unfortunately, a detailed overview would require a rather lengthy book, hence we will restrict ourselves to foundational material and in-depth examples which we believe, hopefully without controversy, are "archetypes" for the discipline. We hope that our choice will give the reader a good overview of what structures logarithmic conformal field theory relies upon and what one can do with it.

In particular, we work almost entirely in the continuum, expecting that the reader is familiar with rational conformal field theory as described (for example) in [14], eschewing approaches based on statistical lattice models and conjectured scaling limits (see [15–18]). We also work, for the most part, with chiral algebras, even though it is well known that the holomorphic factorisation principle of rational conformal field theory fails in the logarithmic setting. Instead, we will see how a natural proposal allows one to construct physically satisfactory non-chiral fields, even when logarithmic behaviour is present. For other approaches to logarithmic conformal field theory, as well as condensed matter physics and string-theoretic applications, discussions of logarithmic vertex operator algebras and other relations to mathematics, we refer to the other articles that constitute this special issue of the Journal of Physics A.

We will outline what we cover in this review shortly. First however, we quickly remind the reader how logarithmic singularities arise in correlation functions as consequences of a non-diagonalisable action of the Virasoro zero-mode L_0. Then, we digress slightly in recalling the (non-logarithmic) theory known as the free boson, in particular, its characters, their modular transformations and the relation between these and the fusion rules (the Verlinde formula [19]). This is in order to set the scene for the analysis of the "archetypal" logarithmic theories that follow. We also mention simple currents for the free boson and the corresponding extended algebra theories as these ideas are also going to play an important role for us.

1.1. **Correlators and Logarithmic Singularities.**

Conformal field theory is relatively tractable among physical models due to its infinite-dimensional algebra of symmetries. As is well-known, this always includes the Virasoro algebra, the infinite-dimensional Lie algebra spanned by modes L_n, $n \in \mathbb{Z}$, and C with commutation relations

$$\big[L_m, L_n\big] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n=0}C, \qquad \big[L_m, C\big] = 0. \tag{1.1}$$

The central mode C will act on all representations as a fixed multiple of the identity, known as the central charge c. We will identify C with c in what follows. The field-theoretic version of these commutation relations is the operator product expansion

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.2}$$

in which the energy-momentum tensor is related to the Virasoro modes by $T\big(z\big) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

In this section, we recall how the global conformal invariance of the vacuum $\big|0\big\rangle$, meaning its annihilation by L_{-1}, L_0 and L_1, fixes the two-point functions of (chiral) fields and gives rise to logarithmic singularities when the

corresponding Virasoro representations admit a non-diagonalisable action of L_0. Given any chiral field $\phi(z)$, the natural action of the Virasoro modes is given by

$$\big[L_n,\phi(w)\big] = \oint_w T(z)\phi(w)z^{n+1}\,\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{1.3}$$

If $\phi(z)$ is a chiral primary field of conformal weight h, then this action gives

$$\big[L_{-1},\phi(w)\big] = \partial\phi(w), \quad \big[L_0,\phi(w)\big] = h\phi(w)+w\partial\phi(w), \quad \big[L_1,\phi(w)\big] = 2hw\phi(w)+w^2\partial\phi(w) \tag{1.4}$$

and the invariance of $\big|0\big\rangle$ then leads to the following differential equations for the two-point functions:

$$\begin{gathered}(\partial_z+\partial_w)\big\langle\phi(z)\phi(w)\big\rangle = 0, \qquad (z\partial_z+w\partial_w+2h)\big\langle\phi(z)\phi(w)\big\rangle = 0,\\ \big(z^2\partial_z+w^2\partial_w+2h(z+w)\big)\big\langle\phi(z)\phi(w)\big\rangle = 0.\end{gathered} \tag{1.5}$$

It is a straight-forward exercise to show that the general solution of these equations has the form

$$\big\langle\phi(z)\phi(w)\big\rangle = \frac{A}{(z-w)^{2h}}, \tag{1.6}$$

for some constant A (which could be zero). We may identify A with $\big\langle\phi\big|\phi\big\rangle$.

So far, we have repeated a standard textbook computation (see [14] for example). We now ask what happens if the primary field $\phi(z)$ corresponds to a state $\big|\phi\big\rangle$ which has a *Jordan partner* $\big|\Phi\big\rangle$ under the L_0-action: $L_0\big|\Phi\big\rangle = h\big|\Phi\big\rangle + \big|\phi\big\rangle$. Then, the constant A in (1.6) is

$$A = \big\langle\phi\big|\phi\big\rangle = \big\langle\phi\big|L_0-h\big|\Phi\big\rangle = 0, \tag{1.7}$$

since $L_0\big|\phi\big\rangle = h\big|\phi\big\rangle$. Moreover, the partner field $\Phi(z)$ has the operator product expansion[1]

$$T(z)\Phi(w) \sim \frac{h\Phi(w)+\phi(w)}{(z-w)^2}+\frac{\partial\Phi(w)}{z-w}, \tag{1.8}$$

so that the Virasoro modes act as

$$\begin{gathered}\big[L_{-1},\Phi(w)\big] = \partial\Phi(w), \qquad \big[L_0,\Phi(w)\big] = h\Phi(w)+w\partial\Phi(w)+\phi(w),\\ \big[L_1,\Phi(w)\big] = 2hw\Phi(w)+w^2\partial\Phi(w)+2w\phi(w).\end{gathered} \tag{1.9}$$

We therefore obtain a set of inhomogeneous differential equations for the two-point functions:

$$\begin{gathered}(\partial_z+\partial_w)\big\langle\phi(z)\Phi(w)\big\rangle = 0, \qquad (z\partial_z+w\partial_w+2h)\big\langle\phi(z)\Phi(w)\big\rangle = -\big\langle\phi(z)\phi(w)\big\rangle,\\ \big(z^2\partial_z+w^2\partial_w+2h(z+w)\big)\big\langle\phi(z)\Phi(w)\big\rangle = -2w\big\langle\phi(z)\phi(w)\big\rangle,\\ (\partial_z+\partial_w)\big\langle\Phi(z)\Phi(w)\big\rangle, \qquad (z\partial_z+w\partial_w+2h)\big\langle\Phi(z)\Phi(w)\big\rangle = -\big\langle\Phi(z)\phi(w)\big\rangle-\big\langle\phi(z)\Phi(w)\big\rangle,\\ \big(z^2\partial_z+w^2\partial_w+2h(z+w)\big)\big\langle\Phi(z)\Phi(w)\big\rangle = -2z\big\langle\phi(z)\Phi(w)\big\rangle-2w\big\langle\Phi(z)\phi(w)\big\rangle.\end{gathered} \tag{1.10}$$

If we assume that $\phi(z)$ and $\Phi(z)$ are mutually bosonic, meaning that $\big\langle\phi(z)\Phi(w)\big\rangle = \big\langle\Phi(z)\phi(w)\big\rangle$, then solving these equations leads to two-point functions of the form

$$\big\langle\phi(z)\phi(w)\big\rangle = 0, \qquad \big\langle\phi(z)\Phi(w)\big\rangle = \frac{B}{(z-w)^{2h}}, \qquad \big\langle\Phi(z)\Phi(w)\big\rangle = \frac{C-2B\log(z-w)}{(z-w)^{2h}}, \tag{1.11}$$

where B and C are constants. This demonstrates that combining global conformal invariance with a non-diagonalisable L_0-action leads to logarithmic singularities in correlation functions.

We remark that $\Phi(z)$ is not uniquely specified because we may, for example, add a multiple of $\phi(z)$ to $\Phi(z)$ without affecting the latter's defining properties. However, adding such a multiple will change the constant C in (1.11), though B will remain invariant. Because of this, C may be tuned to any desired value, so is not expected to be physical. The constant $B = \big\langle\phi\big|\Phi\big\rangle$, on the other hand, is expected to be physically meaningful.

1.2. **The Free Boson.** The free boson is a $c=1$ conformal field theory with chiral algebra $\widehat{\mathfrak{gl}}(1) = \widehat{\mathfrak{u}}(1)$ generated by modes a_n, $n\in\mathbb{Z}$, and a central element K:

$$\big[a_m,a_n\big] = m\delta_{m+n,0}K. \tag{1.12}$$

As usual, K is identified with a real number k times the identity when acting on representations and the Virasoro modes L_n then follow from the standard Sugawara construction. Moreover, when a highest weight state in such a

[1]Here, we assume for simplicity that $L_n\big|\Phi\big\rangle = 0$ for all $n>0$. See Sections 2.4 and 6.3 for a more general discussion.

representation has weight (a_0-eigenvalue) λ, its conformal dimension is $\lambda^2/2k$. Note that the algebra for $k \neq 0$ is almost always rescaled via $a_m \to a_m/\sqrt{k}$ so as to set k to 1.[2]

The irreducible ($k = 1$) highest weight modules $\mathcal{F}_\lambda$, called Fock spaces, have characters given by

$$\mathrm{ch}\big[\mathcal{F}_\lambda\big](q) = \mathrm{tr}_{\mathcal{F}_\lambda} q^{L_0 - c/24} = \frac{q^{\lambda^2/2}}{\eta(q)}. \tag{1.13}$$

It is well known (see [20] for example) that the S-transformations of these characters amount to a Fourier transform and that one can recover non-negative integer fusion multiplicities from them using a continuum version of the Verlinde formula. The only problem with this is that the characters (1.13) do not completely distinguish the irreducible modules: $\mathrm{ch}\big[\mathcal{F}_\lambda\big]$ and $\mathrm{ch}\big[\mathcal{F}_{-\lambda}\big]$ are identical. Consequently, the application of the Verlinde formula cannot, strictly speaking, reproduce the structure constants of the fusion ring, but only of a quotient of the fusion ring by an action of the two-element group $\mathbb{Z}_2$.

The obvious fix is to include the affine weight λ in the character. Of course, then the S-transformation will produce an unwanted factor for which the standard remedy is to include k in the character. In this way, we arrive at the full character (for general k):

$$\mathrm{ch}\big[\mathcal{F}_\lambda\big](y;z;q) = \mathrm{tr}_{\mathcal{F}_\lambda} y^k z^{a_0} q^{L_0 - c/24} = \frac{y^k z^\lambda q^{\lambda^2/2k}}{\eta(q)}. \tag{1.14}$$

Writing $y = \mathrm{e}^{2\pi \mathrm{i} t}$, $z = \mathrm{e}^{2\pi \mathrm{i} u}$ and $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, the modular S-transformation of the characters (1.14) acts via $\mathsf{S}\colon (t|u|\tau) \to \big(t - u^2/2\tau\big|u/\tau\big|-1/\tau\big)$, leading to

$$\mathrm{ch}\big[\mathcal{F}_\lambda\big]\Big|_{\mathsf{S}} = \int_{-\infty}^{\infty} \mathsf{S}_{\lambda\mu}\mathrm{ch}\big[\mathcal{F}_\mu\big]\,\frac{\mathrm{d}\mu}{\sqrt{k}}, \qquad \mathsf{S}_{\lambda\mu} = \mathrm{e}^{-2\pi\mathrm{i}\lambda\mu/k}. \tag{1.15}$$

This follows from a standard gaussian integration, convergent for $k > 0$ (when $k < 0$, we have to assume the standard result through an analytic continuation):

$$\int_{-\infty}^{\infty} \mathsf{S}_{\lambda\mu}\mathrm{ch}\big[\mathcal{F}_\mu\big]\,\frac{\mathrm{d}\mu}{\sqrt{k}} = \frac{\mathrm{e}^{2\pi\mathrm{i}kt}}{\eta(\tau)} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi\tau\mu^2/k + 2\pi\mathrm{i}(u-\lambda/k)\mu}\,\frac{\mathrm{d}\mu}{\sqrt{k}} = \frac{\mathrm{e}^{2\pi\mathrm{i}kt - \mathrm{i}\pi k(u-\lambda/k)^2/\tau}}{\sqrt{-\mathrm{i}\tau}\eta(\tau)} = \mathrm{ch}\big[\mathcal{F}_\lambda\big]\Big|_{\mathsf{S}}. \tag{1.16}$$

We remark that the measure $\mathrm{d}\mu/\sqrt{k}$ is natural given the rescaling property of the a_n.

For $\mathsf{T}\colon (t|u|\tau) \to (t|u|\tau+1)$, the transformation is

$$\mathrm{ch}\big[\mathcal{F}_\lambda\big]\Big|_{\mathsf{T}} = \int_{-\infty}^{\infty} \mathsf{T}_{\lambda\mu}\mathrm{ch}\big[\mathcal{F}_\mu\big]\,\frac{\mathrm{d}\mu}{\sqrt{k}}, \qquad \mathsf{T}_{\lambda\mu} = \mathrm{e}^{\mathrm{i}\pi\left(\lambda^2/k - 1/12\right)}\delta\Big(\frac{\lambda}{\sqrt{k}} = \frac{\mu}{\sqrt{k}}\Big). \tag{1.17}$$

It is straight-forward to check that S^2 and $(\mathsf{ST})^3$ are the conjugation permutation $\lambda \to -\lambda$, hence that the characters span a representation of the modular group $\mathsf{SL}\big(2;\mathbb{Z}\big)$ (of uncountably-infinite dimension).

The S-matrix (or S-density) is symmetric and unitary with respect to the rescaled weights $\lambda/\sqrt{k}$:

$$\int_{-\infty}^{\infty} \mathsf{S}_{\lambda\mu}\mathsf{S}^\dagger_{\mu\nu}\,\frac{\mathrm{d}\mu}{\sqrt{k}} = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}(\lambda-\nu)\mu/k}\,\frac{\mathrm{d}\mu}{\sqrt{k}} = \delta\Big(\frac{\lambda}{\sqrt{k}} = \frac{\nu}{\sqrt{k}}\Big). \tag{1.18}$$

It immediately follows that the diagonal partition function $\mathbf{Z}_{\text{diag.}} = \int_{-\infty}^{\infty} \overline{\mathrm{ch}\big[\mathcal{F}_\lambda\big]}\mathrm{ch}\big[\mathcal{F}_\lambda\big]\,\mathrm{d}\lambda/\sqrt{k}$ is a modular invariant (T-invariance is manifest). Similarly, the invariance of the charge conjugation partition function $\mathbf{Z}_{\text{c.c.}} = \int_{-\infty}^{\infty} \overline{\mathrm{ch}\big[\mathcal{F}_\lambda\big]}\mathrm{ch}\big[\mathcal{F}_{-\lambda}\big]\,\mathrm{d}\lambda/\sqrt{k}$ follows from unitarity and the symmetry $\mathsf{S}_{\lambda,\mu} = \mathsf{S}_{-\lambda,-\mu}$.

The continuum Verlinde formula states that the fusion coefficients are given by

$$\mathsf{N}_{\lambda\mu}{}^{\nu} = \int_{-\infty}^{\infty} \frac{\mathsf{S}_{\lambda\rho}\mathsf{S}_{\mu\rho}\mathsf{S}^*_{\nu\rho}}{\mathsf{S}_{0\rho}}\,\frac{\mathrm{d}\rho}{\sqrt{k}} = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}(\lambda+\mu-\nu)\rho/k}\,\frac{\mathrm{d}\rho}{\sqrt{k}} = \delta\Big(\frac{\nu}{\sqrt{k}} = \frac{\lambda}{\sqrt{k}} + \frac{\mu}{\sqrt{k}}\Big), \tag{1.19}$$

where we recognise that the vacuum module is $\mathcal{F}_0$. The predicted fusion rules are therefore

$$\mathcal{F}_\lambda \times \mathcal{F}_\mu = \int_{-\infty}^{\infty} \mathsf{N}_{\lambda\mu}{}^{\nu}\mathcal{F}_\nu\,\frac{\mathrm{d}\nu}{\sqrt{k}} = \mathcal{F}_{\lambda+\mu}, \tag{1.20}$$

agreeing perfectly with the known fusion rules. Actually, what the Verlinde formula computes is the fusion rules at the level of the characters. However, the free boson theory has the property that its irreducible modules have linearly independent characters, if we use (1.14), and every module in the spectrum is completely reducible. It follows that character fusion and module fusion coincide for this theory.

[2] If we wish to preserve the adjoint (reality condition) $a_m^\dagger = a_{-m}$, then we may only rescale k to ± 1. Free bosons with $k > 0$ are often called *euclidean* whereas those with $k < 0$ are called *lorentzian*.

An important feature of the spectrum of the free boson is that every irreducible module is a simple current, meaning that they have inverses in the fusion ring [21, 22]. Moreover, if we exclude the fusion identity $\mathcal{F}_0$, then the simple current $\mathcal{F}_r$ has no (irreducible) fixed points.[3] We can use the group generated by a simple current $\mathcal{F}_r$ to construct extended algebras in a canonical fashion (see [23, 24] for example): The extension is obtained by promoting the fields associated to the fusion orbit $\bigoplus_{j\in\mathbb{Z}}\mathcal{F}_{jr}$ to symmetry generators. In other words, this direct sum of $\widehat{\mathfrak{gl}}(1)$-irreducibles becomes the (irreducible) vacuum module of the extended algebra. We restrict attention to extended algebras which are integer-moded (hence bosonic). Comparing conformal dimensions of the states of $\mathcal{F}_{jr}$ shows that this is equivalent to demanding that $r^2\in 2\mathbb{Z}$ (we have scaled k to 1 for simplicity).[4]

The irreducible modules of the extended algebra are also obtained as fusion orbits. We denote them by

$$\mathbb{F}_{[\lambda]} = \bigoplus_{j\in\mathbb{Z}} \mathcal{F}_{\lambda+jr}, \tag{1.21}$$

where $[\lambda]=\lambda \bmod r$. Requiring that the extended algebra act with integer moding leads to a finite set of (untwisted) extended algebra modules labelled by $\lambda = m/r$, with $m\in\{0,1,\dots,r^2-1\}$. The S-transformations of the characters of these modules follow readily from (1.15):

$$\begin{aligned}\mathrm{ch}\big[\mathbb{F}_{[m/r]}\big]\Big|_{\mathsf{S}} &= \sum_{j\in\mathbb{Z}} \mathrm{ch}\big[\mathcal{F}_{m/r+jr}\big]\Big|_{\mathsf{S}} = \int_{-\infty}^{\infty} \mathsf{e}^{-2\pi\mathrm{i}m\mu/r} \sum_{j\in\mathbb{Z}} \mathsf{e}^{-2\pi\mathrm{i}jr\mu}\mathrm{ch}\big[\mathcal{F}_\mu\big]\,\mathrm{d}\mu \\ &= \frac{1}{r}\sum_{n=0}^{r^2-1}\sum_{j\in\mathbb{Z}} \mathsf{e}^{-2\pi\mathrm{i}mn/r^2}\mathrm{ch}\big[\mathcal{F}_{n/r+jr}\big] = \frac{1}{r}\sum_{n=0}^{r^2-1}\mathsf{e}^{-2\pi\mathrm{i}mn/r^2}\mathrm{ch}\big[\mathbb{F}_{[n/r]}\big].\end{aligned} \tag{1.22}$$

Here, we have applied the following summation formula:

$$\sum_{j\in\mathbb{Z}}\mathsf{e}^{-2\pi\mathrm{i}jr\mu} = \sum_{\ell\in\mathbb{Z}}\delta\big(r\mu=\ell\big) = \frac{1}{r}\sum_{\ell\in\mathbb{Z}}\delta\big(\mu=\frac{\ell}{r}\big) = \frac{1}{r}\sum_{n=0}^{r^2-1}\sum_{j\in\mathbb{Z}}\delta\big(\mu=\frac{n}{r}+jr\big). \tag{1.23}$$

The extended algebra's S-matrix is therefore given by

$$\mathbb{S}^{(r)}_{mn} = \frac{1}{r}\mathsf{e}^{-2\pi\mathrm{i}mn/r^2}, \qquad m,n\in\{0,1,\dots,r^2-1\}. \tag{1.24}$$

This is again symmetric and unitary, so one can construct a diagonal modular invariant partition function $\mathbf{Z}_r = \sum_{m=0}^{r^2-1}\overline{\mathrm{ch}\big[\mathbb{F}_{[m/r]}\big]}\mathrm{ch}\big[\mathbb{F}_{[m/r]}\big]$ and its charge conjugate version. Expressing this in terms of Fock space characters, one obtains an infinite set of non-diagonal modular invariants for $\widehat{\mathfrak{gl}}(1)$ with discrete spectra. Finally, it is easy to check that the (standard) Verlinde formula for the extended algebra gives non-negative integer coefficients: $\mathsf{N}_{[m/r][n/r]}^{(r)\quad [p/r]} = \delta_{p=m+n \bmod r^2}$.

Thus far, we have seen that the modular S-transformations of the free boson characters may be used to compute the S-transformations of those of the extended algebras. These in turn can then be used to compute the Verlinde formula for the extended theories. In a sense though, this is overkill because simple current technology makes it possible to reproduce the extended algebra fusion rules from those of the free boson. Naïvely, one might try

$$\mathbb{F}_{[\lambda]}\text{“}\times\text{”}\mathbb{F}_{[\mu]} = \bigoplus_{i,j\in\mathbb{Z}}\big(\mathcal{F}_{\lambda+ir}\times\mathcal{F}_{\mu+jr}\big) = \bigoplus_{i\in\mathbb{Z}}\bigoplus_{j\in\mathbb{Z}}\mathcal{F}_{\lambda+\mu+(i+j)r} = \bigoplus_{j\in\mathbb{Z}}\mathbb{F}_{[\lambda+\mu]}. \tag{1.25}$$

However, this gives an overall multiplicity of infinity, even when $\lambda=\mu=0$. The reason is that each of the $\mathcal{F}_{\lambda+ir}$ are in the same module for the extended algebra, hence each of these Fock spaces gives exactly the same contribution to the fusion product. It is therefore necessary to choose a single representative $\mathcal{F}_{\lambda+ir}$, a convenient one has $i=0$, to avoid multiply counting the same information. This "renormalisation" leads to

$$\mathbb{F}_{[\lambda]}\times\mathbb{F}_{[\mu]} = \bigoplus_{j\in\mathbb{Z}}\big(\mathcal{F}_\lambda\times\mathcal{F}_{\mu+jr}\big) = \bigoplus_{j\in\mathbb{Z}}\mathcal{F}_{\lambda+\mu+jr} = \mathbb{F}_{[\lambda+\mu]}, \tag{1.26}$$

fixing the multiplicity issue. We therefore arrive at a very powerful strategy to compute the fusion rules of extended theories which may be summarised as follows:

- Compute the modular S-transformation of the (non-rational) theory with continuous spectrum.
- Deduce fusion rules using the continuum Verlinde formula.
- Use these fusion rules to identify simple current extensions with discrete (finite) spectrum.

[3] A fixed point of a simple current is a module for which fusion with the simple current reproduces itself.

[4] The extended algebra constructed from $\mathcal{F}_r$ is, of course, the symmetry algebra of the free boson compactified on a circle of radius r.

- Extract the fusion rules of the extended theory from those of the non-rational theory.

We conclude this exercise by checking these extended algebra results at the self-dual radius $r=\sqrt{2}$, for which it is well known that the extended algebra is $\widehat{\mathfrak{sl}}(2)$ at level 1. There are $r^2=2$ extended algebra modules $\mathbb{F}_{[0]}$ and $\mathbb{F}_{[1/\sqrt{2}]}$ which are easily checked to have 1 and 2 ground states of dimensions 0 and $1/4$, respectively. The S-matrix and fusion matrices are found to be

$$\mathbb{S}^{(\sqrt{2})}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix};\qquad \mathbb{N}^{(\sqrt{2})}_{[0]}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \mathbb{N}^{(\sqrt{2})}_{[1/\sqrt{2}]}=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{1.27}$$

which does indeed reproduce the correct data for $\widehat{\mathfrak{sl}}(2)_1$.

1.3. **Outline.** Our review commences in Section 2 with an overview of the $c=0$ logarithmic conformal field theory that describes the critical point of the statistical lattice model known as percolation. We describe enough of the underlying lattice theory to introduce Cardy's celebrated formula [25] for a non-local observable known as the horizontal crossing probability. While it is clear that Cardy's derivation cannot be accommodated within a unitary theory (the only unitary $c=0$ conformal field theory is the trivial minimal model $\mathsf{M}(2,3)$), we show that his derivation actually implies a logarithmic theory, following [26]. This necessitates a brief introduction to the famous fusion algorithm of Nahm, Gaberdiel and Kausch [27, 28]. We compute a few fusion products explicitly before describing the results of more involved calculations that detail the structures of the indecomposable modules so constructed. We then use the results to derive a couple of logarithmic correlators, generalising the analysis of Section 1.1, before briefly discussing other non-local percolation observables and other $c=0$ models.

Section 3 introduces the first of our "archetypal" logarithmic conformal field theories, the symplectic fermions of Kausch [29]. More precisely, we discuss a family of $c=-2$ theories which include symplectic fermions, the triplet model $\mathfrak{W}(1,2)$ studied in [30–32], and the corresponding singlet model $\mathfrak{M}(1,2)$, itself a special case of Zamolodchikov's original W-algebra [33]. We begin with the symplectic fermion algebra, constructing its irreducible and indecomposable (twisted) representations and verifying the non-diagonalisability of L_0 on the latter, before decomposing its representation into those of the subalgebras $\mathfrak{W}(1,2)$ and $\mathfrak{M}(1,2)$. For the singlet, the spectrum is continuous and it is here that we derive character formulae and deduce modular transformations. The S-matrix is found to be symmetric and unitary, so we apply a continuous version of the Verlinde formula and find that the resulting (Grothendieck) fusion coefficients are positive integers.

As far as we are aware, the modular properties of the singlet model's characters are new (the generalisation to $\mathfrak{M}(1,p)$ will be reported in [34]). Assuming that the continuous Verlinde formula does give the correct (Grothendieck) fusion coefficients, we also deduce many fusion rules, in particular concluding that the singlet model possesses a countable infinity of simple currents. We identify the maximal simple current extension as symplectic fermions and the maximal bosonic simple current extension as the triplet model. This also seems to be new. We moreover use our singlet results to determine what the (Grothendieck) fusion rules for the triplet model should be, finding agreement with the fusion computations of [32]. This then provides a stringent consistency check of the continuous Verlinde formula. We also conjecture the existence of certain singlet indecomposable modules before briefly discussing the known issue with obtaining an S-matrix for the triplet model (the S-matrix entries are not constant) and how this is manifested in the simple current extension formalism we have developed.

Finally, we discuss the bulk (non-chiral) aspects of these $c=-2$ theories. Bulk logarithmic conformal field theories are not as well understood as their chiral counterparts, though progress has been steady [35–41]. For this, it has proven useful to study analogous situations in mathematics. For example, the representation of a semisimple finite-dimensional associative algebra, acting on itself by left-multiplication, decomposes as a direct sum of irreducibles, where every irreducible appears with multiplicity equal to its dimension (Wedderburn's theorem). However, the non-semisimple case gives a direct sum of projectives, where the multiplicity of each is now the dimension of the irreducible it covers. The semisimple case is also the result for compact Lie groups G acting on the Hilbert space $L^2(\mathsf{G},\mu)$ (with μ the Haar measure), whereas the non-semisimple case seems to be roughly correct for Lie supergroups and many non-compact groups (this is the *minisuperspace limit* [36, 42]).

This is relevant because the modular invariant partition functions that have been constructed for logarithmic conformal field theories often have the form

$$\mathbf{Z}=\sum_i \overline{\mathrm{ch}\big[L_i\big]}\,\mathrm{ch}\big[P_i\big]=\sum_i \overline{\mathrm{ch}\big[P_i\big]}\,\mathrm{ch}\big[L_i\big], \tag{1.28}$$

where i labels the irreducibles L_i in the spectrum and P_i denotes an indecomposable cover of L_i which one expects to be projective in some category. (In the rational case, each P_i and L_i coincide and this reduces to the standard

diagonal invariant.) Because of this, the state space of a logarithmic theory seems likely to decompose, upon restricting to the chiral or antichiral algebra, into a direct sum of projectives (each with infinite multiplicity).

Digressions over, we conclude our discussion of this family of $c = -2$ logarithmic conformal field theories by noting the well known structure for the bulk state space of the symplectic fermions theory and discuss the structure one obtains by restricting to the triplet algebra [35]. We then indicate how one could have guessed this structure, without *a priori* knowledge of the symplectic fermions structure, based on the form of the diagonal modular invariant partition function and the above analogy. This leads to a simple proposal for constructing bulk module structures from chiral ones which we stress automatically satisfies the physical locality requirement (that bulk correlators are single-valued). Algebraically, this will be met if the chiral and antichiral states have conformal dimensions that differ by an integer and, in a logarithmic theory, if the spin operator $L_0 - \overline{L}_0$ acts diagonalisably. We conclude by computing some correlation function and briefly mentioning what is known about the more general $\mathfrak{W}(1,p)$ and $\mathfrak{W}(q,p)$ theories that have received so much attention in the literature.

Section 4 considers an example of a fractional level Wess-Zumino-Witten model, specifically one whose symmetry algebra is $\widehat{\mathfrak{sl}}(2)$ at level $k = -\frac{1}{2}$. The existence of such fractional level theories was first suggested by Kent [43] in order to provide a unified coset construction of all Virasoro minimal models, unitary and non-unitary. They began to be studied seriously once Kac and Wakimoto discovered [44] that the levels required for Kent's cosets, the *admissible levels*, were the only ones which admitted modules whose characters carried a finite-dimensional representation of the modular group $\mathsf{SL}(2;\mathbb{Z})$. Assuming, naturally enough, that this meant that admissible level models were rational, Koh and Sorba computed the fusion rules given by the Verlinde formula, noting that this sometimes resulted in negative integer fusion coefficients [45]. This puzzle was subsequently addressed by many groups [46–55], without any real progress, before Gaberdiel pointed out [56] that the assumption of rationality was in error (see also [57]). He constructed enough fusion products for $\widehat{\mathfrak{sl}}(2)$ at level $k = -\frac{4}{3}$ to conclude that the theory was logarithmic, but was unable to solve the puzzle of negative fusion multiplicities. The level $k = -\frac{1}{2}$ was subsequently argued to be logarithmic using a free field realisation [58, 59], but a complete picture including indecomposable module structure, characters, modular properties and the Verlinde formula has only recently emerged [60–64]. The purpose of Section 4 is to explain this progress for $k = -\frac{1}{2}$.

We start by introducing the closely related $\beta\gamma$ ghost system and derive the current algebra $\widehat{\mathfrak{sl}}(2)_{-1/2}$ as an orbifold. Instead of considering the representation theory of the ghost algebra, as we did for Section 3, we determine the spectrum of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ directly. As before, we find a continuum of generically irreducible modules, but this time they are neither highest nor lowest weight. At the parameter values where the continuum modules become reducible, four highest weight modules are constructed (these are the admissible modules of Kac and Wakimoto). The characters of these admissibles can be meromorphically continued using Jacobi theta functions, leading to a four-dimensional representation of the modular group. We then illustrate the paradox of negative Verlinde fusion coefficients before indicating its resolution [60] using spectral flow automorphisms.

A very important point here is that one must be careful with regions of convergence of characters. Indeed, certain non-isomorphic modules, related by spectral flow, have (up to a sign) exactly the same meromorphically-continued character. However, the regions where these characters converge are disjoint, being separated by a common pole. We then interpret the sum of these characters as a distribution supported at this pole. With this formalism, we obtain modular properties, Verlinde formula and a discrete series of modular invariants. Here, the story is very similar to the previous section and we are again able to propose the structure of the local bulk modules. Finally, we use a free field realisation to compute correlation functions and give an example of a three-point correlator which exhibits singularities at certain module parameters. As in the previous section, this result can be regularised to obtain logarithmic correlators. We conclude with a brief discussion of how all this generalises to the level $k = -\frac{4}{3}$.

Section 5 contains the last of our "archetypal" examples, the Wess-Zumino-Witten theory on the Lie supergroup $\mathsf{GL}(1|1)$. As usual, supergroup models depend upon a level k and the symmetry algebra is an affine Lie superalgebra. But, in contrast to (integer level) bosonic Wess-Zumino-Witten models, our understanding of these superanalogues is still rather rudimentary. Aside from the *rational* theories associated with $\mathsf{OSP}(1|2n)$, see [65] for example, only the theories associated to the Lie supergroups $\mathsf{GL}(1|1)$ and $\mathsf{PSL}(1|1)$ (which is just symplectic fermions) are completely understood. Indeed, these were the first logarithmic conformal field theories investigated over two decades ago: By Rozansky and Saleur [9, 10] for $\mathsf{GL}(1|1)$ and by Gurarie [11] for the fermionic *bc* ghosts that are closely related to symplectic fermions.

We structure this section so as to bring out the analogy with the previous two examples. We start with the algebra and representation theory, then continue with modular data and correlation functions following [36, 37, 66].

This example, like the two that preceded it, exhibit all the features of the simplest known logarithmic conformal field theories. There are certainly many more logarithmic theories that should be considered, some with similar indecomposable structures to our examples, and some which are more complicated. We mention that there are many applications which involve supergroup theories of the latter class as e.g. in statistical physics [67, 68] and the AdS/CFT correspondence [69] — these therefore need a detailed investigation in the near future.

Section 6 aims to briefly outline a reasonably general approach to understanding the mathematical structures that underlie logarithmic conformal field theory. It commences with a somewhat technical discussion which introduces the important idea of a *staggered module*, familiar from Virasoro studies [70, 71], for a large class of associative algebras. Some very basic results are proven at this level of generality (these results have not before been published) before restricting to a discussion of the *logarithmic couplings* that parametrise the isomorphism classes of staggered modules for the Virasoro algebra. We emphasise that these numbers are as important to logarithmic conformal field theory as the three-point constants are to rational theories and we detail how they arise when computing two-point functions. We conclude with a brief analysis of an example of a Virasoro indecomposable whose structure is more complicated than that of a staggered module because L_0 acts with a *rank* 3 *Jordan block*.

Section 7 then summarises what we have presented, describing what we believe is a reasonably general approach to understanding logarithmic conformal field theories. Finally, there is a short appendix in which we have collected some of the necessary basic information about homological algebra, a very useful tool (and language) for describing the structure of indecomposable but reducible representations.

2. Percolation as a Logarithmic Conformal Field Theory

Percolation may be loosely defined as a collection of closely related probabilistic models whose observed behaviour is believed to be reasonably typical for more general classes of statistical theories. In particular, these models exhibit phase transitions as their defining parameters pass through certain critical values [72]. Moreover, percolation is particularly easy to simulate numerically, so it is a popular choice for testing predictions such as conformal invariance and universality [73, 74]. In this section, we discuss how the hypothesis of conformal invariance, which led Cardy to his celebrated formula [25] for the horizontal crossing formula, can be accommodated within the standard framework of (boundary) conformal field theory. It has long been suspected (see [15] for example) that the conformal invariance of percolation requires a logarithmic theory. Here, we follow [26] to deduce from the assumptions underlying Cardy's derivation that the spectrum of percolation contains indecomposable modules on which the Virasoro mode L_0 acts non-diagonalisably, hence that critical percolation is described by a logarithmic conformal field theory.

2.1. **Critical Percolation and the Crossing Formula.**

As with many other statistical models, the primary consideration of percolation is the degree to which a very large number of identical objects tend to cluster together when distributed in a random fashion. The setup for one of the basic percolation models is as follows: Consider a square lattice with a given edge length and choose a fixed rectangular subdomain whose sides are a union of lattice edges. A percolation configuration is then obtained by declaring that each edge within the subdomain is open with probability p and closed with probability $1-p$. The idea is that the subdomain represents a porous material and that open edges permit the flow of a liquid medium whereas closed edges do not. When $p=0$, all edges are closed and material is impermeable to the liquid. When $p=1$, all edges are open and there is no obstruction to the liquid's flow. For $0<p<1$, one is then led to question whether the liquid is able to percolate through the material, whence the model's name.

To be more precise, we may ask for the probability that a randomly chosen configuration of edges in our rectangular subdomain contains a path of open edges connecting a chosen side of the rectangle with the opposite side. Such a path is called a *crossing* and Figure 1 shows an example of a configuration in which one (of the many) crossings has been drawn. Computing this crossing probability analytically is a hopeless task, though simulation can approximate it extremely well. However, one can ask the question again in the continuum limit where the edge length tends to 0 while the size and shape of the rectangular subdomain is kept fixed. In this case, one has the result [72] that the limit of the crossing probabilities is 0 if p is less than a critical value, which turns out to be $p_c=\frac{1}{2}$ for a square lattice, and is 1 if p is greater than p_c. The only interesting value is then the limit of the crossing probabilities when p is precisely this critical value.[5]

[5]Curiously, it seems that the existence of this limit when $p=p_c$ was not known until Cardy's crossing formula (see (2.1)) was rigorously proven [75, 76].

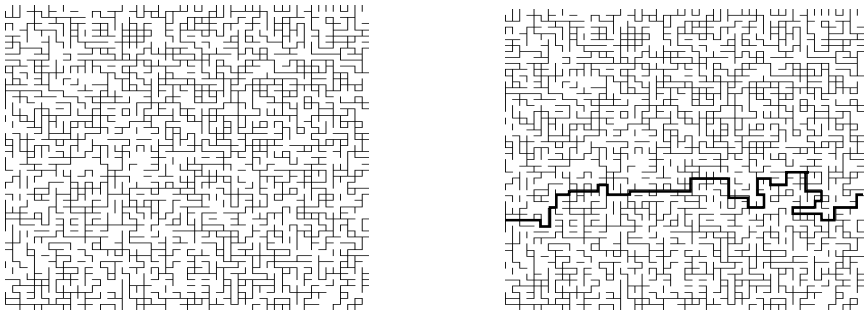

FIGURE 1. A typical percolation configuration (left) for a rectangular subdomain of a square lattice showing only the open edges (closed edges are omitted). This lattice has several crossings from left to right, one of which is indicated in bold (right).

This probability of a crossing being present when $p = p_c$ was famously derived by Cardy [25] within the formalism of boundary conformal field theory and his result is generally recognised as one of the most striking confirmations of the conjecture that the continuum limit of a statistical model is conformally invariant at its critical points. Cardy combined the well known description of percolation as the $Q \to 1$ limit of the Q-state Potts model with an inspired identification of certain boundary-changing operators in these Potts models to write the crossing probability as the four-point correlation function of a Virasoro primary field $\phi_{1,2}$, where the subscript indicates the field's Kac labels. To apply the machinery of conformal field theory, one now maps the rectangular subdomain conformally onto (a compactification of) the upper half-plane so that the fields' insertion points (the corners of the rectangle) are mapped to points z_i, $i = 1,\dots,4$, lying on the real axis (or to ∞).

