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Logarithmic corrections to $\mathcal{N} = 4$ and $\mathcal{N} = 8$ black hole entropy: a one loop test of quantum gravity

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Abstract: We compute logarithmic corrections to the entropy of supersymmetric extremal black holes in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric string theories and find results in perfect agreement with the microscopic results. In particular these logarithmic corrections vanish for quarter BPS black holes in $\mathcal{N} = 4$ supersymmetric theories, but has a finite coefficient for 1/8 BPS black holes in the $\mathcal{N} = 8$ supersymmetric theory. On the macroscopic side these computations require evaluating the one loop determinant of massless fields around the near horizon geometry, and include, in particular, contributions from dynamical four dimensional gravitons propagating in the loop. Thus our analysis provides a test of one loop quantum gravity corrections to the black hole entropy, or equivalently of the AdS_2/CFT_1 correspondence. We also extend our analysis to $\mathcal{N} = 2$ supersymmetric STU model and make a prediction for the logarithmic correction to the black hole entropy in that theory.

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Contents

1 Introduction

Wald's formula gives a generalization of the Bekenstein-Hawking entropy in a classical theory of gravity with higher derivative terms, possibly coupled to other matter fields [1–4]. In the extremal limit the near horizon geometry contains an AdS_2 factor, and Wald's formula leads to a simple algebraic procedure for determining the near horizon field configurations and the entropy [5, 6]. A proposal for computing quantum corrections to this formula was suggested in [7]. In this formulation, called the quantum entropy function formalism, the degeneracy associated with the black hole horizon is given by the string theory partition function Z_{AdS_2} in the near horizon geometry of the black hole. Such a partition function is divergent due to the infinite volume of AdS_2, but the rules of AdS_2/CFT_1 correspondence gives a precise procedure for removing this divergence. While in the classical limit this prescription gives us back the exponential of the Wald entropy, it can in principle be used to systematically calculate the quantum corrections to the entropy of an extremal black hole.

Given this prescription one would like to test this by comparing with some microscopic results. For $\mathcal{N} = 8$ supersymmetric string theories obtained by compactifying type II string theory on T^6 and a class of $\mathcal{N} = 4$ supersymmetric string theories obtained by compactifying type II string theory on $K3 \times T^2$ and its various orbifolds, the exact formula for the microscopic index is known [8–23]. Furthermore it has been argued in [24, 25] that for extremal supersymmetric black holes preserving four supercharges the black hole entropy also gives the index, and hence can be directly compared with the microscopic index. Thus these theories provide us with an ideal ground for testing the macroscopic formula for the black hole entropy.

The microscopic formula for the index in these theories shows that in the limit in which all the components of the charge are large, the logarithm of the index is given by [9, 13, 18]

$$S_{micro} = \pi\sqrt{\Delta} + \mathcal{O}(1) \quad \text{for} \quad \mathcal{N} = 4, \tag{1.1}$$

and [26]

$$S_{micro} = \pi\sqrt{\Delta} - 2\ln\Delta + \mathcal{O}(1) \quad \text{for} \quad \mathcal{N} = 8\,, \tag{1.2}$$

where in both theories Δ is the unique quartic combination of the charges which is invariant under all the continuous duality transformations. Using the equality between index and degeneracy for a black hole, eqs. (1.1), (1.2) should be equal to the entropies of the corresponding black holes. Now for a classical black hole solution carrying these charges, the radius of curvature a of the near horizon $AdS_2 \times S^2$ geometry is related to Δ via

$$\sqrt{\Delta} = a^2/G_N\,, \tag{1.3}$$

where G_N is the four dimensional Newton's constant. Thus the leading contribution $\pi\sqrt{\Delta} = 4\pi a^2/4G_N$ is the classical Bekenstein-Hawking entropy [27, 28]. This leads to a natural question: can we reproduce the logarithmic corrections from the macroscopic side?[1]

This question was partially analyzed in [46] where it was argued that such corrections, if present, must arise from one loop quantum correction to Z_{AdS_2} due to massless fields of the supergravity theory. The stringy modes, massive Kaluza-Klein excitations along compact directions and/or higher derivative corrections to the effective action play no role in this analysis and can be safely ignored. Furthermore [46] computed the contribution to Z_{AdS_2} due to the massless matter multiplets in $\mathcal{N} = 4$ supersymmetric string theories, and found that the net contribution vanishes, in agreement with the fact that the result (1.1) is independent of the number of matter multiplets in the theory. In this paper we complete the computation by including the contribution from the gravity multiplet of $\mathcal{N} = 4$ supersymmetric theories and also extend the analysis to $\mathcal{N} = 8$ supersymmetric string theories. In both cases our results are in perfect agreement with the microscopic results (1.1) and (1.2). Since the computation on the macroscopic side involves one loop determinant of dynamical gravitons propagating on $AdS_2 \times S^2$, our results can be taken as a non-trivial confirmation of quantum gravity contribution to the black hole entropy, or equivalently of AdS_2/CFT_1 correspondence [7] on which the prescription for computing quantum corrections to the entropy is based.[2]

We would like to emphasize that the limit of charges we consider is different from the Cardy limit which, in the present context, would amount to taking one of the charges representing momentum along a compact direction to infinity keeping the other charges

[1] Earlier approaches to computing logarithmic corrections to black hole entropy can be found in [29–45].

[2] The analysis of [47–49] for partition function of (super-) gravity and higher spin theory in AdS_3 also includes the effect of graviton loops. However in 3 dimensions there are no dynamical degrees of freedom in the graviton and hence only the boundary modes associated with asymptotic symmetries contribute to the partition function.

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fixed. This was studied in detail in [25]. The coefficient of the $\log\Delta$ term in this limit can also be read out from the general expression for the microscopic entropy, and is given by -2 for $\mathcal{N}=8$ supersymmetric string theory, and $-(m+2)/4$ for $\mathcal{N}=4$ supersymmetric string theories, m being the total number of matter multiplets in the theory. Thus for type IIB string theory on $K3\times T^2$ the coefficient will be $-(22+2)/4=-6$.

Since our analysis will be somewhat technical we shall now give a brief description of our analysis and the results. The one loop contribution to Z_{AdS_2} arises from two sources. First, the integration over each eigenmode of the kinetic operator carrying non-zero eigenvalue gives a contribution to Z_{AdS_2} through the determinant of the kinetic operator. The logarithm of this determinant can be expressed as integral over the proper time parameter s, with the integrand given by the trace of the heat kernel [50, 51] after removing the contribution due to the zero modes. This typically will be proportional to the infinite volume of $AdS_2\times S^2$ and hence is apparently infrared divergent. But we use the trick of [7, 52] to express the AdS_2 volume as $c_1\beta+c_2$ where c_1 and c_2 are finite constants and β is the (divergent) length of the boundary. The coefficient of β can be absorbed into a redefinition of the ground state energy and the β independent term gives the correction to the black hole entropy S_{BH}. This leaves us with an infrared finite contribution to the entropy. There is also ultraviolet divergence which comes from the lower limit of integration of the parameter s. This is regulated by setting the lower limit of s integration to be the string scale l_s^2. The resulting integration over s goes as ds/s in the range $l_s^2\ll s\ll a^2$ where a is the radius of curvature of AdS_2 and S^2. This gives a term of order $\ln(a^2/l_s^2)$ which can be reinterpreted as $\ln\sqrt{\Delta}$ using (1.3).

The other source of logarithmic corrections is the integration over the zero modes. These zero modes represent eigenmodes of the kinetic operator with zero eigenvalues and arise due to the asymptotic symmetries of the euclidean near horizon geometry. To find the result of integration over these zero modes we first make a change of integration variable from the zero modes to parameters labelling the supergroup of asymptotic symmetries. The supergroup is parametrized in a way that its volume is manifestly independent of a, — the radius of curvature of AdS_2 and S^2. Thus the net a dependence of the zero mode integration arises from the Jacobian associated with the change of variables from the field modes to the supergroup parameters. The a dependence of the Jacobian can be calculated explicitly and leads to additional corrections to the entropy proportional to $\ln a$.[3]

For both $\mathcal{N}=4$ and $\mathcal{N}=8$ supergravity theories we find that the final results of the macroscopic analysis, after adding up the contribution from the zero mode and the non-zero mode integration, are in perfect agreement with the microscopic results (1.1) and (1.2). Even for the $\mathcal{N}=4$ supersymmetric theory where the net contribution vanishes, the individual contributions from the zero modes and the non-zero modes are non-trivial. This has been illustrated in table 1 where we have displayed separately the contributions from the zero modes and the non-zero modes.

[3]The left-over integration over the supergroup includes integration over both, fermionic and bosonic modes. With the help of localization [53–56] it can be shown that the infinite contribution from integration over the bosonic variables cancel the zero contribution due to integration over the fermionic modes, leaving behind a finite result [54].

The theory	non-zero mode contribution	zero mode contribution	total contribution
$\mathcal{N}=4$	$\frac{1}{4}(6+m)\ln\Delta$	$-\frac{1}{4}(6+m)\ln\Delta$	0
$\mathcal{N}=8$	$5\ln\Delta$	$-7\ln\Delta$	$-2\ln\Delta$
STU model	$2\ln\Delta$	$-\ln\Delta$	$\ln\Delta$

Table 1. The logarithmic correction to the black hole entropy in $\mathcal{N}=4$ and $\mathcal{N}=8$ supersymmetric string theories in four dimensions and the STU model. We have displayed separately the contributions from the non-zero modes and the zero modes. m denotes the number of matter multiplet fields in the $\mathcal{N}=4$ supergravity theory. The difference between the zero mode contributions in the different theories arise solely due to the different number of gauge fields they have, — the contributions from the graviton and gravitino zero modes cancel in all the theories.

Our analysis can also be extended to compute the logarithmic correction to the entropy of half BPS black holes in $\mathcal{N}=2$ supersymmetric STU model [57, 58]. The low energy effective action of this theory is a truncation of the $\mathcal{N}=4$ supergravity theory, and furthermore the black hole of the $\mathcal{N}=4$ supergravity theory for which we carry out the analysis can be embedded in this theory. Thus eigenmodes and the eigenvalues of the kinetic operator in the near horizon geometry of the black hole solution in the STU model are a subset of the corresponding eigenmodes and eigenvalues in the $\mathcal{N}=4$ supersymmetric string theory. As a result the coefficient of the $\ln\Delta$ term in the STU model can be found by examining the contribution to the logarithmic correction to $\mathcal{N}=4$ black hole entropy from this subset. From this we arrive at the following prediction for the asymptotic growth of black hole entropy in the STU model:

$$\pi\sqrt{\Delta}+\ln\Delta+\mathcal{O}(1)\,. \tag{1.4}$$

This logarithmic correction is in apparent violation of the proposal of [59] for the index of half BPS states in the STU model, and more generally, with the one loop correction to OSV integral [60] proposed in [61].[4] However (1.4) is consistent with the measure proposed in [62] if we assume that this formula is valid for weak topological string coupling. This perhaps indicates that although the result of [62] was derived in the limit of strong topological string couping, its range of validity, interpreted as the index of single centered black holes, may extend even to weak topological string coupling — the regime in which our analysis is valid since we scale all the charges uniformly. More discussion on logarithmic correction to $\mathcal{N}=2$ black hole entropy, as well as its relation to the results of [61] and [62], can be found in [63].

The rest of the paper is organized as follows. In section 2 we review some results on the eigenfunctions and eigenvalues of the Laplacian operator in $AdS_2\times S^2$ acting on fields carrying different spins. Section 3 we review the general procedure for computing the logarithmic correction to the black hole entropy. Section 4 we describe the action, to

[4]This is different from the 'non-holomorphic corrections' discussed e.g. in [61] since the latter involves logarithm of moduli fields and not of the size of the horizon. Furthermore the analysis of [61], based on the requirement of duality invariance, does not put any constraint on the $\ln\Delta$ terms we find since Δ is manifestly duality invariant.

quadratic order, of the fluctuations of the gravity multiplet fields of $\mathcal{N} = 4$ supergravity around the near horizon geometry of the black hole. Most of the results in these sections were already discussed in [46]. Section 5, section 6, section 7, section 8 and section 9 contain the new results. In section 5 we compute the contribution to the heat kernel (and hence to the logarithmic correction to the entropy) due to the bosonic non-zero modes of the gravity multiplet of the $\mathcal{N} = 4$ supergravity theory. Section 6 contains the contribution from the fermionic non-zero modes of the gravity multiplet of the $\mathcal{N} = 4$ supergravity theory. In section 7 we augment the results by computing the contribution due to the integration over the zero modes, and show that the net contribution to the coefficient of $\ln \Delta$ vanishes. In section 8 we include the contribution due to the extra fields which are present in the $\mathcal{N} = 8$ supergravity theory and show that they give the result $-2 \ln \Delta$. These results are in agreement with the microscopic results (1.1) and (1.2). In section 9 we compute logarithmic corrections to the black hole entropy in the STU model leading to (1.4).

We conclude this introduction by commenting on the method we use to compute the heat kernel, and an alternative. We compute the heat kernel by explicitly constructing the eigenstates and eigenvalues of the kinetic operator in the near horizon geometry, but we could also compute the relevant terms by simply computing the one loop contribution to the trace anomaly in the near horizon geometry [50, 51]. Indeed in the presence of background gravitational field the contribution to the heat kernel in supersymmetric theories was computed in [64–67]. In order to apply it to the present problem we either need to repeat the analysis in the presence of background gauge fields, or use supersymmetry to determine the possible structure of the one loop counterterms and hence the trace anomaly. Neither of this is completely straightforward. Furthermore the trace anomaly method includes the contribution from the zero modes as well which, as we have described, need to be analyzed separately. Thus even if we use the trace anomaly methods for computing the heat kernel, we need to find separately the zero modes of the kinetic operator, remove their contribution from the heat kernel and then separately evaluate their contribution to the entropy. To whatever extent we have tested, the two methods lead to the same result.

2 Eigenfunctions of Laplacians on AdS_2 and S^2

In this section we shall review the results on eigenfunctions and eigenvalues of the Laplacian operator $\square \equiv g^{\mu\nu} D_\mu D_\nu$ on AdS_2 and S^2 for different tensor and spinor fields. These have been studied extensively in [68–71], and also discussed in the context of near horizon geometry of black holes in [46]. We consider the background $AdS_2 \times S^2$ space with a metric of the form:

$$ds^2 = a^2(d\eta^2 + \sinh^2 \eta \, d\theta^2) + a^2(d\psi^2 + \sin^2 \psi d\phi^2) \,. \tag{2.1}$$

We shall denote by x^m the coordinates (η, θ) on AdS_2 and by x^α the coordinates (ψ, ϕ) on S^2 and introduce the invariant antisymmetric tensors $\varepsilon_{\alpha\beta}$ on S^2 and ε_{mn} on AdS_2 respectively, computed with the background metric (2.1):

$$\varepsilon_{\psi\phi} = a^2 \, \sin \psi, \qquad \varepsilon_{\eta\theta} = a^2 \, \sinh \eta \,. \tag{2.2}$$

All indices will be raised and lowered with the background metric $g_{\mu\nu}$ defined in (2.1).

We shall first review the construction of the eigenstates and eigenvalues of the Laplacian acting on individual fields in AdS_2 and S^2 separately. First consider the Laplacian acting on the scalar fields. On S^2 the normalized eigenfunctions of $-\Box$ are just the usual spherical harmonics $Y_{lm}(\psi,\phi)/a$ with eigenvalues $l(l+1)/a^2$. On the other hand on AdS_2 the δ-function normalized eigenfunctions of $-\Box$ are given by [69][5]

$$f_{\lambda,\ell}(\eta,\theta) = \frac{1}{\sqrt{2\pi\, a^2}}\,\frac{1}{2^{|\ell|}(|\ell|)!}\left|\frac{\Gamma\left(i\lambda+\frac{1}{2}+|\ell|\right)}{\Gamma(i\lambda)}\right| e^{i\ell\theta}\sinh^{|\ell|}\eta$$
$$\times F\left(i\lambda+\frac{1}{2}+|\ell|, -i\lambda+\frac{1}{2}+|\ell|; |\ell|+1; -\sinh^2\frac{\eta}{2}\right),$$
$$\ell\in\mathbb{Z}, \qquad 0<\lambda<\infty\,, \tag{2.3}$$

with eigenvalue $\left(\frac{1}{4}+\lambda^2\right)/a^2$. Here F denotes hypergeometric function.

The normalized basis of vector fields on S^2 may be taken as

$$\frac{1}{\sqrt{\kappa_1^{(k)}}}\,\partial_\alpha U_k, \qquad \frac{1}{\sqrt{\kappa_1^{(k)}}}\,\varepsilon_{\alpha\beta}\partial^\beta U_k,\,, \tag{2.4}$$

where $\{U_k\}$ denote normalized eigenfunctions of the scalar Laplacian with eigenvalue $\kappa_1^{(k)}$. The basis states given in (2.4) have eigenvalue of $-\Box$ equal to $\kappa_1^{(k)}-a^{-2}$. Note that for $\kappa_1^{(k)}=0$, i.e. for $l=0$, U_k is a constant and $\partial_m U_k$ vanishes. Hence these modes do not exist for $l=0$.

Similarly a normalized basis of vector fields on AdS_2 may be taken as

$$\frac{1}{\sqrt{\kappa_2^{(k)}}}\,\partial_m W_k, \qquad \frac{1}{\sqrt{\kappa_2^{(k)}}}\,\varepsilon_{mn}\partial^n W_k,\,, \tag{2.5}$$

where W_k are the δ-function normalized eigenfunctions of the scalar Laplacian with eigenvalue $\kappa_2^{(k)}$. The basis states given in (2.5) has eigenvalues of $-\Box$ equal to $\kappa_2^{(k)}+a^{-2}$. There are also additional square integrable modes of eigenvalue a^{-2}, given by [69]

$$A = d\Phi, \qquad \Phi = \frac{1}{\sqrt{2\pi|\ell|}}\left[\frac{\sinh\eta}{1+\cosh\eta}\right]^{|\ell|} e^{i\ell\theta}, \quad \ell=\pm1,\pm2,\pm3,\cdots\,. \tag{2.6}$$

These are not included in (2.5) since the Φ given in (2.6) is not normalizable. $d\Phi$ given in (2.6) is self-dual or anti-self-dual depending on the sign of ℓ. Thus we do not get independent eigenfunctions from $*d\Phi$. However we can also work with a real basis in which we take $dRe(\Phi)$ and $dIm(\Phi)\propto *dRe(\Phi)$ as the independent basis states for $\ell>0$.

A similar choice of basis can be made for a symmetric rank two tensor representing the graviton fluctuation. For example on S^2 we can choose a basis of these modes to be

$$\frac{1}{\sqrt{2}}\,g_{\alpha\beta}U_k, \qquad \frac{1}{\sqrt{2(\kappa_1^{(k)}-2a^{-2})}}\left[D_\alpha\xi_\beta+D_\beta\xi_\alpha-D^\gamma\xi_\gamma\, g_{\alpha\beta}\right], \tag{2.7}$$

[5] Although often we shall give the basis states in terms of complex functions, we can always work with a real basis by choosing the real and imaginary parts of the function.

where ξ_α denotes one of the two vectors given in (2.4). Note that for $\kappa_1^{(k)} = 2a^{-2}$, i.e. for $l = 1$, the second set of states given in (2.7) vanishes since the corresponding ξ_α's label the conformal Killing vectors of the sphere.