The central charge c of the continuum limit of the Q-state Potts model is well known [77], assuming of course that the limit is conformally invariant. For percolation ($Q \to 1$), one obtains $c = 0$ and it therefore follows that $\phi_{1,2}$ has conformal dimension 0. Moreover, $\phi_{1,2}$ will have a singular descendant field at grade 2 and so, according to standard conformal field theory dogma, the four-point correlator representing the crossing probability will satisfy a second-order linear differential equation. The obvious behaviour of the crossing probability as the aspect ratio of the rectangle tends to 0 and ∞ then picks out a unique solution:

$$\mathrm{Pr} = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\eta^{1/3}{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\eta\right), \qquad \text{where} \quad \eta = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}. \tag{2.1}$$

The agreement between this computation and numerical data from simulations [73] is impressive.

The precise formula for the crossing probability is not important for what follows. Rather, what we wish to emphasise is that the derivation is performed with the aid of a limit $Q \to 1$ which hides a remarkable amount of subtlety. Indeed, one might guess that the percolation conformal field theory is a minimal model, based on the usual identification of the Q-state Potts models for $Q = 2$ and 3 with $\mathsf{M}(3,4)$ and $\mathsf{M}(5,6)$, respectively. However, the minimal model with $c = 0$ is $\mathsf{M}(2,3)$ which is trivial in the sense that its field content is limited to constant multiples of the identity. Obviously, four-point functions in $\mathsf{M}(2,3)$ will be constant, so this model cannot accommodate Cardy's derivation. On the other hand, it would be distressing if Cardy's derivation turned out to be inconsistent with the principles of conformal field theory. We will therefore assume that a description of critical percolation can be accommodated within conformal field theory. This will require the consideration of reducible, yet indecomposable, representations.

2.2. **The Necessity of Indecomposability.** Before embarking on our explorations, let us pause to recall some useful facts concerning Virasoro modules. This will also serve to introduce our notation. Verma modules will be denoted by $\mathcal{V}_h$, where h is the conformal dimensional of the highest weight state, and their irreducible quotients by $\mathcal{L}_h$. For $c = 0$, we recall that the Verma module is itself irreducible unless $h = h_{r,s}$ for some $r,s \in \mathbb{Z}_+$, where

$$h_{r,s} = \frac{(3r-2s)^2-1}{24}. \tag{2.2}$$

In the latter case, $\mathcal{V}_h = \mathcal{V}_{h_{r,s}}$ will have a submodule generated by a singular vector at grade rs (its conformal dimension will be $h_{r,s}+rs$). If r is even or s is a multiple of 3, then the maximal proper submodule of $\mathcal{V}_{h_{r,s}}$ is

0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	5	7	$\frac{28}{3}$	12	$\cdots$
$\frac{5}{8}$	$\frac{1}{8}$	$\frac{-1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	$\frac{21}{8}$	$\frac{33}{8}$	$\frac{143}{24}$	$\frac{65}{8}$	$\cdots$
2	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	5	$\cdots$
$\frac{33}{8}$	$\frac{21}{8}$	$\frac{35}{24}$	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{-1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	$\frac{21}{8}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 1. A part of the extended Kac table for $c = 0$, displaying the conformal dimensions $h_{r,s}$ for which the Verma modules $\mathcal{V}_{h_{r,s}}$ are reducible. The rows of the table are labelled by $r = 1,2,3,\ldots$ and the columns by $s = 1,2,3,\ldots$.

generated by the singular vector of lowest (positive) grade.[6] Otherwise, it is generated by the two singular vectors of lowest and next-to-lowest grades. It is convenient to collate the $h_{r,s}$ with $r,s \in \mathbb{Z}_+$ into an *extended Kac table*, a part of which we reproduce in Table 1. Finally, we introduce notation for certain Verma module quotients that will frequently arise in what follows: $\mathcal{Q}_{r,s} = \mathcal{V}_{h_{r,s}}/\mathcal{V}_{h_{r,s}+rs}$.

We begin by postulating that the conformal field theory describing the continuum limit of critical percolation contains a vacuum $\big|0\big\rangle$. Equivalently, by the state-field correspondence, the identity field I is present in the theory. As $h_{1,1} = h_{1,2} = 0$, the vacuum Verma module $\mathcal{V}_0$ has singular vectors at grades 1 and 2 and these turn out to be independent in the sense that the latter is not descended from the former.[7] In fact, the maximal proper submodule of $\mathcal{V}_0$ is generated by these two singular vectors. Since $L_{-1}\big|0\big\rangle$ corresponds to the field $\partial I = 0$, we have to set $L_{-1}\big|0\big\rangle = 0$ (by quotienting $\mathcal{V}_0$ by the submodule it generates). However, the grade 2 singular vector then corresponds to the energy-momentum tensor $T(z)$. If this is set to 0, then each of its modes, the Virasoro generators L_n, must all act as the zero operator on the states of the theory, and this leads us to the (trivial) minimal model $\mathsf{M}\big(2,3\big)$ (or a direct sum of copies of this model).

To get a non-trivial $c = 0$ theory, we must abandon the idea that singular vectors are always set to 0. Instead of assuming that the vacuum $\big|0\big\rangle$ generates the irreducible Virasoro module $\mathcal{L}_0$, we are led to propose that the vacuum module is the reducible, but indecomposable, quotient $\mathcal{Q}_{1,1} = \mathcal{V}_0/\mathcal{V}_1$. This is, in fact, the only remaining option because the only singular vector to survive in $\mathcal{Q}_{1,1}$ is of grade 2 (it corresponds to T) and setting it to 0 leads back to the irreducible vacuum module $\mathcal{L}_0$. In the language of Appendix A.1, our proposed vacuum module $\mathcal{Q}_{1,1}$ is an extension of $\mathcal{L}_0$ by the submodule generated by $\big|T\big\rangle = L_{-2}\big|0\big\rangle$, which is itself irreducible and isomorphic to $\mathcal{L}_2$. This is summarised by the exact sequence

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{Q}_{1,1} \longrightarrow \mathcal{L}_0 \longrightarrow 0. \tag{2.3}$$

This argument shows that there is a unique choice for the vacuum module which leads to a non-trivial theory. Moreover, this choice is reducible, but indecomposable. To accommodate Cardy's derivation, there should also exist in the theory a primary field $\phi_{1,2}$ with a vanishing grade 2 descendant. This last requirement stems from the fact that the crossing probability is derived as a solution to a second order differential equation and this equation is derived from the vanishing of a grade 2 descendant. Because $h_{1,2} = 0$, the corresponding Verma module is again $\mathcal{V}_0$ with singular vectors at grades 1 and 2. This time, we cannot set the grade 1 singular vector to 0 because it would lead to a first order differential equation for Cardy's crossing probability (one can check that the solutions to this equation are all constant). We therefore conclude that the reducible, but indecomposable, module $\mathcal{Q}_{1,2} = \mathcal{V}_0/\mathcal{V}_2$ is present. Again, this is the only possibility compatible with Cardy's derivation; the corresponding exact sequence is

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{Q}_{1,2} \longrightarrow \mathcal{L}_0 \longrightarrow 0. \tag{2.4}$$

This concludes the basic setup for a conformal field theory which is consistent with Cardy's derivation of the crossing formula (2.1). One can therefore declare with confidence that the percolation (boundary) conformal field theory, whatever it may be, must include the indecomposable vacuum module $\mathcal{Q}_{1,1}$ and the indecomposable module $\mathcal{Q}_{1,2}$ in appropriate boundary sectors. It remains to explore the consequences of this conclusion. As usual, one can

[6] We will often use the term "singular vector" to indicate a highest weight state which is a proper descendant. Similarly, the term "highest weight state" will often be used to indicate the one of lowest conformal dimension in a given module.

[7] There are also singular vectors at grades $5,7,12,15,\ldots$ which are each descended from both the grade 1 and grade 2 singular vectors.

try to generate new field content through fusion. It is natural to expect that the identity field I will act as the fusion identity ($I \times I = I$ and $I \times \phi_{1,2} = \phi_{1,2}$) and this is indeed the case. One also expects that the vanishing of the grade 2 singular descendant of $\phi_{1,2}$ will imply that

$$\phi_{1,2} \times \phi_{1,2} = I + \phi_{1,3}, \tag{2.5}$$

where $\phi_{1,3}$ is a Virasoro primary field of conformal dimension $h_{1,3} = \frac{1}{3}$. This also turns out to be true. However, the natural sequel to this computation,

$$\phi_{1,2} \times \phi_{1,3} = \phi_{1,2} + \phi_{1,4}, \tag{2.6}$$

where $\phi_{1,4}$ is primary of dimension $h_{1,4} = 1$, is *false* as we shall see.

2.3. **The Nahm-Gaberdiel-Kausch Fusion Algorithm.** In standard conformal field theory, where the modules are completely reducible, it is permissible to regard fusion as an operation on primary fields, remembering that the fusion rules in fact also apply to the entire family of fields descended from the respective primaries. However, we have already surmised that there are reducible, but indecomposable, modules in the percolation spectrum. Therefore, one needs to be much more precise about fusion and regard it not as an operation on primaries, but rather as an operation on the modules themselves. We also need to be more careful about how fusion rules are computed. The usual method of examining the effect of setting singular vectors to zero on three-point functions might not be practical if we do not know what type of fields to insert in the three-point functions (as we shall see, primary fields do not suffice in general).

The standard method of computing fusion rules when reducible, but indecomposable, modules are involved is known as the *Nahm-Gaberdiel-Kausch* algorithm. This was originally introduced by Nahm [27] in a limited setting and was extended (and applied to indecomposable Virasoro modules at $c = -2$) by Gaberdiel and Kausch [28]. The key insight behind this algorithm is the realisation that one can concretely realise the fusion product of two modules M and N as a quotient of the vector space tensor product $M \otimes_{\mathbb{C}} N$. To demonstrate this, one needs to know how the action of the symmetry algebra (here, the Virasoro algebra) on $M \times N$ is derived from the actions on M and on N. This takes the form of coproduct formulae [78]:

$$\Delta(L_n) = \sum_{m=-1}^{n} \binom{n+1}{m+1} L_m \otimes \mathrm{id} + \mathrm{id} \otimes L_n \qquad (n \geqslant -1), \tag{2.7a}$$

$$\Delta(L_{-n}) = \sum_{m=-1}^{\infty} (-1)^{m+1} \binom{n+m-1}{m+1} L_m \otimes \mathrm{id} + \mathrm{id} \otimes L_{-n} \qquad (n \geqslant 2), \tag{2.7b}$$

$$L_{-n} \otimes \mathrm{id} = \sum_{m=n}^{\infty} \binom{m-2}{m-n} \Delta(L_{-m}) + (-1)^n \sum_{m=-1}^{\infty} \binom{n+m-1}{m+1} \mathrm{id} \otimes L_m \qquad (n \geqslant 2). \tag{2.7c}$$

Actually, one derives two distinct coproducts which should coincide — (2.7c) is then deduced by imposing this equality. Of course, there are generalisations of these formulae for other symmetry algebras [79].

Practically, one does not compute explicitly with the entire fusion module $M \times N$. Rather, one restricts attention to a subspace by setting all states of sufficiently high grade to 0. More precisely, if g is the cutoff grade, then any state which can be written as a linear combination of states of the form $L_{-n_1} \cdots L_{-n_k} |v\rangle$, with $n_1 + \cdots n_k > g$, is set to 0. We will denote the result of this grade g truncation of a Virasoro module N by $N^{(g)}$. This truncation not only replaces the infinite-dimensional fusion product by a finite-dimensional subspace, thereby facilitating explicit computation, but it also renders the first sum in (2.7c) finite (the other sums in (2.7) are already effectively finite if we assume that the conformal dimensions of the states of M and N are bounded below). The point is that this truncation is compatible with fusion computations because (2.7) may be used to prove that $(M \times N)^{(g)}$ can be realised as a quotient of $M' \otimes_{\mathbb{C}} N^{(g)}$ [28]. Here, M' denotes the *special subspace*, a truncation of M in which any state of the form $L_{-n_1} \cdots L_{-n_k} |v\rangle$, with $\max\{n_1, \ldots, n_k\} > 1$, is set to 0. Finally, the quotient of $M' \otimes_{\mathbb{C}} N^{(g)}$ which realises the truncated fusion product may be identified by determining those elements of the tensor space, the so-called *spurious states*, that we are forced to set to 0 as a consequence of setting singular vectors to 0 when forming M and N.

It is always best to illustrate an algorithm with examples. Let us consider the fusion of the percolation ($c = 0$) module $\mathcal{Q}_{1,2}$ of (2.4) with itself, setting the cutoff grade to 0. Then, $\mathcal{Q}'_{1,2}$ is spanned by $|v\rangle$ (the highest weight state of $\mathcal{Q}_{1,2}$) and $L_{-1}|v\rangle$, because $L_{-1}^2 |v\rangle = \frac{2}{3} L_{-2} |v\rangle$, and $\mathcal{Q}^{(0)}_{1,2}$ is spanned by $|v\rangle$. There are no spurious states to find,

so $(\mathcal{Q}_{1,2}\times\mathcal{Q}_{1,2})^{(0)}$ is two-dimensional. Applying (2.7a) with $n=0$, we obtain

$$\Delta(L_0)\big(|v\rangle\otimes|v\rangle\big) = L_{-1}|v\rangle\otimes|v\rangle + L_0|v\rangle\otimes|v\rangle + |v\rangle\otimes L_0|v\rangle = L_{-1}|v\rangle\otimes|v\rangle, \tag{2.8a}$$

$$\begin{aligned}\Delta(L_0)\big(L_{-1}|v\rangle\otimes|v\rangle\big) &= L_{-1}^2|v\rangle\otimes|v\rangle + L_0L_{-1}|v\rangle\otimes|v\rangle + L_{-1}|v\rangle\otimes L_0|v\rangle\\ &= L_{-1}|v\rangle\otimes|v\rangle + \tfrac{2}{3}L_{-2}|v\rangle\otimes|v\rangle = L_{-1}|v\rangle\otimes|v\rangle + \tfrac{2}{3}|v\rangle\otimes L_{-1}|v\rangle\\ &= \tfrac{1}{3}L_{-1}|v\rangle\otimes|v\rangle.\end{aligned} \tag{2.8b}$$

In the course of this calculation, we have combined $\Delta(L_{-1})=\Delta(L_{-2})=0$ with (2.7a) and (2.7c) to obtain

$$L_{-2}|v\rangle\otimes|v\rangle = |v\rangle\otimes L_{-1}|v\rangle = -L_{-1}|v\rangle\otimes|v\rangle. \tag{2.9}$$

It follows from (2.8) that $\Delta(L_0)$ is diagonalisable with eigenvalues $h_{1,1}=0$ and $h_{1,3}=\frac{1}{3}$. From this, we deduce that the fusion product $\mathcal{Q}_{1,2}\times\mathcal{Q}_{1,2}$ decomposes as the direct sum of two highest weight modules whose highest weight states have conformal dimensions 0 and $\frac{1}{3}$, respectively.

To identify the highest weight modules appearing in this decomposition unambiguously, we need to compute to higher cutoff grades. At grade 1, $\mathcal{Q}'_{1,2}\otimes_{\mathbb{C}}\mathcal{Q}^{(1)}_{1,2}$ is four-dimensional, spanned by $|v\rangle\otimes|v\rangle$, $L_{-1}|v\rangle\otimes|v\rangle$, $|v\rangle\otimes L_{-1}|v\rangle$ and $L_{-1}|v\rangle\otimes L_{-1}|v\rangle$, and one uncovers a spurious state as follows:

$$\begin{aligned}0 = \Delta\big(L_{-1}^2\big)\big(|v\rangle\otimes|v\rangle\big) &= L_{-1}^2|v\rangle\otimes|v\rangle + 2L_{-1}|v\rangle\otimes L_{-1}|v\rangle + |v\rangle\otimes L_{-1}^2|v\rangle\\ &= \tfrac{2}{3}L_{-2}|v\rangle\otimes|v\rangle + 2L_{-1}|v\rangle\otimes L_{-1}|v\rangle + \tfrac{2}{3}|v\rangle\otimes L_{-2}|v\rangle\\ &= \tfrac{2}{3}|v\rangle\otimes L_{-1}|v\rangle + 2L_{-1}|v\rangle\otimes L_{-1}|v\rangle - \tfrac{2}{3}L_{-1}|v\rangle\otimes|v\rangle.\end{aligned} \tag{2.10}$$

This time, we have used $\Delta\big(L_{-1}^2\big)=\Delta(L_{-2})=0$, (2.7a) and (2.7c) to obtain the relations

$$L_{-2}|v\rangle\otimes|v\rangle = |v\rangle\otimes L_{-1}|v\rangle \qquad\text{and}\qquad |v\rangle\otimes L_{-2}v = -L_{-1}|v\rangle\otimes|v\rangle. \tag{2.11}$$

There are no other spurious states, so the truncated fusion product is three-dimensional. Computing $\Delta(L_0)$ as before, we find that it is diagonalisable with eigenvalues 0, $\frac{1}{3}$ and $\frac{4}{3}$. This refines the grade 0 conclusion in that we now know that the highest weight module of conformal dimension 0 has its singular vector at grade 1 set to 0, a fact which may be confirmed by checking that $\Delta(L_{-1})$ annihilates the eigenstate with eigenvalue 0. This highest weight module is therefore either $\mathcal{L}_0$ or $\mathcal{Q}_{1,1}$.

To decide which, we compute to grade 2, finding no spurious states in the six-dimensional truncated product $\mathcal{Q}'_{1,2}\otimes_{\mathbb{C}}\mathcal{Q}^{(2)}_{1,2}$. Calculating as before gives $\Delta(L_0)$ as diagonalisable with eigenvalues 0, 2, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{7}{3}$ and $\frac{7}{3}$. The grade 2 state may be checked to be obtained by acting with $\Delta(L_{-2})$ on the eigenvalue 0 state, thereby identifying one of the direct summands of the fusion product as $\mathcal{Q}_{1,1}$. Identifying the other summand requires computing to grade 3. This time, there is a single spurious state and $\Delta(L_0)$ is diagonalisable with eigenvalues 0, 2, 3, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{7}{3}$, $\frac{7}{3}$, $\frac{10}{3}$ and $\frac{10}{3}$. We see that the grade 3 singular descendant of the eigenvalue $\frac{1}{3}$ state has been set to 0, so the remaining summand is the irreducible highest weight module $\mathcal{Q}_{1,3}=\mathcal{L}_{1/3}$.

To summarise, we have used the Nahm-Gaberdiel-Kausch algorithm to compute the fusion rule

$$\mathcal{Q}_{1,2}\times\mathcal{Q}_{1,2} = \mathcal{Q}_{1,1}\oplus\mathcal{L}_{1/3}. \tag{2.12}$$

The computations beyond grade 1 quickly become tedious and are best done using an computer (we implemented the algorithm in MAPLE). Nevertheless, this example shows that fusion products can be identified from a finite amount of computation (although this would not be true if the result involved modules with infinitely many composition factors, Verma modules for instance). On the other hand, the Virasoro mode L_0 acts diagonalisably on this fusion product, so the result is not particularly interesting so far as logarithmic conformal field theory is concerned.

A more interesting computation is the fusion of $\mathcal{Q}_{1,2}$ with the newly discovered percolation module $\mathcal{L}_{1,3}$. At grade 0, $\Delta(L_0)$ is diagonalisable with eigenvalues 0 and 1. Because these eigenvalues differ by an integer, we cannot conclude that the result decomposes as a direct sum of two highest weight modules. Our wariness in this matter is justified by the grade 1 computation in which a new feature is uncovered: $\Delta(L_0)$ is seen to have eigenvalues 0, 1, 1 and 2, but is *not diagonalisable* — the eigenspace of eigenvalue 1 corresponds to a Jordan block of rank 2. This is the sign of logarithmic structure that we have been looking for.

To clarify this structure, note that the eigenvalue 0 state $|\xi\rangle$ is necessarily a highest weight state. We can check that $\Delta(L_{-1})|\xi\rangle$ is non-zero and is (necessarily) the $\Delta(L_0)$-eigenstate of the Jordan block. Its Jordan partner $|\theta\rangle$ is then uniquely determined by $\big(\Delta(L_0)-\mathrm{id}\big)|\theta\rangle = \Delta(L_{-1})|\xi\rangle$, up to adding multiples of $\Delta(L_{-1})|\xi\rangle$. Finally, the eigenvalue 2 state is realised by $\Delta(L_{-1})|\theta\rangle$. All this amounts to defining (and normalising) the states appearing at

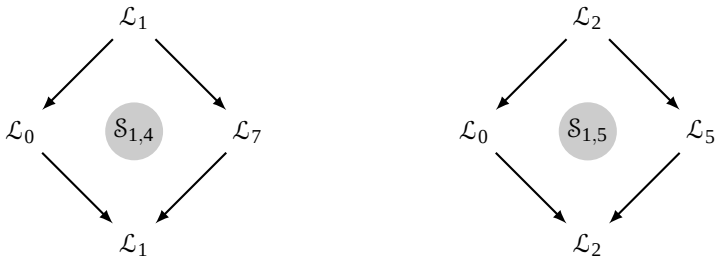

FIGURE 2. Loewy diagrams illustrating the socle series (see Appendix A.4) for the indecomposable Virasoro modules $\mathcal{S}_{1,4}$ and $\mathcal{S}_{1,5}$ constructed using the Nahm-Gaberdiel-Kausch fusion algorithm.

grade 1. What remains to be determined is the action of L_1:[8]

$$\Delta(L_1)\big|\theta\big\rangle = -\tfrac{1}{2}\big|\xi\big\rangle. \tag{2.13}$$

Because $\Delta(L_{-1})\big|\xi\big\rangle$ is a singular vector, this equation holds for *any* choice of Jordan partner state $\big|\theta\big\rangle$.

This grade 1 fusion calculation shows that the product $\mathcal{Q}_{1,2}\times\mathcal{L}_{1,3}$ is an indecomposable non-highest weight module which we shall denote by $\mathcal{S}_{1,4}$. The highest weight state $\big|\xi\big\rangle$ of dimension 0 generates a highest weight submodule of $\mathcal{S}_{1,4}$ whose singular vector of dimension 1, $L_{-1}\big|\xi\big\rangle$, is non-vanishing. Using the fusion algorithm at grade 2, we find that the singular dimension 2 descendant of $\big|\xi\big\rangle$ vanishes, thereby identifying this highest weight submodule as $\mathcal{Q}_{1,2}$. In the quotient module $\mathcal{S}_{1,4}/\mathcal{Q}_{1,2}$, the equivalence class $\big|\theta\big\rangle+\mathcal{Q}_{1,2}$ is a highest weight state of dimension 1. Checking its singular descendants of dimensions 5 and 7 therefore requires fusing to grade 6 and examining the $\mathcal{Q}_{1,2}$-quotient.[9] The results — the first singular descendant is found to vanish whereas the second does not — indicate that the corresponding highest weight module is isomorphic to $\mathcal{Q}_{1,4}=\mathcal{V}_1/\mathcal{V}_5$. This then establishes the exactness of the sequence

$$0\longrightarrow\mathcal{Q}_{1,2}\longrightarrow\mathcal{S}_{1,4}\longrightarrow\mathcal{Q}_{1,4}\longrightarrow 0. \tag{2.14}$$

The Loewy diagram for the indecomposable $\mathcal{S}_{1,4}$ is given in Figure 2 (left). The bottom composition factor $\mathcal{L}_1$ (the socle) is generated by $\Delta(L_{-1})\big|\xi\big\rangle$. By taking appropriate quotients, the $\mathcal{L}_0$ and the top $\mathcal{L}_1$ may be similarly associated with (equivalence classes of) $\big|\xi\big\rangle$ and $\big|\theta\big\rangle$, respectively. The $\mathcal{L}_7$ corresponds to the non-vanishing singular descendant of $\big|\theta\big\rangle+\mathcal{Q}_{1,2}$.

This demonstrates that the percolation conformal field theory necessarily contains indecomposable modules (in some boundary sectors) on which the Virasoro zero-mode acts non-diagonalisably. As we saw in Section 1.1, this leads to logarithmic singularities in correlation functions. Before discussing this in more detail, let us pause to explore further what fusion can tell us about the spectrum of modules. The Nahm-Gaberdiel-Kausch algorithm may be applied to the fusion of $\mathcal{L}_{1/3}$ with itself and computing to grade 5 establishes that the result is the direct sum of $\mathcal{L}_{1/3}$ and a new indecomposable $\mathcal{S}_{1,5}$ whose structure is described by the exact sequence

$$0\longrightarrow\mathcal{Q}_{1,1}\longrightarrow\mathcal{S}_{1,5}\longrightarrow\mathcal{Q}_{1,5}\longrightarrow 0. \tag{2.15}$$

Its Loewy diagram is illustrated in Figure 2 (right). The highest weight submodule is the (indecomposable) vacuum module containing the vacuum $\big|0\big\rangle$ and $\big|T\big\rangle=L_{-2}\big|0\big\rangle$. The latter state (corresponding to the energy-momentum tensor) has a Jordan partner, unique up to adding multiples of $\big|T\big\rangle$, which we will denote by $\big|t\big\rangle$. If we normalise this partner by $\big(\Delta(L_0)-2\,\mathrm{id}\big)\big|t\big\rangle=\big|T\big\rangle$, then explicit computation gives

$$\Delta(L_2)\big|t\big\rangle = -\tfrac{5}{8}\big|0\big\rangle. \tag{2.16}$$

Again, this equation is independent of the choice of $\big|t\big\rangle$.

[8]There is a subtlety to this computation worth mentioning. The action of $\Delta(L_n)$, $n>0$, at grade g should be understood to map into the grade $g-n$ fusion space. However, the latter is always a subspace (quotient) of the former. We may therefore compute $\Delta(L_1)\big|\theta\big\rangle$ in the grade 1 fusion product and project onto the grade 0 subspace by setting all terms with $\Delta(L_0)$-eigenvalue 1 to zero.

[9]This requires finding two spurious states in a 46-dimensional vector space.

It is possible to identify the result of many more fusion rules including [26, 80]

$$\begin{aligned}
\mathcal{Q}_{1,2}\times\mathcal{S}_{1,4} &= 2\,\mathcal{L}_{1/3}\oplus\mathcal{S}_{1,5}, & \mathcal{S}_{1,4}\times\mathcal{S}_{1,4} &= 4\,\mathcal{L}_{1/3}\oplus 2\,\mathcal{S}_{1,5}\oplus\mathcal{S}_{1,7},\\
\mathcal{Q}_{1,2}\times\mathcal{S}_{1,5} &= \mathcal{S}_{1,4}\oplus\mathcal{L}_{10/3}, & \mathcal{S}_{1,4}\times\mathcal{S}_{1,5} &= 2\,\mathcal{S}_{1,4}\oplus 2\,\mathcal{L}_{10/3}\oplus\mathcal{S}_{1,8},\\
\mathcal{L}_{1/3}\times\mathcal{S}_{1,4} &= 2\,\mathcal{S}_{1,4}\oplus\mathcal{L}_{10/3}, & \mathcal{S}_{1,5}\times\mathcal{S}_{1,5} &= \mathcal{L}_{1/3}\oplus 2\,\mathcal{S}_{1,5}\oplus\mathcal{S}_{1,7}\oplus\mathcal{L}_{28/3}.\\
\mathcal{L}_{1/3}\times\mathcal{S}_{1,5} &= 2\,\mathcal{L}_{1/3}\oplus\mathcal{S}_{1,7}, &&
\end{aligned} \tag{2.17}$$

Here, the modules $\mathcal{S}_{1,7}$ and $\mathcal{S}_{1,8}$ are new indecomposables with exact sequences

$$0\longrightarrow\mathcal{Q}_{1,5}\longrightarrow\mathcal{S}_{1,7}\longrightarrow\mathcal{Q}_{1,7}\longrightarrow 0,\qquad 0\longrightarrow\mathcal{Q}_{1,4}\longrightarrow\mathcal{S}_{1,8}\longrightarrow\mathcal{Q}_{1,8}\longrightarrow 0. \tag{2.18}$$

We remark that fusing $\mathcal{S}_{1,4}$ or $\mathcal{S}_{1,5}$ with another module requires knowing the explicit form of the (generalised) singular vectors which have been set to 0. This will be discussed in Section **??**. Fusion computations with the new modules generated here have met with only partial success, chiefly because the computational intensity of the algorithm increases very quickly as the grade required to completely identify the fusion product grows. Nevertheless, all such computations are consistent with the following conjecture for the fusion rules, presented algorithmically for simplicity:[10]

(1) The spectrum includes the irreducibles $\mathcal{Q}_{1,3k}=\mathcal{L}_{(3k-1)(3k-2)/6}$ and the indecomposables $\mathcal{S}_{1,3k-1}$ and $\mathcal{S}_{1,3k-2}$, for $k\in\mathbb{Z}_+$ (we let $\mathcal{S}_{1,1}=\mathcal{Q}_{1,1}$ and $\mathcal{S}_{1,2}=\mathcal{Q}_{1,2}$.) To fuse any of these modules, first break any indecomposables into their constituent highest weight modules ($\mathcal{Q}_{1,-2}=\mathcal{Q}_{1,-1}\equiv\{0\}$):

$$\mathcal{S}_{1,3k-1}\longrightarrow\mathcal{Q}_{1,3k-1}\oplus\mathcal{Q}_{1,3k-5},\quad \mathcal{S}_{1,3k-2}\longrightarrow\mathcal{Q}_{1,3k-2}\oplus\mathcal{Q}_{1,3k-4}. \tag{2.19}$$

(2) Compute the "fusion" using distributivity and

$$\mathcal{Q}_{1,s}\text{ ``}\times\text{''}\,\mathcal{Q}_{1,s'}=\mathcal{Q}_{1,|s-s'|+1}\oplus\mathcal{Q}_{1,|s-s'|+3}\oplus\cdots\oplus\mathcal{Q}_{1,s+s'-3}\oplus\mathcal{Q}_{1,s+s'-1}. \tag{2.20}$$

(We have enclosed the fusion operation in quotes to emphasise that this is not a true fusion rule).

(3) In the result, reverse (2.19) by replacing the combinations $\mathcal{Q}_{1,3k-1}\oplus\mathcal{Q}_{1,3k-5}$ and $\mathcal{Q}_{1,3k-2}\oplus\mathcal{Q}_{1,3k-4}$ by $\mathcal{S}_{1,3k-1}$ and $\mathcal{S}_{1,3k-2}$, respectively. There is always a unique way of doing this.

For example, if we wished to fuse $\mathcal{S}_{1,5}$ with $\mathcal{L}_{10/3}=\mathcal{Q}_{1,6}$, we would instead compute that

$$(\mathcal{Q}_{1,1}\oplus\mathcal{Q}_{1,5})\text{ ``}\times\text{''}\,\mathcal{Q}_{1,6}=\mathcal{Q}_{1,6}\oplus(\mathcal{Q}_{1,2}\oplus\mathcal{Q}_{1,4}\oplus\mathcal{Q}_{1,6}\oplus\mathcal{Q}_{1,8}\oplus\mathcal{Q}_{1,10}) \tag{2.21}$$

from which we read off that

$$\mathcal{S}_{1,5}\times\mathcal{L}_{10/3}=\mathcal{S}_{1,4}\oplus 2\,\mathcal{L}_{10/3}\oplus\mathcal{S}_{1,10}. \tag{2.22}$$

2.4. **Logarithmic Correlators Again.** Consider first the structure of the indecomposable module $\mathcal{S}_{1,5}$. It has a submodule generated by the vacuum $\big|0\big\rangle$, while $\mathcal{S}_{1,5}$ is itself generated by the state $\big|t\big\rangle$ satisfying

$$L_0\big|t\big\rangle=2\big|t\big\rangle+\big|T\big\rangle,\qquad L_1\big|t\big\rangle=0,\qquad L_2\big|t\big\rangle=-\tfrac{5}{8}\big|0\big\rangle,\qquad L_n\big|t\big\rangle=0\quad\text{for } n>2. \tag{2.23}$$

We recall that $\big|T\big\rangle=L_{-2}\big|0\big\rangle$. The operator product expansion of the corresponding fields $T(z)$ and $t(w)$ is therefore slightly different to those considered in Section 1.1:

$$T(z)t(w)\sim-\frac{5}{8}\frac{1}{(z-w)^4}+\frac{2t(w)+T(w)}{(z-w)^2}+\frac{\partial t(w)}{z-w}. \tag{2.24}$$

Normalising so that $\big\langle 0\big|0\big\rangle=1$, we note that $\big\langle T(z)T(w)\big\rangle=0$ because $\big|T\big\rangle$ is singular. The global invariance of the vacuum then leads to the usual three partial differential equations for $\big\langle T(z)t(w)\big\rangle$ whose solution is

$$\big\langle T(z)t(w)\big\rangle=\frac{B}{(z-w)^4},\qquad B=\big\langle 0\big|L_2\big|t\big\rangle=-\tfrac{5}{8}\big\langle 0\big|0\big\rangle=-\tfrac{5}{8}. \tag{2.25}$$

As $T(z)$ and $t(w)$ can be shown to be mutually bosonic [82, App. B], we also obtain

$$\big\langle t(z)t(w)\big\rangle=\frac{A+\frac{5}{4}\log(z-w)}{(z-w)^4}, \tag{2.26}$$

confirming the existence of logarithmic singularities in percolation correlators. We emphasise that, unlike B in (2.25), the value of the constant A depends upon the precise choice we make for $\big|t\big\rangle$.

[10] One can convert this into a general formula, see [81] for example. However, the result seems cumbersome and not particularly illuminating to us.

As a second example, we consider the other module with non-diagonalisable L_0-action that we have studied: $\mathcal{S}_{1,4}$. This module is generated by a state $\big|\theta\big\rangle$ satisfying

$$L_0\big|\theta\big\rangle = \big|\theta\big\rangle + L_{-1}\big|\xi\big\rangle, \qquad L_1\big|\theta\big\rangle = -\tfrac{1}{2}\big|\xi\big\rangle, \qquad L_n\big|\theta\big\rangle = 0 \quad \text{for } n>1. \tag{2.27}$$

Here, $\big|\xi\big\rangle$ is a dimension 0 highest weight state generating a submodule isomorphic to $\mathcal{Q}_{1,2}$. The field $\theta(w)$ corresponding to $\big|\theta\big\rangle$ therefore has operator product expansion

$$T(z)\theta(w) \sim -\frac{1}{2}\frac{\xi(w)}{(z-w)^3} + \frac{\theta(w)+\partial\xi(w)}{(z-w)^2} + \frac{\partial\theta(w)}{z-w}. \tag{2.28}$$

Again, we take $\big\langle\xi\big|\xi\big\rangle = 1$ and arrive at

$$\big\langle\partial x(z)\theta(w)\big\rangle = \frac{B}{(z-w)^2}. \tag{2.29}$$

The determination of B is, however, subtle [82]. Naïvely, we might expect that $\big|\partial\xi\big\rangle = L_{-1}\big|\xi\big\rangle$ implies that $B = \big\langle\partial\xi\big|\theta\big\rangle = \big\langle\xi\big|L_1\big|\theta\big\rangle = -\frac{1}{2}\big\langle\xi\big|\xi\big\rangle = -\frac{1}{2}$, but this turns out to be incorrect. To see why, recall that the standard definition of the outgoing state corresponding to a *primary* field $\phi(z) = \sum_n \phi_n z^{-n-h}$ is

$$\big\langle\phi\big| = \lim_{z\to\infty} z^{2h}\big\langle 0\big|\phi(z) \qquad \Longleftrightarrow \qquad \phi_n^\dagger = \phi_{-n}. \tag{2.30}$$

We certainly want this definition to apply to $\xi(z)$, a dimension 0 primary field. But then,

$$B = \big\langle 0\big|(\partial\xi)_1\big|\theta\big\rangle = -\big\langle 0\big|\xi_1\big|\theta\big\rangle = -\big\langle 0\big|\xi_0 L_1\big|\theta\big\rangle = -\big\langle\xi\big|L_1\big|\theta\big\rangle = \tfrac{1}{2}\big\langle\xi\big|\xi\big\rangle = \tfrac{1}{2}. \tag{2.31}$$

This is the correct conclusion (see Section 6.3 for a more general discussion). In any case, once B is correctly determined, the computation of $\big\langle\theta(z)\theta(w)\big\rangle$ proceeds as before and one obtains

$$\big\langle\theta(z)\theta(w)\big\rangle = \frac{A-\log(z-w)}{(z-w)^2}. \tag{2.32}$$

Once again, A depends upon the precise choice of $\big|\theta\big\rangle$ whereas B does not.

2.5. **Further Developments.** We have seen that the boundary conformal field theory describing critical percolation is logarithmic and that the spectrum includes the modules $\mathcal{Q}_{1,1}$, $\mathcal{Q}_{1,2}$, $\mathcal{L}_{(3k-1)(3k-2)/6}$, $\mathcal{S}_{3k+1}$ and $\mathcal{S}_{3k+2}$, for $k\in\mathbb{Z}_+$. A natural question to ask now is whether there is more to the spectrum. One way to look for additional modules is to consider other measurable quantities in percolation. The most famous generalisation of Cardy's crossing probability is that which asks for the probability that a random configuration of edges (with $p=p_c$) contains a connected cluster of open edges connecting all four sides of the rectangular subdomain. In [83], Watts notes that the four-point functions that solve the second order differential equations that lead to Cardy's formula (2.1) do not satisfy the properties one expects for this more general crossing probability. However, a field of dimension 0 has, at $c=0$, a singular descendant of grade 5 and the corresponding fifth order differential equation not only has a unique solution satisfying Watts' properties, but it also beautifully interpolates the numerical data known [73] for this crossing probability. Watts' proposed solution has since been rigorously proven by Dubédat [84].

Given what we have learned in Section 2.2, the natural interpretation to propose [85] is that the field appearing in Watts' four-point function corresponds to the highest weight state of the module $\mathcal{V}_0/\mathcal{V}_5$.[11] It is rather interesting to note that this quotient module does not have the form $\mathcal{Q}_{r,s}$ for any positive integers r and s. Instead, one may identify it using *fractional* Kac labels: $\mathcal{V}_0/\mathcal{V}_5 = \mathcal{Q}_{2,5/2} = \mathcal{Q}_{5/3,3}$. Perhaps surprisingly, denoting this module by $\mathcal{Q}_{2,5/2}$ is convenient for discussing the modules one subsequently generates by fusing with $\mathcal{Q}_{1,2}$. For example, one finds [85] that

$$\mathcal{Q}_{1,2}\times\mathcal{Q}_{2,5/2} = \mathcal{Q}_{2,3/2}\oplus\mathcal{Q}_{2,7/2}; \qquad \mathcal{Q}_{2,3/2} = \mathcal{V}_{1/3}/\mathcal{V}_{10/3} = \mathcal{L}_{1/3}, \quad \mathcal{Q}_{2,7/2} = \mathcal{V}_0/\mathcal{V}_7. \tag{2.33}$$

Unfortunately, fusing $\mathcal{Q}_{2,5/2}$ with itself leads to indecomposable modules which have significantly more complicated structures and are rather poorly characterised (see [85] for further details). We remark that more general percolation crossing probabilities are considered in [86] from a different perspective.