On AdS_2 the basis states for a symmetric rank two tensor may be chosen as

$$\frac{1}{\sqrt{2}} g_{mn} W_k, \qquad \frac{1}{\sqrt{2(\kappa_2^{(k)} + 2a^{-2})}} \left[D_m \widehat{\xi}_n + D_n \widehat{\xi}_m - D^p \widehat{\xi}_p \, g_{mn} \right], \tag{2.8}$$

where $\widehat{\xi}_m$ denotes one of the two vectors given in (2.5), or the vector given in (2.6). Besides these there is another set of square integrable modes of eigenvalue $2a^{-2}$ of $-\Box$, given by [69]

$$h_{mn} dx^m dx^n = \frac{a}{\sqrt{\pi}} \left[\frac{|\ell|(\ell^2-1)}{2} \right]^{1/2} \frac{(\sinh\eta)^{|\ell|-2}}{(1+\cosh\eta)^{|\ell|}} e^{i\ell\theta} \left(d\eta^2 + 2\, i \, \sinh\eta \, d\eta d\theta - \sinh^2\eta \, d\theta^2 \right)$$
$$\ell \in \mathbb{Z}, \quad |\ell| \geq 2 \,. \tag{2.9}$$

Locally these can be regarded as deformations generated by a diffeomorphism on AdS_2, but these diffeomorphisms themselves are not square integrable.

We can construct the basis states of various fields on $AdS_2 \times S^2$ by taking the product of the basis states on S^2 and AdS_2. For example for a scalar field the basis states will be given by the product of $Y_{lm}(\psi, \phi)$ with the states given in (2.3), and satisfy

$$\Box \, f_{\lambda,k}(\eta,\theta) \, Y_{lm}(\psi,\phi) = -\frac{1}{a^2} \left\{ l(l+1) + \lambda^2 + \frac{1}{4} \right\} f_{\lambda,k}(\eta,\theta) \, Y_{lm}(\psi,\phi) \,. \tag{2.10}$$

For a vector field on $AdS_2 \times S^2$ the basis states will contain two sets. One set will be given by the product of $Y_{lm}(\psi, \phi)$ and (2.5) or (2.6). The other set will contain the product of the functions (2.3) on AdS_2 and the vector fields (2.4) on S^2.

Finally we turn to the basis states for the fermion fields. Consider a Dirac spinor[6] on $AdS_2 \times S^2$. It decomposes into a product of a Dirac spinor on AdS_2 and a Dirac spinor on S^2. We use the following conventions for the vierbeins and the gamma matrices

$$e^0 = a \, \sinh\eta \, d\theta, \qquad e^1 = a \, d\eta, \qquad e^2 = a \, \sin\psi \, d\phi, \qquad e^3 = a \, d\psi \,, \tag{2.11}$$

$$\gamma^0 = -\sigma_3 \otimes \tau_2, \qquad \gamma^1 = \sigma_3 \otimes \tau_1, \qquad \gamma^2 = -\sigma_2 \otimes I_2, \qquad \gamma^3 = \sigma_1 \otimes I_2 \,, \tag{2.12}$$

where σ_i and τ_i are two dimensional Pauli matrices acting on different spaces and I_2 is 2×2 identity matrix. In this convention the Dirac operator on $AdS_2 \times S^2$ can be written as

$$\not{D}_{AdS_2 \times S^2} = \not{D}_{S^2} + \sigma_3 \, \not{D}_{AdS_2} \,, \tag{2.13}$$

where

$$\not{D}_{S^2} = a^{-1} \left[-\sigma^2 \frac{1}{\sin\psi} \partial_\phi + \sigma^1 \, \partial_\psi + \frac{1}{2} \sigma^1 \, \cot\psi \right] , \tag{2.14}$$

and

$$\not{D}_{AdS_2} = a^{-1} \left[-\tau^2 \frac{1}{\sinh\eta} \partial_\theta + \tau^1 \, \partial_\eta + \frac{1}{2} \tau^1 \, \coth\eta \right] . \tag{2.15}$$

[6] Even if the spinors satisfy Majorana/Weyl condition, we shall compute their heat kernel by first computing the result for a Dirac spinor and then taking appropriate square roots.

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First let us analyze the eigenstates of $\not{D}_{S^2}$. They are given by [72]

$$\chi^{\pm}_{l,m} = \frac{1}{\sqrt{4\pi a^2}} \frac{\sqrt{(l-m)!(l+m+1)!}}{l!} e^{i(m+\frac{1}{2})\phi} \begin{pmatrix} i \sin^{m+1}\frac{\psi}{2} \cos^m \frac{\psi}{2} P^{(m+1,m)}_{l-m}(\cos\psi) \\ \pm \sin^m \frac{\psi}{2} \cos^{m+1} \frac{\psi}{2} P^{(m,m+1)}_{l-m}(\cos\psi) \end{pmatrix},$$
$$\eta^{\pm}_{l,m} = \frac{1}{\sqrt{4\pi a^2}} \frac{\sqrt{(l-m)!(l+m+1)!}}{l!} e^{-i(m+\frac{1}{2})\phi} \begin{pmatrix} \sin^m \frac{\psi}{2} \cos^{m+1} \frac{\psi}{2} P^{(m,m+1)}_{l-m}(\cos\psi) \\ \pm i \sin^{m+1} \frac{\psi}{2} \cos^m \frac{\psi}{2} P^{(m+1,m)}_{l-m}(\cos\psi) \end{pmatrix},$$
$$l, m \in \mathbb{Z}, \quad l \geq 0, \quad 0 \leq m \leq l\,, \tag{2.16}$$

satisfying

$$\not{D}_{S^2} \chi^{\pm}_{l,m} = \pm i\, a^{-1}\, (l+1)\, \chi^{\pm}_{l,m}\,, \qquad \not{D}_{S^2} \eta^{\pm}_{l,m} = \pm i\, a^{-1}\, (l+1)\, \eta^{\pm}_{l,m}\,. \tag{2.17}$$

Here $P^{\alpha,\beta}_n(x)$ are the Jacobi Polynomials:

$$P^{(\alpha,\beta)}_n(x) = \frac{(-1)^n}{2^n\, n!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]. \tag{2.18}$$

$\chi^{\pm}_{l,m}$ and $\eta^{\pm}_{l,m}$ provide an orthonormal set of basis functions, e.g.

$$a^2 \int_{S^2} \left(\chi^{\pm}_{l,m}\right)^\dagger \chi^{\pm}_{l',m'} \sin\psi\, d\psi\, d\phi = \delta_{ll'}\delta_{mm'} \tag{2.19}$$

etc.

The eigenstates of $\not{D}_{AdS_2}$ are given by the analytic continuation of the eigenstates given in (2.16) [72], making the replacement $\psi \to i\eta$, $l \to -i\lambda - 1$,

$$\chi^{\pm}_k(\lambda) = \frac{1}{\sqrt{4\pi a^2}} \left| \frac{\Gamma(1+k+i\lambda)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}+i\lambda\right)} \right| e^{i(k+\frac{1}{2})\theta}$$
$$\times \begin{pmatrix} i \frac{\lambda}{k+1} \cosh^k \frac{\eta}{2} \sinh^{k+1} \frac{\eta}{2} F\left(k+1+i\lambda, k+1-i\lambda; k+2; -\sinh^2\frac{\eta}{2}\right) \\ \pm \cosh^{k+1} \frac{\eta}{2} \sinh^k \frac{\eta}{2} F\left(k+1+i\lambda, k+1-i\lambda; k+1; -\sinh^2\frac{\eta}{2}\right) \end{pmatrix},$$

$$\eta^{\pm}_k(\lambda) = \frac{1}{\sqrt{4\pi a^2}} \left| \frac{\Gamma(1+k+i\lambda)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}+i\lambda\right)} \right| e^{-i(k+\frac{1}{2})\theta}$$
$$\times \begin{pmatrix} \cosh^{k+1} \frac{\eta}{2} \sinh^k \frac{\eta}{2} F\left(k+1+i\lambda, k+1-i\lambda; k+1; -\sinh^2\frac{\eta}{2}\right) \\ \pm i \frac{\lambda}{k+1} \cosh^k \frac{\eta}{2} \sinh^{k+1} \frac{\eta}{2} F\left(k+1+i\lambda, k+1-i\lambda; k+2; -\sinh^2\frac{\eta}{2}\right) \end{pmatrix},$$
$$k \in \mathbb{Z}, \quad 0 \leq k < \infty, \quad 0 < \lambda < \infty\,, \tag{2.20}$$

satisfying

$$\not{D}_{AdS_2} \chi^{\pm}_k(\lambda) = \pm i\, a^{-1}\, \lambda\, \chi^{\pm}_k(\lambda)\,, \qquad \not{D}_{AdS_2} \eta^{\pm}_k(\lambda) = \pm i\, a^{-1}\, \lambda\, \eta^{\pm}_k(\lambda)\,. \tag{2.21}$$

$\chi^{\pm}_k(\lambda)$ and $\eta^{\pm}_k(\lambda)$ provide an orthonormal set of basis functions on AdS_2, e.g.

$$a^2 \int \sinh\eta\, d\eta\, d\theta\, (\chi^{\pm}_k(\lambda))^\dagger\, \chi^{\pm}_{k'}(\lambda') = \delta_{kk'}\delta(\lambda-\lambda')\,, \tag{2.22}$$

etc.

The basis of spinors on $AdS_2 \times S^2$ can be constructed by taking the direct product of the spinors given in (2.16) and (2.20). Suppose that ψ_1 denotes an eigenstate of $\not{D}_{S^2}$ with eigenvalue $i\zeta_1$ and ψ_2 denotes an eigenstate of $\not{D}_{AdS_2}$ with eigenvalue $i\zeta_2$:

$$\not{D}_{S^2}\psi_1 = i\zeta_1\,\psi_1, \qquad \not{D}_{AdS_2}\psi_2 = i\zeta_2\,\psi_2. \tag{2.23}$$

We have $\zeta_1 = \pm a^{-1}(l+1)$ and $\zeta_2 = \pm a^{-1}\lambda$. Since σ_3 anti-commutes with $\not{D}_{S^2}$ and commutes with $\not{D}_{AdS_2}$, we have, using (2.13),

$$\begin{aligned} \not{D}_{AdS_2\times S^2}\,\psi_1\otimes\psi_2 &= i\zeta_1\psi_1\otimes\psi_2 + i\zeta_2\sigma_3\,\psi_1\otimes\psi_2\,, \\ \not{D}_{AdS_2\times S^2}\,\sigma_3\,\psi_1\otimes\psi_2 &= i\zeta_2\,\psi_1\otimes\psi_2 - i\zeta_1\sigma_3\,\psi_1\otimes\psi_2\,. \end{aligned} \tag{2.24}$$

Diagonalizing the 2×2 matrix we see that $\not{D}_{AdS_2\times S^2}$ has eigenvalues $\pm i\sqrt{\zeta_1^2+\zeta_2^2}$. Thus the square of the eigenvalue of $\not{D}_{AdS_2\times S^2}$ is given by the sum of squares of the eigenvalues of $\not{D}_{AdS_2}$ and $\not{D}_{S^2}$.

By introducing the 'charge conjugation operator' $\widetilde{C} = \sigma_2\otimes\tau_1$ and defining $\bar{\psi} = \psi^T\widetilde{C}$, we can express the orthonormality relations (2.19), (2.22) as

$$\int d^4x\,\sqrt{\det g}\,\overline{\left(\chi^+_{l,m}\otimes\chi^+_k(\lambda)\right)}\,\left(\eta^+_{l',m'}\otimes\eta^-_{k'}(\lambda')\right) = i\,\delta_{l,l'}\delta_{m,m'}\delta_{k,k'}\delta(\lambda-\lambda')\,, \tag{2.25}$$

etc. This is important since eventually we shall be dealing with fields satisfying appropriate reality conditions for which $\bar{\psi}$ will be defined as $\psi^T\widetilde{C}$ as far as the SO(4) Clifford algebra associated with $AdS_2\times S^2$ is concerned (see (4.8)).

In our analysis we shall also need to find a basis in which we can expand the Rarita-Schwinger field Ψ_μ. Let us denote by χ the spinor $\psi_1\otimes\psi_2$ where ψ_1 and ψ_2 are eigenstates of $\not{D}_{S^2}$ and $\not{D}_{AdS_2}$ with eigenvalues $i\zeta_1$ and $i\zeta_2$ respectively. Then a (non-orthonormal set of) basis states for expanding Ψ_μ on $AdS_2\times S^2$ can be chosen as follows:

$$\begin{aligned} &\Psi_\alpha = \gamma_\alpha\chi, &&\Psi_m = 0\,, \\ &\Psi_\alpha = 0, &&\Psi_m = \gamma_m\chi, \\ &\Psi_\alpha = D_\alpha\chi, &&\Psi_m = 0, \\ &\Psi_\alpha = 0, &&\Psi_m = D_m\chi\,. \end{aligned} \tag{2.26}$$

By including all possible eigenstates χ of $\not{D}_{S^2}$ and $\not{D}_{AdS_2}$ we shall generate the complete set of basis states for expanding the Rarita-Schwinger field barring the subtleties mentioned below.

There are two additional points which will be important for our analysis. First of all we have the relations

$$D_\alpha\chi^\pm_{0,0} = \pm\frac{i}{2}\,a^{-1}\,\gamma_\alpha\chi^\pm_{0,0}, \qquad D_\alpha\eta^\pm_{0,0} = \pm\frac{i}{2}\,a^{-1}\,\gamma_\alpha\eta^\pm_{0,0}\,. \tag{2.27}$$

Thus if we take $\chi = \psi_1\otimes\psi_2$ where ψ_1 corresponds to any of the states $\chi^\pm_{0,0}$ or $\eta^\pm_{0,0}$, and ψ_2 is any eigenstate of $\not{D}_{AdS_2}$, then the basis vectors appearing in (2.26) are not all independent, — the modes in the third row of (2.26) are related to those in the first row. The second point is that the modes given in (2.26) do not exhaust all the modes of the Rarita Schwinger operator; there are some additional discrete modes of the form

$$\xi_m^{(k)\pm} \equiv \psi_1 \otimes \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m \right) \chi_k^\pm(i), \quad \widehat{\xi}_m^{(k)\pm} \equiv \psi_1 \otimes \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m \right) \eta_k^\pm(i), \quad k=1,\cdots\infty\,, \tag{2.28}$$

where $\chi_k^\pm(\lambda)$ and $\eta_k^\pm(\lambda)$ have been defined in (2.20). Since $\chi_k^\pm(i)$ and $\eta_k^\pm(i)$ are not square integrable, these states are not included in the set given in (2.26). However the modes described in (2.28) are square integrable and hence they must be included among the eigenstates of the Rarita-Schwinger operator. These modes can be shown to satisfy the chirality projection condition

$$\begin{aligned}
\tau_3 \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m \right) \chi_k^\pm(i) &= -\left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m \right) \chi_k^\pm(i), \\
\tau_3 \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m \right) \eta_k^\pm(i) &= \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m \right) \eta_k^\pm(i)\,.
\end{aligned} \tag{2.29}$$

3 Logarithmic correction to the black hole entropy

In this section we shall review the general procedure for computing the logarithmic correction to the extremal black hole entropy. Suppose we have an extremal black hole with near horizon geometry $AdS_2 \times S^2$, with equal radius of curvature a of AdS_2 and S^2. Then the Euclidean near horizon metric takes the form given in (2.1). As in [46], we shall make use of the flat directions of the classical entropy function to choose the near horizon parameters such that a is the only parametrically large number, all other parameters e.g. the string coupling or the size of the compact space remains fixed as we take the large charge limit. Let Z_{AdS_2} denote the partition function of string theory in the near horizon geometry, evaluated by carrying out functional integral over all the string fields weighted by the exponential of the Euclidean action $\mathcal{S}$, with boundary conditions such that asymptotically the field configuration approaches the near horizon geometry of the black hole.[7] Then AdS_2/CFT_1 correspondence tells us that the full quantum corrected entropy S_{BH} is related to Z_{AdS_2} via [7, 24]:

$$e^{S_{BH} - E_0\beta} = Z_{AdS_2}\,, \tag{3.1}$$

where E_0 is the energy of the ground state of the black hole carrying a given set of charges, and β denotes the length of the boundary of AdS_2 in a regularization scheme that renders the volume of AdS_2 finite by putting an infrared cut-off $\eta \leq \eta_0$. Let $\Delta\mathcal{L}_{\rm eff}$ denote the one loop correction to the four dimensional effective lagrangian density evaluated in the background geometry (2.1). Then the one loop correction to Z_{AdS_2} is given by

$$\exp\left[\int \sqrt{\det g}\, d\eta\, d\theta\, d\psi\, d\phi\, \Delta\mathcal{L}_{\rm eff} \right] = \exp\left[8\pi^2\, a^4\, (\cosh\eta_0 - 1)\, \Delta\mathcal{L}_{\rm eff} \right]\,. \tag{3.2}$$

[7]Since in AdS_2 the asymptotic boundary conditions fix the electric fields, or equivalently the charges carried by the black hole, and let the constant modes of the gauge fields to fluctuate, we need to include in the path integral a boundary term $\exp(-i\oint \sum_k q_k A_\mu^{(k)} dx^\mu)$ where $A_\mu^{(k)}$ are the gauge fields and q_k are the corresponding electric charges carried by the black hole [7]. This term plays a crucial role in establishing that the classical contribution to the black hole entropy computed via (3.1) gives us the Wald entropy, but will not play any role in the computation of logarithmic corrections.

The term proportional to $\cosh\eta_0$ in the exponent has the interpretation of $-\beta\Delta E_0 + \mathcal{O}\left(\beta^{-1}\right)$ where $\beta = 2\pi a \sinh\eta_0$ is the length of the boundary of AdS_2 parametrized by θ and $\Delta E_0 = -4\pi a^3 \Delta\mathcal{L}_{\rm eff}$ is the shift in the ground state energy. The rest of the contribution in the exponent can be interpreted as the one loop correction to the black hole entropy [7, 24]. Thus we have

$$\Delta S_{BH} = -8\pi^2 a^4\, \Delta\, \mathcal{L}_{\rm eff}\,. \tag{3.3}$$

While the term in the exponent proportional to β and hence ΔE_0 can get further corrections from boundary terms in the action, the finite part ΔS_{BH} is defined unambiguously. This reduces the problem of computing one loop correction to the black hole entropy to that of computing one loop correction to $\mathcal{L}_{\rm eff}$. We shall now describe the general procedure for calculating $\Delta\mathcal{L}_{\rm eff}$.