From a more abstract point of view, we have seen that Cardy's crossing probability leads to indecomposable modules which may be associated with the first row of the (extended) Kac table (Table 1), so one is led to ask

[11] The discussion makes it clear that the singular vector at grade 5 must be set to zero, but it is not *a priori* clear why its grade 7 partner should not be set to zero. It is straight-forward, but tedious, to check that the seventh order differential equation that would result from setting this partner to zero does not admit Watts' crossing formula as a solution.

whether there is a complementary observable quantity that can be associated to the first column. In percolation, this is not so clear. However, the statistical model known as *dilute polymers* (or the *self-avoiding walk*) also has a continuum limit that is (believed to be) described by a $c=0$ conformal field theory. An old proposal of Gurarie and Ludwig [87] associates this latter conformal field theory with modules from the first column of the extended Kac table.[12] We will not detail this polymer theory or its observables here, instead mentioning only that a field corresponding to the module $\mathcal{Q}_{2,1}=\mathcal{L}_{5/8}$ is relevant and that fusing this module with itself leads to an indecomposable module which we denote by $\mathcal{S}_{3,1}$:

$$\mathcal{L}_{5/8}\times\mathcal{L}_{5/8}=\mathcal{S}_{3,1},\qquad 0\longrightarrow\mathcal{Q}_{1,1}\longrightarrow\mathcal{S}_{3,1}\longrightarrow\mathcal{Q}_{3,1}\longrightarrow 0. \tag{2.34}$$

The Loewy diagram of $\mathcal{S}_{3,1}$ is identical to that of $\mathcal{S}_{1,5}$ (illustrated in Figure 2) except that the composition factor $\mathcal{L}_5$ is replaced by $\mathcal{L}_7$. If we regard the submodule $\mathcal{Q}_{1,1}$ as being generated by the vacuum, then $\big|T\big\rangle$ has a Jordan partner $\big|t'\big\rangle\in\mathcal{S}_{3,1}$ which can be distinguished from $\big|t\big\rangle\in\mathcal{S}_{1,5}$ by

$$L_2\big|t\big\rangle=-\tfrac{5}{8}\big|0\big\rangle,\qquad L_2\big|t'\big\rangle=\tfrac{5}{6}\big|0\big\rangle. \tag{2.35}$$

We remark that these coefficients $b_{1,5}=-\frac{5}{8}$ and $b_{3,1}=\frac{5}{6}$, called *anomaly numbers* in [89], have recently been measured directly in the respective lattice theories (through numerical simulation) [90]. This confirms experimentally that percolation corresponds to first row modules and dilute polymers to first column modules, at least in their formulation as boundary conformal field theories.

One thing worth mentioning here is the observation (see [88, App. A]) that the otherwise reasonable-looking two-point function $\big\langle t(z)t'(w)\big\rangle$ is inconsistent with conformal invariance. More precisely, the three inhomogeneous partial differential equations for this correlator that are derived from the global conformal invariance of the vacuum admit no simultaneous solution. This appears [26, 88] to rule out the possibility that both $\mathcal{S}_{1,5}$ and $\mathcal{S}_{3,1}$ can belong to the spectrum. However, a more careful conclusion [85] is that the presence of one of these indecomposables in a boundary sector labelled by boundary conditions B_1 and B_2 precludes the presence of the other in any boundary sector with label B_1 or B_2. This does not prove that $\mathcal{S}_{1,5}$ and $\mathcal{S}_{3,1}$ can coexist in a boundary conformal field theory, but it does provide a loophole whereby inconsistent two-point functions may be avoided. Such a loophole appears to be at work in the results of [81] in which boundary conditions corresponding to all the extended Kac labels (r,s) are constructed for a loop model variant of critical percolation.[13] An extremely important open problem, in our opinion, is to determine if the conformal invariance of the vacuum leads to further, more stringent, constraints on the boundary (and bulk) spectra of logarithmic conformal field theories.

3. Symplectic Fermions and the Triplet Model

The triplet theories $\mathcal{W}\big(q,p\big)$, with $p,q\in\mathbb{Z}_+$, $p>q$ and $\gcd\{p,q\}=1$, form a family of logarithmic extensions of the minimal Virasoro models. When $q=1$, the minimal model is empty, but the logarithmic theory is non-trivial (these are the original triplet models of [30]). We will concentrate on the simplest of these models, that with $q=1$ and $p=2$,[14] which has a free field realisation known as symplectic fermions. We start with this free theory before turning to the triplet algebra $\mathcal{W}\big(1,2\big)$ and then to its subalgebra, the singlet algebra $\mathcal{M}\big(1,2\big)$. The theories associated to these algebras are extremely closely related as we illustrate in Figure 3. We also note that $\mathcal{M}\big(1,2\big)$ is, in fact, a special case of Zamolodchikov's original higher-spin algebra $W(2,3)$ [33]. We then detail the modular transformations of the singlet characters and compute Grothendieck fusion rules for $\mathcal{M}\big(1,2\big)$ using a continuum Verlinde formula (see Section 1.2). This is then lifted to the triplet model and symplectic fermions and compared with their known fusion rules [32].

3.1. **Symplectic Fermions.**

Symplectic fermions were first introduced by Kausch [29] in order to study the fermionic $\xi\eta$ ghost system of central charge $c=-2$. They also describe the Wess-Zumino-Witten model on the abelian supergroup $\mathsf{PSL}(1|1)$ and should be regarded as the simplest fermionic analogue of the free boson. The action involves two non-chiral fermionic fields $\theta^{\pm}\big(z,\overline{z}\big)$:

$$S\big[\theta^{\pm}(z,\overline{z})\big]=\frac{1}{4\pi}\int\Big[\partial\theta^{+}(z,\overline{z})\overline{\partial}\theta^{-}(z,\overline{z})-\partial\theta^{-}(z,\overline{z})\overline{\partial}\theta^{+}(z,\overline{z})\Big]\,\mathrm{d}z\mathrm{d}\overline{z}. \tag{3.1}$$

[12] Actually, the proposal of [87] was that percolation should be associated to the first column and dilute polymers to the first row, though this statement is not repeated in the sequel [88]. This was corrected in [26] for the boundary theory relevant here.

[13] Interestingly, the so-called *Kac modules* $\mathcal{K}_{r,s}$ which appear here generalise the $\mathcal{Q}_{r,1}$ and $\mathcal{Q}_{1,s}$ as quotients of Feigin-Fuchs modules, rather than quotients of Verma modules, see [91, 92].

[14] The model with $p=q=1$ is just $\widehat{\mathfrak{sl}}(2)$ at level 1.

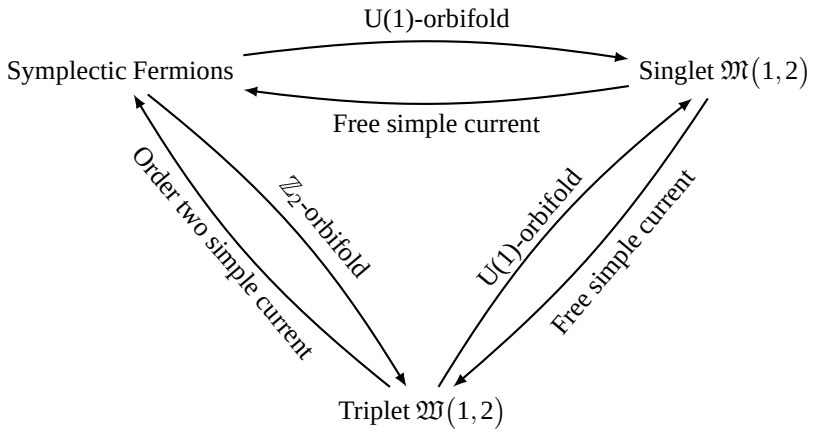

FIGURE 3. The symplectic fermion algebra, the triplet algebra $\mathfrak{W}(1,2)$ and the singlet algebra $\mathfrak{M}(1,2)$ are all related by simple current extensions and orbifolds.

This action is invariant under shifts by holomorphic and anti-holomorphic fields, and this implies, as with the free boson, the operator product expansion

$$\theta^+(z,\overline{z})\theta^-(w,\overline{w}) = A + \log|z-w|^2 + \cdots, \tag{3.2}$$

where A is a constant of integration.

The equations of motion state that $J^\pm(z) = \partial\theta^\pm(z,\overline{z})$ and $\overline{J}^\pm(\overline{z}) = \overline{\partial}\theta^\pm(z,\overline{z})$ are holomorphic and antiholomorphic, respectively, and we will take the former to generate the chiral algebra. These are the symplectic fermion currents and their operator product expansions are

$$J^+(z)J^-(w) \sim \frac{1}{(z-w)^2}, \qquad J^\pm(z)J^\pm(w) \sim 0. \tag{3.3}$$

Their modes then satisfy the anticommutation relations of the affine Lie superalgebra $\widehat{\mathfrak{psl}}(1|1)$ at level $k=1$ (as in the case of the free boson, any non-zero level may be rescaled to 1):

$$\{J^+_m, J^-_n\} = m\delta_{m+n=0}, \qquad \{J^\pm_m, J^\pm_n\} = 0. \tag{3.4}$$

The Virasoro field is $T(z) = :J^-(z)J^+(z):$ and its central charge is $c=-2$.

Let us turn to representations. As usual, we start with highest weight modules and these can be quickly analysed by expanding the double integral

$$\oint_0 \oint_w J^+(z)J^-(w) z^{m+1} w^n (z-w)^{-1} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathrm{d}w}{2\pi\mathrm{i}}$$

in the usual fashion, so as to obtain the generalised commutation relation

$$\sum_{j=0}^{\infty} \left[J^+_{m-j} J^-_{n+j} - J^-_{n-1-j} J^+_{m+1+j} \right] = \frac{1}{2} m(m+1)\delta_{m+n=0} - L_{m+n}. \tag{3.5}$$

From this, we see that the mode indices m and n must satisfy $m+n \in \mathbb{Z}$ so that the Virasoro algebra will be integer-moded. For now, we will assume that the moding for J^+ is the same as that of J^-. Then, m and n must be either both integers or both half-integers.

We apply (3.5), with $m=n=0$, to a state $|\phi\rangle$ which is annihilated by positive modes, obtaining

$$L_0|\phi\rangle = -J^+_0 J^-_0 |\phi\rangle \qquad \Rightarrow \qquad L_0^2|\phi\rangle = 0. \tag{3.6}$$

It follows that $|\phi\rangle$ belongs to a Jordan block for L_0 with eigenvalue 0 and rank at most 2. Repeating this, with $m=-\frac{1}{2}$ and $n=\frac{1}{2}$, gives $L_0|\phi\rangle = -\frac{1}{8}|\phi\rangle$, hence the only half-integer moded highest weight state has conformal dimension $-\frac{1}{8}$. It follows that we have only one module[15] in the half-integer moded sector, necessarily irreducible, that we shall denote by $\mathbb{L}_{1/2}$, the label corresponding to the moding. In the integer-moded sector, we have an irreducible $\mathbb{L}_0$ (the vacuum module) as well as an indecomposable $\mathbb{S}_0$ generated by a dimension 0 generalised

[15]Technically, there are two (graded) modules according as to the parity of its generating state. We shall usually ignore this distinction.

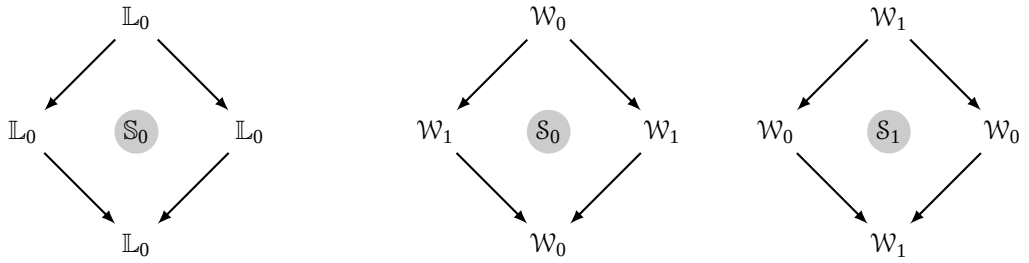

FIGURE 4. The Loewy diagrams for the socle series of the indecomposable symplectic fermion module $\mathbb{S}_0$ (left) and the indecomposable $\mathfrak{W}(1,2)$-modules $\mathcal{S}_0$ and $\mathcal{S}_1$ (right). In each case, the non-diagonalisable action of L_0 maps the top factor (the head) onto the bottom factor (the socle).

eigenvector $\big|\Omega\big\rangle$ of L_0.[16] Its Loewy diagram is given in Figure 4, where we remark that the four composition factors may be associated to the states $\big|\Omega\big\rangle$, $J_0^+\big|\Omega\big\rangle$, $J_0^-\big|\Omega\big\rangle$ and $J_0^-J_0^+\big|\Omega\big\rangle = \big|0\big\rangle$. Finally, we note that if J_0^- is regarded as a creation operator and J_0^+ as an annihilation operator, then $J_0^+\big|\Omega\big\rangle$ generates the vacuum Verma module $\mathbb{V}_0$ (the Verma module for half-integer moding is already irreducible: $\mathbb{V}_{1/2} = \mathbb{L}_{1/2}$) and the indecomposable $\mathbb{S}_0$ is characterised by the exact sequence

$$0 \longrightarrow \mathbb{V}_0 \longrightarrow \mathbb{S}_0 \longrightarrow \mathbb{V}_0 \longrightarrow 0. \tag{3.7}$$

The fusion ring generated by the irreducibles is particularly easy to work out using (a variant [94] of) the Nahm-Gaberdiel-Kausch algorithm. The vacuum module $\mathbb{L}_0$ is the fusion identity and one finds that

$$\mathbb{L}_{1/2} \times \mathbb{L}_{1/2} = \mathbb{S}_0, \qquad \mathbb{L}_{1/2} \times \mathbb{S}_0 = 4\,\mathbb{L}_{1/2}, \qquad \mathbb{S}_0 \times \mathbb{S}_0 = 4\,\mathbb{S}_0. \tag{3.8}$$

The characters $\mathrm{ch}\big[\mathbb{L}_\lambda\big](q) = \mathrm{tr}_{\mathbb{L}_\lambda}\, q^{L_0 - c/24}$ of the irreducibles are likewise easily obtained:

$$\mathrm{ch}\big[\mathbb{L}_0\big] = q^{1/12}\prod_{n=1}^{\infty}(1+q^n)^2 = \frac{\vartheta_2(1;q)}{2\eta(q)}, \qquad \mathrm{ch}\big[\mathbb{L}_{1/2}\big] = q^{-1/24}\prod_{n=1}^{\infty}\Big(1+q^{n-1/2}\Big)^2 = \frac{\vartheta_3(1;q)}{\eta(q)}. \tag{3.9a}$$

Note that the factor of 2 for $\mathbb{L}_0$ would disappear if we instead considered its Verma cover $\mathbb{V}_0$. As symplectic fermions are described by an affine Lie superalgebra, it is natural to also consider the supercharacters in which fermionic states are counted with negative multiplicity (we assume a bosonic ground state):

$$\mathrm{sch}\big[\mathbb{L}_0\big] = q^{1/12}\prod_{n=1}^{\infty}(1-q^n)^2 = \eta(q)^2, \qquad \mathrm{sch}\big[\mathbb{L}_{1/2}\big] = q^{-1/24}\prod_{n=1}^{\infty}\Big(1-q^{n-1/2}\Big)^2 = \frac{\vartheta_4(1;q)}{\eta(q)}. \tag{3.9b}$$

Excluding $\mathrm{sch}\big[\mathbb{L}_0\big]$, whose S-transformation involves factors of τ (we write $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ as usual), these characters and supercharacters have good modular properties. However, the S-matrix one obtains has no strictly positive row or column, hence the Verlinde formula is inapplicable.

Finally, we consider general modings for the currents J^+ and J^-. Representations on which the algebra acts with modings different to that of the vacuum module, for example $\mathbb{L}_{1/2}$, are said to be *twisted*.[17] For symplectic fermions, the general twisted module $\mathbb{V}_\lambda = \mathbb{L}_\lambda$ is the irreducible generated by a highest weight state $\big|\mu_\lambda\big\rangle$ upon which the symplectic fermion currents act with mode decomposition

$$J^\pm(z) = \sum_{n\in\mathbb{Z}\mp\lambda} J_n^\pm z^{-n-1}. \tag{3.10}$$

Taking $0 < \lambda < 1$ and applying (3.5), the conformal dimension of $\big|\mu_\lambda\big\rangle$ is found to be $\Delta_\lambda = -\frac{1}{2}\lambda(1-\lambda)$. The corresponding primary field $\mu_\lambda(z)$ is called a *twist field* [95]. The character of $\mathbb{L}_\lambda$ is given by

$$\mathrm{ch}\big[\mathbb{L}_\lambda\big] = q^{-\lambda(1-\lambda)/2+1/12}\prod_{n=0}^{\infty}\Big(1+q^{n-\lambda}\Big)\Big(1+q^{n-1+\lambda}\Big) = \frac{1}{\eta(q)}\sum_{m\in\mathbb{Z}} q^{(m+\lambda-1/2)^2/2} \tag{3.11}$$

and the supercharacter is obtained by inserting a factor of $(-1)^m$ into the sum. We will return to twisted modules when we consider the singlet algebra $\mathfrak{M}(1,2)$ in Section 3.3.

[16] The existence of this indecomposable module follows upon realising that it may be obtained from the universal enveloping algebra of $\mathfrak{psl}(1|1)$, considered as a four-dimensional $\mathfrak{psl}(1|1)$-module, through the induced module construction. Other indecomposables with integer moding may similarly be constructed [93].

[17] Strictly speaking, these are only modules for an orbifold of the chiral algebra by some cyclic group, but there is usually little harm in neglecting this.

3.2. **The Triplet Algebra $\mathfrak{W}(1,2)$.** The triplet algebra is defined to be the bosonic subalgebra of the symplectic fermion algebra. It is generated by the three fields

$$W^{\pm}(z) = :J^{\pm}(z)\partial J^{\pm}(z):, \qquad W^{0}(z) = :J^{+}(z)\partial J^{-}(z): - :\partial J^{+}(z)J^{-}(z): \tag{3.12}$$

and the energy-momentum tensor $T(z)$. The triplet fields $W^{\pm}(z)$ and $W^{0}(z)$ are Virasoro primaries (with respect to $T(z)$) and the conformal dimension of each is 3. Their operator product expansions are rather unpleasant and may be found, for example, in [29]. A complete set of (untwisted) irreducible $\mathfrak{W}(1,2)$-modules [96] may be obtained by decomposing the symplectic fermion irreducibles $\mathbb{L}_0$ and $\mathbb{L}_{1/2}$ into their bosonic and fermionic subspaces:

$$\mathbb{L}_0 = \mathcal{W}_0 \oplus \mathcal{W}_1, \qquad \mathbb{L}_{1/2} = \mathcal{W}_{-1/8} \oplus \mathcal{W}_{3/8}. \tag{3.13}$$

Here, we have labelled the triplet modules by the conformal dimension of their ground states. The space of ground states is one-dimensional for $\mathcal{W}_0$ and $\mathcal{W}_{-1/8}$, but two-dimensional for $\mathcal{W}_1$ and $\mathcal{W}_{3/8}$. The indecomposable $\mathbb{S}_0$ also becomes a direct sum of two indecomposables when viewed as a $\mathfrak{W}(1,2)$-module:

$$\mathbb{S}_0 = \mathcal{S}_0 \oplus \mathcal{S}_1. \tag{3.14}$$

The Loewy diagrams of $\mathcal{S}_0$ and $\mathcal{S}_1$ are given in Figure 4.

The fusion ring generated by the irreducibles was first determined in [32]. The vacuum module $\mathcal{W}_0$ is again the fusion identity and the other rules are

$$\begin{gathered}
\mathcal{W}_1 \times \mathcal{W}_1 = \mathcal{W}_0, \qquad \mathcal{W}_1 \times \mathcal{W}_{-1/8} = \mathcal{W}_{3/8}, \qquad \mathcal{W}_1 \times \mathcal{W}_{3/8} = \mathcal{W}_{-1/8}, \\
\mathcal{W}_{-1/8} \times \mathcal{W}_{-1/8} = \mathcal{S}_0, \qquad \mathcal{W}_{-1/8} \times \mathcal{W}_{3/8} = \mathcal{S}_1, \qquad \mathcal{W}_{3/8} \times \mathcal{W}_{3/8} = \mathcal{S}_0, \\
\mathcal{W}_1 \times \mathcal{S}_0 = \mathcal{S}_1, \qquad \mathcal{W}_1 \times \mathcal{S}_1 = \mathcal{S}_0, \\
\mathcal{W}_{-1/8} \times \mathcal{S}_0 = \mathcal{W}_{3/8} \times \mathcal{S}_0 = \mathcal{W}_{-1/8} \times \mathcal{S}_1 = \mathcal{W}_{3/8} \times \mathcal{S}_1 = 2\,\mathcal{W}_{-1/8} \oplus 2\,\mathcal{W}_{3/8}, \\
\mathcal{S}_0 \times \mathcal{S}_0 = \mathcal{S}_0 \times \mathcal{S}_1 = \mathcal{S}_1 \times \mathcal{S}_1 = 2\,\mathcal{S}_0 \oplus 2\,\mathcal{S}_1.
\end{gathered} \tag{3.15}$$

We note that $\mathcal{W}_1$ is a simple current of order 2. The corresponding extended algebra is, of course, the symplectic fermion algebra (the ground states of $\mathcal{W}_1$ correspond to the currents $J^{\pm}(z)$). It is easy to check that the symplectic fermion fusion rules (3.8) are consistent with (3.15) and this observation. The irreducible triplet characters were first obtained in [29, 31], but follow easily from averaging the characters and supercharacters of the symplectic fermions irreducibles, given in (3.9):

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{W}_0\big] &= \frac{1}{2}\big(\mathrm{ch}\big[\mathbb{L}_0\big] + \mathrm{sch}\big[\mathbb{L}_0\big]\big) = \frac{\vartheta_2(1;q)}{4\eta(q)} + \frac{\eta(q)^2}{2}, \\
\mathrm{ch}\big[\mathcal{W}_1\big] &= \frac{1}{2}\big(\mathrm{ch}\big[\mathbb{L}_0\big] - \mathrm{sch}\big[\mathbb{L}_0\big]\big) = \frac{\vartheta_2(1;q)}{4\eta(q)} - \frac{\eta(q)^2}{2}, \\
\mathrm{ch}\big[\mathcal{W}_{-1/8}\big] &= \frac{1}{2}\big(\mathrm{ch}\big[\mathbb{L}_{1/2}\big] + \mathrm{sch}\big[\mathbb{L}_{1/2}\big]\big) = \frac{\vartheta_3(1;q) + \vartheta_4(1;q)}{2\eta(q)}, \\
\mathrm{ch}\big[\mathcal{W}_{3/8}\big] &= \frac{1}{2}\big(\mathrm{ch}\big[\mathbb{L}_{1/2}\big] - \mathrm{sch}\big[\mathbb{L}_{1/2}\big]\big) = \frac{\vartheta_3(1;q) - \vartheta_4(1;q)}{2\eta(q)}.
\end{aligned} \tag{3.16}$$

The modular properties of $\mathcal{W}_{-1/8}$ and $\mathcal{W}_{3/8}$ are seen to be good, but those of $\mathcal{W}_0$ and $\mathcal{W}_1$ are not as satisfactory because the $\eta(q)^2$ gives rise to coefficients involving $\log q = 2\pi\mathrm{i}\tau$. For example,

$$\mathrm{ch}\big[\mathcal{W}_0\big]\Big|_{\mathsf{S}} = \frac{\vartheta_4(1;q)}{4\eta(q)} - \frac{\mathrm{i}\tau\eta(q)^2}{2} = \frac{1}{4}\big(\mathrm{ch}\big[\mathcal{W}_{-1/8}\big] - \mathrm{ch}\big[\mathcal{W}_{3/8}\big]\big) - \frac{\mathrm{i}\tau}{2}\big(\mathrm{ch}\big[\mathcal{W}_0\big] - \mathrm{ch}\big[\mathcal{W}_1\big]\big). \tag{3.17}$$

Attempts have been made to interpret this, see [31, 97] for example.

3.3. **The Singlet Algebra $\mathfrak{M}(1,2)$.** To define the singlet algebra at $c = -2$, it is convenient to extend the $\mathbb{Z}_2$-grading of the symplectic fermion algebra, given by parity, to a $\mathbb{Z}$-grading. This may be regarded as the ghost number in the $\xi\eta$ ghost system realisation or as the eigenvalue of a derivation N extending $\widehat{\mathfrak{psl}}(1|1)$. In any case, acting with $J^{\pm}_n$ increases this grade by ± 1. We can now define the singlet algebra $\mathfrak{M}(1,2)$ as the subalgebra of symplectic fermions whose $\mathbb{Z}$-grade matches that of the vacuum. This is therefore a subalgebra of the triplet algebra and it is generated by $T(z)$ and $W^0(z)$. In contrast to the symplectic fermion currents $J^{\pm}(z)$ and the triplet fields $W^{\pm}(z)$, the singlet generators act with integer moding on every twisted symplectic fermion module. Decomposing the symplectic fermion's twisted Verma modules $\mathbb{V}_\lambda$, which coincide with the irreducibles $\mathbb{L}_\lambda$ when $\lambda \neq 0$, into $\mathbb{Z}$-graded subspaces shows that the singlet algebra possesses an uncountable set of non-isomorphic

(untwisted) modules $\mathcal{F}_\mu$, $\mu \in \mathbb{R}$:

$$\mathbb{V}_\lambda = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{\lambda+m} \qquad (0 \leqslant \lambda < 1). \tag{3.18}$$

The minimal conformal dimension for states of $\mathcal{F}_\mu$ is $\Delta_\mu = \frac{1}{2}\mu(\mu-1)$.

For $\lambda \neq 0$, it turns out that the $\mathcal{F}_{\lambda+m}$ so obtained are irreducible. By analogy with the case of superalgebras [98], the $\mathcal{F}_\mu$ with $\mu \notin \mathbb{Z}$ will therefore be referred to as *typical*. When $\lambda = 0$, irreducible $\mathfrak{M}(1,2)$-modules are obtained by decomposing the irreducible vacuum module $\mathbb{L}_0 \neq \mathbb{V}_0$ instead:

$$\mathbb{L}_0 = \bigoplus_{r\in\mathbb{Z}} \mathcal{M}_r. \tag{3.19}$$

These irreducibles will be referred to as *atypical*. Note that the singlet vacuum module is $\mathcal{M}_0$ and that the minimal conformal dimension for states of $\mathcal{M}_r$ is $\frac{1}{2}|r|(|r|+1)$. We also remark that the $\mathcal{F}_\mu$ with $\mu \notin \mathbb{Z}$ and the $\mathcal{M}_r$ with $r \in \mathbb{Z}$ exhaust the irreducible $\mathfrak{M}(1,2)$-modules [96, 99].

The characters of the typical irreducibles $\mathcal{F}_\mu$, $\mu \notin \mathbb{Z}$, are easily extracted from the twisted symplectic fermion characters (3.11) once the $\mathbb{Z}$-charge is taken into account:

$$\mathrm{ch}\big[\mathcal{F}_\mu\big] = \frac{q^{(\mu-1/2)^2/2}}{\eta(q)}. \tag{3.20}$$

This formula also applies to the indecomposables $\mathcal{F}_r$, $r \in \mathbb{Z}$. Note that up to shifting μ by $-\frac{1}{2}$, these characters coincide with the free boson characters (1.13) discussed in Section 1.2. It is also straight-forward to obtain the characters of the atypical irreducibles $\mathcal{M}_r$, $r \in \mathbb{Z}$. However, we will not need their explicit form in what follows and we only mention that these forms involve an interesting number theoretical object called a false theta function.

Instead, we study the structure of the indecomposable modules $\mathcal{F}_r$, for $r \in \mathbb{Z}$. These are defined as the subspaces of $\mathbb{V}_0$ of constant $\mathbb{Z}$-grade. It follows from (3.7) that $\mathcal{F}_r$ is an indecomposable sum of two atypicals and a little thought leads us to the (non-split) exact sequence

$$0 \longrightarrow \mathcal{M}_{r+1} \longrightarrow \mathcal{F}_{r+1} \longrightarrow \mathcal{M}_r \longrightarrow 0. \tag{3.21}$$

Splicing the short exact sequence for $\mathcal{M}_r$ with that for $\mathcal{M}_{r+1}$ and iterating (see Appendix A.2), we arrive at resolutions of the atypical irreducible modules:

$$\cdots \longrightarrow \mathcal{F}_{r+5} \longrightarrow \mathcal{F}_{r+4} \longrightarrow \mathcal{F}_{r+3} \longrightarrow \mathcal{F}_{r+2} \longrightarrow \mathcal{F}_{r+1} \longrightarrow \mathcal{M}_r \longrightarrow 0. \tag{3.22}$$

These imply that the characters of the irreducible atypical modules may be expressed as infinite alternating sums over the typical characters:

$$\mathrm{ch}\big[\mathcal{M}_r\big] = \sum_{j=0}^{\infty}\Big(\mathrm{ch}\big[\mathcal{F}_{r+2j+1}\big] - \mathrm{ch}\big[\mathcal{F}_{r+2j+2}\big]\Big) = \sum_{j=0}^{\infty}(-1)^j \mathrm{ch}\big[\mathcal{F}_{r+j+1}\big]. \tag{3.23}$$

In particular, we conclude that the characters of the $\mathcal{F}_\mu$ — the irreducible typicals as well as the indecomposable atypicals — form a (topological) basis for the vector space spanned by the characters. We will use this to apply the Verlinde formula to irreducible $\mathfrak{M}(1,2)$-modules and thereby deduce the (unknown) fusion rules of the singlet theory.

3.4. **Modular Transformations and the Verlinde Formula.** Before deriving the modular transformations, we remark that the typical singlet characters (3.20) suffer from the same deficiency as the standard free boson characters (1.13) in that they do not completely distinguish the representations: $\mathrm{ch}\big[\mathcal{F}_\mu\big] = \mathrm{ch}\big[\mathcal{F}_{1-\mu}\big]$. As in Section 1.2, the fix is to include the $\mathbb{Z}$-grading and the $\widehat{\mathfrak{psl}}(1|1)$ level k:

$$\mathrm{ch}\big[\mathcal{F}_\mu\big] = \mathrm{tr}_{\mathcal{F}_\mu} y^k z^{\mu-1/2} q^{L_0-c/24} = \frac{y^k z^{\mu-1/2} q^{(\mu-1/2)^2/2k}}{\eta(q)}. \tag{3.24}$$

Here, we have finally fixed our choice for the $\mathbb{Z}$-grading used to define singlet modules: $\mathcal{F}_\mu$ is assigned the grade $\mu - \frac{1}{2}$ in $\mathbb{V}_{\mu'}$ (where $\mu' = \mu \bmod 1$). We do this because the typical singlet character (3.24) then takes the same form as the free boson character (1.14), up to the shifts by $-\frac{1}{2}$.

Writing $y = \mathrm{e}^{2\pi\mathrm{i}t}$, $z = \mathrm{e}^{2\pi\mathrm{i}u}$ and $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, as in Section 1.2, the modular S-transformation for the typical (and indecomposable atypical) characters is then immediate from (1.15):

$$\mathrm{ch}\big[\mathcal{F}_\lambda\big]\Big|_{\mathsf{S}} = \int_{-\infty}^{\infty} \mathsf{S}_{\lambda\mu}\mathrm{ch}\big[\mathcal{F}_\mu\big]\,\mathrm{d}\mu, \qquad \mathsf{S}_{\lambda\mu} = \mathrm{e}^{-2\pi\mathrm{i}(\lambda-1/2)(\mu-1/2)} \tag{3.25}$$

(we have set k back to 1 for convenience here). This S-matrix is again symmetric and unitary. Moreover, with the T-matrix $\mathsf{T}_{\lambda\mu} = \mathrm{e}^{\mathrm{i}\pi(\lambda(\lambda-1)+1/6)}\delta(\lambda=\mu)$, it defines a representation of $\mathsf{SL}(2;\mathbb{Z})$ in which the conjugation permutation is $\lambda \to 1-\lambda$.

By analogy with Section 1.2, we expect that a continuum version of the Verlinde formula will be valid. However, we must now take into account the fact that the vacuum $\mathfrak{M}(1,2)$-module is the *atypical* irreducible $\mathcal{M}_0$. As there is no "atypicality" with the free boson's representations, our story now deviates from that of Section 1.2. Using the character formula (3.23) for atypical irreducibles, we easily obtain their S-transformations expressed in terms of the topological basis $\{\mathrm{ch}[\mathcal{F}_\mu] \ : \ \mu\in\mathbb{R}\}$:

$$\begin{aligned}\mathrm{ch}[\mathcal{M}_r]\Big|_{\mathsf{S}} &= \sum_{j=0}^{\infty}(-1)^j \mathrm{ch}[\mathcal{F}_{r+j+1}]\Big|_{\mathsf{S}} = \sum_{j=0}^{\infty}(-1)^j \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}(r+j+1/2)(\mu-1/2)}\mathrm{ch}[\mathcal{F}_\mu]\,\mathrm{d}\mu \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}(r+1/2)(\mu-1/2)}}{1+e^{-2\pi\mathrm{i}(\mu-1/2)}}\mathrm{ch}[\mathcal{F}_\mu]\,\mathrm{d}\mu = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}r(\mu-1/2)}}{2\cos[\pi(\mu-1/2)]}\mathrm{ch}[\mathcal{F}_\mu]\,\mathrm{d}\mu. \end{aligned} \tag{3.26}$$

The corresponding S-matrix entry is therefore

$$\mathsf{S}_{\underline{r}\mu} = \frac{\mathrm{e}^{-2\pi\mathrm{i}r(\mu-1/2)}}{2\cos[\pi(\mu-1/2)]}, \tag{3.27}$$

where we have indicated when a label corresponds to an atypical irreducible by underlining it.

We can now apply the continuum Verlinde formula to compute fusion coefficients. Actually, because characters cannot distinguish an indecomposable from the direct sum of its composition factors, what the Verlinde formula gives is the structure constants of the *Grothendieck* fusion ring (Appendix A.3). The easiest computation is that for the fusion of an atypical and a typical:

$$\mathsf{N}_{\underline{r}\mu}{}^{\nu} = \int_{-\infty}^{\infty}\frac{\mathsf{S}_{\underline{r}\rho}\mathsf{S}_{\mu\rho}\mathsf{S}^*_{\nu\rho}}{\mathsf{S}_{\underline{0}\rho}}\,\mathrm{d}\rho = \int_{-\infty}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}(r+\mu-\nu)(\rho-1/2)}\,\mathrm{d}\rho = \delta(\nu=\mu+r). \tag{3.28}$$

Fusing two typicals is only slightly more involved because the denominator of $\mathsf{S}_{\underline{0}\rho}$ no longer cancels:

$$\begin{aligned}\mathsf{N}_{\lambda\mu}{}^{\nu} &= \int_{-\infty}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}(\lambda+\mu-\nu-1/2)(\rho-1/2)}\left(\mathrm{e}^{\mathrm{i}\pi(\rho-1/2)}+\mathrm{e}^{-\mathrm{i}\pi(\rho-1/2)}\right)\mathrm{d}\rho \\ &= \delta(\nu=\lambda+\mu)+\delta(\nu=\lambda+\mu-1).\end{aligned} \tag{3.29}$$

Fusing two atypicals is a little more subtle however. Computing naïvely, one quickly arrives at a divergent integral. The problem here may be traced back to the derivation of (3.27) in which we summed a geometric series at its radius of convergence. The fix is obvious: Expand the geometric series once again (in the right region) as continue to integrate. From this perspective, the dubious summation may be simply regarded as a placeholder that simplifies some computations. With this proviso, we quickly obtain

$$\begin{aligned}\mathsf{N}_{\underline{r}\underline{s}}{}^{\nu} &= \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-2\pi\mathrm{i}(r+s-\nu+1/2)(\rho-1/2)}}{2\cos[\pi(\rho-1/2)]}\,\mathrm{d}\rho = \sum_{j=0}^{\infty}\int_{-\infty}^{\infty}(-1)^j\mathrm{e}^{-2\pi\mathrm{i}(r+s+j+1-\nu)\sigma}\,\mathrm{d}\sigma \\ &= \sum_{j=0}^{\infty}(-1)^j\delta(\nu=r+s+j+1).\end{aligned} \tag{3.30}$$

This seems to say that the fusion of two atypicals leads to negative multiplicities (for j odd), but in fact, this infinite alternating sum corresponds to an atypical with positive multiplicity.

The Grothendieck ring of characters is now obtained by integrating these coefficients as in (1.20):

$$\begin{gathered}\mathrm{ch}[\mathcal{M}_r]\,\dot\times\,\mathrm{ch}[\mathcal{F}_\mu] = \mathrm{ch}[\mathcal{F}_{\mu+r}], \qquad \mathrm{ch}[\mathcal{F}_\lambda]\,\dot\times\,\mathrm{ch}[\mathcal{F}_\mu] = \mathrm{ch}[\mathcal{F}_{\lambda+\mu}]+\mathrm{ch}[\mathcal{F}_{\lambda+\mu-1}], \\ \mathrm{ch}[\mathcal{M}_r]\,\dot\times\,\mathrm{ch}[\mathcal{M}_s] = \sum_{j=0}^{\infty}(-1)^j\mathrm{ch}[\mathcal{F}_{r+s+j+1}] = \mathrm{ch}[\mathcal{M}_{r+s}].\end{gathered} \tag{3.31}$$

When one is sure that the characters cannot describe indecomposable modules, these Grothendieck fusion rules may be lifted to genuine fusion rules. In particular, we deduce that

$$\mathcal{M}_r\times\mathcal{M}_s = \mathcal{M}_{r+s}, \quad \mathcal{M}_r\times\mathcal{F}_\mu = \mathcal{F}_{\mu+r}, \quad \mathcal{F}_\lambda\times\mathcal{F}_\mu = \mathcal{F}_{\lambda+\mu}\oplus\mathcal{F}_{\lambda+\mu-1} \qquad (\lambda,\mu,\lambda+\mu\notin\mathbb{Z}), \tag{3.32}$$

the last constraint arising because the conformal dimensions of the states of $\mathcal{F}_\lambda$ and $\mathcal{F}_{\lambda-1}$ differ by λ mod 1. This means that the fusion of two irreducibles is known in every case except $\mathcal{F}_\lambda\times\mathcal{F}_\mu$ when $\lambda+\mu\in\mathbb{Z}$.

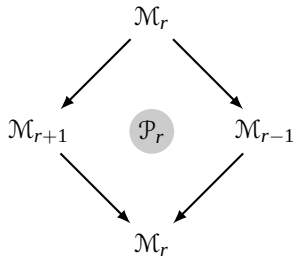

FIGURE 5. Conjectured Loewy diagram for the proposed indecomposable $\mathfrak{M}(1,2)$-module $\mathcal{P}_r$.

Observe now that (3.32) identifies the $\mathcal{M}_r$ as simple currents of infinite order (with no fixed points). It is easy to show that the maximal simple current extension, meaning the algebra generated by $\mathcal{M}_1$ and $\mathcal{M}_{-1}$, is precisely the symplectic fermion algebra $\widehat{\mathfrak{psl}}(1|1)$ (indeed, the generators of $\mathcal{M}_1$ and $\mathcal{M}_{-1}$ have conformal dimension 1). From this, we may conclude that the extension by $\mathcal{M}_3$ and $\mathcal{M}_{-3}$ is the triplet algebra $\mathfrak{W}(1,2)$. It is therefore the maximal *bosonic* simple current extension.

Let us now perform a consistency check on the continuum Verlinde formula by deriving (some of) the $\mathfrak{W}(1,2)$ fusion rules from (3.32). Our identification of the triplet algebra as a simple current extension of the singlet algebra leads to the restriction rules

$$\mathcal{W}_0 = \bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{2m}, \quad \mathcal{W}_1 = \bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{2m+1}, \quad \mathcal{W}_{-1/8} = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{2m+1/2}, \quad \mathcal{W}_{3/8} = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{2m-1/2}. \tag{3.33}$$

As in Section 1.2, we can use these rules to compute fusion, remembering to take a single $\mathfrak{M}(1,2)$-representative for one of the $\mathfrak{W}(1,2)$-modules being fused. To illustrate:

$$\mathcal{W}_0 \times \mathcal{W}_0 = \mathcal{M}_0 \times \Big(\bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{2m}\Big) = \bigoplus_{m\in\mathbb{Z}} (\mathcal{M}_0 \times \mathcal{M}_{2m}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{2m} = \mathcal{W}_0. \tag{3.34}$$

This procedure therefore correctly normalises the extended algebra fusion. Applying this, we can reproduce all the triplet fusion rules (3.15) except for those of $\mathcal{W}_{-1/8}$ and $\mathcal{W}_{3/8}$ with one another. For these latter rules, we can only compute the Grothendieck fusion, for example

$$\begin{aligned}
\operatorname{ch}\big[\mathcal{W}_{-1/8}\big] \dot\times \operatorname{ch}\big[\mathcal{W}_{3/8}\big] &= \operatorname{ch}\big[\mathcal{F}_{2\ell+1/2}\big] \dot\times \Big(\sum_{m\in\mathbb{Z}} \operatorname{ch}\big[\mathcal{F}_{2m-1/2}\big]\Big) = \sum_{m\in\mathbb{Z}} \Big(\operatorname{ch}\big[\mathcal{F}_{2(\ell+m)}\big] + \operatorname{ch}\big[\mathcal{F}_{2(\ell+m)-1}\big]\Big) \\
&= \sum_{m\in\mathbb{Z}} \Big(\operatorname{ch}\big[\mathcal{M}_{2(\ell+m)}\big] + 2\operatorname{ch}\big[\mathcal{M}_{2(\ell+m)-1}\big] + \operatorname{ch}\big[\mathcal{M}_{2(\ell+m)-2}\big]\Big) \\
&= 2\operatorname{ch}\big[\mathcal{W}_0\big] + 2\operatorname{ch}\big[\mathcal{W}_1\big],
\end{aligned} \tag{3.35}$$

where we have used (3.21). We note that this is consistent with $\mathcal{W}_{-1/8} \times \mathcal{W}_{3/8} = \mathcal{S}_1$ because (see Figure 4)

$$\operatorname{ch}\big[\mathcal{S}_0\big] = \operatorname{ch}\big[\mathcal{S}_1\big] = 2\operatorname{ch}\big[\mathcal{W}_0\big] + 2\operatorname{ch}\big[\mathcal{W}_1\big]. \tag{3.36}$$

Of course, the Grothendieck fusion of the $\mathfrak{M}(1,2)$-typicals $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$, $\lambda+\mu \in \mathbb{Z}$, gives us the composition factors of the fusion product:

$$\operatorname{ch}\big[\mathcal{F}_\lambda \times \mathcal{F}_\mu\big] = \operatorname{ch}\big[\mathcal{F}_\lambda\big] \dot\times \operatorname{ch}\big[\mathcal{F}_\mu\big] = \operatorname{ch}\big[\mathcal{M}_{\lambda+\mu}\big] + 2\operatorname{ch}\big[\mathcal{M}_{\lambda+\mu-1}\big] + \operatorname{ch}\big[\mathcal{M}_{\lambda+\mu-2}\big]. \tag{3.37}$$

Because the corresponding fusion products for the triplet and symplectic fermion algebras are indecomposable, we propose that this is true for the singlet algebra as well. We therefore conjecture that

$$\mathcal{F}_\lambda \times \mathcal{F}_{r+1-\lambda} = \mathcal{P}_r \qquad (r \in \mathbb{Z}), \tag{3.38}$$

where $\mathcal{P}_r$ is an indecomposable $\mathfrak{M}(1,2)$-module whose Loewy diagram is given in Figure 5. To prove this, one would have to either construct the fusion product explicitly (which seems very demanding), or deduce the existence of such indecomposables abstractly and show that they describe the restriction of the $\mathfrak{W}(1,2)$-indecomposables $\mathcal{S}_0$ and $\mathcal{S}_1$ to $\mathfrak{M}(1,2)$. Either approach is beyond the scope of this review.

3.5. **Bulk Modular Invariants.** Because of the symmetries of the S-matrix, the singlet theory has two obvious bulk modular invariants, corresponding to the diagonal and charge-conjugate partition functions:

$$\mathbf{Z}_{\text{diag.}}(q,\overline{q}) = \int_{-\infty}^{\infty} \overline{\operatorname{ch}\big[\mathcal{F}_\lambda\big]} \operatorname{ch}\big[\mathcal{F}_\lambda\big]\, \mathrm{d}\lambda, \qquad \mathbf{Z}_{\text{c.c.}}(q,\overline{q}) = \int_{-\infty}^{\infty} \overline{\operatorname{ch}\big[\mathcal{F}_{1-\lambda}\big]} \operatorname{ch}\big[\mathcal{F}_\lambda\big]\, \mathrm{d}\lambda. \tag{3.39}$$

As usual, simple current extensions allow one to construct more. Specifically, we have seen that each atypical $\mathfrak{M}(1,2)$-module $\mathcal{M}_n$ is a simple current, so an extended algebra $\mathbb{W}^{(n)}$ may be constructed by promoting all the fields in the fusion orbit of the vacuum module to symmetry generators:

$$\mathbb{W}^{(n)} = \bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{mn}. \tag{3.40}$$

We have already remarked that $\mathbb{W}^{(1)}$ is the symplectic fermion algebra and $\mathbb{W}^{(2)}$ is the triplet algebra. The other extended algebras likewise give rise to rational logarithmic conformal field theories and may be described as orbifolds of the symplectic fermion theory [29]. The fusion orbits through other $\mathfrak{M}(1,2)$-modules likewise give rise to (twisted) $\mathbb{W}^{(n)}$-modules, for example

$$\mathbb{M}^{(n)}_r = \bigoplus_{m\in\mathbb{Z}} \mathcal{M}_{r+mn}, \qquad \mathbb{F}^{(n)}_\lambda = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}_{\lambda+mn}. \tag{3.41}$$

The untwisted extended algebra modules are precisely the $\mathbb{M}^{(n)}_r$ and the $\mathbb{F}^{(n)}_\lambda$ with $\lambda \in \frac{1}{n}\mathbb{Z}$. By restricting to these, we arrive at new modular invariants.

The modular S-transformation of an untwisted typical extended algebra module is easy to calculate:

$$\begin{aligned}
\operatorname{ch}\big[\mathbb{F}^{(n)}_{j/n}\big]\Big|_{\mathsf{S}} &= \sum_{m\in\mathbb{Z}} \operatorname{ch}\big[\mathcal{F}_{j/n+mn}\big]\Big|_{\mathsf{S}} = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathsf{e}^{-2\pi\mathrm{i}(j/n+mn-1/2)(\mu-1/2)} \operatorname{ch}\big[\mathcal{F}_\mu\big]\,\mathrm{d}\mu \\
&= \int_{-\infty}^{\infty} \sum_{m\in\mathbb{Z}} \mathsf{e}^{-2\pi\mathrm{i}mn(\mu-1/2)} \mathsf{e}^{-2\pi\mathrm{i}(j/n-1/2)(\mu-1/2)} \operatorname{ch}\big[\mathcal{F}_\mu\big]\,\mathrm{d}\mu \\
&= \frac{1}{n}\int_{-\infty}^{\infty} \sum_{\ell\in\mathbb{Z}} \delta(\mu-1/2=\ell/n) \mathsf{e}^{-2\pi\mathrm{i}(j/n-1/2)(\mu-1/2)} \operatorname{ch}\big[\mathcal{F}_\mu\big]\,\mathrm{d}\mu \\
&= \frac{1}{n}\sum_{\ell\in\mathbb{Z}} \mathsf{e}^{-2\pi\mathrm{i}(j/n-1/2)\ell/n} \operatorname{ch}\big[\mathcal{F}_{\ell/n+1/2}\big] \\
&= \frac{1}{n}\sum_{m\in\mathbb{Z}} \sum_{k=0}^{n^2-1} \mathsf{e}^{-2\pi\mathrm{i}(j/n-1/2)(k/n+mn)} \operatorname{ch}\big[\mathcal{F}_{k/n+mn+1/2}\big].
\end{aligned} \tag{3.42}$$

Here, we pause to note that if n is even, then the exponential factor in the above sum is independent of m. We may therefore perform the m-summation and then shift k to $k-\frac{1}{2}n$, obtaining

$$\operatorname{ch}\big[\mathbb{F}^{(n)}_{j/n}\big]\Big|_{\mathsf{S}} = \sum_{k=0}^{n^2-1} \mathbb{S}^{(n)}_{jk} \operatorname{ch}\big[\mathbb{F}^{(n)}_{k/n}\big], \quad \mathbb{S}^{(n)}_{jk} = \frac{1}{n}\mathsf{e}^{-2\pi\mathrm{i}(j/n-1/2)(k/n-1/2)} \qquad (n \text{ even}). \tag{3.43}$$

The *typical* extended characters, for n even, therefore carry a finite-dimensional representation of the modular group. If n is odd however, then the exponential factor includes a factor $(-1)^m$ and the $\mathcal{F}_\lambda$ do not combine to give an untwisted extended algebra module. One is instead forced to consider supercharacters and twisted modules, the final result being that the modular invariant one constructs from $\mathbb{W}^{(n)}$, with n odd, is equivalent to that constructed from $\mathbb{W}^{(2n)}$. This is consistent with the observation that the extended algebra generators are fermionic for n odd and bosonic for n even.

As the (n even) S-matrix of (3.43) is unitary, the diagonal and charge-conjugate partition functions are modular invariants:

$$\mathbf{Z}^{(n)}_{\text{diag.}} = \sum_{j=0}^{n^2-1} \overline{\operatorname{ch}\big[\mathbb{F}^{(n)}_{j/n}\big]} \operatorname{ch}\big[\mathbb{F}^{(n)}_{j/n}\big], \qquad \mathbf{Z}^{(n)}_{\text{c.c.}} = \sum_{j=0}^{n^2-1} \overline{\operatorname{ch}\big[\mathbb{F}^{(n)}_{j/n}\big]} \operatorname{ch}\big[\mathbb{F}^{(n)}_{1-j/n}\big]. \tag{3.44}$$

Expressing these in terms of singlet characters finally gives new modular invariants for $\mathfrak{M}(1,2)$. We will not write them out in generality, noting only that for the triplet algebra $\mathfrak{W}(1,2) = \mathbb{W}^{(2)}$, the diagonal and charge conjugate modular invariant coincide. Since $\operatorname{ch}\big[\mathbb{F}^{(2)}_0\big] = \operatorname{ch}\big[\mathbb{F}^{(2)}_1\big] = \operatorname{ch}\big[\mathcal{W}_0\big] + \operatorname{ch}\big[\mathcal{W}_1\big]$, they are

$$\mathbf{Z}^{(2)}_{\text{diag.}} = \big|\operatorname{ch}\big[\mathcal{W}_{-1/8}\big]\big|^2 + \big|\operatorname{ch}\big[\mathcal{W}_{3/8}\big]\big|^2 + 2\,\big|\operatorname{ch}\big[\mathcal{W}_0\big] + \operatorname{ch}\big[\mathcal{W}_1\big]\big|^2. \tag{3.45}$$

Note that the obvious candidate for the non-chiral vacuum module $\mathcal{W}_0 \otimes \mathcal{W}_0$ contributes with multiplicity 2. We will see that this is explained by the bulk vacuum $\big|0\big\rangle \otimes \big|0\big\rangle$ having a non-chiral logarithmic partner $\big|\Omega\big\rangle$.

Finally, we remark that the modular transformations of the atypical extended characters are problematic. Repeating the calculation that led to the typical extended S-matrix (3.43), assuming again that n is even, leads to the

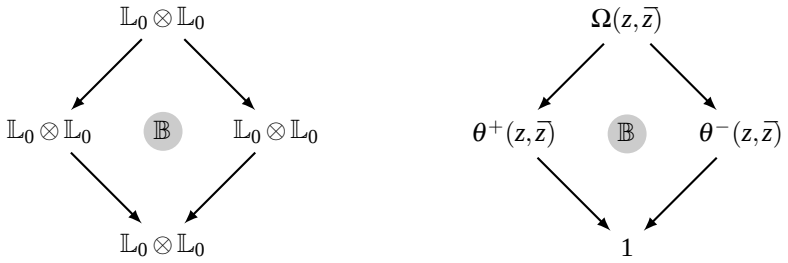

FIGURE 6. On the left, the Loewy diagram of the bulk indecomposable module $\mathbb{B}$ (over the direct sum of two copies of the symplectic fermion algebra). On the right, the same diagram but with the bulk composition factors replaced by the fields naturally associated to them.

sum

$$\mathrm{ch}\big[\mathbb{M}_r^{(n)}\big]\Big|_{\mathsf{S}} = \frac{1}{n}\sum_{k=0}^{n^2-1}\frac{\mathrm{e}^{-2\pi\mathrm{i}rk/n}}{2\cos[\pi k/n]}\mathrm{ch}\big[\mathbb{F}^{(n)}_{k/n+1/2}\big]. \tag{3.46}$$

The pole at $k=\frac{1}{2}n$ cannot be swept aside in this summation as it was when we were integrating. Of course, we know for the case $n=2$ ($\mathfrak{W}(1,2)$) that the S-transformation of the atypical characters involves factors of $\log q$, so we should not expect that the above approach will work. This can be traced back to the fact that the atypical extended characters can, unlike their singlet counterparts, no longer be written as an (infinite) linear combination of the typical extended characters. The extended version of the resolution (3.22) is periodic in the typical labels and, consequently, the character formula (3.23) is divergent.

3.6. **Bulk State Spaces.** Let us consider the symplectic fermions once again. Recall that the action (3.1) is defined in terms of (non-chiral) fermions $\theta^{\pm}(z,\overline{z})$. From these, we can construct the following field:

$$\Omega(z,\overline{z}) = :\theta^+(z,\overline{z})\theta^-(z,\overline{z}):. \tag{3.47}$$

The symplectic fermion currents, both holomorphic and antiholomorphic, act on Ω as follows:

$$\begin{aligned} J^{\pm}(z)\Omega(w,\overline{w}) &\sim \mp\frac{\theta^{\pm}(w,\overline{w})}{z-w}, \qquad & J^{\pm}(z)\theta^{\mp}(w,\overline{w}) &\sim \frac{1}{z-w},\\ \overline{J}^{\pm}(\overline{z})\Omega(w,\overline{w}) &\sim \mp\frac{\theta^{\pm}(w,\overline{w})}{\overline{z}-\overline{w}}, \qquad & \overline{J}^{\pm}(\overline{z})\theta^{\mp}(w,\overline{w}) &\sim \frac{1}{\overline{z}-\overline{w}}. \end{aligned} \tag{3.48}$$

From this, we deduce the structure of the bulk indecomposable symplectic fermion module $\mathbb{B}$, see Figure 6. We remark that its character is twice the atypical contribution to the $\mathfrak{W}(1,2)$ modular invariant (3.45).

The symplectic fermion bulk module is graded by the (total) fermion number, hence it decomposes into the direct sum of two bulk modules over $\mathfrak{W}(1,2)$ (or rather over two copies of it). Only one of these modules $\mathcal{B}$ contains the triplet vacuum module $\mathcal{W}_0\otimes\mathcal{W}_0$ as a submodule. We illustrate its structure in Figure 7. As its character coincides with the atypical contribution to the partition function (3.45), we conclude that the bulk space of states corresponding to this modular invariant is

$$\mathcal{H}_{\text{bulk}} = \big(\mathcal{W}_{-1/8}\otimes\mathcal{W}_{-1/8}\big)\oplus\big(\mathcal{W}_{3/8}\otimes\mathcal{W}_{3/8}\big)\oplus\mathcal{B}. \tag{3.49}$$

This conclusion essentially defines the bulk triplet theory as the bosonic subtheory of the bulk symplectic fermions theory in which the only non-local fields admitted are those on which the fermions act with half-integer moding. This confirms the triplet model as the $\mathbb{Z}_2$-orbifold of symplectic fermions.

It is worth thinking for a few moments how one could have arrived at the triplet model's bulk state space structure (3.49) if the non-chiral information concerning symplectic fermions was not so readily available. First, the modular invariant in (3.45) is very suggestive, especially when we may rewrite the atypical contribution, using the character identity (3.36), as

$$2\big|\mathrm{ch}\big[\mathcal{W}_0\big]+\mathrm{ch}\big[\mathcal{W}_1\big]\big|^2 = \overline{\mathrm{ch}\big[\mathcal{W}_0\big]}\mathrm{ch}\big[\mathcal{S}_0\big]+\overline{\mathrm{ch}\big[\mathcal{W}_1\big]}\mathrm{ch}\big[\mathcal{S}_1\big] = \overline{\mathrm{ch}\big[\mathcal{S}_0\big]}\mathrm{ch}\big[\mathcal{W}_0\big]+\overline{\mathrm{ch}\big[\mathcal{S}_1\big]}\mathrm{ch}\big[\mathcal{W}_1\big]. \tag{3.50}$$

This ties in nicely with the idea discussed in the introduction that natural representations often decompose in a manner whereby each irreducible is paired with its projective cover. Indeed, it is known [100] that the irreducibles $\mathcal{W}_{-1/8}$ and $\mathcal{W}_{3/8}$ are projective and that the projective covers of the irreducibles $\mathcal{W}_0$ and $\mathcal{W}_1$ are $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively (in an appropriate category of vertex algebra modules).

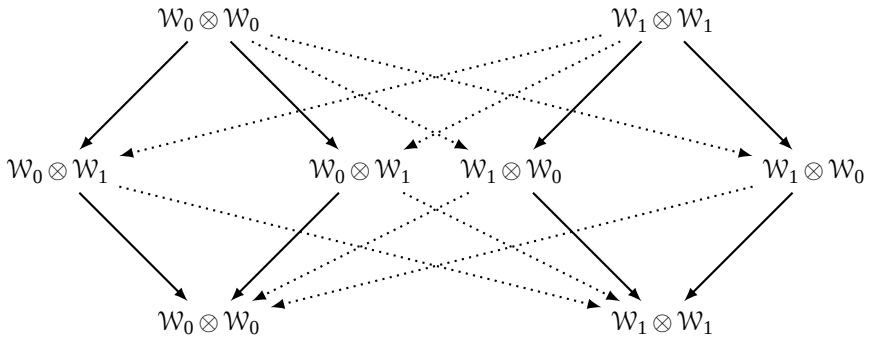

FIGURE 7. The Loewy diagram for the indecomposable bulk module $\mathcal{B}$ of $\mathfrak{W}(1,2)\oplus\mathfrak{W}(1,2)$. The solid/dotted arrows indicate the action of the holomorphic/antiholomorphic triplet algebra. It is clear that restricting to each chiral subalgebra results in $\mathcal{B}$ decomposing as $(\mathcal{W}_0\otimes\mathcal{S}_0)\oplus(\mathcal{W}_1\otimes\mathcal{S}_1)$ and $(\mathcal{S}_0\otimes\mathcal{W}_0)\oplus(\mathcal{S}_1\otimes\mathcal{W}_1)$, respectively.

A simple guess, which works in this case, is therefore to draw the Loewy diagram of the tensor product $(\mathcal{W}_0\otimes\mathcal{S}_0)\oplus(\mathcal{W}_1\otimes\mathcal{S}_1)$. This gives a holomorphic module structure to the bulk module $\mathcal{B}$. Then, complete the diagram by adding (dotted) arrows representing the antiholomorphic module structure so that they trace out the Loewy diagram of $(\mathcal{S}_0\otimes\mathcal{W}_0)\oplus(\mathcal{S}_1\otimes\mathcal{W}_1)$. One quickly finds that there is a unique way to do this. Moreover, one can check that the resulting diagram is manifestly *local*, meaning that the corresponding bulk correlation functions will all be single-valued. For this, we recall [35] that field locality has an algebraic reformulation which requires that $L_0-\overline{L}_0$ be diagonalisable with integer eigenvalues. The second constraint is clearly met and the first follows from the locality of the non-diagonalisable action for symplectic fermions (see (3.5) and (3.48)):

$$L_0\big|\Omega\big\rangle = J_0^-J_0^+\big|\Omega\big\rangle = -J_0^-\big|\Omega\big\rangle = -\big|0\big\rangle, \qquad \overline{L}_0\big|\Omega\big\rangle = \overline{J}_0^-\overline{J}_0^+\big|\Omega\big\rangle = -\overline{J}_0^-\big|\Omega\big\rangle = -\big|0\big\rangle. \tag{3.51}$$

3.7. **Correlation Functions.** Correlation functions and operator product expansions for symplectic fermions, including twist fields, were computed by Kausch [95]. The operator product expansion of the logarithmic partner $\Omega\big(z,\overline{z}\big)$ of the vacuum with itself is logarithmic,

$$\Omega(z,\overline{z})\Omega(w,\overline{w}) = -\Big(A+\log|z-w|^2\Big)^2 - 2\Big(A+\log|z-w|^2\Big)\Omega(w,\overline{w}) + \cdots, \tag{3.52}$$

and the same is true for the twist field $\mu_{1/2}\big(z,\overline{z}\big)$ with itself:

$$\mu_{1/2}(z,\overline{z})\mu_{1/2}(w,\overline{w}) = -|z-w|^{1/2}\Big(\Omega(w,\overline{w}) + \log|z-w|^2 + \text{const}\Big) + \cdots \tag{3.53}$$

In the triplet theory, the twist field generates the bulk triplet module $\mathcal{W}_{-1/8}\otimes\mathcal{W}_{-1/8}$, while the logarithmic partner of the identity is associated to the top composition factor $\mathcal{W}_0\otimes\mathcal{W}_0$ of the bulk staggered module. These operator product expansions are, of course, consistent with the fusion rules and the bulk state space explained in last section. We will now outline an efficient means of computing the correlation function that implies (3.53) (using a different approach to Kausch).

Viewing symplectic fermions as the Wess-Zumino-Witten model of the Lie supergroup $\mathsf{PSL}(1|1)$, it is natural to describe the theory by passing to a first order formulation as in, for example, [101]. For symplectic fermions, this idea has been applied in [102]. The picture is sketched as follows:

$$\partial\theta^+\overline{\partial}\theta^- \xrightarrow{1^{\text{st}}\text{ order}} b^+\partial\theta^+ + b^-\overline{\partial}\theta^- + b^+b^- \xrightarrow{\text{bosonisation}} -\partial\varphi\overline{\partial}\varphi + \mathrm{e}^{-\varphi} + \text{linear dilaton}. \tag{3.54}$$

The first order formulation, or equivalently its bosonisation, is best used to compute correlation functions perturbatively. The symplectic fermion fields are recovered by decomposing $\mathrm{e}^{\pm\varphi(z,\overline{z})} = \mathrm{e}^{\pm\varphi_L(z)}\mathrm{e}^{\pm\varphi_R(\overline{z})}$ with

$$e^{\varphi_L(z)}e^{-\varphi_L(w)} \sim \frac{1}{(z-w)} \tag{3.55}$$

and so on. The fields $J^+\big(z\big) = \mathrm{e}^{\varphi_L(z)}$ and $J^-\big(z\big) = \partial\mathrm{e}^{-\varphi_L(z)}$ commute with the zero-mode of $\mathrm{e}^{-\varphi_L(z)}$ and they have the same operator product expansion as symplectic fermions. The (holomorphic) Virasoro field is

$$T(z) = \frac{1}{2}:\partial\varphi(z,\overline{z})\partial\varphi(z,\overline{z}): + \frac{1}{2}\partial^2\varphi(z,\overline{z}). \tag{3.56}$$

The twist fields μ_λ are identified with the $\mathrm{e}^{\lambda\varphi_L(z)}$ because the corresponding states have dimension $-\lambda(1-\lambda)/2$ and the $J^\pm$ act on them with moding in $\mathbb{Z}\pm\lambda$. The corresponding bulk field will be denoted by $V_\lambda\big(z,\overline{z}\big)=\mathrm{e}^{\lambda\varphi(z,\overline{z})}$.

In this formalism, correlation functions are defined by

$$\big\langle V_{\alpha_1}(z_1,\overline{z}_1)\cdots V_{\alpha_n}(z_n,\overline{z}_n)\big\rangle=\big\langle V_{\alpha_1}(z_1,\overline{z}_1)\cdots V_{\alpha_n}(z_n,\overline{z}_n)\mathrm{e}^{-Q}\big\rangle_0, \tag{3.57}$$

where

$$Q=\int_{\mathbb{C}}\mathrm{e}^{-\varphi(z,\overline{z})}\,\frac{\mathrm{d}z\mathrm{d}\overline{z}}{2\pi} \tag{3.58}$$

and the correlators in the free theory, denoted with the subscript 0, are standard free boson correlators subject to the charge conservation condition

$$\big\langle V_{\alpha_1}(z_1,\overline{z}_1)\cdots V_{\alpha_n}(z_n,\overline{z}_n)\big\rangle_0=-\delta(\alpha_1+\cdots+\alpha_n=1)\prod_{i<j}\big|z_i-z_j\big|^{2\alpha_i\alpha_j}. \tag{3.59}$$

In this manner, we obtain the non-zero one-point and two-point functions

$$\big\langle V_1(z_1,\overline{z}_1)\big\rangle=-1,\qquad \big\langle V_\alpha(z_1,\overline{z}_1)V_{1-\alpha}(z_2,\overline{z}_2)\big\rangle=-|z_1-z_2|^{2\alpha(1-\alpha)}. \tag{3.60}$$

The logarithmic partner of the identity $V_0\big(z,\overline{z}\big)$ is the field $\Omega\big(z,\overline{z}\big)=V_1(z,\overline{z})$. This can be verified by computing three-point functions involving this field. For this, consider first a three-point function of the form

$$C_{\alpha\beta\gamma}(z,\overline{z})=\big\langle V_\alpha(z,\overline{z})V_\beta(1,1)V_\gamma(0,0)\big\rangle, \tag{3.61}$$

with $\alpha+\beta+\gamma=2$. The famous Fateev-Dotsenko integral [103] gives

$$\begin{aligned}C_{\alpha\beta\gamma}(z,\overline{z})&=-|z|^{2\alpha\gamma}|z-1|^{2\alpha\beta}\int_{\mathbb{C}}|z-w|^{-2\alpha}|1-w|^{-2\beta}|w|^{-2\gamma}\,\frac{\mathrm{d}z\mathrm{d}\overline{z}}{2\pi}\\&=-|z|^{2(\alpha\gamma+\beta-1)}|1-z|^{2(\alpha\beta+\gamma-1)}\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}.\end{aligned} \tag{3.62}$$

We see that the three-point function diverges if one or more of its fields coincide with $\Omega\big(z,\overline{z}\big)$ and that, if this happens, then a regularisation will be required. The correct regularisation mimics the addition of the identity to its logarithmic partner:

$$\begin{aligned}C_{\alpha,1-\alpha,1}(z,\overline{z})_{\text{reg.}}&=-\lim_{\varepsilon\to0}\Big(C_{\alpha,1-\alpha-\varepsilon,1+\varepsilon}(z,\overline{z})-\Gamma(-\varepsilon)C_{\alpha,1-\alpha-\varepsilon,\varepsilon}(z,\overline{z})\Big)\\&=-\lim_{\varepsilon\to0}\bigg(|z|^{2\alpha\varepsilon}|1-z|^{2\alpha(1-\alpha-\varepsilon)}\Big(\frac{\Gamma(1-\alpha)\Gamma(\alpha+\varepsilon)\Gamma(-\varepsilon)}{\Gamma(\alpha)\Gamma(1-\alpha-\varepsilon)\Gamma(1+\varepsilon)}\frac{|1-z|^{2\varepsilon}}{|z|^{2\varepsilon}}-\Gamma(-\varepsilon)\Big)\bigg)\\&=-\lim_{\varepsilon\to0}\bigg(|1-z|^{2\alpha(1-\alpha)}\varepsilon\Gamma(-\varepsilon)\Big(\psi(\alpha)+\psi(1-\alpha)-\psi(1)+(\alpha-1)\log|z|^2\\&\qquad\qquad+(1-\alpha)\log|1-z|^2-\alpha\log|z|^2+\alpha\log|1-z|^2\Big)\bigg)\\&=-|1-z|^{2\alpha(1-\alpha)}\Big(\psi(\alpha)+\psi(1-\alpha)-\psi(1)-\log|z|^2+\log|1-z|^2\Big).\end{aligned} \tag{3.63}$$

Here, $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of $\Gamma(x)$. Setting α to $\frac{1}{2}$, we recover the correlator which gives the coefficient of the identity in the operator product expansion (3.53).

3.8. **Further Developments.** The singlet algebras $\mathfrak{M}\big(1,p\big)$ and triplet algebras $\mathfrak{W}\big(1,p\big)$, with $p=2,3,\ldots$ and central charge $c=1-6(p-1)^2/p$, were first discovered by Kausch [30]. They are generated by 1 and 3 fields of dimension $2p-1$, respectively (along with the energy-momentum tensor). The singlet, being a subalgebra of the triplet, has received relatively little attention, but the $\mathfrak{W}\big(1,p\big)$-models have been considered by a variety of groups [38, 104–109]. In particular, we now have a complete picture of the $\mathfrak{W}\big(1,p\big)$ spectrum, characters and fusion rules [100, 110, 111]: There are two projective irreducibles as well as $2(p-1)$ non-projective irreducibles whose projective covers have Loewy diagrams similar to those in Figure 4. These projective covers carry a non-diagonalisable action of L_0, so the $\mathfrak{W}\big(1,p\big)$-models are logarithmic conformal field theories.

The representations of the singlet algebras $\mathfrak{M}\big(1,p\big)$ were considered in [112, 113]. Here, the results are a little less comprehensive: There is a continuum of irreducibles whose characters are known, but fusion rules and indecomposable structures do not seem to have been settled. Some modules with non-diagonalisable L_0-actions have been constructed, so the $\mathfrak{M}\big(1,p\big)$-models are also logarithmic. The modular properties of the singlet models will appear elsewhere [34].

The $(1,p)$ triplet algebras were generalised to $\mathcal{W}(q,p)$, with $p,q\in\mathbb{Z}_+$, $p>q\geqslant 2$, $\gcd\{p,q\}=1$ and $c=1-6(p-q)^2/pq$, in [114]. This is the central charge of the minimal model $\mathsf{M}(q,p)$ and $\mathcal{W}(q,p)$ has, unlike $\mathcal{W}(1,p)$, a reducible but indecomposable vacuum module. Indeed, the unique irreducible quotient of the triplet vacuum module is the minimal model vacuum module. This is appropriate for these central charges, as we have seen in Section 2. Perhaps because of this indecomposable structure, these triplet models have been intensively studied [39, 115–121]. Again, there are two projective irreducibles, but now the (conjectured) projective covers of the other irreducibles can have Loewy diagrams which are significantly more complicated than the diamonds we have seen here, see [39, App. A.1] for example. One interesting feature here is that fusion rules involving certain irreducibles, notably the irreducible quotient of the vacuum, do not behave as expected [118] (fusing with these does not define an "exact functor"). This means that the fusion operation does not define a ring structure on the Grothendieck *group* generated by the irreducible characters. Instead, it seems that one can quotient this group by the ideal of minimal model irreducibles and only then impose fusion. It would be interesting to try to reconstruct this Grothendieck ring (or group) using the continuum Verlinde approach advocated here.

4. The Fractional Level Wess-Zumino-Witten Model $\widehat{\mathfrak{sl}}(2)_{-1/2}$

Our next example is based on an affine Kac-Moody symmetry. As is well known, affine algebras give rise to rational conformal field theories when the level k is a non-negative integer. To get a logarithmic theory, we turn to non-integer levels. In particular, we consider the Kac-Moody algebra $\widehat{\mathfrak{sl}}(2)$ at fractional level $k=-\frac{1}{2}$. This is one of the simplest of the admissible levels introduced by Kac and Wakimoto [44] and, like the triplet model of Section 3, the theory may be described as a subalgebra of a ghost theory. We will start from these ghosts before turning to the relevant representations of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ and their characters. Modular transformations are derived, leading once more to Grothendieck fusion rules which are compared to the known fusion rules computed using the Nahm-Gaberdiel-Kausch algorithm in [62].

4.1. The $\beta\gamma$ Ghost System and its $\mathbb{Z}_2$-Orbifold $\widehat{\mathfrak{sl}}(2)_{-1/2}$.

The $\beta\gamma$ ghost system is generated by four bosonic fields β, γ, $\overline{\beta}$ and $\overline{\gamma}$ whose action takes the form

$$S\left[\beta,\gamma;\overline{\beta},\overline{\gamma}\right]=g\int\left(\beta\overline{\partial}\gamma+\overline{\beta}\partial\overline{\gamma}\right)\mathrm{dzd}\overline{\mathrm{z}}. \tag{4.1}$$

The equations of motion require β and γ to be holomorphic, while $\overline{\beta}$ and $\overline{\gamma}$ become antiholomorphic. The usual symmetries under shifting by (anti)holomorphic fields lead to the operator product expansions

$$\beta(z)\beta(w)\sim 0,\qquad \beta(z)\gamma(w)\sim\frac{1}{z-w},\qquad \gamma(z)\gamma(w)\sim 0 \tag{4.2}$$

and their antiholomorphic analogues (which we shall mostly ignore for now). The (holomorphic) energy-momentum tensor is given by

$$T(z)=\frac{1}{2}[:\beta(z)\partial\gamma(z):-:\partial\beta(z)\gamma(z):] \tag{4.3}$$

and the central charge is $c=-1$.

The affine algebra $\widehat{\mathfrak{sl}}(2)$ is recovered as the $\mathbb{Z}_2$-orbifold of the ghost theory. We define

$$e(z)=\frac{1}{2}:\beta(z)\beta(z):,\qquad h(z)=-:\beta(z)\gamma(z):,\qquad f(z)=\frac{1}{2}:\gamma(z)\gamma(z): \tag{4.4}$$

and note that these composite fields obey the operator product expansions

$$\begin{aligned} &e(z)e(w)\sim 0, && h(z)e(w)\sim\frac{2e(w)}{z-w}, && h(z)h(w)\sim\frac{-1}{(z-w)^2},\\ &f(z)f(w)\sim 0, && h(z)f(w)\sim\frac{-2f(w)}{z-w}, && e(z)f(w)\sim\frac{1/2}{(z-w)^2}-\frac{h(w)}{z-w}. \end{aligned} \tag{4.5}$$

This indeed corresponds to $\widehat{\mathfrak{sl}}(2)$ at level $k=-\frac{1}{2}$, but with respect to a basis $\{e,h,f\}$ of $\mathfrak{sl}(2)$ which differs from the standard basis in that $[e,f]=-h$ and $\kappa(e,f)=-1$ (where $\kappa(\cdot,\cdot)$ denotes the Killing form in the fundamental representation).[18] Whereas the standard basis defines a triangular decomposition of $\mathfrak{sl}(2)$ with the adjoint induced by the real form $\mathfrak{su}(2)$, this basis corresponds to $\mathfrak{sl}(2;\mathbb{R})$ [60].

The automorphisms of $\widehat{\mathfrak{sl}}(2)$ which preserve the Cartan subalgebra spanned by h_0, k and the Virasoro zero-mode L_0 are generated by the conjugation automorphism $\mathfrak{w}$ and the *spectral flow* automorphism σ. The former is the

[18] We also note that the energy-momentum tensor (4.3) may be recovered from the standard Sugawara construction and (4.4).

lift of the non-trivial Weyl reflection of $\mathfrak{sl}(2)$, whereas the latter is a square root of the translation subgroup of the affine Weyl group (it corresponds to translating by a dual root rather than a coroot). Both leave the level $k=-\frac{1}{2}$ invariant and the action on the generators of $\widehat{\mathfrak{sl}}(2)$ is as follows:

$$\begin{aligned} &\mathsf{w}(e_n)=f_n, && \mathsf{w}(h_n)=-h_n, && \mathsf{w}(f_n)=e_n, && \mathsf{w}(L_0)=L_0,\\ &\sigma^\ell(e_n)=e_{n-\ell}, && \sigma^\ell(h_n)=h_n+\tfrac{1}{2}\ell\delta_{n,0}, && \sigma^\ell(f_n)=f_{n+\ell}, && \sigma^\ell(L_0)=L_0-\tfrac{1}{2}\ell h_0-\tfrac{1}{8}\ell^2 \end{aligned} \tag{4.6}$$

(σ is referred to as spectral flow because it does not preserve conformal dimensions). We remark that σ is induced from the spectral flow of the $\beta\gamma$ ghost algebra: $\sigma^2(\beta_n)=\beta_{n-1}$, $\sigma^2(\gamma_n)=\gamma_{n+1}$. The map σ would take the ghost algebra to a twisted sector, consistent with $\widehat{\mathfrak{sl}}(2)_{-1/2}$ being an orbifold.