The near horizon geometry of the black hole has background flux of various electromagnetic fields through S^2 and AdS_2. In this section we shall ignore the effect of this background flux, leaving the full problem for later sections. Then the dynamics of various fields is controlled essentially by their coupling to the background metric (2.1). First consider the case of a massless scalar field. If we denote the eigenvalues of the scalar laplacian by $\{-\kappa_n\}$ and the corresponding normalized eigenfunctions by $f_n(x)$ then the heat kernel $K^s(x,x';s)$ of the scalar Laplacian is defined as (see [50, 51] and references therein)

$$K^s(x,x';s) = \sum_n e^{-\kappa_n\, s}\, f_n(x)\, f_n(x')\,. \tag{3.4}$$

The superscript s on K reflects that the laplacian acts on the scalar fields. In (3.4) we have assumed that we are working in a basis in which the eigenfunctions are real; if this is not the case then we need to replace $f_n(x')$ by $f_n^*(x')$. The contribution of this scalar field to the one loop effective action can now be expressed as

$$\Delta\mathcal{S} = -\frac{1}{2}\sum_n \ln\kappa_n = \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\sum_n e^{-\kappa_n s} = \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\int d^4x\,\sqrt{\det g}\,K^s(x,x;s)\,, \tag{3.5}$$

where $g_{\mu\nu}$ is the $AdS_2\times S^2$ metric and ϵ is an ultraviolet cut-off. Identifying this as $\int d^4x\sqrt{\det g}\,\Delta\mathcal{L}_{\rm eff}$ we get

$$\Delta\mathcal{L}_{\rm eff} = \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\,K^s(0;s)\,, \tag{3.6}$$

where $K^s(0;s)\equiv K^s(x,x;s)$. Note that using the fact that AdS_2 and S^2 are homogeneous spaces we have dropped the dependence on x from $K^s(x,x;s)$.

For higher spin fields the field will carry an extra index (say a). Then we can define the heat kernel $K_{ab}(x,x';s)$ by generalizing (3.4) and the contribution to $\Delta\mathcal{L}_{\rm eff}$ from these fields will be given by

$$\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\,K_{aa}(x,x;s)\,. \tag{3.7}$$

For notational simplicity we shall refer to K_{aa} as the heat kernel and denote it by K, but it should be kept in mind that for higher spin fields this refers to the trace of the heat kernel. For fermions there will be an extra minus sign since the fermionic integral produces a positive power of the determinant. We shall choose the convention in which this extra factor is absorbed into the definition of K. Also the fermionic kinetic operator is linear in derivatives; we shall find it convenient to define the heat kernel using the square of the fermionic kinetic operator, and then include an extra factor of half in the definition of K to account for the final square root that we need to take.

Let us now return to the computation of $K^s(0;s)$. It follows from (3.4) and the fact that $\Box_{AdS_2\times S^2} = \Box_{AdS_2} + \Box_{S^2}$ that the heat kernel of a massless scalar field on $AdS_2 \times S^2$ is given by the product of the heat kernels on AdS_2 and S^2, and in the $x' \to x$ limit takes the form [68]

$$K^s(0;s) = K^s_{AdS_2}(0;s) K^s_{S^2}(0;s)\,. \tag{3.8}$$

$K^s_{S^2}$ and $K^s_{AdS_2}$ in turn can be calculated using (3.4) since we know the eigenfunctions and the eigenvalues of the Laplace operator on these respective spaces. The eigenfunction on AdS_2 are described in (2.3). Since $f_{\lambda,\ell}$ vanish at $\eta = 0$ for $\ell \neq 0$, only the $\ell = 0$ eigenfunctions will contribute to $K^s_{AdS_2}(0;s)$. At $\eta = 0$ the $\ell = 0$ eigenfunction has the value $\sqrt{\lambda \tanh(\pi\lambda)}/\sqrt{2\pi a^2}$. Thus (3.4) gives

$$K^s_{AdS_2}(0;s) = \frac{1}{2\pi\, a^2} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, \exp\left[-s\left(\lambda^2 + \frac{1}{4}\right)/a^2\right]\,. \tag{3.9}$$

On S^2 the eigenfunctions are $Y_{lm}(\psi,\phi)/a$ and the corresponding eigenvalues are $-l(l+1)/a^2$. Since Y_{lm} vanishes at $\psi = 0$ for $m \neq 0$, and $Y_{l0} = \sqrt{2l+1}/\sqrt{4\pi}$ at $\psi = 0$ we have

$$K^s_{S^2}(0;s) = \frac{1}{4\pi a^2} \sum_l e^{-sl(l+1)/a^2} (2l+1)\,. \tag{3.10}$$

We can bring this to a form similar to (3.9) by expressing it as

$$\frac{1}{4\pi i\, a^2}\, e^{s/4a^2} \oint d\widetilde{\lambda}\, \widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, e^{-s\widetilde{\lambda}^2/a^2}\,, \tag{3.11}$$

where $\oint$ denotes integration along a contour that travels from ∞ to 0 staying below the real axis and returns to ∞ staying above the real axis. By deforming the integration contour to a pair of straight lines through the origin — one at an angle κ below the positive real axis and the other at an angle κ above the positive real axis — we get

$$K^s_{S^2}(0;s) = \frac{1}{2\pi a^2} e^{s/4a^2}\, Im \int_0^{e^{i\kappa}\times\infty} \widetilde{\lambda}\, d\widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, e^{-s\widetilde{\lambda}^2/a^2}\,, \qquad 0 < \kappa \ll 1\,. \tag{3.12}$$

Combining (3.10) and (3.9) we get the heat kernel of a scalar field on $AdS_2 \times S^2$:

$$\begin{aligned} K^s(0;s) &= \frac{1}{8\pi^2 a^4} \sum_{l=0}^\infty (2l+1) \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, \exp\left[-\bar{s}\lambda^2 - \bar{s}\left(l + \frac{1}{2}\right)^2\right] \\ &= \frac{1}{4\pi^2 a^4} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\; Im \int_0^{e^{i\kappa}\times\infty} \widetilde{\lambda}\, d\widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, \exp\left[-\bar{s}\lambda^2 - \bar{s}\widetilde{\lambda}^2\right], \end{aligned} \tag{3.13}$$

where

$$\bar{s} = s/a^2 \,. \tag{3.14}$$

We can in principle evaluate the full one loop correction to S_{BH} due to massless fields using (3.3), (3.6) and (3.13), but our goal is to extract the piece proportional to $\ln a$ for large a. Such contributions come from the region of integration $1 \ll s \ll a^2$ or equivalently $a^{-2} \ll \bar{s} \ll 1$. Thus we need to study the behaviour of (3.9), (3.10) for small $\bar{s}$. We shall now describe a general procedure for carrying out this small $\bar{s}$ expansion, not just for the integrals appearing in (3.13) but for a more general class of integrals where we insert some powers of λ and $\widetilde{\lambda}$ into the integrand. For this we first write

$$\tanh(\pi\lambda) = 1 + 2\sum_{k=1}^{\infty}(-1)^k\, e^{-2\pi k\lambda}, \qquad \tan(\pi\widetilde{\lambda}) = i\left[1 + 2\sum_{k=1}^{\infty}(-1)^k\, e^{2\pi i k\widetilde{\lambda}}\right]. \tag{3.15}$$

In the term proportional to 1 in the expression for $\tanh(\pi\lambda)$ ($\tan\pi\lambda$) we change the integration variable in (3.13) from λ ($\widetilde{\lambda}$) to $u \equiv \bar{s}\lambda^2$ ($v = \bar{s}\widetilde{\lambda}^2$). These integrals can be performed exactly in terms of Γ functions. On the other hand in the term proportional to $e^{-2\pi k\lambda}$ ($e^{2\pi i k\widetilde{\lambda}}$) in the expansion of $\tanh(\pi\lambda)$ ($\tan(\pi\widetilde{\lambda})$) we change the variable of integration to $u = 2\pi k\lambda$ ($v = 2\pi i k\widetilde{\lambda}$) and then expand the $e^{-\bar{s}\lambda^2}$ ($e^{-\bar{s}\widetilde{\lambda}^2}$) term in a power series in u (or v). After performing the integrals and a resummation over k we get

$$\begin{aligned}
&\int_0^\infty d\lambda\, \lambda \, \tanh(\pi\lambda)\, e^{-\bar{s}\lambda^2}\, \lambda^{2n} \\
&= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^{\infty} \bar{s}^m \frac{(2m+2n+1)!}{m!}\,(2\pi)^{-2(m+n+1)}\,(-1)^m \\
&\qquad (2^{-2m-2n-1}-1)\,\zeta(2(m+n+1))\,.
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&Im\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\tan(\pi\widetilde{\lambda})\, e^{-\bar{s}\widetilde{\lambda}^2}\,\widetilde{\lambda}^{2n} \\
&= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^{\infty} \bar{s}^m \frac{(2m+2n+1)!}{m!}\,(2\pi)^{-2(m+n+1)}(-1)^{n+1} \\
&\qquad (2^{-2m-2n-1}-1)\,\zeta(2(m+n+1))\,.
\end{aligned} \tag{3.17}$$

This leads to the following expression for $K^s_{AdS_2}(0;s)$ and $K^s_{S^2}(0;s)$:

$$\begin{aligned}
K^s_{AdS_2}(0;s) &= \frac{1}{4\pi a^2\,\bar{s}}\, e^{-\bar{s}/4}\left[1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(2n+1)!\frac{\bar{s}^{n+1}}{\pi^{2n+2}}\frac{1}{2^{2n}}\left(2^{-2n-1}-1\right)\zeta(2n+2)\right] \\
&= \frac{1}{4\pi a^2\,\bar{s}}\, e^{-\bar{s}/4}\left(1 - \frac{1}{12}\bar{s} + \frac{7}{480}\bar{s}^2 + \mathcal{O}(\bar{s}^3)\right),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
K^s_{S^2}(0;s) &= \frac{1}{4\pi a^2\,\bar{s}}\, e^{\bar{s}/4}\left[1 - \sum_{n=0}^{\infty}\frac{1}{n!}(2n+1)!\frac{\bar{s}^{n+1}}{\pi^{2n+2}}\frac{1}{2^{2n}}\left(2^{-2n-1}-1\right)\zeta(2n+2)\right] \\
&= \frac{1}{4\pi a^2\,\bar{s}}\, e^{\bar{s}/4}\left(1 + \frac{1}{12}\bar{s} + \frac{7}{480}\bar{s}^2 + \mathcal{O}(\bar{s}^3)\right).
\end{aligned} \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.8) we get

$$K^s(0;s) = \frac{1}{16\pi^2 a^4\,\bar{s}^2}\left(1 + \frac{1}{45}\bar{s}^2 + \mathcal{O}(s^4)\right). \tag{3.20}$$

eq. (3.6) now gives

$$\Delta\mathcal{L}_{\text{eff}} = \frac{1}{32\pi^2 a^4}\int_{\epsilon/a^2}^{\infty}\frac{d\bar{s}}{\bar{s}^3}\left(1+\frac{1}{45}\bar{s}^2+\mathcal{O}(\bar{s}^4)\right). \tag{3.21}$$

This integral has a quadratically divergent piece proportional to $1/\epsilon^2$. This can be thought of as a renormalization of the cosmological constant and will cancel against contribution from other fields in a supersymmetric theory in which the cosmological constant is not renormalized. Even otherwise in string theory there is a physical cut-off set by the string scale.[8] Our main interest is in the logarithmically divergent piece which comes from the order $\bar{s}^2$ term inside the parentheses. This is given by

$$\frac{1}{1440\pi^2 a^4}\ln(a^2/\epsilon)\,, \tag{3.22}$$

and, according to (3.3) gives a contribution to the entropy

$$\Delta S_{BH} = -\frac{1}{180}\ln(a^2/\epsilon)\,. \tag{3.23}$$

Computation for the higher spin fields follows in a similar manner. We use the basis described in section 2 to construct the heat kernel. For evaluating $K(0;s)$ we need to compute $u(x)^2$ at $x=0$ where u is a generic basis element for the higher spin fields. This can of course be done using the explicit form of the basis functions given in section 2 but here we shall suggest a useful shortcut. Consider for example the state of the form $(\kappa_2^{(k)})^{-1}\partial_m W_k$ given in (2.5). Since $K_{\alpha\alpha}(x,x;s)$ is independent of x due to the homogeneity of AdS_2 and S^2, we can replace the contribution from every term to $K_{\alpha\alpha}(x,x;s)$ by the volume average of the term. Now since W_k and $(\kappa_2^{(k)})^{-1}\partial_m W_k$ are both δ-function normalized states, the volume average of the square of $(\kappa_2^{(k)})^{-1}\partial_m W_k$ over AdS_2 is the same as that of the square of W_k; hence we can replace the square of $(\kappa_2^{(k)})^{-1}\partial_m W_k$ by $W_k(x)^2$ while computing $K_{\alpha\alpha}(x,x;s)$. The contribution to the heat kernel from this set of modes will have the same form as (3.9), except that the $\exp[-s(\lambda^2+1/4)/a^2]$ term will be replaced by $\exp[-s\gamma(\lambda)]$ where $\gamma(\lambda)$ is a function of λ that gives the eigenvalue of the kinetic operator acting on this state. Similar remark holds for all other basis states which are obtained by acting suitable differential operators on the eigenfunctions of the scalar Laplacian. This will be illustrated in detail in section 5.

As we shall see in the later sections, in the presence of non-trivial background gauge fields the individual basis states introduced e.g. in (2.10) and similar basis states for higher spin fields no longer remain eigenstates of the kinetic operators. Instead the kinetic operator is represented as a matrix on such basis states for fields of different spin. The matrix however is still block diagonal, with each block spanned by basis states built by the action of various differential operators on the $Y_{lm}(\psi,\phi)f_{\lambda,k}(\eta,\theta)$ for fixed (l,λ,m,k). In

[8]Typically in a string theory there are multiple scales e.g. string scale, Planck scale, scale set by the mass of the D-branes etc. We shall consider near horizon background where the string coupling constant as well as all the other parameters describing the shape, size and the various background fields along the six compact directions are of order unity. In this case all these length scales will be of the same order.

this case we have to replace the $e^{-\bar{s}\lambda^2-\bar{s}l(l+1)}$ factor in the integrand by $\sum_i \exp[-s\gamma_i(l,\lambda)]$ where the sum over i runs over all the eigenvectors of this matrix and $\gamma_i(l,\lambda)$ represent the corresponding eigenvalues.

For the discrete modes given in (2.6) and (2.9) we need to evaluate the contribution explicitly. This can be done by noting that at $\eta = 0$ only the $\ell = \pm 1$ modes in (2.6) are non-vanishing and only the $\ell = \pm 2$ modes in (2.9) are non-vanishing. This allows us to explicitly evaluate the contribution from the discrete modes to $K_{AdS_2}(0;s)$ for the vector and the symmetric tensor fields:

$$\begin{aligned} \text{vector} &: \frac{1}{2\pi a^2} \\ \text{symmetric tensor} &: \frac{3}{2\pi a^2}\,. \end{aligned} \tag{3.24}$$

Again in the presence of non-trivial background field there can be mixing between the discrete modes of various fields, carrying the same l label, under the action of the kinetic operator. In this case we have to find the eigenvalues $\gamma_i(l)$ of the corresponding matrix, and include factors of $\exp[-s\gamma_i(l)]$ in the summand in computing the contribution to the heat kernel from the discrete modes.

This procedure for computing the heat kernel for higher spin fields from that of scalars does not work for fermions since the eigenfunctions of the fermionic kinetic operator are not given by simple differential operators acting on the eigenfunctions of the scalar kinetic operator. However since the eigenfunctions of the fermionic kinetic operator are given in (2.16), (2.20), we can use this to explicitly compute the heat kernel of a fermion on $AdS_2 \times S^2$. This was done in [46] and the result for a Dirac fermion is

$$-\frac{1}{2\pi^2 a^4}\int_0^\infty d\lambda e^{-\bar{s}\lambda^2}\,\lambda\coth(\pi\lambda)\sum_{l=0}^{\infty}(2l+2)\,e^{-s(l+1)^2/a^2}\,. \tag{3.25}$$

Since the basis for the expansion of a spin 3/2 field is given by various operators acting on the eigenmodes of the spin 1/2 Dirac operator, we can use the previous trick to compute the heat kernel for spin 3/2 field in terms of the heat kernel of the spin 1/2 field. This will be illustrated in section 6.

One final issue that enters the computation is the following. Typically for higher spin fields the heat kernel on $AdS_2 \times S^2$ also receives contribution from zero modes, — discrete modes representing eigenfunctions of the kinetic operator with zero eigenvalue.[9] These give s independent contribution to the heat kernel. Integration over these zero modes cannot be represented as a determinant of the kinetic operator and must be computed separately. For this reason we need to identify in the final expression for the heat kernel on $AdS_2 \times S^2$ the s-independent contribution from the discrete modes and subtract it from the full heat kernel. We then have to evaluate separately the contribution due to integraton over these zero modes.

[9]Note that here we are refering to the zero eigenvalues of the full kinetic operator on $AdS_2 \times S^2$, taking into acount the effect of background gauge fields, and not eigenvalues of the kinetic operators on AdS_2 and S^2 separately.

It follows from (3.3), (3.6) and (1.3) that if the total contribution to $K(0;s)$ after removing the contribution due to the zero modes is given by $c/\pi^2 a^4$ for some constant c, then the net logarithmic correction to the black hole entropy from the non-zero modes will be given by

$$-4c\ln a^2 = -2c\ln\Delta\,. \tag{3.26}$$

4 Quadratic action of gravity multiplet fields in $\mathcal{N}=4$ supergravity

We consider type II string theory compactified on $K3\times T^2$ or equivalently heterotic string theory on T^6. In the language of heterotic string theory the black hole solution we consider contains momentum and winding charge along one of the circles of T^6 denoted by the co-ordinate x^4, and Kaluza-Klein monopole and H-monopole charges associated with another circle of T^6 denoted by x^5. The other compact directions will be denoted by $x^6,\cdots x^9$. The quadratic action involving fluctuations of the various massless fields of $\mathcal{N}=4$ supergravity around the near horizon geometry of the black hole was analyzed in [46]. The result of this paper shows that the dependence on the charges can be scaled out by a simple field redefinition and the final action depends on the charges only through an overall length parameter a describing the radius of curvature of AdS_2 and S^2. The relation between a and the charges has been given in (1.3). It was further found that the net logarithmic correction to the black hole entropy due to fields in the matter multiplet vanish. Thus we shall focus on fields in the gravity multiplet.