Twisting the action of $\widehat{\mathfrak{sl}}(2)$ on a module $\mathcal{M}$ by w or σ^ℓ, we obtain new modules denoted by $\mathcal{M}^\star$ and $\mathcal{M}^{(\ell)}$, respectively. The first is precisely the module conjugate to $\mathcal{M}$ and we shall refer to the second as a "spectral flow" of $\mathcal{M}$. Explicitly, the twisted algebra action defining $\mathcal{M}^\star$ and $\mathcal{M}^{(\ell)}$ is given by defining new states $\mathsf{w}\big(\big|v\big\rangle\big)$ and $\sigma^\ell\big(\big|v\big\rangle\big)$, for each $\big|v\big\rangle\in\mathcal{M}$, and letting $\widehat{\mathfrak{sl}}(2)$ act via

$$J\cdot\mathsf{w}\big(\big|v\big\rangle\big)=\mathsf{w}\big(\mathsf{w}^{-1}(J)\big|v\big\rangle\big),\qquad J\cdot\sigma^\ell\big(\big|v\big\rangle\big)=\sigma^\ell\Big(\sigma^{-\ell}(J)\big|v\big\rangle\Big)\qquad (J\in\widehat{\mathfrak{sl}}(2)). \tag{4.7}$$

4.2. **Representation Theory of $\widehat{\mathfrak{sl}}(2)_{-1/2}$.** The vacuum module of the theory is taken to be the irreducible highest weight module of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ with highest weight (h_0-eigenvalue) 0. The other highest weight states consistent with this may be found using the singular vectors of the corresponding Verma module [122]. One such is $f_0\big|0\big\rangle$ which never constrains the spectrum. The only other singular vector which is not a descendant of $f_0\big|0\big\rangle$ turns out to have weight 4 and conformal dimension 4. It is given explicitly by

$$\big|\chi\big\rangle=\big(156e_{-3}e_{-1}-71e_{-2}^2+44e_{-2}h_{-1}e_{-1}-52h_{-2}e_{-1}^2+16f_{-1}e_{-1}^3-4h_{-1}^2e_{-1}^2\big)\big|0\big\rangle. \tag{4.8}$$

Because we insist upon the irreducibility of the vacuum module, the corresponding field and its modes must therefore decouple from the theory (vanish identically on physical states). For the calculation to follow, it is convenient to consider instead the descendant $f_0^2\big|\chi\big\rangle$ for which the corresponding field is

$$\begin{aligned} &64:eeff:+16:ehhf:-136:eh\partial f:+128:e\partial hf:-12:e\partial^2 f:-8:hhhh:\\ &\quad+200:\partial ehf:-108:\partial e\partial f:+8:\partial hhh:-38:\partial h\partial h:+156:\partial^2 ef:+24:\partial^2 hh:-\partial^3 h. \end{aligned} \tag{4.9}$$

Acting with the zero-mode of this field on a highest weight state $\big|v_\lambda\big\rangle$ of weight λ, we arrive at a constraint on such states to be physical:

$$h_0(h_0-1)(2h_0+1)(2h_0+3)\big|v_\lambda\big\rangle=0. \tag{4.10}$$

It follows that a physical highest weight must be one of $\lambda=0$, 1, $-\frac{1}{2}$ or $-\frac{3}{2}$. As this analysis also applies to singular vectors, one can conclude that the physical highest weight modules must also be irreducible.

The physical highest weight modules are therefore characterised by the $\mathfrak{sl}(2)$-module spanned by their ground states. This ground state module has dimension 1 and 2 for $\lambda=0$ and 1, respectively. We denote the corresponding irreducible $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules by $\mathcal{L}_0$ and $\mathcal{L}_1$. For $\lambda=-\frac{1}{2}$ and $-\frac{3}{2}$, the ground states span an infinite-dimensional (irreducible) module of $\mathfrak{sl}(2)$, a discrete series representation in fact, hence we will denote the $\widehat{\mathfrak{sl}}(2)_{-1/2}$-irreducibles by $\mathcal{D}_{-1/2;+}$ and $\mathcal{D}_{-3/2;+}$, with the "+" serving to indicate that the space of ground states is generated by a highest weight state (for $\mathfrak{sl}(2)$). We remark that the four irreducibles $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{D}_{-1/2;+}$ and $\mathcal{D}_{-3/2;+}$ exhaust the *admissible* modules of Kac and Wakimoto [44] for $k=-\frac{1}{2}$. The conformal dimensions of their ground states are 0, $\frac{1}{2}$, $-\frac{1}{8}$ and $-\frac{1}{8}$, respectively.

There are several reasons to be dissatisfied with this spectrum of physical highest weight modules. The first is that while the characters of these admissible modules have good modular properties [123], an application of the Verlinde formula leads to negative fusion coefficients [45] (see Section 4.4). The second, which is far more elementary, is that this spectrum is not closed under conjugation. Indeed, while $\mathcal{L}_0$ and $\mathcal{L}_1$ are readily seen to be self-conjugate, the application of w to $\mathcal{D}_{-1/2;+}$ and $\mathcal{D}_{-3/2;+}$ leads to new modules:

$$\mathcal{D}^\star_{-1/2;+}=\mathcal{D}_{1/2;-},\qquad \mathcal{D}^\star_{-3/2;+}=\mathcal{D}_{3/2;-}. \tag{4.11}$$

These new modules are labelled with "$-$" symbols to indicate that their ground states span lowest weight discrete series representations. A third reason for dissatisfaction is that, as we shall see in Section 4.3, the spectrum of admissible highest weight modules is not closed under fusion.

To check the physicality of the highest weight modules $\mathcal{D}_{1/2;-}$ and $\mathcal{D}_{3/2;-}$, we should let the zero-mode of the null field (4.9) act on their ground states. In fact, we may as well analyse this constraint for a general space of ground states, allowing the possibility of principal series $\mathfrak{sl}(2)$-representations as well.[19] For this, we parametrise the ground states representation by $\mu \in \mathbb{R}$ and label its states by $\lambda \in \mathbb{R}$. The action of the zero-modes on these states is given by

$$\begin{aligned} e_0\big|v_\lambda^\mu\big\rangle &= \tfrac{1}{2}(\lambda-\mu)\big|v_{\lambda+2}^\mu\big\rangle, \qquad & h_0\big|v_\lambda^\mu\big\rangle &= \lambda\big|v_\lambda^\mu\big\rangle, \\ f_0\big|v_\lambda^\mu\big\rangle &= \tfrac{1}{2}(\lambda+\mu)\big|v_{\lambda-2}^\mu\big\rangle, \qquad & L_0\big|v_\lambda^\mu\big\rangle &= \tfrac{1}{6}\mu(\mu+2)\big|v_\lambda^\mu\big\rangle. \end{aligned} \tag{4.12}$$

We remark that ground states with $\lambda = \mu$ are highest weight, whereas those with $\lambda = -\mu$ are lowest weight. It follows that if $\lambda \neq \pm\mu \bmod 2$, then the $\big|v_\lambda^\mu\big\rangle$, with μ fixed and λ fixed modulo 2, span an irreducible principal series representation of $\mathfrak{sl}(2)$. It is not hard to verify that for a given $\lambda \neq \pm\mu \bmod 2$, the parameters μ and $-2-\mu$ define isomorphic representations. Thus, principal series irreducibles are distinguished by the value of λ mod 2 and the eigenvalue of L_0 (or the quadratic Casimir).

Applying the zero-mode of the null field (4.9) to the state $\big|v_\lambda^\mu\big\rangle$ now gives

$$\begin{aligned} 0 &= \big(64f_0^2e_0^2 + 16f_0h_0^2e_0 - 192f_0h_0e_0 + 180f_0e_0 - 8h_0^4 - 8h_0^3 + 10h_0^2 + 6h_0\big)\big|v_\lambda^\mu\big\rangle \\ &= (2\mu+1)(2\mu+3)\big(\mu(\mu+2) - 3\lambda^2\big)\big|v_\lambda^\mu\big\rangle. \end{aligned} \tag{4.13}$$

If $3\lambda^2 = \mu(\mu+2)$, we note that each choice for μ yields at most two consistent possibilities for λ. The space of ground states is therefore one-dimensional, in which case $\lambda = 0$ and $\mu = 0$ or -2, or two-dimensional, in which case $\lambda = \pm 1$ and $\mu = 1$ or -3. The conformal dimensions are 0 and $\frac{1}{2}$, respectively. If $\mu = -\frac{1}{2}$ or $-\frac{3}{2}$, then $\lambda \in \mathbb{R}/2\mathbb{Z}$ is unconstrained while the conformal dimension of each of the ground states is fixed to be $\frac{1}{6}\mu(\mu+2) = -\frac{1}{8}$.

In this way, we recover the admissible $\widehat{\mathfrak{sl}}(2)_{-1/2}$-irreducibles $\mathcal{L}_0$ and $\mathcal{L}_1$. We also deduce the physicality of *any* irreducible module whose ground states have conformal dimension $-\frac{1}{8}$. This includes the admissibles $\mathcal{D}_{-1/2;+}$ and $\mathcal{D}_{-3/2;+}$ as well as their conjugates $\mathcal{D}_{1/2;-}$ and $\mathcal{D}_{3/2;-}$. However, we also obtain an uncountable family of physical relaxed highest weight modules whose ground states form a principal series $\mathfrak{sl}(2)$-representation of conformal dimension $-\frac{1}{8}$. These modules are characterised by $\lambda \in \mathbb{R}/2\mathbb{Z}$ and will be denoted by $\mathcal{E}_\lambda$. They are irreducible when $\lambda \neq \mu = \pm\frac{1}{2} \bmod 2$, so we will refer to the $\mathcal{E}_\lambda$ with this range of parameters λ as being *typical*. The remaining $\mathcal{E}_\lambda$, along with the other physical $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{D}_{-1/2;+}$, $\mathcal{D}_{-3/2;+}$, $\mathcal{D}_{1/2;-}$ and $\mathcal{D}_{3/2;-}$, are defined to be *atypical*.

Finally, we remark that the atypical $\mathcal{E}_\lambda$ (with $\lambda = \pm\frac{1}{2} \bmod 2$) are indecomposable and their structure is not completely fixed by λ and the conformal dimension of their ground states. This deficiency may be overcome by affixing a label "+" or "−" to communicate whether the indecomposable has a highest or lowest weight $\mathfrak{sl}(2)$-state among its ground states. In this way, we arrive at four distinct atypical indecomposables whose structures are specified by the following exact sequences:

$$\begin{aligned} &0 \longrightarrow \mathcal{D}_{-1/2;+} \longrightarrow \mathcal{E}_{-1/2;+} \longrightarrow \mathcal{D}^-_{+3/2} \longrightarrow 0, \quad 0 \longrightarrow \mathcal{D}_{-3/2;+} \longrightarrow \mathcal{E}_{+1/2;+} \longrightarrow \mathcal{D}^-_{+1/2} \longrightarrow 0, \\ &0 \longrightarrow \mathcal{D}^-_{+1/2} \longrightarrow \mathcal{E}_{+1/2;-} \longrightarrow \mathcal{D}_{-3/2;+} \longrightarrow 0, \quad 0 \longrightarrow \mathcal{D}^-_{+3/2} \longrightarrow \mathcal{E}_{-1/2;-} \longrightarrow \mathcal{D}_{-1/2;+} \longrightarrow 0. \end{aligned} \tag{4.14}$$

4.3. **Spectral Flow and Fusion.** At positive-integer level k, the spectral flow automorphism σ acts on the set of integrable modules as the involution $\lambda \to k-\lambda$. This is no longer true for fractional levels as we shall see. For $k = -\frac{1}{2}$, one can check that the atypical irreducibles are related by spectral flow as follows:

$$\mathcal{L}_0^{(1)} = \mathcal{D}_{-1/2;+}, \qquad \mathcal{L}_1^{(1)} = \mathcal{D}_{-3/2;+}, \qquad \mathcal{L}_0^{(-1)} = \mathcal{D}_{1/2;-}, \qquad \mathcal{L}_1^{(-1)} = \mathcal{D}_{3/2;-}. \tag{4.15}$$

This is most easy checked by computing the action of $\sigma^{\pm 1}$ on the *extremal* states of a module, though one can also use the character formulae discussed in Section 4.4. These are defined to be the states whose conformal dimension is minimal among those sharing their weight. For example, $e_{-1}^j|0\rangle$ and $f_{-1}^j|0\rangle$ are the extremal states of $\mathcal{L}_0$ and (4.6) and (4.7) give

$$\begin{aligned} h_0\sigma^{\pm1}\Big(e_{-1}^j|0\rangle\Big) &= \sigma^{\pm1}\Big(\big(h_0 \mp \tfrac{1}{2}\big)e_{-1}^j|0\rangle\Big) = \big(2j \mp \tfrac{1}{2}\big)\sigma^{\pm1}\Big(e_{-1}^j|0\rangle\Big), \\ L_0\sigma^{\pm1}\Big(e_{-1}^j|0\rangle\Big) &= \sigma^{\pm1}\Big(\big(L_0 \pm \tfrac{1}{2}h_0 - \tfrac{1}{8}\big)e_{-1}^j|0\rangle\Big) = \big(j \pm j - \tfrac{1}{8}\big)\sigma^{\pm1}\Big(e_{-1}^j|0\rangle\Big). \end{aligned} \tag{4.16}$$

[19] The corresponding $\widehat{\mathfrak{sl}}(2)$-modules are sometimes known as *relaxed* highest weight modules [124,125]. We will see that these relaxed modules are essential to the consistency of the theory. For now, we only remark that relaxed modules are natural for (twisted) $\beta\gamma$ representations because the ghost zero-modes β_0 and γ_0 are bosonic, hence do not square to 0.

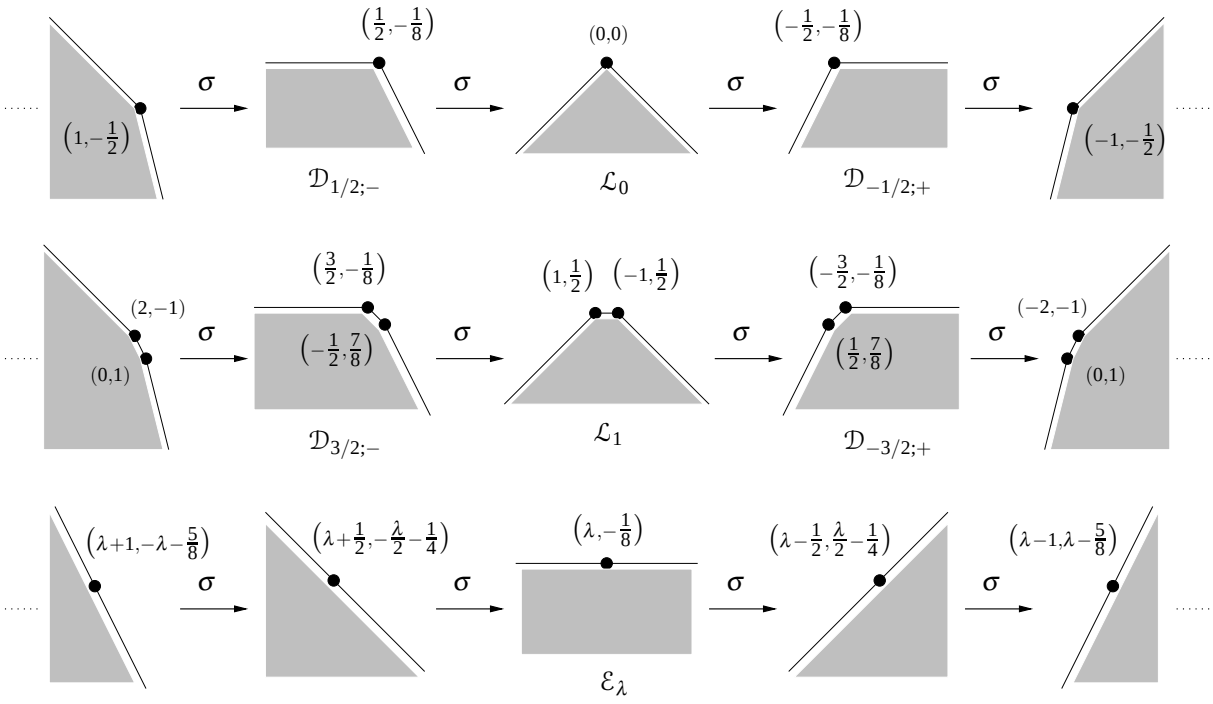

FIGURE 8. Depictions of the physical irreducible $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules. Each labelled state declares its weight and conformal dimension (in that order). Conformal dimensions increase from top to bottom and $\mathfrak{sl}(2)$-weights increase from right to left.

Thus, $\sigma^{\pm1}$ shifts the weight uniformly by $\mp\frac{1}{2}$, whereas σ increases the conformal dimension of $e^j_{-1}|0\rangle$ by $2j-\frac{1}{8}$ and σ^{-1} increases it by $-\frac{1}{8}$. A similar calculation describes the spectral flow of the states $f^j_{-1}|0\rangle$.

More interesting is the question of what happens if we iterate the spectral flow. For example,

$$L_0\sigma^2\left(f^j_{-1}|0\rangle\right) = -\left(j+\tfrac{1}{2}\right)\sigma^2\left(f^j_{-1}|0\rangle\right) \tag{4.17}$$

shows that the conformal dimensions of the states of $\mathcal{L}_0^{(2)}$ are *unbounded below*. The same is true for $\mathcal{L}_0^{(\ell)}$ and $\mathcal{L}_1^{(\ell)}$, when $\ell\neq 0,\pm1$, and $\mathcal{E}_\lambda^{(\ell)}$, when $\ell\neq 0$. Because algebra automorphisms map physical modules to physical modules, we conclude that almost all of the physical modules of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ have the property that the conformal dimensions of their states have no lower bound. We illustrate the weights and conformal dimensions of the states of these modules in Figure 8.

The full spectrum of irreducible modules then consists of the spectral flow images $\mathcal{L}_0^{(\ell)}$, $\mathcal{L}_1^{(\ell)}$ and $\mathcal{E}_\lambda^{(\ell)}$, where $\ell\in\mathbb{Z}$ and $\lambda\neq\pm\frac{1}{2}\in\mathbb{R}/2\mathbb{Z}$. We extend the notion of typicality and atypicality so that it is preserved by spectral flow. We then also have four families of atypical indecomposables $\mathcal{E}^{(\ell)}_{\pm1/2;\pm}$. As twisting by automorphisms preserves module structure, the members of these families are described by applying σ^ℓ to each module in the appropriate exact sequence of (4.14). This is a rather large collection of modules, but it is still not quite complete.

We turn now to the fusion of the irreducible $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules identified above. Because there are uncountably many of these, this seems a rather daunting undertaking. However, two things work in our favour: First, fusion and spectral flow are strongly believed to play nicely together in the sense that

$$\mathcal{M}^{(\ell_1)}\times\mathcal{N}^{(\ell_2)} = (\mathcal{M}\times\mathcal{N})^{(\ell_1+\ell_2)} \tag{4.18}$$

for all (physical) modules $\mathcal{M}$ and $\mathcal{N}$. We know of no proof for this relation despite much evidence in its favour, but we will assume it in the following, so that we may restrict the fusion rules to the "untwisted" sector $\ell_1=\ell_2=0$. The second boon is that the singular vectors of the typical modules $\mathcal{E}_\lambda$ are expressible as polynomial functions of λ. This allows us to compute their fusion decompositions as functions of λ.

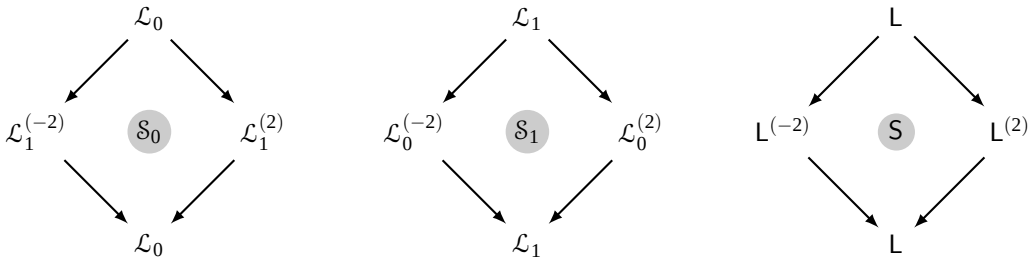

FIGURE 9. Loewy diagrams for the indecomposable $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules $\mathcal{S}_0$ and $\mathcal{S}_1$ (left) and the indecomposable $\beta\gamma$ ghost module S (right).

The fusion rules of the untwisted irreducibles were computed in [62] using the Nahm-Gaberdiel-Kausch algorithm. As expected, $\mathcal{L}_0$ is the fusion identity and the other products are

$$\mathcal{L}_1 \times \mathcal{L}_1 = \mathcal{L}_0, \qquad \mathcal{L}_1 \times \mathcal{E}_\lambda = \mathcal{E}_{\lambda+1},$$
$$\mathcal{E}_\lambda \times \mathcal{E}_\mu = \begin{cases} \mathcal{S}_{\lambda+\mu} & \text{if } \lambda+\mu = 0,1 \bmod 2, \\ \mathcal{E}^{(1)}_{\lambda+\mu+1/2} \oplus \mathcal{E}^{(-1)}_{\lambda+\mu-1/2} & \text{otherwise.} \end{cases} \tag{4.19}$$

Here, the labels on the typicals $\mathcal{E}_\lambda$ and the new atypicals $\mathcal{S}_\lambda$ must be taken modulo 2. These two additional modules $\mathcal{S}_0$ and $\mathcal{S}_1$ are indecomposables whose Loewy diagrams are given in Figure 9. Their fusion rules are

$$\mathcal{L}_1 \times \mathcal{S}_\lambda = \mathcal{S}_{\lambda+1}, \qquad \begin{aligned} \mathcal{E}_\lambda \times \mathcal{S}_\mu &= \mathcal{E}^{(-2)}_{\lambda+\mu+1} \oplus 2\,\mathcal{E}_{\lambda+\mu} \oplus \mathcal{E}^{(2)}_{\lambda+\mu+1}, \\ \mathcal{S}_\lambda \times \mathcal{S}_\mu &= \mathcal{S}^{(-2)}_{\lambda+\mu+1} \oplus 2\,\mathcal{S}_{\lambda+\mu} \oplus \mathcal{S}^{(2)}_{\lambda+\mu+1}. \end{aligned} \tag{4.20}$$

This shows that the fusion ring generated by the irreducibles closes upon adding the indecomposables $\mathcal{S}_0$ and $\mathcal{S}_1$ (and their spectral flows).

We remark that $\mathcal{S}_0$ has the vacuum module $\mathcal{L}_0$ as its socle. One can check from Figure 8 that the composition factors $\mathcal{L}_1^{(\mp 2)}$ (which lie immediately above $\mathcal{L}_0$ in the Loewy diagram of $\mathcal{S}_0$) have states $\big|\theta^\pm\big\rangle$ of weight ± 2 and conformal dimension -1. Indeed, the explicit construction of $\mathcal{S}_0$ as a fusion product allows us to choose normalisations for states so that $f_{-1}\big|\theta^+\big\rangle = e_{-1}\big|\theta^-\big\rangle = \big|0\big\rangle$. We also note from this construction that the vacuum $\big|0\big\rangle$ is part of a rank 2 Jordan block for L_0. We denote the generalised eigenvector in this block by $\big|\Omega\big\rangle$, normalising it so that $L_0\big|\Omega\big\rangle = \big|0\big\rangle$. One can show that this fixes the structure of the indecomposable $\mathcal{S}_0$ uniquely [62]. In particular, one derives that $e_1\big|\Omega\big\rangle = -\frac{1}{4}\big|\theta^+\big\rangle$ and $f_1\big|\Omega\big\rangle = -\frac{1}{4}\big|\theta^-\big\rangle$. In any case, the non-diagonalisable action of L_0 on $\mathcal{S}_0$ leads to logarithmic singularities in the two-point function of $\Omega(z)$. L_0 is similarly non-diagonalisable when acting upon $\mathcal{S}_1$. This confirms that the fractional level Wess-Zumino-Witten model $\widehat{\mathfrak{sl}}(2)_{-1/2}$ is a logarithmic conformal field theory.

Note that $\mathcal{L}_1$, with its two ground states of dimension $\frac{1}{2}$, is observed to be a simple current of order 2. The corresponding simple current extension of $\widehat{\mathfrak{sl}}(2)_{-1/2}$ is, of course, the $\beta\gamma$ ghost algebra [60]. Combining fusion orbits under $\mathcal{L}_1$, we deduce that the ghost vacuum module $\mathsf{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ has an indecomposable cover $\mathsf{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ upon which the ghosts act with half-integer moding (this is the untwisted sector). This is analogous to the case of symplectic fermions analysed in Section 3.1 and we give the Loewy diagram of S in Figure 9. Again, L_0 acts non-diagonalisably on S, hence the $\beta\gamma$ ghost theory is also logarithmic.

However, the untwisted sector for $\beta\gamma$ ghosts also contains other modules thanks to the existence of spectral flow. In fact, it contains $\mathsf{L}^{(2m)}$, $\mathsf{S}^{(2m)}$ and $\mathsf{E}^{(2m+1)}_\lambda$, for $\lambda \in \mathbb{R}/\mathbb{Z}$ and $m \in \mathbb{Z}$ (and $\mathsf{E}_\lambda = \mathcal{E}_\lambda \oplus \mathcal{E}_{\lambda+1}$). Similarly, the twisted sector with integer moding contains $\mathsf{L}^{(2m+1)}$, $\mathsf{S}^{(2m+1)}$ and $\mathsf{E}^{(2m)}_\lambda$, again for $\lambda \in \mathbb{R}/\mathbb{Z}$ and $m \in \mathbb{Z}$. If one allows more general moding, then there is a continuum of sectors, each with a similar spectrum obtained by relaxing the constraint on m.

4.4. **Modular Transformations and the Verlinde Formula.** The $\beta\gamma$ ghost theory is free, so it is easy to write the characters of its modules using a Poincaré-Birkhoff-Witt basis. Indeed, the vacuum character is

$$\mathrm{ch}\big[\mathsf{L}\big] = \mathrm{tr}_{\mathsf{L}}\, y^k z^{h_0} z^{L_0 - c/24} = y^{-1/2} \prod_{n=1}^{\infty} \frac{1}{\big(1 - z^{-1}q^{n-1/2}\big)\big(1 - zq^{n-1/2}\big)} = y^{-1/2} \frac{\eta(q)}{\vartheta_4\big(z;q\big)}, \tag{4.21}$$

where we have included the $\widehat{\mathfrak{sl}}(2)$-weight and level in preparation for deriving the corresponding affine characters. The characters of its images under spectral flow may be obtained from

$$\mathrm{ch}\big[\mathcal{M}^{(\ell)}\big]\big(y;z;q\big) = \mathrm{ch}\big[\mathcal{M}\big]\big(yz^{\ell}q^{\ell^2/4};zq^{\ell/2};q\big), \tag{4.22}$$

which holds for an arbitrary module $\mathcal{M}$. Combining this with the identifications (4.15) and the assertion that $\widehat{\mathfrak{sl}}(2)_{-1/2}$ is the $\mathbb{Z}_2$-orbifold of the $\beta\gamma$ ghosts, we obtain the characters of the admissible $\widehat{\mathfrak{sl}}(2)_{-1/2}$-modules:

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{L}_0\big] &= \frac{y^{-1/2}}{2}\left[\frac{\eta(q)}{\vartheta_4\big(z;q\big)} + \frac{\eta(q)}{\vartheta_3\big(z;q\big)}\right], \qquad \mathrm{ch}\big[\mathcal{D}_{-1/2;+}\big] &= \frac{y^{-1/2}}{2}\left[\frac{-\mathrm{i}\eta(q)}{\vartheta_1\big(z;q\big)} + \frac{\eta(q)}{\vartheta_2\big(z;q\big)}\right], \\
\mathrm{ch}\big[\mathcal{L}_1\big] &= \frac{y^{-1/2}}{2}\left[\frac{\eta(q)}{\vartheta_4\big(z;q\big)} - \frac{\eta(q)}{\vartheta_3\big(z;q\big)}\right], \qquad \mathrm{ch}\big[\mathcal{D}_{-3/2;+}\big] &= \frac{y^{-1/2}}{2}\left[\frac{-\mathrm{i}\eta(q)}{\vartheta_1\big(z;q\big)} - \frac{\eta(q)}{\vartheta_2\big(z;q\big)}\right].
\end{aligned} \tag{4.23}$$

Unlike the triplet algebra characters studied in Section 3.2, these characters have good modular properties.

Indeed, with $y = \mathrm{e}^{2\pi\mathrm{i}t}$, $z = \mathrm{e}^{2\pi\mathrm{i}u}$ and $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ as usual, the S-matrix with respect to the (ordered) basis $\big\{\mathrm{ch}\big[\mathcal{L}_0\big],\mathrm{ch}\big[\mathcal{L}_1\big],\mathrm{ch}\big[\mathcal{D}_{-1/2;+}\big],\mathrm{ch}\big[\mathcal{D}_{-3/2;+}\big]\big\}$ is found to be symmetric and unitary:

$$\mathsf{S} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & \mathrm{i} & \mathrm{i} \\ -1 & -1 & \mathrm{i} & \mathrm{i} \end{pmatrix}. \tag{4.24}$$

However, when one computes the conjugation matrix $\mathsf{C} = \mathsf{S}^2$ and the fusion matrices (using the standard Verlinde formula), trouble arises in the form of negative multiplicities [45]:

$$\mathsf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathsf{N}_{\mathcal{D}_{-1/2;+}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{N}_{\mathcal{D}_{-3/2;+}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{4.25}$$

To understand this paradox of negative multiplicities, one should first recall that the fusion ring generated by the admissibles and their conjugates contains all their spectral flows $\mathcal{L}_0^{(\ell)}$ and $\mathcal{L}_1^{(\ell)}$ as well. However, the well known periodicities of theta functions lead to the following character identities:

$$\mathrm{ch}\big[\mathcal{L}_0^{(\ell-1)}\big] + \mathrm{ch}\big[\mathcal{L}_1^{(\ell+1)}\big] = \mathrm{ch}\big[\mathcal{L}_1^{(\ell-1)}\big] + \mathrm{ch}\big[\mathcal{L}_0^{(\ell+1)}\big] = 0. \tag{4.26}$$

The characters of the $\mathcal{L}_\lambda^{(\ell)}$ are therefore linearly dependent [58] and a basis for the span of these characters is precisely given by the characters of the admissibles. The resolution to the paradox [60] is then that the characters do not completely specify the module, because of (4.26), hence the map from modules to characters is a projection. For example, the conjugate of $\mathcal{D}_{-1/2;+}$ is $\mathcal{D}_{1/2;-}$, but we have

$$\mathrm{ch}\big[\mathcal{D}_{1/2;-}\big] = \mathrm{ch}\big[\mathcal{L}_0^{(-1)}\big] = -\mathrm{ch}\big[\mathcal{L}_1^{(1)}\big] = -\mathrm{ch}\big[\mathcal{D}_{-3/2;+}\big], \tag{4.27}$$

which explains why C asserts that the conjugate to $\mathcal{D}_{-1/2;+}$ is $-\mathcal{D}_{-3/2;+}$ (it is, as far as the characters are concerned). Similarly, the fusion rule $\mathcal{D}_{-3/2;+}\times\mathcal{D}_{-3/2;+} = \mathcal{L}_0^{(2)}$ translates into the Grothendieck rule

$$\mathrm{ch}\big[\mathcal{D}_{-3/2;+}\big] \mathbin{\dot\times} \mathrm{ch}\big[\mathcal{D}_{-3/2;+}\big] = \mathrm{ch}\big[\mathcal{L}_0^{(2)}\big] = -\mathrm{ch}\big[\mathcal{L}_1\big], \tag{4.28}$$

which explains why the Verlinde formula gives $\mathsf{N}_{\mathcal{D}_{-3/2;+}\mathcal{D}_{-3/2;+}}{}^{\mathcal{L}_1} = -1$.

Having recognised the source of negative multiplicities (the characters are not linearly independent), we can turn to fixing it. As the reader may have guessed, this will involve the typical modules $\mathcal{E}_\lambda$ introduced in Section 4.2. The characters of these may be computed [61] from those of the $\beta\gamma$ ghost modules E_λ which are, again, easily deduced from Poincaré-Birkhoff-Witt bases:

$$\mathrm{ch}\big[\mathcal{E}_\lambda\big] = \frac{y^{-1/2}z^\lambda}{\eta(q)^2}\sum_{n\in\mathbb{Z}} z^{2n} \qquad \Rightarrow \qquad \mathrm{ch}\big[\mathcal{E}_\lambda^{(\ell)}\big] = \frac{y^{-1/2}z^{\lambda-\ell/2}q^{\ell(\lambda-\ell/4)/2}}{\eta(q)^2}\sum_{n\in\mathbb{Z}} z^{2n}q^{\ell n}. \tag{4.29}$$

It is important to note that this character is not convergent for any $z \neq 0$, hence it does not define a meromorphic function. Instead, we shall treat it as a distribution in the variables t, u and τ:

$$\begin{aligned}\mathrm{ch}\big[\mathcal{E}_\lambda^{(\ell)}\big] &= \frac{\mathrm{e}^{-\mathrm{i}\pi t}\mathrm{e}^{2\pi\mathrm{i}(\lambda-\ell/2)u}\mathrm{e}^{\mathrm{i}\pi\ell(\lambda-\ell/4)\tau}}{\eta(\tau)^2}\sum_{n\in\mathbb{Z}}\mathrm{e}^{2\pi\mathrm{i}(2u+\ell\tau)n}\\ &= \frac{\mathrm{e}^{-\mathrm{i}\pi t}\mathrm{e}^{\mathrm{i}\pi\ell^2\tau/4}}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\pi(\lambda-\ell/2)m}\delta(m=2u+\ell\tau).\end{aligned} \tag{4.30}$$

This has a rather satisfying interpretation [63]: The character of $\mathcal{L}_\lambda^{(\ell)}$ is, as a meromorphic function of z, only convergent in the annulus $|q|^{-\ell+1} < |z| < |q|^{-\ell-1}$ (we also need $|q|<1$) because of poles at the annulus' boundaries [58]. One therefore realises that the linear dependencies (4.26) amount to summing characters which are defined on *disjoint* annuli of convergence and obtaining 0. On the other hand, applying spectral flow to the exact sequences (4.14) gives the character relations

$$\begin{aligned}\mathrm{ch}\big[\mathcal{L}_0^{(\ell-1)}\big] + \mathrm{ch}\big[\mathcal{L}_1^{(\ell+1)}\big] &= \mathrm{ch}\big[\mathcal{E}_{+1/2;\pm}^{(\ell)}\big],\\ \mathrm{ch}\big[\mathcal{L}_1^{(\ell-1)}\big] + \mathrm{ch}\big[\mathcal{L}_0^{(\ell+1)}\big] &= \mathrm{ch}\big[\mathcal{E}_{-1/2;\pm}^{(\ell)}\big],\end{aligned} \tag{4.31}$$

from which we deduce that, in the distributional setting, summing these characters does not give 0, but rather gives a distribution whose support is precisely the pole separating the two annuli of convergence.

In any case, (4.31) shows that the atypical irreducible characters are not linearly dependent when treated as distributions. To obtain distributional formulae for these characters, we splice the exact sequences (4.14) together to obtain resolutions for the atypical irreducibles $\mathcal{L}_0^{(\ell)}$ and $\mathcal{L}_1^{(\ell)}$ in terms of the atypical indecomposables $\mathcal{E}_{\pm1/2;+}^{(\ell)}$. As with the singlet algebra $\mathfrak{M}\big(1,2\big)$ in Section 3.3, these resolutions translate into alternating sums for the atypical characters:

$$\mathrm{ch}\big[\mathcal{L}_\lambda^{(\ell)}\big] = \sum_{\ell'=0}^{\infty}(-1)^{\ell'}\mathrm{ch}\big[\mathcal{E}_{\lambda+\ell'+1/2;+}^{(\ell+2\ell'+1)}\big] \qquad (\lambda=0,1). \tag{4.32}$$

Our (topological) basis for the space of characters will therefore be chosen to consist of those of the typical irreducibles $\mathcal{E}_\lambda^{(\ell)}$, $\lambda \neq \pm\frac{1}{2} \bmod 2$, supplemented by those of the atypical indecomposables $\mathcal{E}_{\pm1/2;+}^{(\ell)}$.

We therefore turn to S-transforming the characters (4.30) of the $\mathcal{E}_\lambda^{(\ell)}$. We find that

$$\mathrm{ch}\big[\mathcal{E}_\lambda^{(\ell)}\big]\Big|_{\mathsf{S}} = \sum_{\ell'\in\mathbb{Z}}\int_{-1}^{1}\mathsf{S}_{(\ell,\lambda)(\ell',\lambda')}\mathrm{ch}\big[\mathcal{E}_{\lambda'}^{(\ell')}\big]\,\mathrm{d}\lambda', \quad \mathsf{S}_{(\ell,\lambda)(\ell',\lambda')} = \frac{1}{2}\frac{|\tau|}{-\mathrm{i}\tau}\mathrm{e}^{\mathrm{i}\pi\left(\ell\ell'/2-\ell\lambda'-\ell'\lambda\right)}, \tag{4.33}$$

which is easily verified by expanding both sides, integrating and then summing. Once again, the S-matrix is seen to be symmetric and unitary. The τ-dependent factor in the S-matrix entries originates from the homogeneity of delta functions, $\delta(ax=0)=|a|^{-1}\delta(x=0)$, and is not particularly worrisome because it will cancel when transforming bulk characters (partition functions) and when applying the Verlinde formula. Mathematically, it signifies that the space of characters carries a *projective* representation of the modular group $\mathsf{SL}\big(2;\mathbb{Z}\big)$ rather than a genuine one.

The modular S-transformation of the atypical irreducible characters is now computed from (4.32) exactly as we did for the atypical singlet modules in Section 3.4. The resulting S-matrix entries take the form

$$\mathsf{S}_{\underline{(\ell,\lambda)}(\ell',\lambda')} = \sum_{\ell'=0}^{\infty}(-1)^{\ell'}\mathsf{S}_{(\ell+2\ell'+1)(\lambda+\ell'+1/2)} = \frac{1}{2}\frac{|\tau|}{-\mathrm{i}\tau}\frac{\mathrm{e}^{\mathrm{i}\pi\left(\ell\ell'/2-\ell\lambda'-\ell'\lambda\right)}}{2\cos(\pi\lambda')}. \tag{4.34}$$

Here, as before, we distinguish atypical from typical labels by underlining the former (so $\underline{(\ell,\lambda)}$ stands for $\mathcal{L}_\lambda^{(\ell)}$ with $\lambda=0,1$). Applying the continuous Verlinde formula, we can now easily obtain the Grothendieck fusion coefficients. The calculations are straight-forward and very similar to those presented in Section 3.4, so we only report the resulting Grothendieck fusion rules:

$$\begin{aligned}\mathrm{ch}\big[\mathcal{L}_\lambda^{(\ell)}\big] \mathbin{\dot\times} \mathrm{ch}\big[\mathcal{L}_\mu^{(m)}\big] &= \mathrm{ch}\big[\mathcal{L}_{\lambda+\mu}^{(\ell+m)}\big],\\ \mathrm{ch}\big[\mathcal{L}_\lambda^{(\ell)}\big] \mathbin{\dot\times} \mathrm{ch}\big[\mathcal{E}_\mu^{(m)}\big] &= \mathrm{ch}\big[\mathcal{E}_{\lambda+\mu}^{(\ell+m)}\big],\\ \mathrm{ch}\big[\mathcal{E}_\lambda^{(\ell)}\big] \mathbin{\dot\times} \mathrm{ch}\big[\mathcal{E}_\mu^{(m)}\big] &= \mathrm{ch}\big[\mathcal{E}_{\lambda+\mu+1/2}^{(\ell+m+1)}\big] + \mathrm{ch}\big[\mathcal{E}_{\lambda+\mu-1/2}^{(\ell+m-1)}\big].\end{aligned} \tag{4.35}$$

These agree perfectly with the Grothendieck versions of the fusion rules (4.19) for the irreducibles and may be checked to imply the Grothendieck versions of those (4.20) of the indecomposables. We remark that these decompositions also confirm the Grothendieck version of the conjectured relation (4.18).