We shall use the convention in which the indices μ,ν run over all the four coordinates of $AdS_2\times S^2$, the indices α,β run over the coordinates of S^2 and the indices m,n run over the coordinates of AdS_2. In this convention the gravity multiplet fluctuations around the near horizon geometry are labelled by a set of six vector fields $\mathcal{A}^{(a)}_\mu$ $(1\le a\le 6)$, a spin two field $h_{\mu\nu}$, two scalars χ_1 and χ_2 describing fluctuations of the axion-dilaton field, and four gravitino and four dilatino fields. It is natural to combine the four dilatino fields into a 16 component right-handed Majorana-Weyl spinor Λ of the ten dimensional Lorentz group, and the four gravitino fields into a set of fields $\{\psi_\mu\}$ where for each μ $(0\le\mu\le 3)$ ψ_μ is a 16 component left-handed Majorana-Weyl spinor of the ten dimensional Lorentz group.[10] In the harmonic gauge the quadratic part of the action involving these fluctuating fields is given by [46]

$$S = S_b + S_f = \int d^4x\,\sqrt{\det g}(\mathcal{L}_b+\mathcal{L}_f)\,, \tag{4.1}$$

$$\mathcal{L}_b = -\frac{1}{4}h_{\mu\nu}\left(\widetilde{\Delta}h\right)^{\mu\nu} + \frac{1}{2}\chi_1\Box\chi_1 + \frac{1}{2}\chi_2\Box\chi_2 + \frac{1}{2}\sum_{a=1}^{6}\mathcal{A}^{(a)}_\mu(g^{\mu\nu}\Box - R^{\mu\nu})\mathcal{A}^{(a)}_\nu$$
$$+\frac{1}{2}a^{-2}\left(h^{mn}h_{mn} - h^{\alpha\beta}h_{\alpha\beta} + 2\chi_2\,(h^m_{\ m} - h^\alpha_{\ \alpha})\right) + \frac{\sqrt{2}}{a}\left[i\varepsilon^{mn}\,f^{(1)}_{\alpha m}h^\alpha_{\ n} + \varepsilon^{\alpha\beta}\,f^{(2)}_{\alpha m}h_\beta^{\ m}\right]$$

[10] Our conventions here are somewhat different from that of [46], where Λ refered to the dilatino field in ten dimensions, and the four dimensional dilatino, obtain after dimensional reduction, was denoted by λ. The latter field is being called Λ here, and we shall not make any reference to the ten dimensional fields before dimensional reduction.

$$+\frac{1}{2\sqrt{2}\,a}\left[i\varepsilon^{mn}f^{(1)}_{mn}\left(-2\chi_2+h^{\gamma}{}_{\gamma}-h^{p}{}_{p}\right)-\varepsilon^{\alpha\beta}f^{(2)}_{\alpha\beta}\left(-2\chi_2+h^{p}{}_{p}-h^{\gamma}{}_{\gamma}\right)\right]$$
$$+\frac{1}{a\sqrt{2}}\chi_1\left(i\varepsilon^{mn}f^{(2)}_{mn}+\varepsilon^{\alpha\beta}f^{(1)}_{\alpha\beta}\right), \tag{4.2}$$

and

$$\begin{aligned}\mathcal{L}_f=&-\frac{1}{2}\Bigg[\bar\psi_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho+\bar\Lambda\Gamma^\mu D_\mu\Lambda\\&+\frac{1}{4\sqrt{2}}\bar\psi_\mu\left[-\Gamma^{\mu\nu\rho\sigma}+2g^{\mu\sigma}g^{\nu\rho}+2\Gamma^{\mu\rho\nu}\Gamma^\sigma+\Gamma^{\mu\nu}\Gamma^{\rho\sigma}\right]\left(\bar F^1_{\rho\sigma}\Gamma^4+\bar F^2_{\rho\sigma}\Gamma^5\right)\psi_\nu\\&+\frac{1}{4}\left[\bar\psi_\mu\Gamma^{\rho\sigma}\Gamma^\mu\left(\bar F^1_{\rho\sigma}\Gamma^4+\bar F^2_{\rho\sigma}\Gamma^5\right)\Lambda-\bar\Lambda\left(\bar F^1_{\rho\sigma}\Gamma^4+\bar F^2_{\rho\sigma}\Gamma^5\right)\Gamma^\mu\Gamma^{\rho\sigma}\psi_\mu\right]\\&-\frac{1}{2}\bar\psi_\mu\Gamma^\mu\Gamma^\nu D_\nu\Gamma^\rho\psi_\rho\Bigg].\end{aligned} \tag{4.3}$$

Here

$$f^{(a)}_{\mu\nu}\equiv\partial_\mu\mathcal{A}^{(a)}_\nu-\partial_\nu\mathcal{A}^{(a)}_\mu, \tag{4.4}$$

$$\begin{aligned}\left(\widetilde\Delta h\right)_{\mu\nu}=&-\Box h_{\mu\nu}-R_{\mu\tau}h^\tau{}_\nu-R_{\nu\tau}h_\mu{}^\tau-2R_{\mu\rho\nu\tau}h^{\rho\tau}+\frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Box h_{\rho\sigma}\\&+R\,h_{\mu\nu}+\left(g_{\mu\nu}R^{\rho\sigma}+R_{\mu\nu}g^{\rho\sigma}\right)h_{\rho\sigma}-\frac{1}{2}R\,g_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma},\end{aligned} \tag{4.5}$$

and $\varepsilon^{\alpha\beta}$ and ε^{mn} have been defined in (2.2). All indices are raised and lowered by the background metric $g_{\mu\nu}$ given in (2.1). $R_{\mu\nu\rho\sigma}$ is the Riemann tensor on $AdS_2\times S^2$ constructed from the background metric (2.1) and $\bar F^1_{\rho\sigma}$ and $\bar F^2_{\rho\sigma}$ are background gauge field strengths whose non-vanishing components are

$$\bar F^1_{mn}=-\frac{i}{\sqrt{2}a}\varepsilon_{mn},\qquad \bar F^2_{\alpha\beta}=\frac{1}{\sqrt{2}a}\varepsilon_{\alpha\beta}. \tag{4.6}$$

Γ^M are ten dimensional gamma matrices chosen as follows:

$$\Gamma^\mu=\gamma^\mu\otimes I_8,\qquad \Gamma^m=\sigma_3\otimes\tau_3\otimes\widehat\Gamma^m,\qquad 0\le\mu\le3,\quad 4\le m\le 9, \tag{4.7}$$

where γ^μ's have been defined in (2.12) and $\widehat\Gamma^m$ are 8×8 SO(6) gamma matrices. $\bar\psi_\mu$, $\bar\Lambda$ are defined as

$$\bar\psi_\mu\equiv\psi_\mu^T\,C,\quad \bar\Lambda\equiv\Lambda^T\,C, \tag{4.8}$$

where T denotes transpose and C is the SO(10) charge conjugation matrix satisfying

$$(C\Gamma^A)^T=C\Gamma^A,\qquad C^T=-C. \tag{4.9}$$

Our choice for C will be:

$$C=\sigma_2\otimes\tau_1\otimes\widehat C, \tag{4.10}$$

where $\widehat C$ is the SO(6) charge conjugation matrix satisfying

$$(\widehat C\widehat\Gamma^p)^T=-\widehat C\widehat\Gamma^p,\qquad \widehat C^T=\widehat C. \tag{4.11}$$

We can use the vielbeins to convert the tangent space indices to coordinate indices and vice versa. We shall use the same symbol Γ for labelling the gamma matrices carrying coordinate indices.

The Lagrangian densities given in (4.2) and (4.3) includes gauge fixing terms of the form:

$$-\frac{1}{2}g^{\rho\sigma}\left(D^{\mu}h_{\mu\rho}-\frac{1}{2}D_{\rho}\,h^{\mu}{}_{\mu}\right)\left(D^{\nu}\,h_{\nu\sigma}-\frac{1}{2}D_{\sigma}h^{\nu}{}_{\nu}\right)-\frac{1}{2}D^{\mu}\mathcal{A}^{(a)}_{\mu}D^{\nu}\mathcal{A}^{(a)}_{\nu}+\frac{1}{4}\bar{\psi}_{\mu}\Gamma^{\mu}\Gamma^{\nu}D_{\nu}\Gamma^{\rho}\psi_{\rho}\,. \tag{4.12}$$

Gauge fixing also leads to a set of ghost fields. Let us denote by b_μ and c_μ the ghosts associated with diffeomorphism invariance, by $b^{(a)}$ and $c^{(a)}$ the ghosts associated with the U(1) gauge invariances, and by $\widetilde{b}$, $\widetilde{c}$ the ten dimensional left-handed Majorana-Weyl bosonic ghosts associated with local supersymmetry. Quantization of the gravitino also requires the introduction of a third ten dimensional right-handed Majorana-Weyl bosonic ghost field which we shall denote by $\tilde{e}$.[11] Then the total ghost action is given by [46]

$$\mathcal{L}_{\text{ghost}}=\left[b^{\mu}\left(g_{\mu\nu}\Box+R_{\mu\nu}\right)c^{\nu}+b^{(a)}\Box c^{(a)}-2\,b^{(a)}\bar{F}^{a}_{\mu\nu}\,D^{\mu}c^{\nu}\right]+\bar{\tilde{b}}\,\Gamma^{\mu}D_{\mu}\tilde{c}+\bar{\tilde{e}}\,\Gamma^{\mu}D_{\mu}\tilde{e}\,. \tag{4.13}$$

Our goal will be to compute the one loop contribution to $\mathcal{L}_{\text{eff}}$ due to these fields and use this to compute the correction to the black hole entropy.

5 Contribution from the integer spin fields

In this section we shall compute the contribution to the heat kernel due to the gravity multiplet fields of integer spin — both physical fields and the ghosts. We begin with the physical bosonic fields which include the fluctuations $h_{\mu\nu}$, $\mathcal{A}^{(a)}_{\mu}$ for $1\le a\le 6$ and the scalar fields χ_1 and χ_2. From the structure of $\mathcal{L}_b$ given in (4.2) we see that the fields $\mathcal{A}^{(a)}_{\mu}$ for $3\le a\le 6$ are not affected by the presence of the background flux. Hence their contribution to the heat kernel is given by that of four regular vector fields in $AdS_2\times S^2$. This was computed in [46], but we shall review the analysis since the method it uses will be of use for other fields as well. As explained in section 3, the general strategy is to express various fields as derivatives of scalar fields and then express the scalar fields as linear combinations of complete set eigenstates of the $-\Box_{S^2}$ and $-\Box_{AdS_2}$ operator. For example we can write[12]

$$\mathcal{A}^{(a)}_{\alpha}=\sum_k\left[\frac{1}{\sqrt{\kappa^{(k)}_1}}\left(P^{(k)}_a\partial_\alpha\,u_k+Q^{(k)}_a\varepsilon_{\alpha\beta}\partial^\beta\,u_k\right)\right],$$

[11] This comes from the special nature of the gauge fixing term given in (4.12); to get this term we first insert into the path integral the gauge fixing term $\delta(\Gamma^\mu\psi_\mu-\xi(x))$ for some arbitrary space-time dependent spinor $\xi(x)$; and then average over all $\xi(x)$ with a weight factor of $\exp(-\int\sqrt{\det g}\,\bar{\xi}\not{D}\xi)$. The integration over ξ introduces an extra factor of $\det\not{D}$ which needs to be canceled by an additional spin half bosonic ghost with the standard kinetic operator proportional to $\not{D}$.

[12] Note that we are pretending that the eigenvalues are discrete whereas in reality the eigenvalues of $-\Box_{AdS_2}$ are continuous and hence the u's are delta function normalized. But this does not affect the diagonalization of the kinetic operator.

$$\mathcal{A}_m^{(a)} = \sum_k \left[\frac{1}{\sqrt{\kappa_2^{(k)}}} \left(R_a^{(k)} \partial_m u_k + S_a^{(k)} \varepsilon_{mn} \partial^n u_k \right) \right], \quad \text{for } 3 \le a \le 6\,, \tag{5.1}$$

where $\{u_k\}$ are a complete set of scalar functions with eigenvalue $\kappa_1^{(k)} = l(l+1)/a^2$ of $-\Box_{S^2}$ and $\kappa_2^{(k)} = \lambda^2 + \frac{1}{4}$ of $-\Box_{AdS_2}$ and $P_a^{(k)}$'s, $Q_a^{(k)}$'s, $R_a^{(k)}$'s, and $S_a^{(k)}$'s, are constants. Upon substiting (5.1) into (4.2), and integrating over $AdS_2 \times S^2$, we shall get an expression quadratic in the coefficients P, Q, R, S. Orthonormality of the u_k's guarantee that the quadratic term is block diagonal, with different blocks labelled by different k, i.e. different (l, λ). Thus for each (l, λ) we shall have a finite dimensional matrix to diagonalize. If we denote the eigenvalues of this matrix by $\gamma_i(l, \lambda)$, then the net contribution to the heat kernel will be given by

$$\frac{1}{8\pi^2 a^4} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda) \sum_{l=0}^\infty (2l+1) \sum_i e^{-\bar{s}\gamma_i(l,\lambda)} + K_{\text{discrete}}\,, \tag{5.2}$$

where K_{discrete} denotes the contribution from the discrete modes given by the product of $Y_{lm}(\psi, \phi)$ with (2.6). This can be computed in a similar way using the fact that the discrete modes of each vector gives a contribution of $1/2\pi a^2$ to the AdS_2 heat kernel. The corresponding contribution will involve only a sum over l but no integration over λ. In order to avoid proliferation of indices we shall from now on work in a fixed k sector and drop the superscript k and the subscript a from all subsequent formulæ. Then the part of the action involving the coefficients P, Q, R, S is given by

$$-\frac{1}{2}(\kappa_1 + \kappa_2)(P^2 + Q^2 + R^2 + S^2)\,. \tag{5.3}$$

For $\kappa_1 = 0$, i.e. for $l = 0$ the modes P and Q are absent since the corresponding u is constant on S^2 and hence $\partial_\alpha u$ vanishes. Thus we have four eigenvalues of the form $\kappa_1 + \kappa_2 = [\lambda^2 + (l + \frac{1}{2})^2]/a^2$ for $l \ge 1$ and two eigenvalues of the form $\lambda^2 + \frac{1}{4}$. Finally for $\kappa_2 = 0$, i.e. $\lambda = i/2$ we have some additional discrete modes. The net contribution from all these modes to the trace of the heat kernel is given by:

$$\frac{4}{8\pi^2 a^4} \left[e^{-\bar{s}/4} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, e^{-\bar{s}\lambda^2} \sum_{l=0}^\infty (2l+1)\, e^{-\bar{s}l(l+1)}\, (4 - 2\delta_{l,0}) + \sum_{l=0}^\infty (2l+1)\, e^{-\bar{s}l(l+1)} \right]. \tag{5.4}$$

The last term without an integration over λ represents the contribution from the discrete modes.

We now turn to the rest of the physical bosonic fields which include the gauge fields $\mathcal{A}_\mu^{(a)}$ for $a = 1, 2$, the graviton $h_{\mu\nu}$ and the scalars χ_1 and χ_2. The analysis proceeds in a similar manner by expanding various fields in a basis obtained from derivatives of u_k. As before we work in a fixed k sector and drop the index k since there is no mixing between sectors with different k. We take the following expansion for different fields:

$$\mathcal{A}_\alpha^{(1)} = \frac{1}{\sqrt{\kappa_1}} \left(C_1 \partial_\alpha u + C_2 \varepsilon_{\alpha\beta} \partial^\beta u \right), \qquad \mathcal{A}_m^{(1)} = \frac{1}{\sqrt{\kappa_2}} \left(C_3 \partial_m u + C_4 \varepsilon_{mn} \partial^n u \right),$$

$$
\begin{aligned}
\mathcal{A}^{(2)}_\alpha &= \frac{1}{\sqrt{\kappa_1}}\left(C_5\partial_\alpha\, u + C_6\varepsilon_{\alpha\beta}\partial^\beta\, u\right)\,, \qquad \mathcal{A}^{(2)}_m = \frac{1}{\sqrt{\kappa_2}}\left(C_7\partial_m\, u + C_8\varepsilon_{mn}\partial^n\, u\right)\,,\\
h_{m\alpha} &= \frac{1}{\sqrt{\kappa_1\kappa_2}}\left(B_1\,\partial_\alpha\partial_m\, u + B_2\,\varepsilon_{mn}\partial_\alpha\partial^n u + B_3\,\varepsilon_{\alpha\beta}\,\partial^\beta\partial_m u + B_4\,\varepsilon_{\alpha\beta}\,\varepsilon_{mn}\,\partial^\beta\partial^n u\right)\,,\\
h_{\alpha\beta} &= \frac{1}{\sqrt{2}}\left(i\,B_5 + B_6\right) g_{\alpha\beta}\, u + \frac{1}{\sqrt{\kappa_1 - 2a^{-2}}}\left(D_\alpha\xi_\beta + D_\beta\xi_\alpha - g_{\alpha\beta}\, D^\gamma\xi_\gamma\right)\,,\\
h_{mn} &= \frac{1}{\sqrt{2}}\left(i\,B_5 - B_6\right) g_{mn}\, u + \frac{1}{\sqrt{\kappa_2 + 2a^{-2}}}\left(D_m\widehat{\xi}_n + D_n\widehat{\xi}_m - g_{mn}\, D^p\widehat{\xi}_p\right)\,,\\
\xi_\alpha &= \frac{1}{\sqrt{\kappa_1}}\left(B_7\partial_\alpha\, u + B_8\,\varepsilon_{\alpha\beta}\partial^\beta\, u\right)\,, \qquad \widehat{\xi}_m = \frac{1}{\sqrt{\kappa_2}}\left(B_9\partial_m\, u + B_0\,\varepsilon_{mn}\partial^n\, u\right)\,,\\
\chi_1 &= C_9 u, \qquad \chi_2 = C_0 u\,,
\end{aligned}
\tag{5.5}
$$

where $B_0, \cdots B_9$ and $C_0, \cdots C_9$ are arbitrary coefficients. The normalizations of the coefficients have been chosen such that the deformations parametrized by individual coefficients are correctly normalized and the deformations parametrized by different coefficients have vanishing inner product. For reasons to be explained later we shall first consider deformations associated with u_k's for which $\kappa^{(1)}_k > 2a^{-2}$ (i.e. $l > 1$). The need for restricting to modes with $\kappa_1 > 2a^{-2}$ is clear from the denominator factor of $\kappa_1 - 2a^{-2}$ in the expansion (5.5) of $h_{\alpha\beta}$. We also exclude all the discrete modes on AdS_2 corresponding to $\kappa^{(2)}_k = 0$ from the initial analysis; they will be incorporated later. Note the i multiplying B_5, — we have taken into account that the conformal factor of the metric, parametrized by B_5, has wrong sign kinetic term, and hence must be rotated to lie along the imaginary axis to make the path integral well defined. Substituting (5.5) into (4.2) and integrating over $AdS_2 \times S^2$ using the orthonormality of the basis states we get the contribution to the action from the $\kappa_1 > 2a^{-2}$, $\kappa_2 > 0$ modes to be

$$
\begin{aligned}
&-\frac{1}{2}(\kappa_1+\kappa_2)\left[\sum_{i=0}^{9} C_i^2 + \sum_{i=1}^{6} B_i^2\right] - \frac{1}{2}(\kappa_1+\kappa_2 - 4\,a^{-2})(B_7^2+B_8^2)\\
&-\frac{1}{2}(\kappa_1+\kappa_2+4a^{-2})(B_9^2+B_0^2)\\
&+a^{-2}\,(B_9^2+B_0^2) - a^{-2}\,(B_7^2+B_8^2) - 2\,i\,a^{-2}B_5B_6 - 2\sqrt{2}\,a^{-2}B_6C_0\\
&+i\sqrt{2}a^{-1}\left[-\sqrt{\kappa_1}C_3B_2 + \sqrt{\kappa_1}C_4B_1 + \sqrt{\kappa_2}C_1B_2 + \sqrt{\kappa_2}C_2B_4\right]\\
&+\sqrt{2}a^{-1}\left[-\sqrt{\kappa_1}C_7B_3 - \sqrt{\kappa_1}C_8B_4 + \sqrt{\kappa_2}C_5B_3 - \sqrt{\kappa_2}C_6B_1\right]\\
&+i\sqrt{2}a^{-1}\sqrt{\kappa_2}\,(-C_0+\sqrt{2}B_6)C_4 + \sqrt{2}a^{-1}\sqrt{\kappa_1}(C_0+\sqrt{2}B_6)C_6\\
&+\sqrt{2}a^{-1}(\sqrt{\kappa_1}C_2 + i\sqrt{\kappa_2}C_8)C_9\,.
\end{aligned}
\tag{5.6}
$$

This needs to be further integrated over λ and summed over l to get the full action but we shall work in a sector with fixed l and λ as before.

We can now diagonalize the kinetic operator by analyzing various blocks. First of all note that B_7, B_8, B_9 and B_0 do not have any cross terms. Hence the eigenvalues in these sectors can be read out immediately from (5.6). We get

$$
\begin{aligned}
B_7, B_8 : &\qquad \kappa_1+\kappa_2 - 2\,a^{-2}\,,\\
B_9, B_0 : &\qquad \kappa_1+\kappa_2 + 2\,a^{-2}\,.
\end{aligned}
\tag{5.7}
$$

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Next we note that the parameters B_2, C_3 and C_1 mix among themselves but do not mix with any parameter outside this set. In this three dimensional subspace the kinetic operator takes the form

$$\begin{pmatrix} \kappa_1+\kappa_2 & ia^{-1}\sqrt{2\kappa_1} & -ia^{-1}\sqrt{2\kappa_2} \\ ia^{-1}\sqrt{2\kappa_1} & \kappa_1+\kappa_2 & 0 \\ -ia^{-1}\sqrt{2\kappa_2} & 0 & \kappa_1+\kappa_2 \end{pmatrix}. \tag{5.8}$$

Diagonalizing this matrix we find the eigenvalues in this sector to be

$$\kappa_1+\kappa_2, \quad \kappa_1+\kappa_2 \pm ia^{-1}\sqrt{2(\kappa_1+\kappa_2)}\,. \tag{5.9}$$

Similarly we find from (5.6) that the parameters B_3, C_7, C_5 mix among themselves but do not mix with any parameters outside this set. In this subspace the kinetic operator takes the form:

$$\begin{pmatrix} \kappa_1+\kappa_2 & a^{-1}\sqrt{2\kappa_1} & -a^{-1}\sqrt{2\kappa_2} \\ a^{-1}\sqrt{2\kappa_1} & \kappa_1+\kappa_2 & 0 \\ -a^{-1}\sqrt{2\kappa_2} & 0 & \kappa_1+\kappa_2 \end{pmatrix}. \tag{5.10}$$

The eigenvalues of this matrix are given by

$$\kappa_1+\kappa_2, \quad \kappa_1+\kappa_2 \pm a^{-1}\sqrt{2(\kappa_1+\kappa_2)}\,. \tag{5.11}$$

The parameters B_4, C_2, C_8, C_9 mix among themselves but do not mix with any other parameter. In this four dimensional subspace the kinetic operator is given by

$$\begin{pmatrix} \kappa_1+\kappa_2 & -ia^{-1}\sqrt{2\kappa_2} & a^{-1}\sqrt{2\kappa_1} & 0 \\ -ia^{-1}\sqrt{2\kappa_2} & \kappa_1+\kappa_2 & 0 & -a^{-1}\sqrt{2\kappa_1} \\ a^{-1}\sqrt{2\kappa_1} & 0 & \kappa_1+\kappa_2 & -ia^{-1}\sqrt{2\kappa_2} \\ 0 & -a^{-1}\sqrt{2\kappa_1} & -ia^{-1}\sqrt{2\kappa_2} & \kappa_1+\kappa_2 \end{pmatrix}. \tag{5.12}$$

The eigenvalues are

$$\begin{aligned} &\kappa_1+\kappa_2+a^{-1}\sqrt{2(\kappa_1-\kappa_2)}, \quad \kappa_1+\kappa_2+a^{-1}\sqrt{2(\kappa_1-\kappa_2)}, \\ &\kappa_1+\kappa_2-a^{-1}\sqrt{2(\kappa_1-\kappa_2)}, \quad \kappa_1+\kappa_2-a^{-1}\sqrt{2(\kappa_1-\kappa_2)}\,. \end{aligned} \tag{5.13}$$

Finally the remaining parameters B_1, C_4, C_6, C_0, B_6, B_5 all mix among themselves and produce a kinetic operator

$$\begin{pmatrix} \kappa_1+\kappa_2 & -ia^{-1}\sqrt{2\kappa_1} & a^{-1}\sqrt{2\kappa_2} & 0 & 0 & 0 \\ -ia^{-1}\sqrt{2\kappa_1} & \kappa_1+\kappa_2 & 0 & ia^{-1}\sqrt{2\kappa_2} & -2ia^{-1}\sqrt{\kappa_2} & 0 \\ a^{-1}\sqrt{2\kappa_2} & 0 & \kappa_1+\kappa_2 & -a^{-1}\sqrt{2\kappa_1} & -2a^{-1}\sqrt{\kappa_1} & 0 \\ 0 & ia^{-1}\sqrt{2\kappa_2} & -a^{-1}\sqrt{2\kappa_1} & \kappa_1+\kappa_2 & 2\sqrt{2}\,a^{-2} & 0 \\ 0 & -2ia^{-1}\sqrt{\kappa_2} & -2a^{-1}\sqrt{\kappa_1} & 2\sqrt{2}\,a^{-2} & \kappa_1+\kappa_2 & 2ia^{-2} \\ 0 & 0 & 0 & 0 & 2ia^{-2} & \kappa_1+\kappa_2 \end{pmatrix}. \tag{5.14}$$

We shall denote the eigenvalues of this matrix by

$$\kappa_1+\kappa_2+a^{-2}f_i(l,\lambda), \qquad 1\le i\le 6\,, \quad \kappa_1\equiv l(l+1)/a^2, \quad \kappa_2\equiv\frac{1}{4a^2}+\frac{\lambda^2}{a^2}\,. \tag{5.15}$$

For $\kappa_1 = 2a^{-2}$ (i.e. $l = 1$) the modes parametrized by B_7 and B_8 are absent since the vectors $\partial_\alpha u$ and $\varepsilon_{\alpha\beta}\partial^\beta u$ are the conformal Killing vectors of S^2 and hence do not generate any deformation of the metric. The rest of the modes are not affected. Thus we get the same set of eigenvalues except the ones given in the first line of (5.7). The net contribution to the heat kernel from the $l \geq 1$ modes is then given by

$$\frac{1}{8\pi^2 a^4}\, e^{-\bar{s}/4} \int_0^\infty d\lambda\, \lambda\, \tanh(\pi\lambda)\; e^{-\bar{s}\lambda^2} \Bigg[\sum_{l=1}^\infty (2l+1)\, e^{-\bar{s}l(l+1)} \Big\{ 2 + 2e^{2\bar{s}} + 2e^{-2\bar{s}} - 2\, e^{2\bar{s}}\delta_{l,1} \\ + e^{i\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{-i\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{-\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} \\ + 2\, e^{\bar{s}\sqrt{2l(l+1)-2\lambda^2-\frac{1}{2}}} + 2\, e^{-\bar{s}\sqrt{2l(l+1)-2\lambda^2-\frac{1}{2}}} + \sum_{i=1}^{6} e^{-\bar{s} f_i(l,\lambda)} \Big\} \Bigg]\,. \tag{5.16}$$

For $\kappa_1 = 0$ (i.e. $l = 0$) the function u is a constant on S^2, and as a result all the modes which involve a derivative with respect to a coordinate of S^2 are absent. This will require us to set to zero the modes corresponding to C_1, C_2, C_5, C_6, B_1, B_2, B_3, B_4, B_7 and B_8. The net contribution to the action from the rest of the modes is given by

$$-\frac{1}{2}\kappa_2 \sum_{i=0,3,4,7,8,9} C_i^2 - \frac{1}{2}\kappa_2 \sum_{i=5}^{6} B_i^2 - \frac{1}{2}(\kappa_2 + 2a^{-2})(B_9^2 + B_0^2) - 2\sqrt{2}\, a^{-2}\, B_6 C_0 \\ + i\sqrt{2}a^{-1}\sqrt{\kappa_2}\,(-C_0 + \sqrt{2}B_6)C_4 + a^{-1} i\sqrt{2\kappa_2}C_8 C_9 - 2\, i\, a^{-2} B_5 B_6\,. \tag{5.17}$$

Since C_3, C_7, B_9 and B_0 do not mix with other fields, they produce the following eigenvalues of the kinetic operator:

$$\kappa_2, \quad \kappa_2, \quad \kappa_2 + 2\, a^{-2}, \quad \kappa_2 + 2\, a^{-2}\,. \tag{5.18}$$

C_8 and C_9 mix with each other but not with others, producing eigenvalues:

$$\kappa_2 \pm i\, a^{-1}\, \sqrt{2\kappa_2}\,. \tag{5.19}$$

Finally C_4, C_0, B_6, B_5 mix with each other producing the matrix:

$$\begin{pmatrix} \kappa_2 & ia^{-1}\sqrt{2\kappa_2} & -2ia^{-1}\sqrt{\kappa_2} & 0 \\ ia^{-1}\sqrt{2\kappa_2} & \kappa_2 & 2\sqrt{2}\, a^{-2} & 0 \\ -2ia^{-1}\sqrt{\kappa_2} & 2\sqrt{2}\, a^{-2} & \kappa_2 & 2ia^{-2} \\ 0 & 0 & 2ia^{-2} & \kappa_2 \end{pmatrix}. \tag{5.20}$$

We shall denote the eigenvalues of this matrix by

$$\kappa_2 + a^{-2}\, g_i(\lambda), \quad 1 \leq i \leq 4, \quad \kappa_2 \equiv \frac{1}{4a^2} + \frac{\lambda^2}{a^2}\,. \tag{5.21}$$

Thus the net contribution to the heat kernel from the $l = 0$ modes is given by

$$\frac{1}{8\pi^2 a^4}\, e^{-\bar{s}/4} \int_0^\infty d\lambda\, \lambda\, \tanh(\pi\lambda)\; e^{-\bar{s}\lambda^2} \left[2 + 2\, e^{-2\bar{s}} + e^{i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + e^{-i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + \sum_{i=1}^{4} e^{-\bar{s}\, g_i(\lambda)} \right]. \tag{5.22}$$

We can combine the contributions (5.16) and (5.22) as follows. We first extend the sum in (5.16) all the way to $l = 0$ and subtract explicitly the extra contribution due to the $l = 0$ terms. This includes in particular the terms involving $f_i(0, \lambda)$. Now it is easy to see that for $l = 0$, i.e. $\kappa_1 = 0$, the 6×6 matrix given in (5.14) takes a block diagonal form, with B_1 and C_6 forming a 2×2 block with eigenvalues $\kappa_2 \pm a^{-1}\sqrt{2\kappa_2}$, and C_4, C_0, B_6, B_5 forming a 4×4 block that is identical to the matrix given in (5.20). Thus the corresponding $f_i(0, \lambda)$'s coincide with $\kappa_2 \pm a^{-1}\sqrt{2\kappa_2}$ and the four $g_i(\lambda)$'s. Using this result we can express the sum of (5.16) and (5.22) as

$$\begin{aligned}
&\frac{1}{8\pi^2 a^4}\, e^{-\bar{s}/4} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, e^{-\bar{s}\lambda^2} \Bigg[\sum_{l=0}^{\infty} (2l+1)\, e^{-\bar{s}l(l+1)} \bigg\{ 2 + 2e^{2\bar{s}} + 2e^{-2\bar{s}} \\
&\quad + e^{i\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{-i\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{-\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} \\
&\quad + 2\, e^{\bar{s}\sqrt{2l(l+1)-2\lambda^2-\frac{1}{2}}} + 2\, e^{-\bar{s}\sqrt{2l(l+1)-2\lambda^2-\frac{1}{2}}} + \sum_{i=1}^{6} e^{-\bar{s} f_i(l,\lambda)} \bigg\} \Bigg] \\
&- \frac{1}{8\pi^2 a^4}\, e^{-\bar{s}/4} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, e^{-\bar{s}\lambda^2} \Bigg[6 + 2e^{2\bar{s}} + 2e^{i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{-i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} \\
&\quad + 2e^{\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{-\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} \Bigg] . \qquad (5.23)
\end{aligned}$$

Finally we need to consider the discrete modes associated with square integrable wave-functions of various fields on AdS_2. These involve the discrete modes of the vector fields on AdS_2 described in (2.6), and also the discrete modes of the symmetric rank two tensor on AdS_2 described in (2.9). We can take the product of these modes with any mode of S^2 to describe deformations of vector and symmetric rank 2 tensors on $AdS_2 \times S^2$. Let us denote by $\{v_m^{(k)}, \varepsilon_{mn} v^{(k)n}\}$ a real basis of vector fields obtained from the product of the real and imaginary parts of (2.6) and a spherical harmonic on S^2 with eigenvalue $\kappa_1^{(k)}$ of $-\Box_{S^2}$ and by $w_{mn}^{(k)}$ a real basis for symmetric rank two tensors obtained from the product of real and imaginary parts of (2.9) and a spherical harmonic on S^2 with eigenvalue $\kappa_1^{(k)}$ of $-\Box_{S^2}$. Note that the eigenvalues of $\Box_{AdS_2}$ are already fixed for these modes; so we do not need to specify them. We shall choose $w_{mn}^{(k)}$ and $v_m^{(k)}$ to be real. As before we shall drop the superscript (k) and consider the following deformations for each k:

$$\begin{aligned}
\mathcal{A}_m^{(1)} &= E_1 v_m + \widetilde{E}_1 \varepsilon_{mn} v^n, \quad \mathcal{A}_m^{(2)} = E_2 v_m + \widetilde{E}_2 \varepsilon_{mn} v^n, \\
h_{m\alpha} &= \frac{1}{\sqrt{\kappa_1}} \left(E_3 \partial_\alpha v_m + \widetilde{E}_3 \varepsilon_{mn} \partial_\alpha v^n + E_4 \varepsilon_{\alpha\beta} \partial^\beta v_m + \widetilde{E}_4 \varepsilon_{\alpha\beta} \varepsilon_{mn} \partial^\beta v^n \right) \\
h_{mn} &= \frac{a}{\sqrt{2}} \left(D_m \widehat{\xi}_n + D_n \widehat{\xi}_m - g_{mn} D^p \widehat{\xi}_p \right), \quad \widehat{\xi}_m = E_5 v_m + \widetilde{E}_5 \varepsilon_{mn} v^n, \qquad (5.24)
\end{aligned}$$

and

$$h_{mn} = E_6 w_{mn} . \qquad (5.25)$$

Note that for $\kappa_1 = 0$ the modes $E_3, \widetilde{E}_3, E_4, \widetilde{E}_4$ are absent; this will be taken care of in the computation. These parameters describe a set of orthonormal deformations as long as the v_m and w_{mn} are correctly normalized. Also orthonormality of the various modes on AdS_2 guarantee that the modes given in (5.24), (5.25) do not mix with each other and the modes analyzed earlier. Substituting the modes given in (5.24) into (4.2) we arrive at the following contribution to the action from these modes

$$-\frac{1}{2}\kappa_1 \sum_{i=1}^{4}(E_i^2+\widetilde{E}_i^2)-\frac{1}{2}\left(\kappa_1+2a^{-2}\right)(E_5^2+\widetilde{E}_5^2)-a^{-1}\sqrt{2\kappa_1}\,(iE_1\widetilde{E}_3-i\widetilde{E}_1E_3+E_2E_4+\widetilde{E}_2\widetilde{E}_4)\,. \tag{5.26}$$

The ten eigenvalues of the kinetic operator are

$$\begin{aligned}&\kappa_1+2\,a^{-2}, \quad \kappa_1+2\,a^{-2}, \quad \kappa_1\pm a^{-1}\sqrt{2\kappa_1}, \quad \kappa_1\pm a^{-1}\sqrt{2\kappa_1},\\ &\kappa_1\pm i\,a^{-1}\sqrt{2\kappa_1}, \quad \kappa_1\pm i\,a^{-1}\sqrt{2\kappa_1}\,.\end{aligned} \tag{5.27}$$

Note that for $\kappa_1 = 2/a^2$, i.e. for $l = 1$ we have a pair of zero eigenvalues. Physically these arise due to the fact that the dimensional reduction of the metric on S^2 produces a massless SU(2) gauge field on AdS_2, and these, like the U(1) gauge fields, have zero modes on AdS_2. For $\kappa_1 = 0$ the modes E_3, $\widetilde{E}_3$, E_4 and $\widetilde{E}_4$ are absent and we get six eigenvalues

$$0, \quad 0, \quad 0, \quad 0, \quad 2a^{-2}, \quad 2a^{-2}\,. \tag{5.28}$$

Finally the modes described in (5.25) does not mix with anything and describes a mode with eigenvalue κ_1 of the kinetic operator, leading to a contribution

$$-\frac{1}{2}\,\kappa_1\,E_6^2 \tag{5.29}$$

to the action. Combining these results and recalling the coefficient of the contribution from the discrete modes of AdS_2 given in (3.24) ($1/2\pi a^2$ for the discrete mode of the vector[13] and $3/2\pi a^2$ for the discrete mode of the symmetric rank two tensor) we get the net contribution from the discrete modes to be:

$$\begin{aligned}&\frac{1}{8\pi^2a^4}\left[\sum_{l=1}^{\infty}e^{-\bar{s}l(l+1)}\,(2l+1)\left\{e^{-2\bar{s}}+e^{-\bar{s}\sqrt{2l(l+1)}}+e^{\bar{s}\sqrt{2l(l+1)}}+e^{-i\bar{s}\sqrt{2l(l+1)}}+e^{i\bar{s}\sqrt{2l(l+1)}}+3\right\}\right.\\ &\qquad\left.+2+e^{-2\bar{s}}+3\right]\\ &=\frac{1}{8\pi^2a^4}\left[\sum_{l=0}^{\infty}e^{-\bar{s}l(l+1)}(2l+1)\left\{3+e^{-2\bar{s}}+e^{-\bar{s}\sqrt{2l(l+1)}}+e^{\bar{s}\sqrt{2l(l+1)}}+e^{-i\bar{s}\sqrt{2l(l+1)}}+e^{i\bar{s}\sqrt{2l(l+1)}}\right\}-2\right].\end{aligned} \tag{5.30}$$

The $2+e^{2\bar{s}}$ in the second line represents the contribution to the heat kernel from the product of the discrete modes of the vector field with $l = 0$ mode on S^2 (i.e. with eigenvalues given in (5.28)) and the 3 represents the contribution from the modes of the metric given by the product of $l = 0$ modes in S^2 and w_{mn} in AdS_2.