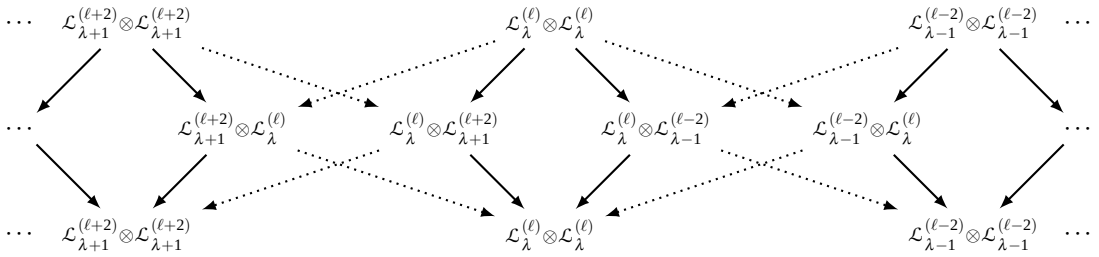

FIGURE 10. A part of the (proposed) Loewy diagrams for the atypical bulk modules. Taking $\lambda = 0 \bmod 2$ and $\ell = 0,1,2,3 \bmod 4$ gives the diagrams for $\mathcal{B}_0$, $\mathcal{B}_{-1/2;+}$, $\mathcal{B}_1$ and $\mathcal{B}_{-3/2;+}$, respectively.

4.5. **Bulk Modular Invariants and State Spaces.** The symmetries of the S-matrix imply, as usual, that the diagonal partition function (and its charge-conjugate variant) is modular invariant:

$$\mathbf{Z}_{\text{diag.}}(q,\overline{q}) = \sum_{\ell\in\mathbb{Z}} \int_{-1}^{1} \overline{\mathrm{ch}\big[\mathcal{E}^{(\ell)}_{\lambda}\big]}\,\mathrm{ch}\big[\mathcal{E}^{(\ell)}_{\lambda}\big]\,\mathrm{d}\lambda\,. \tag{4.36}$$

At the level of the quantum state space, this suggests a splitting into typical and atypical sectors:

$$\mathcal{H}_{\text{bulk}} = \left[\bigoplus_{\ell\in\mathbb{Z}} \oint_{\substack{-1\\ \lambda\neq\pm1/2}}^{1} \left(\mathcal{E}^{(\ell)}_{\lambda}\otimes\mathcal{E}^{(\ell)}_{\lambda}\right)\mathrm{d}\lambda\right] \oplus \mathcal{B}_{\text{atyp}}. \tag{4.37}$$

Once again, the character of the atypical contribution $\mathcal{B}_{\text{atyp}}$ may be put in the suggestive form

$$\begin{aligned}\mathrm{ch}\big[\mathcal{B}_{\text{atyp}}\big] &= \sum_{\ell\in\mathbb{Z}}\left(\overline{\mathrm{ch}\big[\mathcal{L}^{(\ell)}_{0}\big]}\,\mathrm{ch}\big[\mathcal{S}^{(\ell)}_{0}\big] + \overline{\mathrm{ch}\big[\mathcal{L}^{(\ell)}_{1}\big]}\,\mathrm{ch}\big[\mathcal{S}^{(\ell)}_{1}\big]\right)\\ &= \sum_{\ell\in\mathbb{Z}}\left(\overline{\mathrm{ch}\big[\mathcal{S}^{(\ell)}_{0}\big]}\,\mathrm{ch}\big[\mathcal{L}^{(\ell)}_{0}\big] + \overline{\mathrm{ch}\big[\mathcal{S}^{(\ell)}_{1}\big]}\,\mathrm{ch}\big[\mathcal{L}^{(\ell)}_{1}\big]\right),\end{aligned} \tag{4.38}$$

but, unlike the bulk triplet module of Section 3.6, $\mathcal{B}_{\text{atyp}} = \mathcal{B}_0\oplus\mathcal{B}_1\oplus\mathcal{B}_{-1/2;+}\oplus\mathcal{B}_{-3/2;+}$ is decomposable. This follows immediately from considering the weights, modulo 2, and conformal dimensions, modulo 1, of the atypical states. We have chosen to label the bulk atypicals as we did the admissible irreducibles $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{D}_{-1/2;+}$ and $\mathcal{D}_{-3/2;+}$ because the bulk modules are distinguished by which of the four admissibles has its character squared contributing to the bulk character.

A natural proposal for the structure of these bulk atypical modules is then to draw the Loewy diagrams of the contributing indecomposables $\mathcal{S}^{(\ell)}_{\lambda}$, tensoring each factor (on the left) with $\mathcal{L}^{(\ell)}_{\lambda}$. This defines the holomorphic structure of the bulk atypical and the antiholomorphic structure is added by identifying factors which combine to give $\mathcal{S}^{(\ell)}_{\lambda}\otimes\mathcal{L}^{(\ell)}_{\lambda}$. The resulting bulk Loewy diagram differs from that of the triplet model, pictured in Figure 7, in that there are now infinitely many composition factors ($\widehat{\mathfrak{sl}}(2)_{-1/2}$ behaves more like the singlet model in this respect). We illustrate a part of it in Figure 10. Note that the proposed structure suggests that spectral flow acts periodically on the atypical sector:

$$\cdots \xrightarrow{\sigma} \mathcal{B}_0 \xrightarrow{\sigma} \mathcal{B}_{-1/2;+} \xrightarrow{\sigma} \mathcal{B}_1 \xrightarrow{\sigma} \mathcal{B}_{-3/2;+} \xrightarrow{\sigma} \mathcal{B}_0 \xrightarrow{\sigma} \cdots. \tag{4.39}$$

We further remark that these bulk modules are manifestly local ($L_0 - \overline{L}_0$ may be diagonalised).

There are, of course, many simple currents with which we can try to construct additional modular invariants. Indeed, the fusion rules (4.19), coupled with (4.18), show that every atypical irreducible $\mathcal{L}^{(\ell)}_{\lambda}$ is a simple current. Requiring that the simple currents have fields of integer dimension restricts this to two families, $\mathcal{L}^{(4m)}_0$ and $\mathcal{L}^{(4m+2)}_1$, for $m\in\mathbb{Z}$. However, the latter family is generated by the spectral flows of the bosonic ghost fields. Because the ghost fields have dimension $\frac{1}{2}$ and their spectral flows have integer dimension, these flowed fields will be *mutually fermionic*. We do not therefore expect them to give rise to modular invariants.

The construction of the extended algebras proceeds as with the singlet algebra (Section 3.5). For each $n\in\mathbb{Z}$, there is an extended algebra $\mathbb{W}^{(n)} = \bigoplus_{m\in\mathbb{Z}}\mathcal{L}^{(4mn)}_0$ and its untwisted typical modules are the $\mathbb{E}^{(n;\ell)}_{\lambda} = \bigoplus_{m\in\mathbb{Z}}\mathcal{E}^{(4mn+\ell)}_{\lambda}$

with $2n\lambda \in \mathbb{Z}$ and $\ell = 0,1,\dots,4m-1$. The extended S-matrix is then

$$\mathrm{ch}\big[\mathbb{E}^{(n;\ell)}_{j/2n}\big]\Big|_{\mathsf{S}} = \sum_{j',\ell'=0}^{4n-1} \mathbb{S}^{(n)}_{(j,\ell)(j',\ell')}\mathrm{ch}\big[\mathbb{E}^{(n;\ell')}_{j'/2n}\big], \qquad \mathbb{S}^{(n)}_{(j,\ell)(j',\ell')} = \frac{1}{4n}\frac{|\tau|}{-\mathrm{i}\tau}\mathrm{e}^{\mathrm{i}\pi\left(\ell\ell' n - j'\ell - j\ell'\right)/2n}, \tag{4.40}$$

and its evident symmetries immediately imply the modular invariance of the extended partition function

$$\mathbf{Z}^{(n)}_{\mathrm{diag.}} = \sum_{j=0}^{4n-1}\sum_{\ell=0}^{4n-1} \overline{\mathrm{ch}\big[\mathbb{E}^{(n;\ell)}_{j/2n}\big]}\,\mathrm{ch}\big[\mathbb{E}^{(n;\ell)}_{j/2n}\big]. \tag{4.41}$$

Unfortunately, the modular properties of the atypical extended characters again remain out of reach.

We conclude by mentioning that these extended partition functions may describe the level $-\frac{1}{2}$ Wess-Zumino-Witten models on the non-compact simple Lie group $\mathsf{SL}(2;\mathbb{R})$. More precisely, we recall that this group has centre $\mathbb{Z}_2$ and fundamental group $\mathbb{Z}$, so there is an infinite number of Lie groups having $\mathfrak{sl}(2;\mathbb{R})$ as their Lie algebra. This includes the simply-connected universal cover of $\mathsf{SL}(2;\mathbb{R})$, often referred to as AdS_3, and the centreless adjoint group $\mathsf{PSL}(2;\mathbb{R})$. We propose that the partition function $\mathbf{Z}_{\mathrm{diag.}}$ of (4.36) describes strings on AdS_3 with level $-\frac{1}{2}$, whereas $\mathbf{Z}^{(1)}_{\mathrm{diag.}}$ describes strings on $\mathsf{PSL}(2;\mathbb{R})$ (or $\mathsf{SL}(2;\mathbb{R})$).

Whether this proposal is true or not, we remark that we can arrive at a consistent structure for the atypical sector of the quantum state space of these extended theories by identifying factors in Figure 10 which are identical except that their spectral flow indices differ by a multiple of $4n$. This imposes a periodicity on the infinitely wide bulk Loewy diagrams so that the resulting diagrams have a finite number of composition factors. For $n = 1$, it is easy to check that each extended atypical diagram has eight factors and that the structure looks identical to the triplet bulk atypical pictured in Figure 7.

4.6. **Correlation Functions.** Correlation functions for $\widehat{\mathfrak{sl}}(2)_{-1/2}$ may be computed using a strategy that is almost identical to that used for symplectic fermions. We start by recalling the bosonisation of the ghost fields $\beta(z)$ and $\gamma(z)$. Let $\varphi_L(z)$ and $y_L(z)$ be free bosons with operator product expansions

$$\varphi_L(z)\varphi_L(w) \sim \log(z-w), \qquad y_L(z)y_L(w) \sim -\log(z-w). \tag{4.42}$$

Then, the bosonisation amounts to the identifications

$$\beta(z) = :\mathrm{e}^{-\left(\varphi_L(z)+y_L(z)\right)}:, \qquad \gamma(z) = :\partial\varphi_L(z)\mathrm{e}^{\varphi_L(z)+y_L(z)}:. \tag{4.43}$$

We observe that these fields commute with the zero-mode of $\mathrm{e}^{-\varphi_L(z)}$, so we conclude that we can use the same screening charge as for symplectic fermions:

$$Q = \int_{\mathbb{C}} \mathrm{e}^{-\varphi(z,\overline{z})}\,\frac{\mathrm{d}z\mathrm{d}\overline{z}}{2\pi}. \tag{4.44}$$

The $\widehat{\mathfrak{sl}}(2)_{-1/2}$ currents now take the bosonised form

$$\begin{aligned} e(z) &= \frac{1}{2}:\beta(z)\beta(z): = \frac{1}{2}:\mathrm{e}^{-2\left(\varphi_L(z)+y_L(z)\right)}:, \qquad h(z) = \frac{1}{2}:\beta(z)\gamma(z): = \partial y_L(z), \\ f(z) &= \frac{1}{2}:\gamma(z)\gamma(z): = \frac{1}{2}:\big(\partial\varphi(z)\partial\varphi(z) - \partial^2\varphi(z)\big)\mathrm{e}^{2\left(\varphi_L(z)+y_L(z)\right)}:. \end{aligned} \tag{4.45}$$

We will neglect the antiholomorphic currents which are constructed similarly. In this free field realisation, it is natural to consider the following bulk fields:

$$V_{\ell,\lambda;n}(z,\overline{z}) = :\mathrm{e}^{(-\lambda+1/2)\varphi(z,\overline{z})+(-\lambda+\ell/2)y(z,\overline{z})-n\left(\varphi_L(z)+y_L(z)\right)}:. \tag{4.46}$$

Here we have chosen a convenient labelling which requires $\ell \in \mathbb{Z}$ and $n \in 2\mathbb{Z}$. We compute the following operator product expansions with the currents:

$$\begin{aligned} e(z)V_{\ell,\lambda;0}(w,\overline{w}) &= \frac{\frac{1}{2}V_{\ell,\lambda;2}(w,\overline{w})}{(z-w)^{1-\ell}} + \cdots, \qquad h(z)V_{\ell,\lambda;0}(w,\overline{w}) = \frac{(\lambda-\ell/2)V_{\ell,\lambda;0}(w,\overline{w})}{z-w} + \cdots, \\ f(z)V_{\ell,\lambda;0}(w,\overline{w}) &= \frac{\frac{1}{2}\left(-\lambda+\frac{1}{2}\right)\left(-\lambda+\frac{3}{2}\right)V_{\ell,\lambda;-2}(w,\overline{w})}{(z-w)^{1+\ell}} + \cdots, \\ T(z)V_{\ell,\lambda;0}(w,\overline{w}) &= \frac{-\frac{1}{8}\left(1+\ell^2-4\ell\lambda\right)V_{\ell,\lambda;0}(w,\overline{w})}{(z-w)^2} + \frac{\partial V_{\ell,\lambda;0}(w,\overline{w})}{z-w} + \cdots. \end{aligned} \tag{4.47}$$

This means that the field $V_{\ell,\lambda;0}(w,\overline{w})$ transforms as a ground state in the representation $\mathcal{E}_\lambda^{(\ell)}$. In general, n parametrises the ground states of these representations, but we will restrict to $n=0$ for simplicity.

Correlation functions are defined exactly as in the symplectic fermion case,

$$\big\langle V_{\ell_1,\lambda_1;0}(z_1,\overline{z}_1)\cdots V_{\ell_n,\lambda_n;0}(z_n,\overline{z}_n)\big\rangle = \big\langle V_{\ell_1,\lambda_1;0}(z_1,\overline{z}_1)\cdots V_{\ell_n,\lambda_n;0}(z_n,\overline{z}_n)\mathsf{e}^{-Q}\big\rangle_0, \tag{4.48}$$

and correlators in the free theory are standard free boson correlators, subject to the charge conservation condition

$$\begin{aligned}&\big\langle V_{\ell_1,\lambda_1;0}(z_1,\overline{z}_1)\cdots V_{\ell_n,\lambda_n;0}(z_n,\overline{z}_n)\big\rangle_0\\ &\quad= -\delta\big(\lambda_1+\cdots+\lambda_n=\tfrac{1}{2}(n-2)\big)\,\delta_{\ell_1+\cdots+\ell_n=n-2}\prod_{i<j}\big|z_i-z_j\big|^{2\left((\lambda_i-1/2)(\lambda_j-1/2)-(\lambda_i-\ell_i/2)(\lambda_j-\ell_j/2)\right)}.\end{aligned} \tag{4.49}$$

In this way, we arrive at the following non-zero one- and two-point functions:

$$\begin{aligned}\big\langle V_{\ell,\lambda;0}(z,\overline{z})\big\rangle &= -\delta\big(\lambda=-\tfrac{1}{2}\big)\,\delta_{\ell=-1},\\ \big\langle V_{\ell,\lambda;0}(z,\overline{z})V_{\ell',\lambda';0}(w,\overline{w})\big\rangle &= -\delta\big(\lambda+\lambda'=0\big)\,\delta_{\ell+\ell'=0}|z-w|^{2\left((\lambda-\ell/2)^2-\lambda^2+1/4\right)},\end{aligned} \tag{4.50}$$

where

$$h_{\ell,\lambda}=\frac{(\lambda-\frac{1}{2})((\lambda+\frac{1}{2}))}{2}-\frac{(\lambda-\frac{\ell}{2})^2}{2}=\frac{1}{2}\Big(\ell\lambda-\frac{\ell^2}{4}-\frac{1}{4}\Big) \tag{4.51}$$

is the conformal dimension of the ground state of weight $\lambda-\ell/2$ of $\mathcal{E}_\lambda^{(\ell)}$. We remark that this one-point function is consistent with the vacuum appearing as a composition factor of the indecomposables $\mathcal{E}^{(-1)}_{-1/2;\pm}$ and that the two-point function is in line with our notion of conjugation. For the three-point function, we again need the Fateev-Dotsenko integral formula to obtain

$$\begin{aligned}\big\langle V_{\ell,\lambda;0}(z,\overline{z})V_{m,\mu;0}(1,1)V_{n,\nu;0}(0,0)\big\rangle &= \frac{-1}{|z-1|^{2\left(h_{\ell,\lambda}+h_{m,\mu}-h_{n,\nu}\right)}|z|^{2\left(h_{\ell,\lambda}-h_{m,\mu}+h_{n,\nu}\right)}}\\ &\cdot\left[\delta(\lambda+\mu+\nu=1/2)\delta_{\ell+m+n=1}+\frac{\Gamma\big(\frac{1}{2}+\lambda\big)\Gamma\big(\frac{1}{2}+\mu\big)\Gamma\big(\frac{1}{2}+\nu\big)}{\Gamma\big(\frac{1}{2}-\lambda\big)\Gamma\big(\frac{1}{2}-\mu\big)\Gamma\big(\frac{1}{2}-\nu\big)}\delta\big(\lambda+\mu+\nu=-\tfrac{1}{2}\big)\,\delta_{\ell+m+n=-1}\right].\end{aligned} \tag{4.52}$$

We observe that there are singularities in the three-point function whenever $\lambda\in\mathbb{Z}+\frac{1}{2}$. Regularising as in the case of symplectic fermions would give logarithmic correlators, thus confirming the presence of the indecomposable modules $\mathcal{S}_0^{(\ell)}$ and $\mathcal{S}_1^{(\ell)}$. These correlation functions may be checked to be consistent with the fusion rules (4.19).

4.7. **Further Developments.** Unlike the triplet theories discussed in Section 3.8 and the superalgebra theories that we will consider in Section 5.7, the logarithmic conformal field theories with admissible level affine algebra symmetries remain relatively unexplored. Aside from $\widehat{\mathfrak{sl}}(2)_{-1/2}$, reviewed above, the only other admissible theory to have received comparable treatment is $\widehat{\mathfrak{sl}}(2)_{-4/3}$ [56, 63]. Here, there are three admissible highest weight modules, all irreducible, which we denote (with the same conventions as used above) by $\mathcal{L}_0$, $\mathcal{D}_{-2/3;+}$ and $\mathcal{D}_{-4/3;+}$. Again, spectral flow acts and we obtain two infinite families because $\mathcal{D}_{-4/3;+}=\mathcal{L}_0^{(1)}$. There are also typical irreducibles $\mathcal{E}_\lambda$, with $\lambda\neq\pm\frac{2}{3}$ mod 2, and atypical indecomposables $\mathcal{E}_{\pm1/2;\pm}$, both of whose ground states have conformal dimension $-\frac{1}{3}$, as well as their spectral flows.

The Nahm-Gaberdiel-Kausch algorithm gives (untwisted) fusion rules including [56]

$$\mathcal{D}_{2/3;-}\times\mathcal{D}_{-2/3;+}=\mathcal{L}_0\oplus\mathcal{E}_0,\qquad \mathcal{D}_{-2/3;+}\times\mathcal{E}_0=\mathcal{S}_{-2/3;+},\qquad \mathcal{E}_0\times\mathcal{E}_0=\mathcal{E}_0\oplus\mathcal{S}_0, \tag{4.53}$$

with the vacuum module $\mathbb{L}_0$ acting again as the fusion identity. Here, $\mathcal{S}_0$ and $\mathcal{S}_{-2/3;+}$ are indecomposables with respective socles $\mathcal{L}_0$ and $\mathcal{D}_{-2/3;+}$ and the familiar diamond-shaped Loewy diagrams. Both exhibit a non-diagonalisable action of L_0. The modular properties of the characters of the typicals and atypicals are derived as for $k=-\frac{1}{2}$ with the resulting S-matrix entries being [63]

$$\begin{aligned}\mathsf{S}_{(\ell,\lambda)(\ell',\lambda')}&=\frac{1}{2}\frac{|\tau|}{-\mathrm{i}\tau}\mathsf{e}^{\mathrm{i}\pi\left(4\ell\ell'/3-\ell\lambda'-\ell'\lambda\right)},\\ \mathsf{S}_{\underline{(\ell,0)}(\ell',\lambda')}&=\frac{1}{2}\frac{|\tau|}{-\mathrm{i}\tau}\frac{\mathsf{e}^{\mathrm{i}\pi\ell\left(4\ell'/3-\lambda'\right)}}{1+2\cos(\pi\lambda')},\qquad \mathsf{S}_{\underline{(\ell,-2/3)}(\ell',\lambda')}=\frac{|\tau|}{-\mathrm{i}\tau}\frac{\mathsf{e}^{\mathrm{i}\pi(\ell+1/2)\left(4\ell'/3-\lambda'\right)}\cos(\pi\lambda'/2)}{1+2\cos(\pi\lambda')}.\end{aligned} \tag{4.54}$$

Applying the Verlinde formula then leads to

$$\mathcal{E}_\lambda\times\mathcal{E}_\mu=\mathcal{E}^{(-1)}_{\lambda+\mu-4/3}\oplus\mathcal{E}_{\lambda+\mu}\oplus\mathcal{E}^{(1)}_{\lambda+\mu+4/3}\qquad(\lambda+\mu\neq0,\pm\tfrac{2}{3}\bmod 2). \tag{4.55}$$

At the atypical points 0 and $\pm\frac{2}{3}$, the natural prediction is instead that

$$\mathcal{E}_\lambda \times \mathcal{E}_{-\lambda} = \mathcal{E}_0 \oplus \mathcal{S}_0, \qquad \mathcal{E}_\lambda \times \mathcal{E}_{-\lambda+2/3} = \mathcal{E}_0^{(1)} \oplus \mathcal{S}_{2/3;-}, \qquad \mathcal{E}_\lambda \times \mathcal{E}_{-\lambda} = \mathcal{E}_0^{(-1)} \oplus \mathcal{S}_{-2/3;+}, \tag{4.56}$$

where $\mathcal{S}_{2/3;-} = \mathcal{S}^{(-1)}_{-2/3;+} = \mathcal{S}^\star_{-2/3;+}$. Moreover, it also suggests the fusion rule

$$\mathcal{D}_{-2/3;+} \times \mathcal{S}_0 = \mathcal{E}_0^{(-1)} \oplus \mathcal{S}_{-2/3;+} \oplus \mathcal{E}_0^{(2)}, \tag{4.57}$$

from which the remaining rules follow by using associativity.

The fact that Verlinde formulae for $\widehat{\mathfrak{sl}}(2)_{-1/2}$ and $\widehat{\mathfrak{sl}}(2)_{-4/3}$ have been successfully derived using the above formalism suggests that this will generalise to all admissible levels. This will be detailed in [64]. Other affine algebras at admissible levels have not yet received much attention, though $\widehat{\mathfrak{sl}}(3)$ was briefly addressed [126] in the days before spectral flow and indecomposability were realised to be critical and some structure theory for admissible level $\widehat{\mathfrak{sl}}(2|1)$ may be found in [127–129]. The link with the Wess-Zumino-Witten model on $\mathsf{SL}(2;\mathbb{R})$ is interesting because it suggests that these theories may be logarithmic for general values of the level. The famous articles [42, 130, 131] of Maldacena and Ooguri suggest no logarithmic structure for this model, though it could be argued that they did not look for any (see also [132]). Indeed, recent mathematical work [133] suggests that there may be more to this picture than was previously realised.

5. Wess-Zumino-Witten Models with $\widehat{\mathfrak{gl}}(1|1)$ Symmetry

The Wess-Zumino-Witten model on $\mathsf{GL}(1|1)$ is by far the best understood conformal field theory associated to Lie supergroups. It was first studied by Rozansky and Saleur two decades ago [9, 10] in two of the original landmark logarithmic conformal field theory papers. More recently, this model was reconsidered by Saleur and Schomerus [36] who were able to compute correlation functions and propose a structure for the full bulk theory. Their computations revealed a striking similarity to the twist field correlators of the symplectic fermion theory, an observation that was explained in [102], which in turn was motivated by [134]. The correlation functions of Saleur and Schomerus implicitly suggested fusion rules, which were then confirmed in [66]. This theory is also one of the few for which the boundary theory is thoroughly studied [135–137], meaning that D-branes have been classified and boundary three-point correlation functions and bulk-boundary two-point functions are known. We follow [66] in reviewing this example.

5.1. $\mathfrak{gl}(1|1)$ and its Representations.

The Lie superalgebra $\mathfrak{gl}(1|1)$ is generated by two bosonic elements N and E and by two fermionic ones $\psi^\pm$ subject to the relations

$$\big[N, \psi^\pm\big] = \pm\psi^\pm, \qquad \big\{\psi^+, \psi^-\big\} = E. \tag{5.1}$$

It naturally acts on the super vector space $\mathbb{C}^{1|1}$ and its elements are identified with supermatrices as follows:

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.2}$$

The Killing form, corresponding to the supertrace form in the adjoint representation, is degenerate for $\mathfrak{gl}(1|1)$. To obtain a non-degenerate bilinear form $\kappa\big(\cdot,\cdot\big)$, one instead takes the supertrace in the defining representation (5.2):

$$\kappa\big(N,E\big) = \kappa\big(E,N\big) = 1, \qquad \kappa\big(\psi^+,\psi^-\big) = -\kappa\big(\psi^-,\psi^+\big) = 1. \tag{5.3}$$

Verma modules $\overline{\mathcal{V}}_{n,e}$ are constructed from highest weight states $\big|\overline{v}_{n,e}\big\rangle$ satisfying

$$E\big|\overline{v}_{n,e}\big\rangle = e\big|\overline{v}_{n,e}\big\rangle, \qquad N\big|\overline{v}_{n,e}\big\rangle = \big(n+\tfrac{1}{2}\big)\big|\overline{v}_{n,e}\big\rangle, \qquad \psi^+\big|\overline{v}_{n,e}\big\rangle = 0 \tag{5.4}$$

in the usual manner. However, as ψ^- squares to 0, all Verma modules are two-dimensional.[20] Moreover, as $\psi^+\psi^-\big|\overline{v}_{n,e}\big\rangle = e\big|\overline{v}_{n,e}\big\rangle$, the Verma module $\overline{\mathcal{V}}_{n,e}$ is reducible if and only if $e = 0$. Once again, irreducibility is the generic situation and hence modules with $e \neq 0$ are referred to as being typical. In the atypical case where $e = 0$, there is an irreducible one-dimensional submodule spanned by $\psi^-\big|\overline{v}_{n,0}\big\rangle$ — we shall denote it by $\overline{\mathcal{A}}_{n-1/2}$ — and the quotient $\overline{\mathcal{A}}_{n+1/2}$ is also one-dimensional and irreducible:

$$0 \longrightarrow \overline{\mathcal{A}}'_{n-1/2} \longrightarrow \overline{\mathcal{V}}_{n,0} \longrightarrow \overline{\mathcal{A}}_{n+1/2} \longrightarrow 0. \tag{5.5}$$

[20] We remark at this point that the label n of the Verma module $\overline{\mathcal{V}}_{n,e}$ refers to the *average* of the N-eigenvalues for this representation. This average labelling convention for n will be adhered to for all $\mathfrak{gl}(1|1)$-modules.

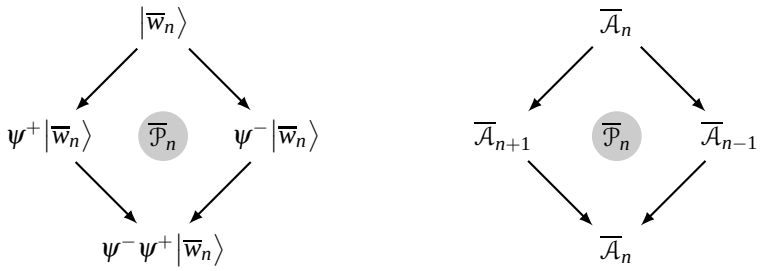

FIGURE 11. The projective cover $\overline{\mathcal{P}}_n$ of the $\mathfrak{gl}(1|1)$-module $\overline{\mathcal{A}}_n$, illustrated by its states on the left and by its Loewy diagram on the right.

Here, the prime attached to the submodule $\overline{\mathcal{A}}_{n-1/2}$ serves to remind us that its highest weight state has the opposite parity to the highest weight generators of the other modules appearing in this sequence.

Typical irreducibles turn out to be projective in the category of all finite-dimensional modules upon which N and E act diagonalisably. The atypical irreducibles have projective covers $\overline{\mathcal{P}}_n$ which are generated by a vector $|\overline{w}_n\rangle$ satisfying $E|\overline{w}_n\rangle = 0$ and $N|\overline{w}_n\rangle = n|\overline{w}_n\rangle$. The fermionic generators ψ^+ and ψ^- act freely on $|\overline{w}_n\rangle$, resulting in the four-dimensional representation illustrated in Figure 11. These atypical projectives naturally appear in the representation ring generated by the irreducibles under the (graded) tensor product:

$$\begin{gathered}
\overline{\mathcal{A}}_n \otimes \overline{\mathcal{A}}_{n'} = \overline{\mathcal{A}}_{n+n'}, \qquad \overline{\mathcal{A}}_n \otimes \overline{\mathcal{V}}_{n',e'} = \overline{\mathcal{V}}_{n+n',e'}, \qquad \overline{\mathcal{A}}_n \otimes \overline{\mathcal{P}}_{n'} = \overline{\mathcal{P}}_{n+n'}, \\
\overline{\mathcal{V}}_{n,e} \otimes \overline{\mathcal{V}}_{n',e'} = \begin{cases} \overline{\mathcal{P}}'_{n+n'} & \text{if } e+e' = 0, \\ \overline{\mathcal{V}}_{n+n'+1/2,e+e'} \oplus \overline{\mathcal{V}}'_{n+n'-1/2,e+e'} & \text{otherwise,} \end{cases} \\
\overline{\mathcal{V}}_{n,e} \otimes \overline{\mathcal{P}}_{n'} = \overline{\mathcal{V}}'_{n+n'+1,e} \oplus 2\,\overline{\mathcal{V}}_{n+n',e} \oplus \overline{\mathcal{V}}'_{n+n'-1,e}, \qquad \overline{\mathcal{P}}_n \otimes \overline{\mathcal{P}}_{n'} = \overline{\mathcal{P}}'_{n+n'+1} \oplus 2\,\overline{\mathcal{P}}_{n+n'} \oplus \overline{\mathcal{P}}'_{n+n'-1}.
\end{gathered} \tag{5.6}$$

The prime on the indecomposables $\overline{\mathcal{P}}_n$ refers to the relative parity of the generating state $|\overline{w}_n\rangle$. We remark that the Casimir $Q = NE + \psi^-\psi^+$ acts non-diagonalisably on $\overline{\mathcal{P}}_n$: $Q|\overline{w}_n\rangle = \psi^-\psi^+|\overline{w}_n\rangle$.

5.2. **The $\mathsf{GL}(1|1)$ Wess-Zumino-Witten Model.** Wess-Zumino-Witten models on compact reductive Lie groups give rise to a natural family of rational conformal field theories. The model on a Lie supergroup may be constructed in the same manner. One starts with a supergroup-valued field g and parametrises it using a "Gauss-like" decomposition. For $\mathsf{GL}(1|1)$, this corresponds to

$$g = \mathrm{e}^{c_-\psi^-}\mathrm{e}^{XE+YN}\mathrm{e}^{-c_+\psi^+}, \tag{5.7}$$

so the fields of the theory are the two bosonic fields $X(z,\overline{z})$ and $Y(z,\overline{z})$ and the two fermionic fields $c_\pm(z,\overline{z})$. The standard Wess-Zumino-Witten action is then reduced, using the Polyakov-Wiegmann identity, to

$$S_{\mathrm{WZW}}[g] = \frac{k}{4\pi}\int\left(-\partial X\overline{\partial}Y - \partial Y\overline{\partial}X + 2\mathrm{e}^Y\partial c_+\overline{\partial}c_-\right)\mathrm{d}z\mathrm{d}\overline{z}, \tag{5.8}$$

where k is the level. Varying this action leads to the expected equations of motion:

$$\overline{\partial}J(z,\overline{z}) = 0, \qquad \partial\overline{J}(z,\overline{z}) = 0. \tag{5.9}$$

Here, $J = k\partial g g^{-1}$ and $\overline{J} = -kg^{-1}\overline{\partial}g$ are Lie superalgebra-valued currents. In component form, they become

$$J^E = -k\partial Y, \qquad J^N = -k\partial X + kc_-\partial c_+\mathrm{e}^Y, \qquad J^- = k\mathrm{e}^Y\partial c_+, \qquad J^+ = -k\partial c_- - kc_-\partial Y, \tag{5.10}$$

and similarly for the anti-holomorphic current $\overline{J}$.

Upon quantising, the modes of the holomorphic current satisfy the relations of the affine Kac-Moody superalgebra $\widehat{\mathfrak{gl}}(1|1)$ at level k (we will mostly ignore the antiholomorphic sector as usual):

$$\left[J^E_r, J^N_s\right] = r\delta_{r+s=0}k, \qquad \left[J^N_r, J^\pm_s\right] = \pm J^\pm_{r+s}, \qquad \left\{J^+_r, J^-_s\right\} = J^E_{r+s} + r\delta_{r+s=0}k. \tag{5.11}$$

As in the case of the free boson (Section 1.2), any non-zero level k can be rescaled to 1. We will assume in what follows that such a rescaling has been made.

The energy-momentum tensor $T(z)$ has central charge zero and is given by a variant of the Sugawara construction[21]

$$T(z) = \frac{1}{2} : J^N J^E + J^E J^N - J^+ J^- + J^- J^+ : (z) + \frac{1}{2} : J^E J^E : (z). \tag{5.12}$$

We note that, as in the case of $\widehat{\mathfrak{sl}}(2)$ discussed in Section 4, $\widehat{\mathfrak{gl}}(1|1)$ possesses a family of spectral flow automorphisms σ^ℓ which are indispensable to understanding its representation theory:

$$\sigma^\ell\big(J^N_r\big) = J^N_r, \qquad \sigma^\ell\big(J^E_r\big) = J^E_r - \ell\delta_{r,0}, \qquad \sigma^\ell\big(J^\pm_r\big) = J^\pm_{r\mp\ell}, \qquad \sigma^\ell(L_0) = L_0 - \ell J^N_0. \tag{5.13}$$

As before, these automorphisms may be used to construct new modules $\mathcal{M}^{(\ell)}$ from an arbitrary $\widehat{\mathfrak{gl}}(1|1)$-module $\mathcal{M}$ by twisting the action on the states as in (4.7).

5.3. **Representation Theory of $\widehat{\mathfrak{gl}}(1|1)$.** The representation theory of the affine algebra $\widehat{\mathfrak{gl}}(1|1)$ is very similar to that of its horizontal subalgebra $\mathfrak{gl}(1|1)$. We define affine Verma modules $\mathcal{V}_{n,e}$ and their irreducible quotients, as before, by defining a highest weight state $\big|v_{n,e}\big\rangle$ to be one satisfying[22]

$$J^N_0\big|v_{n,e}\big\rangle = \big(n+\tfrac{1}{2}\big)\big|v_{n,e}\big\rangle, \qquad J^E_0\big|v_{n,e}\big\rangle = e\big|v_{n,e}\big\rangle, \qquad J^+_0\big|v_{n,e}\big\rangle = J^\pm_r\big|v_{n,e}\big\rangle = 0 \quad (r>0). \tag{5.14}$$

The conformal dimension of the highest weight state $\big|v_{n,e}\big\rangle$ is then (recalling that k has been set to 1)

$$L_0\big|v_{n,e}\big\rangle = \Delta_{n,e}\big|v_{n,e}\big\rangle = \big(ne+\tfrac{1}{2}e^2\big)\big|v_{n,e}\big\rangle. \tag{5.15}$$

It follows that every singular vector of $\mathcal{V}_{n,0}$ has dimension 0, leading to the non-split exact sequence

$$0 \longrightarrow \mathcal{A}'_{n-1/2,0} \longrightarrow \mathcal{V}_{n,0} \longrightarrow \mathcal{A}_{n+1/2,0} \longrightarrow 0, \tag{5.16a}$$

where the prime again indicates that the submodule's highest weight state has parity opposite to those of the other modules. A simple counting argument [66] now shows that the $\mathcal{V}_{n,e}$ with $0<|e|<1$ are irreducible. By employing the spectral flow automorphisms of (5.13), one concludes that the Verma modules $\mathcal{V}_{n,e}$ with $e\notin\mathbb{Z}$ are irreducible (typical) and that the (atypical) case $e\in\mathbb{Z}$ yields reducible Verma modules. Along with (5.16a), the (non-split) exact sequences turn out to be

$$\begin{aligned} 0 &\longrightarrow \mathcal{A}'_{n-1,e} \longrightarrow \mathcal{V}_{n,e} \longrightarrow \mathcal{A}_{n,e} \longrightarrow 0 \quad (e\in\mathbb{Z}_-),\\ 0 &\longrightarrow \mathcal{A}'_{n+1,e} \longrightarrow \mathcal{V}_{n,e} \longrightarrow \mathcal{A}_{n,e} \longrightarrow 0 \quad (e\in\mathbb{Z}_+). \end{aligned} \tag{5.16b}$$

Note that, once again, the vacuum module $\mathcal{A}_{0,0}$ is atypical.