[13] In carrying out this computation we have to take into account the fact that we have chosen a real basis in which v_m and $\varepsilon_{mn}v^n$ are independent vectors. This gives an extra factor of $\frac{1}{2}$, leading to a contribution of $1/4\pi a^2$ per mode from AdS_2. For example the zero modes of a free gauge field which does not couple to the background flux will be described by the modes E_1 and $\widetilde{E}_1$ with action $\kappa_1(E_1^2+\widetilde{E}_1^2)$, and together they will give a contribution of $1/2\pi a^2$ from the AdS_2 part.

We must also include in our list of bosonic fields in the gravity multiplet the ghosts which arise during the gauge fixing of the six U(1) gauge groups and the diffeomorphism group. The Lagrangian density for the ghost fields has been given in (4.13). In particular the kinetic term has the form:

$$- \begin{pmatrix} b^\mu & b^{(a)} \end{pmatrix} \begin{pmatrix} -g_{\mu\nu}\Box - R_{\mu\nu} & 0 \\ 2\,\bar{F}^a_{\rho\nu}\,D^\rho & -\Box \end{pmatrix} \begin{pmatrix} c^\nu \\ c^{(a)} \end{pmatrix} . \tag{5.31}$$

Since this has a lower triangular form, the off diagonal term does not affect the eigenvalues. Thus the scalar ghosts have the standard kinetic operator $-\Box$ and the twelve scalar ghosts arising from U(1) gauge invariance gives a contribution $-12K^s(0;s)$:

$$-12\,\frac{1}{8\pi^2 a^4}\,e^{-\bar{s}/4}\int_0^\infty d\lambda\,\lambda\,\tanh(\pi\lambda)\;e^{-\bar{s}\lambda^2}\left[\sum_{l=0}^\infty (2l+1)\,e^{-\bar{s}l(l+1)}\right]. \tag{5.32}$$

The contribution from the vector ghosts b_μ, c_μ can be analyzed by decomposing them into various modes as e.g. in (5.5) for $\kappa_1 > 0$:

$$\begin{aligned} b_\alpha &= A\frac{1}{\sqrt{\kappa_1}}\,\partial_\alpha u + B\frac{1}{\sqrt{\kappa_1}}\,\varepsilon_{\alpha\beta}\partial^\beta u, \\ b_m &= C\frac{1}{\sqrt{\kappa_2}}\,\partial_m u + D\frac{1}{\sqrt{\kappa_2}}\,\varepsilon_{mn}\partial^n u, \\ c_\alpha &= E\frac{1}{\sqrt{\kappa_1}}\,\partial_\alpha u + F\frac{1}{\sqrt{\kappa_1}}\,\varepsilon_{\alpha\beta}\partial^\beta u, \\ c_m &= G\frac{1}{\sqrt{\kappa_2}}\,\partial_m u + H\frac{1}{\sqrt{\kappa_2}}\,\varepsilon_{mn}\partial^n u\,. \end{aligned} \tag{5.33}$$

Substituting this into (5.31) we get the following action:

$$(\kappa_1 + \kappa_2 - 2a^{-2})(AE + BF) + (\kappa_1 + \kappa_2 + 2a^{-2})(CG + DH)\,. \tag{5.34}$$

This has four eigenvalues of magnitude $(\kappa_1 + \kappa_2 - 2a^{-2})$ and four eigenvalues of magnitude $(\kappa_1 + \kappa_2 + 2a^{-2})$. For $\kappa_1 = 0$ i.e. $l = 0$ the modes corresponding to A, B, E, F are missing and we get the action to be

$$(\kappa_2 + 2a^{-2})(CG + DH)\,. \tag{5.35}$$

This has four eigenvalues of magnitude $(\kappa_2 + 2a^{-2})$. The net contribution to the trace of the heat kernel from these modes is

$$\begin{aligned} &-\frac{1}{8\pi^2 a^4}\,e^{-\bar{s}/4}\int_0^\infty d\lambda\,\lambda\,\tanh(\pi\lambda)\;e^{-\bar{s}\lambda^2}\left[\sum_{l=1}^\infty (2l+1)\,e^{-\bar{s}l(l+1)}\left\{4e^{-2\bar{s}} + 4e^{2\bar{s}}\right\} + 4e^{-2\bar{s}}\right] \\ &= -\frac{1}{8\pi^2 a^4}\,e^{-\bar{s}/4}\int_0^\infty d\lambda\,\lambda\,\tanh(\pi\lambda)\;e^{-\bar{s}\lambda^2}\left[\sum_{l=0}^\infty (2l+1)\,e^{-\bar{s}l(l+1)}\left\{4e^{-2\bar{s}} + 4e^{2\bar{s}}\right\} - 4e^{2\bar{s}}\right]. \end{aligned} \tag{5.36}$$

To this we must include the contribution from the additional modes obtained by taking the product of the discrete modes for vector fields on AdS_2 given in (2.6) and the eigenstates of the scalar Laplacian on S^2. These modes have eigenvalues $\kappa_1 + 2a^{-2}$ and hence gives a contribution to the heat kernel of the form:

$$-\frac{1}{4\pi^2 a^4}\sum_{l=0}^{\infty}(2l+1)\, e^{-\bar{s}l(l+1)}e^{-2\bar{s}}\,. \tag{5.37}$$

Adding (5.4), (5.23), (5.30), (5.32), (5.36) and (5.37) we get the following expression for the total heat kernel from the bosonic sector and the scalar and vector ghost fields of the gravity multiplet:

$$\begin{aligned}
K^B_{gravity}(0;s) \;=\;& \frac{1}{8\pi^2 a^4}\, e^{-\bar{s}/4}\int_0^\infty d\lambda\,\lambda\,\tanh(\pi\lambda)\; e^{-\bar{s}\lambda^2}\Bigg[\sum_{l=0}^{\infty}(2l+1)\, e^{-\bar{s}l(l+1)}\\
&\times\Bigg\{6 - 2e^{2\bar{s}} - 2e^{-2\bar{s}}\\
&+e^{i\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{-i\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}} + e^{-\bar{s}\sqrt{2\lambda^2+2l(l+1)+\frac{1}{2}}}\\
&+2\, e^{\bar{s}\sqrt{2l(l+1)-2\lambda^2-\frac{1}{2}}} + 2\, e^{-\bar{s}\sqrt{2l(l+1)-2\lambda^2-\frac{1}{2}}} + \sum_{i=1}^{6} e^{-\bar{s}f_i(l,\lambda)}\Bigg\}\\
&-\Bigg\{14 - 2e^{2\bar{s}} + 2e^{i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{-i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{-\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}}\Bigg\}\Bigg]\\
&+\frac{1}{8\pi^2 a^4}\sum_{l=0}^{\infty}(2l+1)\, e^{-\bar{s}l(l+1)}\Bigg\{7 - e^{-2\bar{s}} + e^{\bar{s}\sqrt{2l(l+1)}} + e^{-\bar{s}\sqrt{2l(l+1)}}\\
&\qquad + e^{i\bar{s}\sqrt{2l(l+1)}} + e^{-i\bar{s}\sqrt{2l(l+1)}}\Bigg\} - \frac{1}{4\pi^2 a^4}\,.
\end{aligned} \tag{5.38}$$

Following the trick leading to (3.11) we can express this as[14]

$$\begin{aligned}
K^B_{gravity}(0;s) \;=\;& \frac{1}{8\pi^2 a^4}\, e^{-\bar{s}/4}\int_0^\infty d\lambda\,\lambda\,\tanh(\pi\lambda)\; e^{-\bar{s}\lambda^2}\Bigg[e^{\bar{s}/4}Im\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\tan(\pi\widetilde{\lambda})\; e^{-\bar{s}\widetilde{\lambda}^2}\\
&\times 2\times\Bigg\{6 - 2e^{2\bar{s}} - 2e^{-2\bar{s}}\\
&+e^{i\bar{s}\sqrt{2\lambda^2+2\widetilde{\lambda}^2}} + e^{-i\bar{s}\sqrt{2\lambda^2+2\widetilde{\lambda}^2}} + e^{\bar{s}\sqrt{2\lambda^2+2\widetilde{\lambda}^2}} + e^{-\bar{s}\sqrt{2\lambda^2+2\widetilde{\lambda}^2}}\\
&+2\, e^{\bar{s}\sqrt{2\widetilde{\lambda}^2-2\lambda^2-1}} + 2\, e^{-\bar{s}\sqrt{2\widetilde{\lambda}^2-2\lambda^2-1}} + \sum_{i=1}^{6} e^{-\bar{s}f_i(\widetilde{\lambda}-\frac{1}{2},\lambda)}\Bigg\}\\
&-\Bigg\{14 - 2e^{2\bar{s}} + 2e^{i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{-i\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}} + 2e^{-\bar{s}\sqrt{2\lambda^2+\frac{1}{2}}}\Bigg\}\Bigg]\\
&+\frac{1}{8\pi^2 a^4}\, e^{\bar{s}/4} Im\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\tan(\pi\widetilde{\lambda})\; e^{-\bar{s}\widetilde{\lambda}^2}\\
&\Bigg\{14 - 2e^{-2\bar{s}} + 2e^{\bar{s}\sqrt{2\widetilde{\lambda}^2-\frac{1}{2}}} + 2e^{-\bar{s}\sqrt{2\widetilde{\lambda}^2-\frac{1}{2}}} + 2e^{i\bar{s}\sqrt{2\widetilde{\lambda}^2-\frac{1}{2}}} + 2e^{-i\bar{s}\sqrt{2\widetilde{\lambda}^2-\frac{1}{2}}}\Bigg\}\\
&-\frac{1}{4\pi^2 a^4}\,.
\end{aligned} \tag{5.39}$$

[14] Note that although the individual terms in the sum have branch points on the real $\widetilde{\lambda}$ axis, the sum of all the terms inside each curly bracket is free from such branch point singularities.

Our goal is to extract the behavior of this expression in the region $a^{-2} \ll \bar{s} \ll 1$ since the logarithmic correction to the entropy from the non-zero modes come from this domain. This is done using the same trick as in section 3. First we expand all terms in (5.39) other than the $e^{-\bar{s}\lambda^2}$ and $e^{-\bar{s}\widetilde{\lambda}^2}$ factors in a power series expansion in $\bar{s}$. The only additional subtlety in this analysis comes from the fact that the eigenvalues $f_i(l,\lambda)$ are not given explicitly. However we can use the expansion

$$\sum_{i=1}^{6} e^{-\bar{s} f_i} = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \bar{s}^n \sum_{i=1}^{6} (f_i)^n \tag{5.40}$$

to reduce the problem to the computation of $\sum_{i=1}^{6}(f_i)^n$. Since f_i/a^2's are the eigenvalues of the matrix M obtained by removing the diagonal $\kappa_1 + \kappa_2$ terms from the matrix given in (5.14), $\sum_{i=1}^{6}(f_i)^n$ is given by $a^{2n} Tr(M^n)$ which can be easily computed. In our analysis we need the result for $n \leq 4$. The results are

$$\begin{gathered}\sum_i f_i = 0, \quad \sum_i (f_i)^2 = 4(2\widetilde{\lambda}^2 - 2\lambda^2 + 1), \quad \sum_i (f_i)^3 = 48(\widetilde{\lambda}^2 + \lambda^2), \\ \sum_i (f_i)^4 = -28 - 112\lambda^2 + 80\lambda^4 + 112\widetilde{\lambda}^2 + 96\lambda^2\widetilde{\lambda}^2 + 80\widetilde{\lambda}^4 \end{gathered} \tag{5.41}$$

This allows us to express the right hand side of (5.39) in terms of products of factors of the form

$$\int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, e^{-\bar{s}\lambda^2}\, \lambda^{2n}, \quad \text{and} \quad Im \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda} \tan(\pi\widetilde{\lambda})\, e^{-\bar{s}\widetilde{\lambda}^2}\, \widetilde{\lambda}^{2n} \tag{5.42}$$

Using eqs. (3.16), (3.17) we can express the right hand side of (5.39) in a power series expansion in $\bar{s}$. We need to compute up to order s^0 term in this expansion for computation of logarithmic correction to the entropy. Collecting all the terms of order s^0 we get

$$K^B_{gravity}(0;s) = -\frac{13}{90\pi^2 a^4} + \frac{2}{3\pi^2 a^4} + \frac{2}{3\pi^2 a^4} - \frac{1}{4\pi^2 a^4} + \cdots = \frac{169}{180\pi^2 a^4} + \cdots, \tag{5.43}$$

where $\cdots$ denote terms proportional to $\bar{s}^{-2}$ and $\bar{s}^{-1}$ as well as positive powers of $\bar{s}$. In the central expression in (5.43) the four terms represent respectively the contributions from the terms inside the three curly brackets in (5.39) and the last term in (5.39).

Finally we need to remove from this the contribution due to the zero modes. To identify the zero modes we can look for the s independent terms in the contribution to the heat kernel from various discrete modes. These consist of the following:

1. The $l = 0$ modes in the last term in (5.4), giving a contribution of $1/2\pi^2 a^4$ to $K(0;s)$.'These represent the zero modes of the four gauge fields $\mathcal{A}_m^{(a)}$ for $3 \leq a \leq 6$.

2. The third term inside $\{\ \}$ in the first line of (5.30) for $l = 1$, giving a contribution of $3/8\pi^2 a^4$. These represent the zero modes of the SU(2) gauge fields arising out of dimensional reduction on S^2.

3. The 2 and 3 in the second line of (5.30). The first one gives a contribution of $2/8\pi^2 a^4$ and represent the zero modes of the two gauge fields $\mathcal{A}_m^{(a)}$ for $1 \le a \le 2$. The second one gives a contribution of $3/8\pi^2 a^4$ and represent the zero modes of the metric associated with the asymptotic symmetries of AdS_2.

Thus the net contribution to $K^B_{gravity}(0;s)$ from all the zero modes is given by

$$\frac{1}{2\pi^2 a^4} + \frac{3}{8\pi^2 a^4} + \frac{1}{4\pi^2 a^4} + \frac{3}{8\pi^2 a^4} = \frac{3}{2\pi^2 a^4} \,. \tag{5.44}$$

Subtracting (5.44) from (5.43) we get the net contribution to the s independent part of $K^B_{gravity}(0;s)$ from the non-zero modes:

$$-\frac{101}{180\pi^2 a^4} \,. \tag{5.45}$$

6 Contribution from the half integer spin fields

Next we must analyze the fermionic contribution to the heat kernel. For this we express the gravity multiplet part of fermionic action given in (4.3) as

$$\mathcal{L}_f = -\frac{1}{2}\left(\bar\Lambda K^{(1)} + \bar\psi^\alpha K^{(2)}_\alpha + \bar\psi^m K^{(3)}_m\right) \,, \tag{6.1}$$

where

$$\begin{aligned}
K^{(1)} &= (\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2})\Lambda + \frac{1}{2\sqrt{2}\,a}(\sigma_3 \widehat\Gamma^4 - i\tau_3\widehat\Gamma^5)(\Gamma^\beta\psi_\beta - \Gamma^n\psi_n)\,, \\
K^{(2)}_\alpha &= -\frac{1}{2\sqrt{2}\,a}\,\Gamma_\alpha\,(\sigma_3\widehat\Gamma^4 - i\tau_3\widehat\Gamma^5)\,\Lambda - \frac{1}{2}\Gamma^n(\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2})\Gamma_\alpha\psi_n \\
&\quad - \left(\frac{1}{2}\Gamma^\beta\left(\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2}\right)\Gamma_\alpha + \frac{i}{2a}\sigma_3\varepsilon_\alpha{}^\beta\left(\sigma_3\widehat\Gamma^4 - i\tau_3\widehat\Gamma^5\right)\right)\psi_\beta \\
K^{(3)}_m &= \frac{1}{2\sqrt{2}\,a}\,\Gamma_m\,(\sigma_3\widehat\Gamma^4 - i\tau_3\widehat\Gamma^5)\,\Lambda - \frac{1}{2}\Gamma^\beta(\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2})\Gamma_m\psi_\beta \\
&\quad + \left(-\frac{1}{2}\Gamma^n\left(\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2}\right)\Gamma_m + \frac{i}{2a}\tau_3\varepsilon_m{}^n\left(\sigma_3\widehat\Gamma^4 - i\tau_3\widehat\Gamma^5\right)\right)\psi_n \,.
\end{aligned} \tag{6.2}$$

Let us denote by $\mathcal{D}$ the differential opeartor such that (6.2) may be expressed as

$$\begin{pmatrix} K^{(1)} \\ K^{(2)}_\alpha \\ K^{(3)}_m \end{pmatrix} = \mathcal{D}\begin{pmatrix} \Lambda \\ \psi_\alpha \\ \psi_m \end{pmatrix} \,. \tag{6.3}$$

Our goal will be to calculate the eigenvalues of $\mathcal{D}$ (or more precisely $\mathcal{D}^2$) since these will appear in the expression of the heat kernel. For this we follow the same strategy as in the bosonic case, i.e. instead of working in the infinite dimensional space of fermionic deformations, we identify finite dimensional subspaces such that modes inside one subspace do not mix with the modes outside this subspace under the action of $\mathcal{D}$. Let us pick one particular basis state χ for the spinor, given by the direct product of (χ^+_{lm} or η^+_{lm}) with

($\chi^+(\lambda)$ or $\eta^+(\lambda)$) defined in (2.16), (2.20), and an arbitrary spinor in the representation of the Clifford algebra generated by $\widehat\Gamma^4,\cdots\widehat\Gamma^9$ carrying $\widehat\Gamma^{45}$ eigenvalue i. Then χ satisfies

$$\widehat\Gamma^{45}\chi = i\chi, \quad \not{D}_{S^2}\chi = i\zeta_1\,\chi, \quad \not{D}_{AdS_2}\chi = i\zeta_2\,\chi, \quad \zeta_1>0, \quad \zeta_2\geq 0\,. \tag{6.4}$$

From this we can derive the identities:

$$\varepsilon_{\alpha\beta}D^\beta\chi = -i\sigma_3 D_\alpha\chi - \zeta_1\sigma_3\Gamma_\alpha\chi\,, \quad \varepsilon_{mn}D^n\chi = -i\tau_3 D_m\chi - \zeta_2\tau_3\sigma_3\Gamma_m\chi\,. \tag{6.5}$$