We remark that, in contrast with the action of spectral flow on $\widehat{\mathfrak{sl}}(2)$-modules, spectral flows of $\widehat{\mathfrak{gl}}(1|1)$-modules do not have states whose conformal dimensions are unbounded below. This can be traced back to the fact that each of the modes $J^\pm_r$ squares to 0.[23] Indeed, character methods and some analysis of Verma modules and their contragredient duals allow one to identify the result of applying σ:

$$\mathcal{V}^{(1)}_{n,e} = \mathcal{V}'_{n-1,e+1}, \quad \text{if } e\notin\mathbb{Z}, \qquad \mathcal{A}^{(1)}_{n,e} = \begin{cases} \mathcal{A}'_{n-1/2,0}, & \text{if } e=-1,\\ \mathcal{A}_{n-1/2,1}, & \text{if } e=0,\\ \mathcal{A}'_{n-1,e+1}, & \text{otherwise.} \end{cases} \tag{5.17}$$

Note that this immediately explains how the irreducibility of the Verma modules with $e\notin\mathbb{Z}$ could be deduced from that of those with $0<|e|<1$.

[21] The Lie superalgebra $\mathfrak{gl}(1|1)$ is not simple and the space of invariant bilinear forms turns out to be two-dimensional. However, there is a unique choice that leads to a Virasoro field [9].

[22] As with $\mathfrak{gl}(1|1)$-modules, the label n parametrising modules refers to the average J^N_0-eigenvalue of the ground states.

[23] The corresponding statement for integrable $\widehat{\mathfrak{g}}$-modules, with $\widehat{\mathfrak{g}}$ a Kac-Moody algebra, may similarly be traced back to the special form of the singular vectors of the Verma covers of these modules.

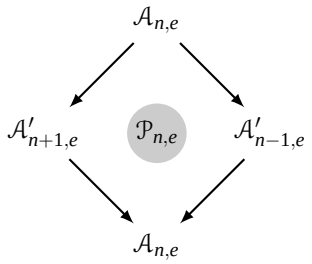

FIGURE 12. The Loewy diagram of the indecomposable $\widehat{\mathfrak{gl}}(1|1)$-module $\mathcal{P}_{n,e}$.

Finally, the fusion rules follow readily from the principles behind the Nahm-Gaberdiel-Kausch algorithm and the tensor product rules (5.6) of $\mathfrak{gl}(1|1)$ (we ignore parity here to avoid an overabundance of cases):

$$\begin{aligned}
\mathcal{A}_{n,e}\times\mathcal{A}_{n',e'} &= \mathcal{A}_{n+n'-\varepsilon(e,e'),e+e'}, \qquad \mathcal{A}_{n,e}\times\mathcal{V}_{n',e'} = \mathcal{V}_{n+n'-\varepsilon(e),e+e'},\\
\mathcal{V}_{n,e}\times\mathcal{V}_{n',e'} &= \begin{cases} \mathcal{P}_{n+n'+\varepsilon(e+e'),e+e'}, & \text{if } e+e'\in\mathbb{Z},\\ \mathcal{V}_{n+n'+1/2,e+e'}\oplus\mathcal{V}_{n+n'-1/2,e+e'} & \text{otherwise,}\end{cases}\\
\mathcal{A}_{n,e}\times\mathcal{P}_{n',e'} &= \mathcal{P}_{n+n'-\varepsilon(e,e'),e+e'},\\
\mathcal{V}_{n,e}\times\mathcal{P}_{n',e'} &= \mathcal{V}_{n+n'+1-\varepsilon(e'),e+e'}\oplus 2\,\mathcal{V}_{n+n'-\varepsilon(e'),e+e'}\oplus\mathcal{V}_{n+n'-1-\varepsilon(e'),e+e'},\\
\mathcal{P}_{n,e}\times\mathcal{P}_{n',e'} &= \mathcal{P}_{n+n'+1-\varepsilon(e,e'),e+e'}\oplus 2\,\mathcal{P}_{n+n'-\varepsilon(e,e'),e+e'}\oplus\mathcal{P}_{n+n'-1-\varepsilon(e,e'),e+e'}.
\end{aligned} \tag{5.18}$$

Here, we have defined

$$\varepsilon(e) = \tfrac{1}{2}\operatorname{sgn}(e), \qquad \varepsilon(e,e') = \varepsilon(e)+\varepsilon(e')-\varepsilon(e+e'), \tag{5.19}$$

with the convention that the sign function satisfies $\operatorname{sgn}(0)=0$.

The modules $\mathcal{P}_{n,e}$, with $e\in\mathbb{Z}$, generated by the above fusion rules may be constructed by inducing the $\mathfrak{gl}(1|1)$-module $\overline{\mathcal{P}}_n$ and applying spectral flow. They are indecomposable and, just as $\overline{\mathcal{P}}_n$ carries a non-diagonalisable action of the Casimir Q, so the action of L_0 on $\mathcal{P}_{n,e}$ is non-diagonalisable. The Loewy diagrams are given in Figure 12. It is often stated that the $\mathcal{P}_{n,e}$ are the projective covers of the $\mathcal{A}_{n,e}$, presumably in the category of vertex algebra modules upon which J_0^N and J_0^E act diagonalisably, but we are not aware of a proof of this statement.

5.4. **Modular Transformations and the Verlinde Formula.** For superalgebras, it is natural to work with supercharacters rather than characters. Those of the Verma modules $\mathcal{V}_{n,e}$ are given by

$$\begin{aligned}
\operatorname{sch}\big[\mathcal{V}_{n,e}\big](x;y;z;q) &= \operatorname{str}_{\mathcal{V}_{n,\ell}} x y^{J_0^E} z^{J_0^N} q^{L_0-c/24} = x y^e z^{n+1/2} q^{\Delta_{n,e}} \prod_{i=1}^{\infty}\frac{\big(1-zq^i\big)\big(1-z^{-1}q^{i-1}\big)}{\big(1-q^i\big)^2}\\
&= \mathrm{i} x y^e z^n q^{\Delta_{n,e}}\frac{\vartheta_1(z;q)}{\eta(q)^3},
\end{aligned} \tag{5.20}$$

where we recall that the level k (of which x is supposed to keep track) has been set to 1. With $x=\mathrm{e}^{2\pi\mathrm{i}t}$, $y=\mathrm{e}^{2\pi\mathrm{i}u}$, $z=\mathrm{e}^{2\pi\mathrm{i}v}$ and $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$ and the S-transformation $\mathsf{S}\colon (t|u|v|\tau)\to(t-uv/\tau|u/\tau|v/\tau|-1/\tau)$, the induced transformation of the supercharacters may be computed as (the analytic continuation of) a double Gaussian integral. The resulting S-matrix coefficients are

$$\operatorname{sch}\big[\mathcal{V}_{n,e}\big]\Big|_{\mathsf{S}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{S}_{(n,e),(n',e')}\operatorname{sch}\big[\mathcal{V}_{n',e'}\big]\,\mathrm{d}n'\mathrm{d}e', \qquad \mathsf{S}_{(n,e),(n',e')} = -\mathrm{i}\omega\mathrm{e}^{-2\pi\mathrm{i}\left(ne'+n'e+ee'\right)}, \tag{5.21}$$

where ω is a square root of minus one that depends upon how the analytic continuation is performed. We note that the sign $-\mathrm{i}\omega$ will cancel in bulk modular invariants and the Verlinde formula.

The exact sequences (5.16) may now be used to construct resolutions for the atypicals in terms of Verma modules. We consider only the vacuum module for brevity:

$$\cdots\longrightarrow\mathcal{V}'_{-7/2,0}\longrightarrow\mathcal{V}_{-5/2,0}\longrightarrow\mathcal{V}'_{-3/2,0}\longrightarrow\mathcal{V}_{-1/2,0}\longrightarrow\mathcal{A}_{0,0}\longrightarrow 0. \tag{5.22}$$

Because the relative parities of the Verma modules alternate, the vacuum supercharacter is not an alternating sum of Verma characters. The vacuum S-matrix entries are therefore

$$\mathsf{S}_{\underline{(0,0)},(n',e')} = \sum_{j=0}^{\infty}\mathsf{S}_{(-j-1/2,0),(n',e')} = -\mathrm{i}\omega\sum_{j=0}^{\infty}\mathrm{e}^{2\pi\mathrm{i}(j+1/2)e'} = \frac{\mathrm{i}\omega}{\mathrm{e}^{\mathrm{i}\pi e'}-\mathrm{e}^{-\mathrm{i}\pi e'}} = \frac{\omega}{2\sin[\pi e']}. \tag{5.23}$$

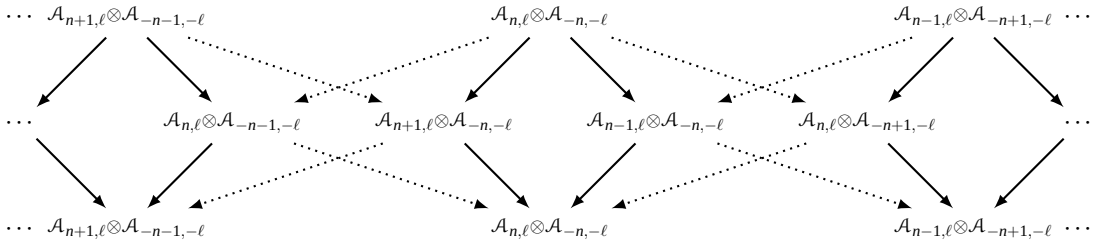

FIGURE 13. A part of the Loewy diagrams for the atypical bulk modules $\mathcal{J}_{n,e}$ corresponding to the charge-conjugate modular invariant partition function. We have neglected to indicate relative parities because the conjugate of an atypical $\mathcal{A}_{n,e}$ changes parity unless $e=0$.

From this, and the continuous Verlinde formula, we can compute the Grothendieck fusion of the typicals:

$$\begin{aligned}\mathsf{N}_{(n,e),(n',e')}{}^{(n'',e'')} &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{e}^{-2\pi\mathsf{i}\left((n+n'-n'')E+(e+e'-e'')(N+E)\right)}\left(\mathsf{e}^{\mathsf{i}\pi E}-\mathsf{e}^{\mathsf{i}\pi E}\right)\mathrm{d}N\mathrm{d}E\\ &= \delta\left(n''=n+n'+1/2\right)-\delta\left(n''=n+n'-1/2\right) \qquad (5.24)\\ \Rightarrow\qquad \mathrm{sch}\left[\mathcal{V}_{n,e}\right]\dot{\times}\mathrm{sch}\left[\mathcal{V}_{n',e'}\right] &= \mathrm{sch}\left[\mathcal{V}_{n+n'+1/2,e+e'}\right]-\mathrm{sch}\left[\mathcal{V}_{n+n'-1/2,e+e'}\right]. \qquad (5.25)\end{aligned}$$

The minus sign appearing in this result indicates that the parity of $\mathcal{V}_{n+n'-1/2,e+e'}$ is opposite to that of $\mathcal{V}_{n+n'+1/2,e+e'}$, hence we would affix a prime to the former module. This is therefore in perfect agreement with the genuine typical fusion rule reported in (5.18), the case when $e+e'\in\mathbb{Z}$ following from the character identity

$$\mathrm{sch}\left[\mathcal{V}_{n+n'+1/2,e+e'}\right]+\mathrm{sch}\left[\mathcal{V}'_{n+n'-1/2,e+e'}\right]=\mathrm{sch}\left[\mathcal{P}_{n+n'+\varepsilon(e+e'),\ell+\ell'}\right], \qquad (5.26)$$

which may be checked using the exact sequences (5.16) and the Loewy diagram in Figure 12. The other Grothendieck fusion products may be checked in a similar fashion. We remark that one can also use the modular properties of the characters, rather than the supercharacters, to compute Grothendieck fusion rules. However, closure under the (projective) modular group action then requires the consideration of the characters and supercharacters of twisted $\widehat{\mathfrak{gl}}(1|1)$-modules on which the generators act with half-integer moding.

5.5. **Bulk Modular Invariants and State Spaces.** The bulk state space of the $\mathsf{GL}(1|1)$ Wess-Zumino-Witten model was first proposed by Saleur and Schomerus in [36]. The result is

$$\mathcal{H}_{\text{bulk}}=\left[\oint_{\substack{\mathbb{R}^2\\ e\notin\mathbb{Z}}}\left[\left(\mathcal{V}_{n,e}\otimes\mathcal{V}'_{-n,-e}\right)\right]\mathrm{d}n\mathrm{d}e\right]\oplus\mathcal{B}_{\text{atyp}}, \qquad (5.27)$$

where the atypical contributions further decompose as

$$\mathcal{B}_{\text{atyp}}=\bigoplus_{e\in\mathbb{Z}}\oint_0^1\mathcal{J}_{n,e}\,\mathrm{d}n. \qquad (5.28)$$

We illustrate the structure of the indecomposable bulk atypicals $\mathcal{J}_{n,e}$ in Figure 13, noting that this proposal corresponds to the charge-conjugate partition function, rather than the diagonal one.

The modular invariance of the corresponding bulk super partition function is now straightforward to verify:

$$\begin{aligned}\mathbf{Z}_{\text{c.c.}}\Big|_{\mathsf{S}} &= \int_{\mathbb{R}^2}\left(\overline{\mathrm{sch}\left[\mathcal{V}'_{-n,-e}\right]\Big|_{\mathsf{S}}}\mathrm{sch}\left[\mathcal{V}_{n,e}\right]\Big|_{\mathsf{S}}\right)\mathrm{d}n\mathrm{d}e\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathsf{S}_{(n,e),(n',e')}\overline{\mathsf{S}_{(-n,-e),(n'',e'')}}\left(\overline{\mathrm{sch}\left[\mathcal{V}'_{n'',e''}\right]}\mathrm{sch}\left[\mathcal{V}_{n',e'}\right]\right)\mathrm{d}n\mathrm{d}e\mathrm{d}n'\mathrm{d}e'\mathrm{d}n''\mathrm{d}e''\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(-\omega^2)\mathsf{e}^{-2\pi\mathsf{i}\left(ne'+n'e+ee'+ne''+n''e+ee''\right)}\left(\overline{\mathrm{sch}\left[\mathcal{V}'_{n'',e''}\right]}\mathrm{sch}\left[\mathcal{V}_{n',e'}\right]\right)\mathrm{d}n\mathrm{d}e\mathrm{d}n'\mathrm{d}e'\mathrm{d}n''\mathrm{d}e''\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\delta\left(e'+e''=0\right)\delta\left(n'+e'+n''+e''=0\right)\left(\overline{\mathrm{sch}\left[\mathcal{V}'_{n'',e''}\right]}\mathrm{sch}\left[\mathcal{V}_{n',e'}\right]\right)\mathrm{d}n'\mathrm{d}e'\mathrm{d}n''\mathrm{d}e''\\ &= \int_{\mathbb{R}^2}\left(\overline{\mathrm{sch}\left[\mathcal{V}'_{-n,-e}\right]}\mathrm{sch}\left[\mathcal{V}_{n,e}\right]\right)\mathrm{d}n'\mathrm{d}e'\\ &= \mathbf{Z}_{\text{c.c.}}.\end{aligned} \qquad (5.29)$$

The diagonal super partition function is also modular invariant. We remark that constructing an ordinary (non-super) partition function that is modular invariant requires the introduction of half-integer moded sectors [66] and the result is the invariant associated to the bosonic subtheory (orbifold) of $\mathsf{GL}(1|1)$.

Other super partition functions which are modular invariant can be found using extended algebras. Every atypical irreducible $\mathcal{A}_{n,e}$ is a simple current, by (5.18). One can therefore construct a large variety of extended algebras, among the most interesting being [66, 138] the tensor product of the $\beta\gamma$ ghost algebra and that of a pair of free fermions, the affine Kac-Moody superalgebra $\widehat{\mathfrak{sl}}(2|1)$ at levels $-\frac{1}{2}$ and 1, as well as an infinite series of (non-Lie) superalgebras containing, as subalgebras, the $N=2$ superconformal algebra, the Bershadsky-Polakov algebra $\mathcal{W}_3^{(2)}$, and its generalisations, the Feigin-Semikhatov algebras $\mathcal{W}_n^{(2)}$.

One very interesting observation is that the supercharacters of the atypical irreducibles of these extended algebras turn out to be *mock modular forms*. These are familiar, but mysterious, objects in number theory whose modular transformations may be expressed in terms of an integral, the Mordell integral [139]. One can evaluate the Verlinde formula directly by using this Mordell integral [140]. It would therefore be extremely interesting if one could rederive this using the method of atypical resolutions. We note that mock modular forms in general seem to be closely tied to atypical characters of affine Lie superalgebras, see [141–143].

5.6. **Correlation Functions.** The three-point functions of the $\mathsf{GL}(1|1)$ Wess-Zumino-Witten model were computed in [36]. The results bear a striking resemblance to those of symplectic fermions, an observation which was explained in [102]. We briefly summarise the computations, generalising the method used for both symplectic fermions and $\widehat{\mathfrak{sl}}(2)_{-1/2}$. First, consider a set of three free bosons $\varphi(z,\bar{z}), y(z,\bar{z})$ and $x(z,\bar{z})$ whose non-regular operator product expansions take the form

$$\varphi(z,\overline{z})\varphi(w,\overline{w}) = \log|z-w|^2 + \cdots, \qquad y(z,\overline{z})x(w,\overline{w}) = \log|z-w|^2 + \cdots. \tag{5.30}$$

As before, we denote the chiral part of fields by a subscript L. Define four holomorphic fields

$$J^E(z) = -\partial y(z), \quad J^N(z) = -\partial x(z), \quad J^-(z) = -:\mathrm{e}^{y_L(z)-\varphi_L(z)}:, \quad J^+(z) = -:\partial\varphi(z)\mathrm{e}^{-y_L(z)+\varphi_L(z)}:, \tag{5.31}$$

and their antiholomorphic analogues similarly. The non-regular operator product expansions of these fields are

$$J^E(z)J^N(w) \sim \frac{1}{(z-w)^2}, \quad J^N(z)J^\pm(w) \sim \frac{\pm J^\pm(w)}{z-w}, \quad J^+(z)J^-(w) \sim \frac{1}{(z-w)^2} + \frac{\pm J^E(w)}{z-w}, \tag{5.32}$$

so we have a free field realisation of the currents of $\widehat{\mathfrak{gl}}(1|1)$. As these fields again commute with the zero-mode of $e^{-\varphi_L(z)}$, we can use the same screening charge as before:

$$Q = \int_{\mathbb{C}} \mathrm{e}^{-\varphi(z,\bar{z})}\,\frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\pi}. \tag{5.33}$$

In this free field realization, one can identify interesting bulk fields, namely

$$\begin{aligned} V^{--}_{-e,-n+1/2} &= :\mathrm{e}^{e(\varphi+x)+ny}:, & V^{+-}_{-e,-n+1/2} &= :\mathrm{e}^{e(\varphi+x)+ny+\varphi_L-y_L}:, \\ V^{-+}_{-e,-n+1/2} &= :\mathrm{e}^{e(\varphi+x)+ny+\varphi_R-y_R}:, & V^{++}_{-e,-n+1/2} &= :\mathrm{e}^{e(\varphi+x)+ny+\varphi-y}:. \end{aligned} \tag{5.34}$$

We find the following non-regular operator product expansions with the currents:

$$\begin{aligned} J^E(z)V^{-\pm}_{-e,-n+1/2}(w,\overline{w}) &\sim \frac{-eV^{-\pm}_{-e,-n+1/2}(w,\overline{w})}{z-w}, & J^E(z)V^{+\pm}_{-e,-n+1/2}(w,\overline{w}) &\sim \frac{-eV^{+\pm}_{-e,-n+1/2}(w,\overline{w})}{z-w}, \\ J^N(z)V^{-\pm}_{-e,-n+1/2}(w,\overline{w}) &\sim \frac{-nV^{-\pm}_{-e,-n+1/2}(w,\overline{w})}{z-w}, & J^N(z)V^{+\pm}_{-e,-n+1/2}(w,\overline{w}) &\sim \frac{-(n-1)V^{+\pm}_{-e,-n+1/2}(w,\overline{w})}{z-w}, \\ J^+(z)V^{-\pm}_{-e,-n+1/2}(w,\overline{w}) &\sim \frac{eV^{+\pm}_{-e,-n+1/2}(w,\overline{w})}{z-w}, & J^-(z)V^{+\pm}_{-e,-n+1/2}(w,\overline{w}) &\sim \frac{V^{-\pm}_{-e,-n+1/2}(w,\overline{w})}{z-w}, \end{aligned} \tag{5.35}$$

which imply that these fields correspond to the primaries of $\mathcal{V}_{-n+1/2,-e}$.

Correlation functions are now defined in almost exactly the same manner as for symplectic fermions and $\widehat{\mathfrak{sl}}(2)_{-1/2}$. The only difference is that there are now two free bosons in addition to the screened boson $\varphi(z,\bar{z})$. The results for the three-point functions include, for example,

$$\begin{aligned} &\left\langle V^{--}_{-e_1,-n_1+1/2}(z,\overline{z})V^{++}_{-e_2,-n_2+1/2}(1,1)V^{++}_{-e_3,-n_3+1/2}(0,0)\right\rangle \\ &\qquad = -\frac{\Gamma(1-e_1)\Gamma(-e_2)\Gamma(-e_3)}{\Gamma(e_1)\Gamma(1+e_2)\Gamma(1+e_3)}\,\frac{\delta(n_1+n_2+n_3=2)\delta(e_1+e_2+e_3=0)}{|z-1|^{2(e_2(1-e_1-n_1)+e_1(1-n_2))}|z|^{2(e_3(1-e_1-n_1)+e_1(1-n_3))}}, \end{aligned} \tag{5.36}$$

from which we again observe singularities at the atypical points $e \in \mathbb{Z}$. As before, regularising leads to logarithms signifying the presence of the indecomposables $\mathcal{P}_{n,e}$ and the results are consistent with the fusion rules (5.18).

5.7. **Further Developments.** Conformal field theories associated to affine Kac-Moody superalgebras provide a rich source of interesting new logarithmic conformal field theories. Unfortunately, the only examples that are understood in great detail are those with $\widehat{\mathfrak{psl}}(1|1)$ and $\widehat{\mathfrak{gl}}(1|1)$ symmetries. However, this does not mean that no progress has been made on more sophisticated superalgebra models.

As with $\widehat{\mathfrak{gl}}(1|1)$, the spectrum of the the bulk theory may be conjectured using a combination of harmonic analysis and a first order formulation. By the latter, one means a perturbative description of the Wess-Zumino-Witten model in terms of its bosonic subtheory coupled to free *bc*-ghosts. One can then compute correlation functions and so on as above. This approach has been considered, with varying degrees of detail, for the bulk theories corresponding to the Wess-Zumino-Witten models on $\mathsf{SL}(2|1)$ [37], $\mathsf{PSL}(2|2)$ [144] and for general type I supergroups in [101]. A different approach determines, and then exploits, correspondences with super-Liouville theories [145, 146]. Both methods appear to generalise to the boundary conformal field theory, but there is an obstacle amounting to identifying the appropriate boundary screening charges. This has so far only been achieved for $\mathsf{GL}(1|1)$ [136] and $\mathsf{OSP}(1|2)$ [147]. In both cases, the boundary screening charge was found to be essentially given by the square root of the bulk screening charge. This behaviour is very similar to that observed for matrix factorization in Landau-Ginzburg theories [148] where the boundary and bulk screening charges seem to be similarly related, at least for $\mathsf{GL}(n|n)$ [149].

Some supergroup Wess-Zumino-Witten theories and their cosets have remarkable properties that lead to interesting applications in physics. The key point is the rather innocent-seeming observation that the Killing form, the supertrace in the adjoint representation, vanishes identically for the simple Lie superalgebras $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(2n+2|2n)$ and $\mathfrak{d}\big(2,1;\alpha\big)$. The corresponding Wess-Zumino-Witten models have been argued to possess exactly marginal perturbations [150–152] including, as a special case, the *principal chiral model*. In the case of $\mathsf{PSU}(1,1|2)$, this describes (the target space supersymmetric part of) superstring theory on $\mathsf{AdS}_3 \times S^3 \times X$, where X is some four-dimensional manifold [69]. One prediction of the celebrated *AdS/CFT correspondence* is that this string theory is dual to the two-dimensional conformal field theory associated with certain symmetric orbifolds of the four-manifold X. The superstring theory dual to four-dimensional conformal gauge theory is likewise described by the conformal field theory associated to the coset [153]

$$\frac{\mathsf{PSU}(2,2|4)}{\mathsf{SU}\big(2,2\big) \times \mathsf{SU}\big(4\big)}. \tag{5.37}$$

This coset differs from the gauged Wess-Zumino-Witten model, but it can still be argued to be conformally invariant due to the fact that the Killing form of the numerator vanishes [152, 154, 155].

Conformal field theories with superalgebra symmetries also appear in statistical physics. Supersymmetric disordered systems are described by perturbations of n pairs of free fermions and $\beta\gamma$ ghosts. The bilinears in these fields are well known to define the currents of $\widehat{\mathfrak{gl}}(n|n)$ at level $k = 1$. The associated disordered system is given by the corresponding current-current perturbation [68] (although in this case it seems that the perturbed theory is not quite conformal). One of the important open questions in this area is finding an effective field theory for the transition of plateaux of the integer quantum Hall effect. Such a theory might have $\mathsf{GL}(n|n)$ symmetry and sigma models related to $\mathsf{PSL}(2|2)$ have also been argued to appear in this context [67].

6. Staggered Modules

We have seen in the previous sections that logarithmic conformal field theories all have certain types of reducible, but indecomposable, modules in their spectra. The action of the Virasoro zero-mode L_0 is not diagonalisable on these modules, leading to logarithmic singularities in appropriate correlators. In this section, we shall discuss the mathematical structure of the simplest class of modules on which L_0 cannot be diagonalised, the *staggered modules*. These were so named by Rohsiepe in his study [70] of indecomposable Virasoro modules formed by glueing several highest weight modules together.[24] Here, we will discuss indecomposables formed by glueing modules from more general, but still structurally well-understood, classes. However, we will restrict to glueing

[24]The "staggering" presumably derives from a useful pictorial representation (see Figure **??** for example) in which the vertical positions of the highest weight states are ordered according as to their conformal dimensions. The result bears a passing similarity to the staggered starting positions customarily used for runners racing around a track.

only two of these well-understood modules together. These are the most common types of indecomposables encountered in logarithmic conformal field theory and the majority of the best understood examples of these theories feature only this type of reducible indecomposable (those studied in Sections 3, 4 and 5 for example). A further advantage is that we expect a classification of such staggered modules to be feasible, see [70, 71] for the Virasoro case.

6.1. **Staggered Modules.** In preparation for defining staggered modules in some generality, we first declare that for a given Lie algebra (or more generally, associative algebra[25]), we will choose a collection of *standard modules* whose structure is reasonably well-understood: For $\widehat{\mathfrak{gl}}(1|1)$, the standard modules are the Verma modules; for $\widehat{\mathfrak{sl}}(2)_{-1/2}$, standard means the spectral flows of the relaxed highest weight modules which we have denoted by $\mathcal{E}_\lambda^{(\ell)}$; for $\mathfrak{M}(1,2)$, standard means the Feigin-Fuchs modules $\mathcal{F}_\mu$. These are the modules whose characters have the most satisfactory modular transformation properties. We remark that these notions of being "standard" may be lifted to the respective simple current extensions by declaring that an extended module is standard whenever it is the orbit, under fusing with the simple current, of a standard module. In this sense, $\mathcal{W}_{-1/8}$ and $\mathcal{W}_{3/8}$ are standard $\mathfrak{W}(1,2)$-modules, while $\mathcal{W}_0$ and $\mathcal{W}_1$ are not.

Note that this notion is more general than logarithmic conformal field theory. For example, we may choose the standard modules of a (type I) Lie superalgebra like $\mathfrak{gl}(1|1)$ to be its Kac modules [98].[26] Similarly, a good choice for quantum groups like $\mathcal{U}_q\big(\mathfrak{sl}(2)\big)$ is that standard means highest weight. Even the diagram algebras, such as the Temperley-Lieb algebra, which crop up in statistical lattice models have standard modules, in this case the right choices are the cell modules of Graham and Lehrer [156] (also known as standard modules; we have borrowed the nomenclature from this example).

The feature that these collections have in common is that standard modules are always indecomposable and, moreover, are naturally parametrised so that they are generically irreducible. In this review, we have referred to irreducible standard modules as typical and reducible ones as atypical. Furthermore, and this is crucial for the next definition, the central elements of the associative algebra all act diagonalisably on standard modules. We now define a staggered module $\mathcal{S}$ to be one which is (isomorphic to) an extension of a standard module by another upon which there is a central element Q acting non-diagonalisably:

$$0 \longrightarrow \mathcal{H}^{\mathrm{L}} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\pi} \mathcal{H}^{\mathrm{R}} \longrightarrow 0. \tag{6.1}$$

In other words, $\mathcal{S}$ has a submodule isomorphic to a standard module $\mathcal{H}^{\mathrm{L}}$, which we shall refer to as the *left module*, and the quotient $\mathcal{S}/\iota\big(\mathcal{H}^{\mathrm{L}}\big)$ is isomorphic to another standard module $\mathcal{H}^{\mathrm{R}}$, which we shall refer to as the *right module*. For $\mathfrak{gl}(1|1)$ and $\mathcal{U}_q\big(\mathfrak{sl}(2)\big)$, we may take Q to be quadratic Casimir; for the Temperley-Lieb algebra, Q can be taken to be the "braid transfer matrix" [157]. For the algebras arising in logarithmic conformal field theory, it is L_0 which acts non-diagonalisably — to get a central element, one can act with $\mathrm{e}^{2\pi\mathrm{i}L_0}$ instead. It is not hard to see that L_0 and $\mathrm{e}^{2\pi\mathrm{i}L_0}$ may be interchanged with only minor modifications to the arguments that follow.[27]

The first result to note about staggered modules is that they only exist if Q acts on both the left and right modules as the same multiple λ of the identity (Appendix A.1). The second result to note is that the non-trivial Jordan blocks of Q all have rank 2. This follows directly from the exactness of (6.1): If $\big|v\big\rangle \in \mathcal{S}$ is an arbitrary element of a non-trivial Jordan block for Q of (generalised) eigenvalue λ, then $(Q-\lambda)\big|v\big\rangle$ need not be 0, but $\pi(Q-\lambda)\big|v\big\rangle = (Q-\lambda)\pi\big|v\big\rangle = 0$, since Q is diagonalisable on $\mathcal{H}^{\mathrm{R}}$. Exactness then gives $(Q-\lambda)\big|v\big\rangle = \iota\big|w\big\rangle$, for some $\big|w\big\rangle \in \mathcal{H}^{\mathrm{L}}$ of Q-eigenvalue λ, whence $(Q-\lambda)^2\big|v\big\rangle = (Q-\lambda)\iota\big|w\big\rangle = \iota(Q-\lambda)\big|w\big\rangle = 0$, because Q is also diagonalisable on $\mathcal{H}^{\mathrm{L}}$.

To further investigate these staggered modules, we introduce some more notation. Let $\big|\theta\big\rangle \in \mathcal{S}$ be a generalised eigenvector of Q of eigenvalue λ, so that $\big|\chi\big\rangle = (Q-\lambda)\big|\theta\big\rangle \neq 0$. Then, $\pi\big|\theta\big\rangle \in \mathcal{H}^{\mathrm{R}}$ is non-zero because $\pi\big|\theta\big\rangle = 0$ implies that $\big|\theta\big\rangle \in \iota\big(\mathcal{H}^{\mathrm{L}}\big)$, by exactness, hence that θ is a genuine eigenvector of Q. Now, suppose that there is an element U of our associative algebra such that $U\pi\big|\theta\big\rangle = 0$. Then, $\pi U\big|\theta\big\rangle = 0$, hence $U\big|\theta\big\rangle \in \iota\big(\mathcal{H}^{\mathrm{L}}\big)$. Because of

[25] It is more convenient to consider the universal enveloping algebra of the Lie algebra in what follows.

[26] Actually, we would have liked to refer to standard modules in general as "Kac modules". However, this term is already in use for logarithmic conformal field theories with Virasoro algebra symmetries [81]. It is not clear to us at present if these Kac modules for the Virasoro algebra are good candidates for standard modules in the sense we wish. Moreover, a theory of Virasoro staggered modules, with standard meaning highest weight, has already been developed [71].

[27] One could also imagine conformal field theories on which other zero-modes act non-diagonalisably, however very few examples appear to be known (see [158] for one).

this, the centrality of Q now gives

$$U\big|\chi\big\rangle = U(Q-\lambda)\big|\theta\big\rangle = (Q-\lambda)U\big|\theta\big\rangle = 0, \tag{6.2}$$

the last equality again following from the fact that elements of $\iota\big(\mathcal{H}^{\mathrm{L}}\big)$ are genuine eigenvectors of Q. In other words, any U annihilating $\pi\big|\theta\big\rangle \in \mathcal{H}^{\mathrm{R}}$ also annihilates $\big|\chi\big\rangle \in \iota\big(\mathcal{H}^{\mathrm{L}}\big)$.

This is a bit abstract, so let us consider an important specialisation that occurs when standard means highest weight and $\big|\theta\big\rangle$ is chosen to project onto the highest weight state of $\mathcal{H}^{\mathrm{R}}$. Then, $\big|\chi\big\rangle = (Q-\lambda)\big|\theta\big\rangle$ is necessarily non-zero because Q would be diagonalisable on all of $\mathcal{S}$ otherwise. But, $\big|\theta\big\rangle$ is annihilated by all positive modes U, hence we can conclude that $\big|\chi\big\rangle$ is too. It follows that in a staggered module over a Lie (super)algebra, $\big|\chi\big\rangle$ is a *non-zero singular vector* of $\mathcal{H}^{\mathrm{L}}$. Similarly, if $\pi\big|\theta\big\rangle$ is a relaxed highest weight state, then so is $\big|\chi\big\rangle$ (we might call it a relaxed singular vector).

Taking this a step further, we may suppose that $\mathcal{H}^{\mathrm{R}}$ is generated by a set $\big\{\big|\Theta^j\big\rangle\big\}_{j\in J}$ and then choose elements $\big|\theta^j\big\rangle \in \mathcal{S}$ so that $\pi\big|\theta^j\big\rangle = \big|\Theta^j\big\rangle$. For each $j \in J$, the elements of our associative algebra which annihilate $\big|\Theta^j\big\rangle$ form an ideal whose generators we denote by U_i^j, $i \in I$. We now define $\big|\omega_i^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$ by

$$\iota\big|\omega_i^j\big\rangle = U_i^j\big|\theta^j\big\rangle \tag{6.3}$$

(applying π shows that $U_i^j\big|\theta^j\big\rangle \in \iota\big(\mathcal{H}^{\mathrm{L}}\big)$). Because there is an ambiguity in our choice of each $\big|\theta^j\big\rangle$ up to adding arbitrary elements $\big|\eta^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$, there is a similar ambiguity in our definition of $\big|\omega_i^j\big\rangle$:

$$\big|\theta^j\big\rangle \longrightarrow \big|\theta^j\big\rangle + \iota\big|\eta^j\big\rangle \qquad \Rightarrow \qquad \big|\omega_i^j\big\rangle \longrightarrow \big|\omega_i^j\big\rangle + U_i^j\big|\eta^j\big\rangle. \tag{6.4}$$

With this setup, we can prove that the $\big|\omega_i^j\big\rangle$ determine the isomorphism class of a staggered module, generalising the Virasoro result given in [71, Prop. 3.6].

Theorem. *Let $\mathcal{S}$ and $\overline{\mathcal{S}}$ be staggered modules with the same left module $\mathcal{H}^{\mathrm{L}}$ and the same right module $\mathcal{H}^{\mathrm{R}}$. Then, there exists an isomorphism $\psi\colon \overline{\mathcal{S}} \to \mathcal{S}$ making the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^{\mathrm{L}} & \xrightarrow{\iota} & \mathcal{S} & \xrightarrow{\pi} & \mathcal{H}^{\mathrm{R}} & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & \mathcal{H}^{\mathrm{L}} & \xrightarrow{\overline{\iota}} & \overline{\mathcal{S}} & \xrightarrow{\overline{\pi}} & \mathcal{H}^{\mathrm{R}} & \longrightarrow & 0 \end{array} \tag{6.5}$$

commute if and only if there exist $\big|\eta^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$, for each $j \in J$, such that

$$\big|\overline{\omega}_i^j\big\rangle = \big|\omega_i^j\big\rangle + U_i^j\big|\eta^j\big\rangle, \tag{6.6}$$

for all $i \in I$.

Proof. If we have such an isomorphism ψ, then

$$\pi\big(\psi\big|\overline{\theta}^j\big\rangle - \big|\theta^j\big\rangle\big) = \overline{\pi}\big|\overline{\theta}^j\big\rangle - \pi\big|\theta^j\big\rangle = \big|\Theta^j\big\rangle - \big|\Theta^j\big\rangle = 0, \tag{6.7}$$

hence $\psi\big|\overline{\theta}^j\big\rangle - \big|\theta^j\big\rangle = \iota\big|\eta^j\big\rangle$ for some $\big|\eta^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$. Applying U_i^j now gives (6.6), as required.

Conversely, if (6.6) holds for some $\big|\eta^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$, then define ψ to be $\iota \circ \overline{\iota}^{-1}$ on $\overline{\iota}\big(\mathcal{H}^{\mathrm{L}}\big)$ and by

$$\psi\big|\overline{\theta}^j\big\rangle = \big|\theta^j\big\rangle + \iota\big|\eta^j\big\rangle, \qquad \psi U\big|\overline{\theta}^j\big\rangle = U\psi\big|\overline{\theta}^j\big\rangle \tag{6.8}$$

otherwise. Now, ψ is clearly a homomorphism and it is easy to invert. All we need to check is that it is well-defined because it may happen that $U\big|\overline{\theta}^j\big\rangle \in \overline{\iota}\big(\mathcal{H}^{\mathrm{L}}\big)$, in which case the two definitions for ψ must agree. But, $U\big|\overline{\theta}^j\big\rangle \in \overline{\iota}\big(\mathcal{H}^{\mathrm{L}}\big)$ leads to $U\big|\overline{\Theta}^j\big\rangle = 0$, hence U belongs to the (left) ideal that annihilates $\big|\overline{\Theta}^j\big\rangle$. Thus, we may write $U = \sum_i V_i^j U_i^j$. The first definition for ψ now gives

$$\psi U\big|\overline{\theta}^j\big\rangle = \iota\overline{\iota}^{-1}\sum_i V_i^j\overline{\iota}\big|\overline{\omega}_i^j\big\rangle = \iota\sum_i V_i^j\Big(\big|\omega_i^j\big\rangle + U_i^j\big|\eta^j\big\rangle\Big) = \iota\sum_i V_i^j\big|\omega_i^j\big\rangle + \iota U\big|\eta^j\big\rangle, \tag{6.9}$$

while the second yields

$$\psi U\big|\overline{\theta}^j\big\rangle = U\big(\big|\theta^j\big\rangle + \iota\big|\eta^j\big\rangle\big) = \sum_i V_i^j\iota\big|\omega_i^j\big\rangle + \iota U\big|\eta^j\big\rangle, \tag{6.10}$$

completing the proof. ∎

We illustrate this theorem with staggered Virasoro modules for which standard means Verma. Then, $\mathcal{H}^{\mathrm{R}}$ is generated by its highest weight state $\big|\Theta\big\rangle$ (so $|J| = 1$) and the annihilator of this state is generated by L_1, L_2 and

$L_0 - h$, where h is the conformal dimension of $\big|\Theta\big\rangle$. We therefore obtain three vectors

$$\big|\omega_0\big\rangle = (L_0 - h)\big|\theta\big\rangle, \qquad \big|\omega_1\big\rangle = L_1\big|\theta\big\rangle, \qquad \big|\omega_2\big\rangle = L_2\big|\theta\big\rangle, \tag{6.11}$$

where $\pi\big|\theta\big\rangle = \big|\Theta\big\rangle$. Up to the ambiguity in choosing $\big|\theta\big\rangle$, these three vectors completely specify the isomorphism class of a staggered module. In fact, because $\big|\omega_0\big\rangle$ is (a rescaling of) $\big|\chi\big\rangle$ and Virasoro singular vectors of a given conformal dimension are unique, this vector is already determined by $\mathcal{H}^{\mathrm{L}}$ and $\mathcal{H}^{\mathrm{R}}$, so the staggered module is characterised by $\big|\omega_1\big\rangle$ and $\big|\omega_2\big\rangle$.[28]

This theorem therefore allows us to reduce the problem of deciding if two staggered modules are isomorphic to computing a set of vectors $\big|\omega_i^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$ and seeing if there exist $\big|\eta^j\big\rangle \in \mathcal{H}^{\mathrm{L}}$ such that (6.6) holds. If one can determine which sets of $\big|\omega_i^j\big\rangle$ do actually correspond to a staggered module — this is the existence problem and it is decidedly difficult in general — then the question of counting the number of isomorphism classes of staggered modules with given left and right modules becomes an exercise in linear algebra. We remark that this is not quite the same as computing $\mathrm{Ext}^1\big(\mathcal{H}^{\mathrm{R}}, \mathcal{H}^{\mathrm{L}}\big)$ (see Appendix A.1) because we are restricting to extensions with a non-diagonalisable action of the centre.

6.2. **Logarithmic Couplings.** Because the counting of staggered module isomorphism classes may be reduced, modulo the existence problem, to a question of linear algebra, it seems plausible that the space of isomorphism classes will be a vector space (or affine space). It seems reasonable to ask if there is a natural means to parametrise this space. This is the idea behind logarithmic couplings: Instead of characterising a staggered module by a collection of vectors $\big|\omega_i^j\big\rangle$, subject to the ambiguities (6.6), we try to find a collection of numbers which likewise characterise the staggered module but which are *invariant* under (6.6).

This programme has only been studied in any detail for the Virasoro algebra (related computations for $\widehat{\mathfrak{sl}}(2)_{-1/2}$ were detailed in [62] and for $\widehat{\mathfrak{sl}}(2)_{-4/3}$ in [56, 63]). When standard means Verma, it turns out [71, Thm. 6.4 and Thm. 6.14] that the vector space of isomorphism classes of staggered modules has dimension 0 when $\big|\chi\big\rangle$ is the (generating) highest weight state of $\mathcal{H}^{\mathrm{L}}$ (staggered modules are unique) and has dimension 1 when $\big|\chi\big\rangle$ is a *principal* singular vector, meaning that it is descended from no other proper singular vector. These are the most useful cases, though it is also possible for the dimension to be 2, and we shall consider the second case exclusively in the remainder of this section.

We therefore need a single number to identify a staggered Virasoro module, up to isomorphism, when $\big|\chi\big\rangle$ is principal. A method to compute this number was originally proposed by Gaberdiel and Kausch [28] in the course of explicitly constructing certain $c = -2$ and $c = -7$ staggered modules using the Nahm-Gaberdiel-Kausch fusion algorithm. They chose $\big|\theta\big\rangle$ to satisfy $L_n\big|\theta\big\rangle = 0$, for all $n > 1$, and then defined $\beta \in \mathbb{C}$ by $L_1^\ell\big|\theta\big\rangle = \beta\big|\xi\big\rangle$, where $\big|\xi\big\rangle$ denotes the highest weight state of $\mathcal{H}^{\mathrm{L}}$ and ℓ is the difference in conformal dimensions of $\big|\theta\big\rangle$ and $\big|\xi\big\rangle$.[29] One can check that β does not depend upon the choice of $\big|\theta\big\rangle$, assuming that we only choose among the $\big|\theta\big\rangle$ which are annihilated by the L_n with $n > 1$. It follows that isomorphic staggered modules will have the same β.

More recently, these numbers β were generalised to other staggered modules, most notably in [80] where it was realised that Gurarie and Ludwig's anomaly numbers (see Equation (2.35)) for percolation and dilute polymers were just (differently normalised versions of) the β for the $c = 0$ staggered modules $\mathcal{S}_{1,5}$ and $\mathcal{S}_{3,1}$ (see Sections 2.3 and 2.5). Unfortunately, attention was not always paid to the crucial requirement that β not depend upon the choice of $\big|\theta\big\rangle$. In particular, when one considers staggered modules more general than those considered in [28], it is not usually possible to find any $\big|\theta\big\rangle$ satisfying $L_n\big|\theta\big\rangle = 0$ for all $n > 1$. It follows that the proposed recipe for computing β does not make sense for general staggered modules.

This was corrected in [26] where a definition of β was given for any staggered Virasoro module with $\big|\chi\big\rangle$ principal (see [71] for the general case). First, we normalise [159] the singular vector $\big|\chi\big\rangle$ so that

$$\big|\chi\big\rangle = U\big|\xi\big\rangle, \quad U = L_{-1}^\ell + \cdots, \tag{6.12}$$

where the omitted terms are Virasoro monomials involving the L_{-n} with $n \geqslant 1$ and at least one $n > 1$. We then define β by

$$U^\dagger\big|\theta\big\rangle = \beta\big|\xi\big\rangle, \tag{6.13}$$

[28] This remains true if standard means, instead, highest weight because the additional generators of the annihilating ideal lead to additional states $\big|\omega_i\big\rangle$ which can be computed given the $\big|\omega_1\big\rangle$ and $\big|\omega_2\big\rangle$ defined in the Verma case [71, Prop. 3.4].

[29] We remark that $\big|\theta\big\rangle$ is naturally restricted to being a generalised eigenvalue of L_0 in this development.

where $L_n^\dagger = L_{-n}$ is the usual adjoint (lifted to the universal enveloping algebra). If $|\xi\rangle$ is given norm 1, then applying $\langle\xi|$ to both sides of this definition leads to

$$\beta = \langle\xi|U^\dagger|\theta\rangle = \langle\chi|\theta\rangle, \tag{6.14}$$

which is obviously invariant under $|\theta\rangle \to |\theta\rangle + \iota|\eta\rangle$, $|\eta\rangle \in \mathcal{H}^{\mathrm{L}}$, because $|\chi\rangle$ is singular in $\iota(\mathcal{H}^{\mathrm{L}})$. The quantities β were christened *logarithmic couplings* in [26], though the terms *beta-invariants* [71] and *indecomposability parameters* [90] have also been used since.

The logarithmic coupling defined in (6.13) has the property that two staggered modules with the same left and right modules will be non-isomorphic if their couplings are different. The converse is also true [71, Thm. 6.15]: If the logarithmic couplings of such staggered modules coincide, then the modules are isomorphic. In other words, β is a complete invariant of the space of isomorphism classes of staggered Virasoro modules. The same is true for staggered Virasoro modules with standard meaning highest weight, though it is then no longer true that every β need correspond to a staggered module.

A downside to this theory is that β is not particularly easy to compute in general. One can explicitly construct the staggered module, for example using the Nahm-Gaberdiel-Kausch algorithm [26, 28, 80] (or as a space of local martingales for a Schramm-Loewner evolution process [160]). In this way, the logarithmic couplings $\beta_{r,s}$ of the $c = 0$ staggered modules $\mathcal{S}_{r,s}$ considered in Section 2 may be computed:[30]

$$\beta_{1,4} = -\tfrac{1}{2}, \quad \beta_{1,5} = -\tfrac{5}{18}, \quad \beta_{1,7} = -420, \quad \beta_{1,8} = -\tfrac{10780000}{243}, \quad \beta_{3,1} = \tfrac{10}{27}. \tag{6.15}$$

A slightly more efficient method [82] is to check if the existence of singular vectors in staggered modules fixes β. However, the most efficient known seems to be the proposal of [161] in which a limit formula is obtained for β as a byproduct of cancelling divergences in Virasoro primary operator product expansions as the conformal dimensions and central charges tend to their required values in a controlled manner. Surprisingly, the logarithmic couplings of certain classes of staggered modules with $h^{\mathrm{R}} - h^{\mathrm{L}}$ small can be computed as a function of the central charge [82, 160, 162, 163]. These functions are reasonably simple and conjectures for more general formulae have been made. However, there appears to have been no progress as yet on resolving these conjectures.

6.3. **More Logarithmic Correlation Functions.** We now briefly reconsider the (chiral) two-point function calculations of Sections 1.1 and 2.4.[31] Given the states $|\theta\rangle$ and $|\chi\rangle = (L_0 - h)|\theta\rangle$, the global invariance of the vacuum always leads to the following form for the two-point functions of the corresponding fields:

$$\langle\chi(z)\chi(w)\rangle = 0, \qquad \langle\chi(z)\theta(w)\rangle = \frac{B}{(z-w)^{2h}}, \qquad \langle\theta(z)\theta(w)\rangle = \frac{A - 2B\log(z-w)}{(z-w)^{2h}}, \tag{6.16}$$

where A and B are constants. If $\ell = 0$, so $|\theta\rangle$ has the same conformal dimension as $|\xi\rangle$, then $|\chi\rangle = |\xi\rangle$ and $B = \langle\xi|\theta\rangle$. In this case, $|\xi\rangle$ has norm zero, so we are free to normalise the hermitian form so that $B = 1$. Note that B does not depend upon the choice of $|\theta\rangle$, whereas A does. We remark that if the vacuum $|0\rangle$ has a logarithmic partner $|\Omega\rangle$, so $L_0|\Omega\rangle = |0\rangle$, then this analysis shows that the one-point function of the identity field vanishes whereas that of its partner will be constant.

When $\ell = 1$, $|\chi\rangle$ can only be $L_{-1}|\xi\rangle$ which is only singular when $|\xi\rangle$ has dimension 0 and $|\theta\rangle$ has dimension 1. It follows that the hermitian form may be normalised via $\langle\xi(z)\xi(w)\rangle = \langle\xi|\xi\rangle = 1$. Because $|\omega_1\rangle = L_1|\theta\rangle = \beta|\xi\rangle$, we can compute B using the partial differential equation derived from the L_1-invariance of the vacuum. This equation has inhomogeneous terms proportional to the logarithmic coupling β of the staggered module which determine the constant of integration in $\langle\xi(z)\theta(w)\rangle$:

$$\langle\xi(z)\theta(w)\rangle = \frac{\beta}{z-w} \qquad \Rightarrow \qquad \langle\chi(z)\theta(w)\rangle = \frac{-\beta}{(z-w)^2} \qquad \Rightarrow \qquad B = -\beta. \tag{6.17}$$

The constant A appearing in $\langle\theta(z)\theta(w)\rangle$ again varies with the choice of $|\theta\rangle$, unless $\beta = 0$.

For $\ell > 1$, computing the proportionality constant between B and β is a little more cumbersome. As above, one way is to determine $\langle\chi(z)\theta(w)\rangle$ from $\langle\xi(z)\theta(w)\rangle$. For this, we derive in the usual fashion, for $n \geqslant 2$ and an

[30] We remark that the discrepancy between the logarithmic couplings $\beta_{1,5} = -\frac{5}{18}$, $\beta_{3,1} = \frac{10}{27}$ and the anomaly numbers $b_{1,5} = -\frac{5}{8}$, $b_{3,1} = \frac{5}{6}$ of Gurarie and Ludwig (see Equation (2.35)) is just a matter of normalisation. We have chosen to (canonically) normalise $|\chi\rangle$ as $(L_{-1}^2 - \frac{2}{3}L_{-2})|\xi\rangle$. Identifying $|\xi\rangle$ with the vacuum $|0\rangle$, we find that $|T\rangle = -\frac{3}{2}|\chi\rangle$ and $|t\rangle = -\frac{3}{2}|\theta\rangle$, hence the anomaly numbers are obtained from the logarithmic couplings by multiplying by $(-\frac{3}{2})^2$.

[31] Three-point correlators may likewise be computed, assuming that one has already determined the three-point coupling constants between primary fields.

arbitrary field $\phi(z)$, the following relation:

$$\big\langle (L_{-n}\phi)(z)\theta(w)\big\rangle = \frac{(-1)^n}{(z-w)^{n-1}}\left[\partial_w + \frac{h(n-1)}{z-w}\right]\big\langle \phi(z)\theta(w)\big\rangle + (-1)^n \sum_{k=1}^{\ell} \binom{k+n-1}{n-2} \frac{\big\langle \phi(z)\omega_k(w)\big\rangle}{(z-w)^{n+k}}. \quad (6.18)$$

Here, we have defined $\big|\omega_k\big\rangle = L_k\big|\theta\big\rangle$. Moreover, we note that when $\ell > 1$, we may always choose $\big|\theta\big\rangle$ so that $\big|\omega_1\big\rangle = 0$.[32] For this choice of $\big|\theta\big\rangle$, we deduce (from the global conformal invariance of the vacuum) that $\big\langle \xi(z)\theta(w)\big\rangle = 0$. (This is certainly not true when $\ell = 1$ — see (6.17).)

To illustrate the method, consider the $c = 0$ staggered module $\mathcal{S}_{1,7}$ that was briefly considered in Section 2.3. Recalling (2.18) and (6.15), we note that $\big|\xi\big\rangle$ has dimension 2, $\big|\theta\big\rangle$ has dimension $h = 5$, $\beta = -420$, and $\big|\chi\big\rangle = \big(L_{-1}^3 - 6L_{-2}L_{-1} + 6L_{-3}\big)\big|\xi\big\rangle$. Setting $\big|\omega_1\big\rangle = 0$ forces $\big|\omega_2\big\rangle = -\frac{1}{48}\beta L_{-1}\big|\xi\big\rangle$ and $\big|\omega_3\big\rangle = \frac{1}{12}\beta\big|\xi\big\rangle$. From $\big\langle \xi(z)\theta(w)\big\rangle = 0$ and (6.18), we now obtain

$$\begin{aligned} \big\langle (L_{-3}\xi)(z)\theta(w)\big\rangle &= -4\frac{\big\langle \xi(z)\omega_2(w)\big\rangle}{(z-w)^5} - 5\frac{\big\langle \xi(z)\omega_3(w)\big\rangle}{(z-w)^6} = \frac{-\beta/12}{(z-w)^{10}}, \\ \big\langle (L_{-2}L_{-1}\xi)(z)\theta(w)\big\rangle &= \frac{\big\langle (L_{-1}\xi)(z)\omega_2(w)\big\rangle}{(z-w)^5} = \frac{5\beta/12}{(z-w)^{10}}, \end{aligned} \quad (6.19)$$

and $\big\langle \big(L_{-1}^3\xi\big)(z)\theta(w)\big\rangle = 0$. Therefore,

$$\big\langle \chi(z)\theta(w)\big\rangle = \frac{-3\beta}{(z-w)^{10}} \quad (6.20)$$

and $B = -3\beta = 1260$. Unfortunately, we are not aware of any general results concerning the ratio B/β. However, the limit formula for β given in [161] has a variant which gives B directly [164].

Whether one prefers β or B, it is clear that such a parameter is mathematically important and physically relevant. For example, the logarithmic couplings for the $c = 0$ staggered modules containing the vacuum distinguish the percolation and dilute polymers theories. In this case, the staggered modules have different right modules, but this might be too difficult to check explicitly in a more general physical situation. Another example occurs in the symplectic fermions theory in which one can identify $c = -2$ staggered Virasoro modules where $\big|\xi\big\rangle$ and $\big|\theta\big\rangle$ have dimensions 0 and 1, respectively. For each such module, one example being $\big|\theta\big\rangle = J_{-1}^+\big|\Omega\big\rangle$ and $\big|\xi\big\rangle = -J_0^+\big|\Omega\big\rangle$ (see Section 3.1 for notation), β is found to be -1. However, there is another $c = -2$ theory, the *abelian sandpile model* (see [165] for example), in which a staggered module with $\big|\xi\big\rangle$ dimension 0 and $\big|\theta\big\rangle$ dimension 1 is present. However, the logarithmic coupling here has been measured as $\beta = \frac{1}{2}$ [166], indicating that this theory is not equivalent to symplectic fermions.

6.4. **Further Developments.** As noted at the beginning, the staggered modules which have occupied us throughout this section are among the simplest, structurally, which give rise to logarithmic singularities in two-point functions. However, more complicated structures may arise: Examples of indecomposable Virasoro modules on which L_0 acts with Jordan blocks of rank 3 were first discovered in [80] through fusion. These appear structurally as a glueing of four highest weight modules and their existence was posited more generally in [81, 167] using a (conjectured) means of analysing Virasoro representations using lattice techniques. Moreover, indecomposables formed by glueing three highest weight modules, but with only rank 2 Jordan blocks, have been shown to arise in the percolation conformal field theory [85]. More recent lattice computations [163] suggest that indecomposables with Jordan blocks of arbitrary rank are physically relevant. Unfortunately, there is almost nothing known about the finer mathematical structure of these more complicated indecomposables.

We conclude with an example illustrating a Virasoro indecomposable with a rank 3 Jordan block for L_0. It may be realised as the fusion product of the $c = 0$ irreducibles $\mathcal{L}_{1/8}$ and $\mathcal{L}_{-1/24}$, but has only been explicitly constructed to grade 6 [80], so its deeper structure remains unknown (see [81] for a conjectured character). There are two ground states of dimension 0 and they form a Jordan block for L_0. We denote the eigenvector by $\big|\xi\big\rangle$ and its partner by $\big|\theta\big\rangle$, normalised so that $L_0\big|\theta\big\rangle = \big|\xi\big\rangle$. At dimension 1, one finds four states with all but one forming a rank 3 Jordan block. $L_{-1}\big|\xi\big\rangle$ is found to be non-zero, hence is the L_0-eigenstate belonging to the Jordan block; the other eigenstate will be denoted by $\big|\xi'\big\rangle$. The generator of the Jordan block will be denoted by $\big|\zeta\big\rangle$ and, along with $L_{-1}\big|\theta\big\rangle$, this completes a basis for the dimension 1 subspace.

[32] This follows from counting arguments. If $p(\ell)$ denotes the number of partitions of ℓ, then $\big|\omega_1\big\rangle$ belongs to a space of dimension $p\big(h^{\mathrm{R}} - h^{\mathrm{L}} - 1\big)$, whereas one has $p\big(h^{\mathrm{R}} - h^{\mathrm{L}}\big) - 1$ effective independent shifts $\big|\theta\big\rangle \to \big|\theta\big\rangle + \big|\eta\big\rangle$. Because $p(\ell) - p(\ell-1) \geqslant 1$ whenever $\ell > 2$, we conclude that $\big|\theta\big\rangle$ may always be shifted so that $\big|\omega_1\big\rangle = 0$. Strictly speaking, we should also allow for the possibility that $\mathcal{H}^{\mathrm{L}}$ has a vanishing singular vector of dimension less than that of $\big|\omega_0\big\rangle$, but this turns out not to change the result.

The states $|\zeta\rangle$ and $|\xi'\rangle$ must satisfy the following relations:

$$(L_0-1)|\zeta\rangle = a_1 L_{-1}|\theta\rangle + a_2|\xi'\rangle + a_3 L_{-1}|\xi\rangle, \quad L_1|\zeta\rangle = b_1|\xi\rangle + b_2|\theta\rangle, \quad L_1|\xi'\rangle = c_1|\xi\rangle + c_2|\theta\rangle. \quad (6.21)$$

However, $a_1 \neq 0$ as $|\zeta\rangle$ generates a rank 3 block, hence we may scale $|\zeta\rangle$ so that $a_1 = 1$. Moreover, $|\xi'\rangle$ is an L_0-eigenvector, hence $0 = L_1(L_0-1)|\xi'\rangle = L_0 L_1|\xi'\rangle = c_1 L_0|\xi\rangle + c_2 L_0|\theta\rangle = c_2|\xi\rangle$, hence $c_2 = 0$. One can check explicitly that c_1 is non-zero, hence we may normalise $|\xi'\rangle$ so that $c_1 = 1$. We now make use of the freedom we have in defining $|\zeta\rangle$: Shifting by multiples of $L_{-1}|\theta\rangle$ and $|\xi'\rangle$ allows us to tune a_3 and b_1 to 0. Finally, $b_2|\xi\rangle = L_0 L_1|\zeta\rangle = L_1(L_0-1)|\zeta\rangle = (a_2+2)|\xi\rangle$ reduces us to a single unknown:

$$(L_0-1)|\zeta\rangle = L_{-1}|\theta\rangle + (b_2-2)|\xi'\rangle, \qquad L_1|\zeta\rangle = b_2|\theta\rangle, \qquad L_1|\xi'\rangle = |\xi\rangle. \quad (6.22)$$

This result may be checked explicitly with the result that the fusion product has $b_2 = -\frac{1}{12}$. It is now straightforward to calculate the corresponding two-point correlators:

$$\begin{aligned}
\langle \xi(z)\xi(w)\rangle = \langle \xi(z)\xi'(w)\rangle = 0, \quad \langle \xi(z)\theta(w)\rangle = 1, \quad \langle \theta(z)\theta(w)\rangle = A - 2\log(z-w),\\
\langle \theta(z)\xi'(w)\rangle = \frac{1}{z-w}, \quad \langle \xi'(z)\xi'(w)\rangle = \frac{1}{(z-w)^2}, \quad \langle \xi(z)\zeta(w)\rangle = \frac{b_2}{z-w},\\
\langle \theta(z)\zeta(w)\rangle = \frac{B-2b_2\log(z-w)}{z-w}, \quad \langle \xi'(z)\zeta(w)\rangle = \frac{C-(b_2-3)\log(z-w)}{(z-w)^2},\\
\langle \zeta(z)\zeta(w)\rangle = \frac{D+E\log(z-w)-(b_2-1)(b_2-6)\log^2(z-w)}{(z-w)^2}.
\end{aligned} \quad (6.23)$$

The constants A, B, C, D and E are all dependent upon the precise choices that we have made for the fields. We remark that a $\log^2$ term in correlators is indicative of a rank 3 Jordan block for L_0.

7. Discussion and Conclusions

In this review, we covered various aspects of logarithmic conformal field theory. We have started with a discussion of a logarithmic theory describing the crossing probability for percolation at the critical point. We then have discussed three well-understood examples of logarithmic conformal field theory. Our approach of study consists of two essential steps.

First, one needs to thoroughly understand the representation theory of the chiral algebra. The pattern is that there is a large set of typical (standard) modules with few non-generic (atypical) points at which modules cease to be irreducible. It is then crucial to understand the structure of projective modules that can be constructed from the atypical irreducibles.

The second step is the modular data. Here, we have a procedure to compute the Verlinde formula and bulk modular invariants which then can be combined to identify extended algebras. This procedure has been outlined already in the introduction in the example of the free boson. As we are convinced of its future usefullness, we summarize it now for logarithmic theories.

(1) Start with a logarithmic conformal field theory with continous spectrum
(2) Compute the modular S-transformation for characters of typical representations.
(3) Splice short exact sequences to obtain a long exact sequence that resolves atypical irreducible modules in terms of typicals. This allows to express atypical characters as infinite sums of typicals.
(4) Use these sums to deduce the S-matrix of characters of atypical irreducible modules.
(5) Apply the natural continuum version of the Verlinde formula.
(6) Use the modular properties to compute bulk modular invariants corresponding to extended algebras.
(7) Use fusion of the parent theory to deduce the fusion rules of the extended theory.

We stress that our method of applying the Verlinde formula requires to have non-periodic resolutions of atypical modules in terms of typical representations. The extended theories usually only have discrete or even finite spectrum and resolutions are usually periodic. In such examples, modular transformations depend on the modular parameter τ, especially one obtains a representation of the modular group with non-diagonalizable T-transformation. In such a case the application of the Verlinde formula is not clear, see however [104] for a proposed generalized Verlinde formula. As we can immediately deduce fusion of the extended theory by its continuous parent, we have a straight forward method that circumvents a generalized Verlinde formula. Nonetheless it is an apparent question whether one can derive the proposal of [104] using our methods.

The three examples of this review are closely related, namely a $\widehat{\mathfrak{u}}(1)$-coset algebra of an extension of $\widehat{\mathfrak{gl}}(1|1)$ is $\widehat{\mathfrak{sl}}(2)$ at level $-1/2$ [102], while the singlet $\mathfrak{M}(1,2)$-algebra and triplet $\mathfrak{W}(1,2)$-algebra are $\widehat{\mathfrak{u}}(1)$-coset algebras of $\widehat{\mathfrak{sl}}(2)$ at level $-1/2$ and its maximal extended algebra [61]. The generalization is that the singlet $\mathfrak{M}(1,p)$-algebra and triplet $\mathfrak{W}(1,p)$-algebra are $\widehat{\mathfrak{u}}(1)$-coset algebras of certain fractional level W-algebras [168].

In this article we only partialy emphasized the power of free field realizations. All three examples allowed for a free field realization inside a certain lattice theory, where the chira; algebra was the kernel of a screening charge. Such a description is a powerful tool in understanding the representation theory and also as explained in the present article it provides an explicit integral for correlation functions.

Presently, the simplest class of logarithmic conformal field theory is finally under control. Hopefully the near future will bring insights into more sophisticated theories. Having applications in both string theory and statistical physics in mind, it seems worth to turn to the theories associated to $\widehat{\mathfrak{gl}}(2|2)$ and $\widehat{\mathfrak{psl}}(2|2)$.

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APPENDIX A. HOMOLOGICAL ALGEBRA: A (VERY BASIC) PRIMER

The quantum state space of a (bulk) conformal field theory is a module (representation space) over a symmetry algebra which must contain two commuting copies of the Virasoro algebra (one copy in the boundary case). The *raison d'être* of logarithmic conformal field theory is that there are occasions for which this module cannot be written as a direct sum of irreducible submodules over the symmetry algebra, but rather that one must include, in the direct sum, some submodules which are reducible but indecomposable. When discussing reducible but indecomposable modules, the language of homological algebra and category theory becomes very convenient, indeed almost unavoidable. In this appendix, we will introduce some of the basic concepts and terminology that are used freely throughout the text.

A.1. **Exact Sequences and Extensions.** First, recall that a module is said to be *reducible* if it contains a non-trivial proper submodule and *decomposable* if it may be written as the direct sum of two non-trivial submodules: $M = M_1 \oplus M_2$ with $M_1, M_2 \neq \{0\}$. We remark that in the latter case, M_1 and M_2 are called *direct summands* of the decomposable module M. An *irreducible* (or *simple*) module is then one which has only the trivial module and itself as submodules. A direct sum of irreducible modules is said to be *completely reducible* (or *semisimple*). Similarly, an *indecomposable* module is one for which $M = M_1 \oplus M_2$ implies that either M_1 or M_2 is trivial. Finally, we define a *proper* submodule of M to be one which is not all of M. A *maximal* proper submodule of M is then a proper submodule N for which $N \subset M' \subseteq M$ (with M' a submodule) implies that $M' = M$.

The structure-preserving maps between modules M and N, the *module homomorphisms* (or intertwiners), are defined to be those linear maps $f\colon M \to N$ which commute with the action of the symmetry algebra A:

$$a \cdot f(m) = f(a \cdot m) \qquad \text{for all } a \in A \text{ and } m \in M. \tag{A.1}$$

Canonical examples include the identity map $\mathrm{id}\colon M \to M$, the inclusion map $\iota\colon M \to N$ of a submodule M of N, and the canonical projection $\pi\colon N \to N/M$ onto the quotient by a submodule. We note that both the kernel and image of a module homomorphism $f\colon M \to N$ are submodules (of M and N, respectively).

One of the central notions of homological algebra is the *exact sequence*. This is a chain of modules connected by module homomorphisms,

$$\cdots \longrightarrow M_{-2} \xrightarrow{f_{-2}} M_{-1} \xrightarrow{f_{-1}} M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots, \tag{A.2}$$

which is *exact*: At each position n of the chain, the kernel of the outgoing homomorphism f_n coincides with the image of the incoming one f_{n-1}, that is, $\ker f_n = \operatorname{im} f_{n-1}$. One may also consider chains which terminate at either end, in which case one does not require exactness at the endpoints. As examples, note that the identity, inclusion and quotient maps give rise to the following exact sequences:

$$0 \longrightarrow M \xrightarrow{\mathrm{id}} M \longrightarrow 0, \qquad 0 \longrightarrow M \xrightarrow{\iota} N, \qquad N \xrightarrow{\pi} N/M \longrightarrow 0. \tag{A.3}$$

Conversely, the first sequence says that id is bijective, the second that ι is injective and the third that π is surjective. We remark that it is standard to abbreviate the trivial module $\{0\}$ by just 0.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\pi} Q \longrightarrow 0 \tag{A.4}$$

(note that ι is automatically an injection and π is automatically a surjection). This concisely summarises the common situtation in which M is a submodule N and the quotient N/M is isomorphic to Q. An obvious question that arises now is whether there is a submodule of N isomorphic to Q such that $N \cong M \oplus Q$ — if so, then we say that the short exact sequence *splits*. Certainly, if N is such a direct sum, then the sequence (A.4) is exact. We are therefore asking if a direct sum is the only possibility. In the very important case in which both M and Q are indecomposable, this amounts to asking if M and Q can be "glued together" to form a new indecomposable N, or whether they may only be combined as $M \oplus Q$.

Answering this question is a very subtle and difficult business in general. The general machinery of homological algebra reduces this to the computation of the first *extension group* $\mathrm{Ext}^1(Q,M)$ (when (A.4) is exact, N is said to be an *extension* of Q by M). In particular, $\mathrm{Ext}^1(Q,M) = 0$ guarantees that (A.4) always splits, hence that $N \cong M \oplus Q$ is the only possibility. We will not need to concern ourselves with the formal machinery required to compute extension groups in general. Instead, we only remark that there is one easy test for deciding when a short exact sequence must split: If some *central* element of the symmetry algebra A acts as a multiple of the identity on M and as a *different* multiple of the identity on Q, then it acts on N as a linear transformation with two distinct eigenspaces which may be easily checked to be submodules isomorphic to M and Q. In other words, it follows that N splits as $M \oplus Q$.

We will typically apply this test on modules that are graded by the (generalised) eigenvalues of a zero-mode such as that of the Virasoro algebra L_0. In this case, the central element should be identified with $\mathrm{e}^{2\pi\mathrm{i}L_0}$ and the test reduces to the remark that if the states of the indecomposable modules M_i, $i = 1,2$, have conformal dimensions (L_0-eigenvalues) in $\mathbb{Z} + h_i$, then $\mathrm{Ext}^1(M_1,M_2) = 0$ will follow if $h_1 \neq h_2 \bmod \mathbb{Z}$.

We also note that there may exist modules Q for which $\mathrm{Ext}^1(Q,M) = 0$ for all modules M (in the category under consideration). Such Q are said to be *projective*. These modules give a rough measure of maximal complexity among indecomposable modules. More precisely, they have the property that if they appear as a quotient of any module, then that module decomposes as the direct sum of Q and something else. We mention that if an irreducible module Q is a quotient of an indecomposable projective module P, then we say that P is the *projective cover* of Q (projectivity guarantees that these covers are unique when they exist). This is a remarkably useful concept. Unfortunately, progress in rigorously identifying projective covers for the module categories of interest in logarithmic conformal field theory is lamentably slow.

A.2. **Splicing Exact Sequences.** One construction that we will take advantage of is that of *splicing* two short exact sequences together. This procedure starts from two short exact sequences of the forms

$$0 \longrightarrow M_1 \xrightarrow{\iota_1} N_1 \xrightarrow{\pi_1} M_0 \longrightarrow 0, \qquad 0 \longrightarrow M_2 \xrightarrow{\iota_2} N_2 \xrightarrow{\pi_2} M_1 \longrightarrow 0 \tag{A.5}$$

and produces the sequence

$$0 \longrightarrow M_2 \xrightarrow{\iota_2} N_2 \xrightarrow{\iota_1\circ\pi_2} N_1 \xrightarrow{\pi_1} M_0 \longrightarrow 0. \tag{A.6}$$

It is a simple exercise to check that the resulting sequence is also exact. Moreover, if one has a short exact sequence with M_2 in the third position, $0 \to M_3 \to N_3 \to M_2 \to 0$ say, then the splicing process may be repeated to obtain a new, longer exact sequence. The most interesting case occurs when one can splice infinitely many times, thereby obtaining a long exact sequence called a *resolution* of M_0 in terms of the N_i:

$$\cdots \longrightarrow N_5 \xrightarrow{\iota_4\circ\pi_5} N_4 \xrightarrow{\iota_3\circ\pi_4} N_3 \xrightarrow{\iota_2\circ\pi_3} N_2 \xrightarrow{\iota_1\circ\pi_2} N_1 \xrightarrow{\pi_1} M_0 \longrightarrow 0. \tag{A.7}$$

This is useful when the N_i belong to a class of well-behaved modules and one wishes to understand the behaviour of M_0.

A.3. **Grothendieck Groups and Rings.** One point that deserves emphasising is that the indecomposable structures that modules may exhibit become irrelevant when modules are replaced by their underlying vector spaces. This is because exact sequences of vector spaces always split. It follows that quantities attached to modules which only depend upon their vector space structure, characters being prime examples, will be blind to any indecomposable structure. One is therefore motivated to consider the effect of forgetting the indecomposable structure of modules such as the quantum state space. Certainly, it will usually be easier to ignore questions of whether a

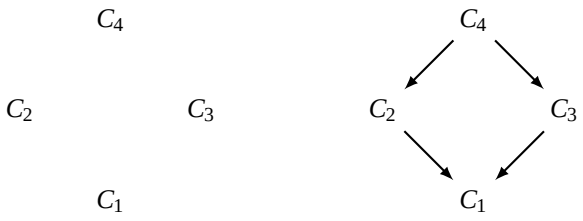

FIGURE 14. An example of a Loewy diagram (left) and its annotated version (right).

module is completely reducible or not and for some applications, computing modular invariant partition functions for example, maintaining such ignorance is perfectly justified.

This "forgetting" may be formalised through the notion of a *Grothendieck group*. Here, one starts with a collection (category) of modules, preferably finitely-generated, and forms an abelian group Gr whose generators are formal elements $\big[M\big]$, where M is a module from the collection, and whose relations are

$$\big[M\big] - \big[N\big] + \big[Q\big] = 0, \qquad \text{whenever} \quad 0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0 \quad \text{is exact.} \tag{A.8}$$

The point of these relations is to ensure that all extensions of Q by M are identified with $M \oplus Q$ in Gr. In favourable circumstances, such as when the characters of the irreducible modules are linearly independent, then the abelian group generated by the characters will be isomorphic to the Grothendieck group. Note that the collections of modules that we will be considering will always carry a tensor product structure (fusion) and that this structure will always induce a well-defined product on Gr. We will therefore usually refer to Gr as a *Grothendieck ring* or Grothendieck fusion ring.

A.4. **Socle Series and Loewy Diagrams.** Finally, it is often convenient to go beyond the formalism of exact sequences in order to visualise the structure of an indecomposable module. One way to do this is through *composition series*. This is a filtration of a module M by submodules,

$$0 = M_0 \subset M_1 \subset \cdots M_{\ell-1} \subset M_\ell = M, \tag{A.9}$$

such that the *subquotients* $C_i = M_i/M_{i-1}$ of M are all irreducible. Composition series are not unique, but if M does possess a composition series, then the *length* ℓ and the *composition factors* C_i are common to every composition series of M (only the ordering of the C_i may change).

A common variation on this theme is the *socle series*. Here, one again has a filtration (A.9) by modules, but the condition that M_i/M_{i-1} be irreducible is replaced by the requirement that M_i/M_{i-1} be the *socle* of M/M_{i-1}. The socle of a module M is defined to be its maximal completely reducible submodule. Equivalently, it is the (direct) sum of the irreducible submodules of M. As the socle is unique (if it exists), the same is true of socle series. In essence, composition series describe how irreducible modules are glued together to form M, whereas socle series describe how completely reducible modules are glued and so are usually more efficient. The dual notion to the socle is the maximal semisimple quotient, sometimes called the *head*.

We conclude with an abstract example illustrating an indecomposable structure which commonly arises in logarithmic conformal field theory (and elsewhere). The socle series is

$$0 = M_0 \subset M_1 \subset M_2 \subset M_3 = M, \qquad \text{with } M_1 \cong C_1,\ M_2/M_1 \cong C_2 \oplus C_3 \text{ and } M_3/M_2 \cong C_4, \tag{A.10}$$

where the C_i are irreducible subquotients (composition factors) of M. A common way of visualising this information is through the associated *Loewy diagram*. This is constructed by "layering" with the i-th layer consisting of the direct summands comprising the i-th filtration quotient M_i/M_{i-1}. In the example at hand, the bottom layer is $\operatorname{soc} M = M_1 \cong C_1$, the middle layer is $\operatorname{soc}(M/M_1) = M_2/M_1 = C_2 \oplus C_3$, and the top layer is $\operatorname{soc}(M/M_2) = M/M_2 = C_4$. This is illustrated in Figure 14 (left).

It is often convenient to annotate Loewy diagrams with arrows detailing the finer structure of the indecomposable module as in Figure 14 (right). Such an arrow will always point down from a composition factor C_j at layer i to another C_k at layer $i-1$. Roughly speaking, both C_j and C_k may be associated to certain states of M and the arrow indicates that the action of the algebra can take a state associated with C_j to a state associated with C_k, but not vice-versa. More precisely, it is possible to isolate C_j and C_k by canonically constructing a length 2 subquotient of M whose composition factors are precisely C_j and C_k. If this subquotient is indecomposable, then we draw an

arrow from C_j to C_k (if $\mathrm{Ext}^1(C_j,C_k)$ has dimension greater than 1, we should also affix a label to the arrow to precisely identify the subquotient). We will not try to demonstrate or verify this precise criterion for drawing arrows on Loewy diagrams here. Suffice to say that it will be clear in the examples considered how composition factors are associated to states and arrows will be drawn on this rough basis.

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