The set of states χ constructed this way do not form a complete set of basis states since we have left out the states with $\zeta_1<0$ and/or $\zeta_2<0$ and those with $\widehat\Gamma^{45}$ eigenvalue $-i$. We shall overcome the first two problems by including in the basis the states $\sigma_3\chi$, $\tau_3\widehat\Gamma^4\widehat\Gamma^5\chi = i\tau_3\chi$ and $\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\chi = i\sigma_3\tau_3\chi$. Since χ and $\sigma_3\chi$ have opposite $\not{D}_{S^2}$ eigenvalues and χ and $\tau_3\chi$ have opposite $\not{D}_{AdS_2}$ eigenvalues, this amounts to including in the basis states with $\zeta_1<0$ and/or $\zeta_2<0$. To overcome the last problem we add four more states in the basis obtained by acting $\widehat\Gamma^4$ on the states already included. We now consider the subspace consisting of the following fermionic deformations:

$$\begin{aligned}
\Lambda &= a_1\chi + a_2\sigma_3\widehat\Gamma^4\chi + a_3\tau_3\widehat\Gamma^5\chi + a_4\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\chi \\
&\quad +\sigma_3\left[a_1'\chi + a_2'\sigma_3\widehat\Gamma^4\chi + a_3'\tau_3\widehat\Gamma^5\chi + a_4'\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\chi\right]\,, \\
\psi_\alpha &= b_1\Gamma_\alpha\chi + b_2\sigma_3\widehat\Gamma^4\Gamma_\alpha\chi + b_3\tau_3\widehat\Gamma^5\Gamma_\alpha\chi + b_4\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\Gamma_\alpha\chi \\
&\quad +b_5D_\alpha\chi + b_6\sigma_3\widehat\Gamma^4D_\alpha\chi + b_7\tau_3\widehat\Gamma^5D_\alpha\chi + b_8\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5D_\alpha\chi \\
&\quad +\sigma_3\Big[b_1'\Gamma_\alpha\chi + b_2'\sigma_3\widehat\Gamma^4\Gamma_\alpha\chi + b_3'\tau_3\widehat\Gamma^5\Gamma_\alpha\chi + b_4'\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\Gamma_\alpha\chi \\
&\quad +b_5'D_\alpha\chi + b_6'\sigma_3\widehat\Gamma^4D_\alpha\chi + b_7'\tau_3\widehat\Gamma^5D_\alpha\chi + b_8'\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5D_\alpha\chi\Big] \\
\psi_m &= c_1\Gamma_m\chi + c_2\sigma_3\widehat\Gamma^4\Gamma_m\chi + c_3\tau_3\widehat\Gamma^5\Gamma_m\chi + c_4\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\Gamma_m\chi \\
&\quad +c_5\sigma_3\,D_m\chi + c_6\widehat\Gamma^4D_m\chi + c_7\sigma_3\tau_3\widehat\Gamma^5D_m\chi + c_8\tau_3\widehat\Gamma^4\widehat\Gamma^5D_m\chi \\
&\quad +\sigma_3\Big[c_1'\Gamma_m\chi + c_2'\sigma_3\widehat\Gamma^4\Gamma_m\chi + c_3'\tau_3\widehat\Gamma^5\Gamma_m\chi + c_4'\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\Gamma_m\chi \\
&\quad +c_5'\sigma_3D_m\chi + c_6'\widehat\Gamma^4D_m\chi + c_7'\tau_3\sigma_3\widehat\Gamma^5D_m\chi + c_8'\tau_3\widehat\Gamma^4\widehat\Gamma^5D_m\chi\Big]
\end{aligned} \tag{6.6}$$

where a_i's, b_i's, c_i's, a_i''s, b_i''s and c_i''s are arbitrary grassman variables and χ is a fixed spinor satisfying (6.4). We shall see that the action of $\mathcal{D}$ keeps us inside this subspace.

Before we proceed some comments are in order. First note that the basis states used in (6.6) are not orthonormal. As we shall discuss shortly, this will not affect our analysis. Second, due to the relation (2.27) the basis states used in the expansion (6.6) are not all independent for $\zeta_1=1/a$. For this reason we shall for now consider the case $\zeta_1>1/a$. The $\zeta_1=1/a$ case will be analyzed separately. Finally there are additional set of states associated with the discrete modes described in (2.28), — these will also be discussed separately.

Using (6.2), (6.6) and (6.5) we can express $K^{(1)}$, $K^{(2)}$ and $K^{(3)}$ in the form

$$K^{(1)} = A_1\chi + A_2\sigma_3\widehat\Gamma^4\chi + A_3\tau_3\widehat\Gamma^5\chi + A_4\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\chi$$

$$
\begin{aligned}
&+\sigma_3\left[A'_1\chi + A'_2\sigma_3\widehat{\Gamma}^4\chi + A'_3\tau_3\widehat{\Gamma}^5\chi + A'_4\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\chi\right],\\
K^{(2)}_\alpha &= B_1\Gamma_\alpha\chi + B_2\sigma_3\widehat{\Gamma}^4\Gamma_\alpha\chi + B_3\tau_3\widehat{\Gamma}^5\Gamma_\alpha\chi + B_4\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\Gamma_\alpha\chi\\
&+B_5 D_\alpha\chi + B_6\sigma_3\widehat{\Gamma}^4 D_\alpha\chi + B_7\tau_3\widehat{\Gamma}^5 D_\alpha\chi + B_8\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5 D_\alpha\chi\\
&+\sigma_3\Big[B'_1\Gamma_\alpha\chi + B'_2\sigma_3\widehat{\Gamma}^4\Gamma_\alpha\chi + B'_3\tau_3\widehat{\Gamma}^5\Gamma_\alpha\chi + B'_4\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\Gamma_\alpha\chi\\
&+B'_5 D_\alpha\chi + B'_6\sigma_3\widehat{\Gamma}^4 D_\alpha\chi + B'_7\tau_3\widehat{\Gamma}^5 D_\alpha\chi + B'_8\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5 D_\alpha\chi\Big]\\
K^{(3)}_m &= C_1\Gamma_m\chi + C_2\sigma_3\widehat{\Gamma}^4\Gamma_m\chi + C_3\tau_3\widehat{\Gamma}^5\Gamma_m\chi + C_4\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\Gamma_m\chi\\
&+C_5\sigma_3 D_m\chi + C_6\widehat{\Gamma}^4 D_m\chi + C_7\sigma_3\tau_3\widehat{\Gamma}^5 D_m\chi + C_8\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5 D_m\chi\\
&+\sigma_3\Big[C'_1\Gamma_m\chi + C'_2\sigma_3\widehat{\Gamma}^4\Gamma_m\chi + C'_3\tau_3\widehat{\Gamma}^5\Gamma_m\chi + C'_4\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\Gamma_m\chi\\
&+C'_5\sigma_3 D_m\chi + C'_6\widehat{\Gamma}^4 D_m\chi + C'_7\tau_3\sigma_3\widehat{\Gamma}^5 D_m\chi + C'_8\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5 D_m\chi\Big]
\end{aligned}
\tag{6.7}
$$

where[15]

$$
\begin{aligned}
A_1 &= i\zeta_1 a_1 - \frac{b_2}{\sqrt{2}a} - \frac{ib_3}{\sqrt{2}a} - \frac{i\zeta_1 b_6}{2\sqrt{2}a} + \frac{\zeta_1 b_7}{2\sqrt{2}a} - \frac{c_2}{\sqrt{2}a} - \frac{ic_3}{\sqrt{2}a} - \frac{i\zeta_2 c_6}{2\sqrt{2}a} + \frac{\zeta_2 c_7}{2\sqrt{2}a} + i\zeta_2 a'_1\\
A_2 &= -i\zeta_1 a_2 + \frac{b_1}{\sqrt{2}a} - \frac{ib_4}{\sqrt{2}a} + \frac{i\zeta_1 b_5}{2\sqrt{2}a} + \frac{\zeta_1 b_8}{2\sqrt{2}a} - \frac{c_1}{\sqrt{2}a} + \frac{ic_4}{\sqrt{2}a} - \frac{i\zeta_2 c_5}{2\sqrt{2}a} - \frac{\zeta_2 c_8}{2\sqrt{2}a} + i\zeta_2 a'_2\\
A_3 &= i\zeta_1 a_3 - \frac{ib_1}{\sqrt{2}a} - \frac{b_4}{\sqrt{2}a} + \frac{\zeta_1 b_5}{2\sqrt{2}a} - \frac{i\zeta_1 b_8}{2\sqrt{2}a} + \frac{ic_1}{\sqrt{2}a} + \frac{c_4}{\sqrt{2}a} - \frac{\zeta_2 c_5}{2\sqrt{2}a} + \frac{i\zeta_2 c_8}{2\sqrt{2}a} - i\zeta_2 a'_3\\
A_4 &= -i\zeta_1 a_4 - \frac{ib_2}{\sqrt{2}a} + \frac{b_3}{\sqrt{2}a} + \frac{\zeta_1 b_6}{2\sqrt{2}a} + \frac{i\zeta_1 b_7}{2\sqrt{2}a} - \frac{ic_2}{\sqrt{2}a} + \frac{c_3}{\sqrt{2}a} + \frac{\zeta_2 c_6}{2\sqrt{2}a} + \frac{i\zeta_2 c_7}{2\sqrt{2}a} - i\zeta_2 a'_4\\
A'_1 &= i\zeta_2 a_1 - i\zeta_1 a'_1 + \frac{b'_2}{\sqrt{2}a} + \frac{ib'_3}{\sqrt{2}a} + \frac{i\zeta_1 b'_6}{2\sqrt{2}a} - \frac{\zeta_1 b'_7}{2\sqrt{2}a} - \frac{c'_2}{\sqrt{2}a} - \frac{ic'_3}{\sqrt{2}a} - \frac{i\zeta_2 c'_6}{2\sqrt{2}a} + \frac{\zeta_2 c'_7}{2\sqrt{2}a}\\
A'_2 &= i\zeta_2 a_2 + i\zeta_1 a'_2 - \frac{b'_1}{\sqrt{2}a} + \frac{ib'_4}{\sqrt{2}a} - \frac{i\zeta_1 b'_5}{2\sqrt{2}a} - \frac{\zeta_1 b'_8}{2\sqrt{2}a} - \frac{c'_1}{\sqrt{2}a} + \frac{ic'_4}{\sqrt{2}a} - \frac{i\zeta_2 c'_5}{2\sqrt{2}a} - \frac{\zeta_2 c'_8}{2\sqrt{2}a}\\
A'_3 &= -i\zeta_2 a_3 - i\zeta_1 a'_3 + \frac{ib'_1}{\sqrt{2}a} + \frac{b'_4}{\sqrt{2}a} - \frac{\zeta_1 b'_5}{2\sqrt{2}a} + \frac{i\zeta_1 b'_8}{2\sqrt{2}a} + \frac{ic'_1}{\sqrt{2}a} + \frac{c'_4}{\sqrt{2}a} - \frac{\zeta_2 c'_5}{2\sqrt{2}a} + \frac{i\zeta_2 c'_8}{2\sqrt{2}a}\\
A'_4 &= -i\zeta_2 a_4 + i\zeta_1 a'_4 + \frac{ib'_2}{\sqrt{2}a} - \frac{b'_3}{\sqrt{2}a} - \frac{\zeta_1 b'_6}{2\sqrt{2}a} - \frac{i\zeta_1 b'_7}{2\sqrt{2}a} - \frac{ic'_2}{\sqrt{2}a} + \frac{c'_3}{\sqrt{2}a} + \frac{\zeta_2 c'_6}{2\sqrt{2}a} + \frac{i\zeta_2 c'_7}{2\sqrt{2}a}\\
B_1 &= -\frac{a_2}{2\sqrt{2}a} + \frac{ia_3}{2\sqrt{2}a}\\
&\quad -i\zeta_1 b_1 + \frac{1}{2a}b_2 - \frac{i}{2a}b_3 + \left(\widetilde{\zeta}_1^2 - \frac{1}{2}\zeta_1^2 + K\right) b_5 + \frac{i\zeta_1}{2a}b_6 + \frac{\zeta_1}{2a}b_7 + \frac{1}{2}\zeta_1\zeta_2 b'_5\\
&\quad +i\zeta_1 c_1 - \frac{1}{2}\zeta_1\zeta_2 c_5 + \left(\widetilde{\zeta}_2^2 - \frac{1}{2}\zeta_2^2\right) c'_5\\
B_2 &= \frac{a_1}{2\sqrt{2}a} + \frac{ia_4}{2\sqrt{2}a}
\end{aligned}
$$

[15]These relations were derived without using the fact that χ has $\widetilde{\Gamma}^{45}$ eigenvalue i or that ζ_1 and ζ_2 are positive.

$$
\begin{aligned}
&+\frac{1}{2a}b_1+i\zeta_1 b_2+\frac{i}{2a}b_4+\frac{i\zeta_1}{2a}b_5-\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b_6-\frac{\zeta_1}{2a}b_8+\frac{1}{2}\zeta_1\zeta_2 b_6'\\
&+i\zeta_1 c_2-\frac{1}{2}\zeta_1\zeta_2 c_6+\left(\frac{1}{2}\zeta_2^2-\widetilde{\zeta}_2^2\right)c_6'\\
B_3 =\ &\frac{ia_1}{2\sqrt{2}a}-\frac{a_4}{2\sqrt{2}a}\\
&-\frac{i}{2a}b_1-i\zeta_1 b_3+\frac{1}{2a}b_4+\frac{\zeta_1}{2a}b_5+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b_7+\frac{i\zeta_1}{2a}b_8-\frac{1}{2}\zeta_1\zeta_2 b_7'\\
&-i\zeta_1 c_3+\frac{1}{2}\zeta_1\zeta_2 c_7+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2\right)c_7'\\
B_4 =\ &\frac{ia_2}{2\sqrt{2}a}+\frac{a_3}{2\sqrt{2}a}\\
&+\frac{i}{2a}b_2+\frac{1}{2a}b_3+i\zeta_1 b_4-\frac{\zeta_1}{2a}b_6+\frac{i\zeta_1}{2a}b_7-\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b_8-\frac{1}{2}\zeta_1\zeta_2 b_8'\\
&-i\zeta_1 c_4+\frac{1}{2}\zeta_1\zeta_2 c_8+\left(\frac{1}{2}\zeta_2^2-\widetilde{\zeta}_2^2\right)c_8'\\
B_5 =\ &-\frac{1}{2a}b_6+\frac{i}{2a}b_7+i\zeta_2 b_5'-2c_1-i\zeta_2 c_5\\
B_6 =\ &-\frac{1}{2a}b_5-\frac{i}{2a}b_8+i\zeta_2 b_6'-2c_2-i\zeta_2 c_6\\
B_7 =\ &\frac{i}{2a}b_5-\frac{1}{2a}b_8-i\zeta_2 b_7'+2c_3+i\zeta_2 c_7\\
B_8 =\ &-\frac{i}{2a}b_6-\frac{1}{2a}b_7-i\zeta_2 b_8'+2c_4+i\zeta_2 c_8\\
B_1' =\ &\frac{a_2'}{2\sqrt{2}a}-\frac{ia_3'}{2\sqrt{2}a}\\
&+\frac{1}{2}\zeta_1\zeta_2 b_5+i\zeta_1 b_1'+\frac{1}{2a}b_2'-\frac{i}{2a}b_3'-\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b_5'+\frac{i\zeta_1}{2a}b_6'+\frac{\zeta_1}{2a}b_7'\\
&+\left(\frac{1}{2}\zeta_2^2-\widetilde{\zeta}_2^2\right)c_5+i\zeta_1 c_1'-\frac{1}{2}\zeta_1\zeta_2 c_5'\\
B_2' =\ &-\frac{a_1'}{2\sqrt{2}a}-\frac{ia_4'}{2\sqrt{2}a}\\
&+\frac{1}{2}\zeta_1\zeta_2 b_6+\frac{1}{2a}b_1'-i\zeta_1 b_2'+\frac{i}{2a}b_4'+\frac{i\zeta_1}{2a}b_5'+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b_6'-\frac{\zeta_1}{2a}b_8'\\
&+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2\right)c_6+i\zeta_1 c_2'-\frac{1}{2}\zeta_1\zeta_2 c_6'\\
B_3' =\ &-\frac{ia_1'}{2\sqrt{2}a}+\frac{a_4'}{2\sqrt{2}a}\\
&-\frac{1}{2}\zeta_1\zeta_2 b_7-\frac{i}{2a}b_1'+i\zeta_1 b_3'+\frac{1}{2a}b_4'+\frac{\zeta_1}{2a}b_5'-\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b_7'+\frac{i\zeta_1}{2a}b_8'\\
&+\left(\frac{1}{2}\zeta_2^2-\widetilde{\zeta}_2^2\right)c_7-i\zeta_1 c_3'+\frac{1}{2}\zeta_1\zeta_2 c_7'\\
B_4' =\ &-\frac{ia_2'}{2\sqrt{2}a}-\frac{a_3'}{2\sqrt{2}a}
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{1}{2}\zeta_1\zeta_2 b_8+\frac{i}{2a}b'_2+\frac{1}{2a}b'_3-i\zeta_1 b'_4-\frac{\zeta_1}{2a}b'_6+\frac{i\zeta_1}{2a}b'_7+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2+K\right)b'_8\\
&+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2\right)c_8-i\zeta_1 c'_4+\frac{1}{2}\zeta_1\zeta_2 c'_8\\
B'_5 &= i\zeta_2 b_5-\frac{1}{2a}b'_6+\frac{i}{2a}b'_7-2c'_1-i\zeta_2 c'_5\\
B'_6 &= i\zeta_2 b_6-\frac{1}{2a}b'_5-\frac{i}{2a}b'_8-2c'_2-i\zeta_2 c'_6\\
B'_7 &= -i\zeta_2 b_7-\frac{1}{2a}b'_8+\frac{i}{2a}b'_5+2c'_3+i\zeta_2 c'_7\\
B'_8 &= -i\zeta_2 b_8-\frac{1}{2a}b'_7-\frac{i}{2a}b'_6+2c'_4+i\zeta_2 c'_8\\
C_1 &= \frac{a_2}{2\sqrt{2}a}-\frac{ia_3}{2\sqrt{2}a}+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2\right)b_5-i\zeta_2 b'_1+\frac{1}{2}\zeta_1\zeta_2 b'_5\\
&-\frac{1}{2a}c_2+\frac{i}{2a}c_3-\frac{1}{2}\zeta_1\zeta_2 c_5-\frac{i\zeta_2}{2a}c_6-\frac{\zeta_2}{2a}c_7-i\zeta_2 c'_1+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2+L\right)c'_5\\
C_2 &= \frac{a_1}{2\sqrt{2}a}+\frac{ia_4}{2\sqrt{2}a}+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2\right)b_6+i\zeta_2 b'_2-\frac{1}{2}\zeta_1\zeta_2 b'_6\\
&-\frac{1}{2a}c_1-\frac{i}{2a}c_4-\frac{i\zeta_2}{2a}c_5+\frac{1}{2}\zeta_1\zeta_2 c_6+\frac{\zeta_2}{2a}c_8-i\zeta_2 c'_2+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2+L\right)c'_6\\
C_3 &= \frac{ia_1}{2\sqrt{2}a}-\frac{a_4}{2\sqrt{2}a}-\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2\right)b_7-i\zeta_2 b'_3+\frac{1}{2}\zeta_1\zeta_2 b'_7\\
&+\frac{i}{2a}c_1-\frac{1}{2a}c_4-\frac{\zeta_2}{2a}c_5-\frac{1}{2}\zeta_1\zeta_2 c_7-\frac{i\zeta_2}{2a}c_8+i\zeta_2 c'_3-\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2+L\right)c'_7\\
C_4 &= -\frac{ia_2}{2\sqrt{2}a}-\frac{a_3}{2\sqrt{2}a}-\left(\widetilde{\zeta}_1^2-\frac{1}{2}\zeta_1^2\right)b_8+i\zeta_2 b'_4-\frac{1}{2}\zeta_1\zeta_2 b'_8\\
&-\frac{i}{2a}c_2-\frac{1}{2a}c_3+\frac{\zeta_2}{2a}c_6-\frac{i\zeta_2}{2a}c_7+\frac{1}{2}\zeta_1\zeta_2 c_8+i\zeta_2 c'_4-\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2+L\right)c'_8\\
C_5 &= 2b'_1+i\zeta_1 b'_5-i\zeta_1 c_5+\frac{1}{2a}c_6-\frac{i}{2a}c_7\\
C_6 &= -2b'_2-i\zeta_1 b'_6+\frac{1}{2a}c_5+i\zeta_1 c_6+\frac{i}{2a}c_8\\
C_7 &= 2b'_3+i\zeta_1 b'_7-\frac{i}{2a}c_5-i\zeta_1 c_7+\frac{1}{2a}c_8\\
C_8 &= -2b'_4-i\zeta_1 b'_8+\frac{i}{2a}c_6+\frac{1}{2a}c_7+i\zeta_1 c_8\\
C'_1 &= \frac{a'_2}{2\sqrt{2}a}-\frac{ia'_3}{2\sqrt{2}a}+i\zeta_2 b_1-\frac{1}{2}\zeta_1\zeta_2 b_5+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\,\zeta_1^2\right)b'_5\\
&-i\zeta_2 c_1+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2+L\right)c_5-\frac{1}{2a}c'_2+\frac{i}{2a}c'_3+\frac{1}{2}\zeta_1\zeta_2 c'_5-\frac{i\zeta_2}{2a}c'_6-\frac{\zeta_2}{2a}c'_7\\
C'_2 &= \frac{a'_1}{2\sqrt{2}a}+\frac{ia'_4}{2\sqrt{2}a}-i\zeta_2 b_2+\frac{1}{2}\zeta_1\zeta_2 b_6+\left(\widetilde{\zeta}_1^2-\frac{1}{2}\,\zeta_1^2\right)b'_6\\
&-i\zeta_2 c_2+\left(\widetilde{\zeta}_2^2-\frac{1}{2}\zeta_2^2+L\right)c_6-\frac{1}{2a}c'_1-\frac{i}{2a}c'_4-\frac{i\zeta_2}{2a}c'_5-\frac{1}{2}\zeta_1\zeta_2 c'_6+\frac{\zeta_2}{2a}c'_8
\end{aligned}
$$

$$
\begin{aligned}
C'_3 &= \frac{i a'_1}{2\sqrt{2}a} - \frac{a'_4}{2\sqrt{2}a} + i\zeta_2 b_3 - \frac{1}{2}\zeta_1\zeta_2 b_7 - \left(\widetilde{\zeta}_1^2 - \frac{1}{2}\,\zeta_1^2\right) b'_7 \\
&\quad + i\zeta_2 c_3 - \left(\widetilde{\zeta}_2^2 - \frac{1}{2}\zeta_2^2 + L\right) c_7 + \frac{i}{2a}c'_1 - \frac{1}{2a}c'_4 - \frac{\zeta_2}{2a}c'_5 + \frac{1}{2}\zeta_1\zeta_2 c'_7 - \frac{i\zeta_2}{2a}c'_8 \\
C'_4 &= -\frac{i a'_2}{2\sqrt{2}a} - \frac{a'_3}{2\sqrt{2}a} - i\zeta_2 b_4 + \frac{1}{2}\zeta_1\zeta_2 b_8 - \left(\widetilde{\zeta}_1^2 - \frac{1}{2}\,\zeta_1^2\right) b'_8 \\
&\quad + i\zeta_2 c_4 - \left(\widetilde{\zeta}_2^2 - \frac{1}{2}\zeta_2^2 + L\right) c_8 - \frac{i}{2a}c'_2 - \frac{1}{2a}c'_3 + \frac{\zeta_2}{2a}c'_6 - \frac{i\zeta_2}{2a}c'_7 - \frac{1}{2}\zeta_1\zeta_2 c'_8 \\
C'_5 &= -2b_1 - i\zeta_1 b_5 + i\zeta_1 c'_5 + \frac{1}{2a}c'_6 - \frac{i}{2a}c'_7 \\
C'_6 &= 2b_2 + i\zeta_1 b_6 + \frac{1}{2a}c'_5 - i\zeta_1 c'_6 + \frac{i}{2a}c'_8 \\
C'_7 &= -2b_3 - i\zeta_1 b_7 - \frac{i}{2a}c'_5 + i\zeta_1 c'_7 + \frac{1}{2a}c'_8 \\
C'_8 &= 2b_4 + i\zeta_1 b_8 + \frac{i}{2a}c'_6 + \frac{1}{2a}c'_7 - i\zeta_1 c'_8
\end{aligned}
\tag{6.8}
$$

where

$$
\widetilde{\zeta}_1^2 = \zeta_1^2 - \frac{1}{2a^2}\,, \qquad \widetilde{\zeta}_2^2 = \zeta_2^2 + \frac{1}{2a^2}, \qquad K = \frac{1}{2a^2}, \qquad L = -\frac{1}{2a^2}\,, \tag{6.9}
$$

and we have used

$$
-D_\alpha D^\alpha \chi = \widetilde{\zeta}_1^2 \chi, \quad -D_m D^m \chi = \widetilde{\zeta}_2^2 \chi, \quad \Gamma^\beta [D_\beta, D_\alpha]\chi = K\,\Gamma_\alpha \chi, \quad \Gamma^m [D_m, D_n]\chi = L\,\Gamma_n \chi\,. \tag{6.10}
$$

We can express (6.8) as

$$
\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \\ \vec{A}' \\ \vec{B}' \\ \vec{C}' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{a}' \\ \vec{b}' \\ \vec{c}' \end{pmatrix}\,, \tag{6.11}
$$

where $\mathcal{M}$ is a 40×40 matrix. The eigenvalues of $\mathcal{M}^2$ will determine the heat kernel in the fermionic sector of the gravity multiplet.

Let us now discuss the possible complication that could arise due to the fact that we have chosen to expand the various fields in a non-orthonormal set of basis functions. If we did use an orthonormal basis then the resulting matrix $\mathcal{M}$ will be related to the one appearing in (6.11) by a similarity transformation. This however will not affect the eigenvalues of $\mathcal{M}^2$. Since our final result will be expressed in terms of the eigenvalues of $\mathcal{M}^2$, the non-orthonormality of our basis vectors will not affect the result.

To proceed we introduce a matrix $\mathcal{M}_1$ through

$$
\mathcal{M}^2 = -(\zeta_1^2 + \zeta_2^2) I_{40} + a^{-2} \mathcal{M}_1\,, \tag{6.12}
$$

where I_{40} denotes the 40×40 identity matrix. It is easy to see that in the limit of large ζ_1, ζ_2 the dominant contribution to the eigenvalues come from the first term. Let us denote by β_k for $1 \le k \le 40$ the 40 eigenvalues of the matrix $\mathcal{M}_1$, and introduce variables λ and l through:

$$\zeta_1 = (l+1)/a, \qquad \zeta_2 = \lambda/a \tag{6.13}$$

Then the contribution to the heat kernel from the fermionic modes for $\zeta_1 > 1/a$, $\zeta_2 \ge 0$ will be given by

$$K^f_{(1)}(0;s) = -\frac{1}{8\pi^2 a^4}\sum_{l=1}^{\infty}(2l+2)\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}(l+1)^2-\bar{s}\lambda^2}\sum_{k=1}^{40}e^{\bar{s}\beta_k}\,. \tag{6.14}$$

The overall minus sign reflects the fact that we are dealing with fermions. The normalization factor is fixed by noting that since the four gravitinoes represented by the sixteen component field ψ_μ for $0 \le \mu \le 3$ give effectively $4\times 4 = 16$ Majorana fermions, and the dilatino, represented by the sixteen component field Λ, gives 4 Majorana fermions in four dimensions, we have in total 20 Majorana or 10 Dirac fermions in four dimensions. Thus the heat kernel should agree with that of 10 free Dirac fermions in the limit of small $\bar{s}$ when the effect of background flux can be ignored, i.e. $\bar{s}\beta_k$ can be set equal to 0. Comparing (6.14) with (3.25) we see that we indeed have the equivalent of ten Dirac fermions.

The contribution from the $\zeta_1 = 1/a$, i.e. $l = 0$ term has to be evaluated separately. For this the basis states used in the (6.6) are not independent, since we have $D_\alpha\chi = \frac{i}{2a}\Gamma_\alpha\chi$. Using this we can choose the coefficients $b_5, \cdots b_8$ and $b'_5, \cdots b'_8$ to zero in (6.6). Furthermore in eqs. (6.7) we can make the replacement $D_\alpha\chi \to \frac{i}{2a}\Gamma_\alpha\chi$, which amounts to replacing in (6.8) the expressions for B_k by that of $B_k + \frac{i}{2a}B_{k+4}$ and of B'_k by that of $B'_k + \frac{i}{2a}B'_{k+4}$ for $1 \le k \le 4$ and then drop the expressions for B_{k+4} and B'_{k+4} for $1 \le k \le 4$. This gives a 32×32 matrix $\widetilde{\mathcal{M}}$ relating $(A_1, \cdots A_4, A'_1, \cdots A'_4, B_1, \cdots B_4, B'_1, \cdots B'_4, C_1, \cdots C_8, C'_1, \cdots C'_8)$ to $(a_1, \cdots a_4, a'_1, \cdots a'_4, b_1, \cdots b_4, b'_1, \cdots b'_4, c_1, \cdots c_8, c'_1, \cdots c'_8)$. Let us now define a matrix $\widetilde{\mathcal{M}}_1$ through

$$\widetilde{\mathcal{M}}^2 = -(a^{-2} + \zeta_2^2)I_{32} + a^{-2}\widetilde{\mathcal{M}}_1\,, \tag{6.15}$$

where I_{32} denotes the 32×32 identity matrix. If $\widetilde{\beta}_k$'s are the eigenvalues of $\widetilde{\mathcal{M}}_1$ then the contribution from the $l = 0$ modes to the heat kernel may be expressed as

$$K^f_{(2)}(0;s) = -\frac{1}{4\pi^2 a^4}\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}-\bar{s}\lambda^2}\sum_{k=1}^{32}e^{\bar{s}\widetilde{\beta}_k}\,. \tag{6.16}$$

We can combine (6.14) and (6.16) to write

$$K^f_{(1)}(0;s) + K^f_{(2)}(0;s) = \widetilde{K}^f_{(1)}(0;s) + \widetilde{K}^f_{(2)}(0;s)\,, \tag{6.17}$$

where

$$\begin{aligned}\widetilde{K}^f_{(1)}(0;s) &= -\frac{1}{8\pi^2 a^4}\sum_{l=0}^{\infty}(2l+2)\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}(l+1)^2-\bar{s}\lambda^2}\sum_{k=1}^{40}e^{\bar{s}\beta_k} \\ &= -\frac{1}{4\pi^2 a^4}\, Im\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}\widetilde{\lambda}^2-\bar{s}\lambda^2}\sum_{k=1}^{40}e^{\bar{s}\beta_k|_{l+1\to\widetilde{\lambda}}}\end{aligned} \tag{6.18}$$

$$\widetilde{K}^f_{(2)}(0;s) = -\frac{1}{4\pi^2 a^4}\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}-\bar{s}\lambda^2}\left[\sum_{k=1}^{32} e^{\bar{s}\widetilde{\beta}_k} - \sum_{k=1}^{40} e^{\bar{s}\beta_k|_{l=0}}\right]. \tag{6.19}$$

In the second step in (6.18) we have used a trick similar to that described in (3.11), (3.12) to convert the sum over l to integral of $\widetilde{\lambda}$.

We also need to compute the contribution due to the discrete modes described in (2.28). For this we set the fields Λ and ψ_α to 0, and expand ψ_m as in (6.6) with $c_{k+4} = 2c_k a$, $c'_{k+4} = 2c'_k a$ for $1 \le k \le 4$, with $\zeta_2 = i/a$, $\zeta_1 \ge 1/a$, i.e. $l \ge 0$.[16] It can be seen that with this choice A_i, A'_i, B_i, B'_i computed from (6.8) vanish and we have $C_{k+4} = 2C_k a$, $C'_{k+4} = 2C'_k a$ for $1 \le k \le 4$. Thus we can express these relations as

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C'_1 \\ C'_2 \\ C'_3 \\ C'_4 \end{pmatrix} = \widehat{\mathcal{M}} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c'_1 \\ c'_2 \\ c'_3 \\ c'_4 \end{pmatrix}, \tag{6.20}$$

for some 8×8 matrix $\widehat{\mathcal{M}}$. If $\widehat{\beta}_k$ denote the eigenvalues of

$$\widehat{\mathcal{M}}_1 \equiv a^2\{\widehat{\mathcal{M}}^2 + (\zeta_1^2 - a^{-2})I_8\}, \tag{6.21}$$

then the contribution to $K(0;s)$ from these modes is given by

$$\begin{aligned} K^f_{(3)}(0;s) &= -\frac{1}{8\pi^2 a^4}\sum_{l=0}^\infty (2l+2)e^{\bar{s}-\bar{s}(l+1)^2}\sum_{k=1}^8 e^{\bar{s}\widehat{\beta}_k} \\ &= -\frac{1}{4\pi^2 a^4}\, Im \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\, e^{\bar{s}-\bar{s}\widetilde{\lambda}^2}\sum_{k=1}^8 e^{\bar{s}\widehat{\beta}_k}|_{l+1\to\widetilde{\lambda}}\,. \end{aligned} \tag{6.22}$$

Finally the three sets of bosonic ghosts $\widetilde{b}$, $\widetilde{c}$ and $\widetilde{e}$ associated with gauge fixing of local supersymmetry, each of which gives rise to four Majorana fermions in four dimensions, contributes

$$\begin{aligned} K^f_{\rm ghost} &= \frac{3}{\pi^2 a^4}\sum_{l=0}^\infty (2l+2)\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}(l+1)^2-\bar{s}\lambda^2} \\ &= \frac{6}{\pi^2 a^4}\, Im\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}\widetilde{\lambda}^2-\bar{s}\lambda^2}, \end{aligned} \tag{6.23}$$

to $K(0;s)$.

[16] This basis is still overcomplete since, as discussed in (2.29), the action of τ_3 on the basis states is fixed once we choose χ to be $\chi^+_k(i)$ or $\eta^+_k(i)$. Thus we could work with either the C_i's or the C'_i's. But we shall proceed by including both sets and include a factor of 1/2 in the expression for the heat kernel.

To evaluate the right hand sides of (6.18), (6.19), and (6.22) we use the relations

$$\sum_k e^{\bar{s}\beta_k} = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{s}^n \sum_k \beta_k^n = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{s}^n Tr(\mathcal{M}_1^n)\,, \tag{6.24}$$

$$\sum_k e^{\bar{s}\widetilde{\beta}_k} = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{s}^n \sum_k \widetilde{\beta}_k^n = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{s}^n Tr(\widetilde{\mathcal{M}}_1^n)\,. \tag{6.25}$$

and

$$\sum_k e^{\bar{s}\widehat{\beta}_k} = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{s}^n \sum_k \widehat{\beta}_k^n = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{s}^n Tr(\widehat{\mathcal{M}}_1^n)\,, \tag{6.26}$$

Explicit computation gives

$$\begin{aligned}
Tr(\mathcal{M}_1) &= 0\\
Tr(\mathcal{M}_1^2) &= 64 + 16(l+1)^2 - 16\lambda^2\\
Tr(\mathcal{M}_1^3) &= -144(l+1)^2 - 144\lambda^2\\
Tr(\mathcal{M}_1^4) &= 256 + 192(l+1)^2 + 80(l+1)^4 - 192\lambda^2 + 32(l+1)^2\lambda^2 + 80\lambda^4\,.
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
Tr(\widetilde{\mathcal{M}}_1) &= -8\\
Tr(\widetilde{\mathcal{M}}_1^2) &= 72 - 8\lambda^2\\
Tr(\widetilde{\mathcal{M}}_1^3) &= -152 - 120\lambda^2\\
Tr(\widetilde{\mathcal{M}}_1^4) &= 520 - 112\lambda^2 + 72\lambda^4\,.
\end{aligned} \tag{6.28}$$

$$\begin{aligned}
Tr(\widehat{\mathcal{M}}_1) &= -8\\
Tr(\widehat{\mathcal{M}}_1^2) &= 8 + 8(l+1)^2\\
Tr(\widehat{\mathcal{M}}_1^3) &= -8 - 24(l+1)^2\\
Tr(\widehat{\mathcal{M}}_1^4) &= 8 + 48(l+1)^2 + 8(l+1)^4\,.
\end{aligned} \tag{6.29}$$

Furthermore we also have the analogs of eqs. (3.16) and (3.17):

$$\begin{aligned}
&\int_0^{\infty} d\lambda\, \lambda\, \coth(\pi\lambda)\, e^{-\bar{s}\lambda^2}\, \lambda^{2n}\\
&= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^{\infty} \bar{s}^m \frac{(2m+2n+1)!}{m!}\,(2\pi)^{-2(m+n+1)}\,(-1)^m\\
&\qquad \zeta(2(m+n+1))\,.
\end{aligned} \tag{6.30}$$

$$\begin{aligned}
&Im\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda}\, \cot(\pi\widetilde{\lambda})\, e^{-\bar{s}\widetilde{\lambda}^2}\, \widetilde{\lambda}^{2n}\\
&= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^{\infty} \bar{s}^m \frac{(2m+2n+1)!}{m!}\,(2\pi)^{-2(m+n+1)}(-1)^{n+1}\\
&\qquad \zeta(2(m+n+1))\,.
\end{aligned} \tag{6.31}$$

Using these relations we get the following order s^0 contributions to various terms when expanded in a power series in s around $s=0$:

$$\widetilde{K}^f_{(1)}(0;s)\,:\,\frac{5}{72\pi^2 a^4}$$

$$\begin{aligned}\widetilde{K}^f_{(2)}(0;s) &: -\frac{1}{3\pi^2 a^4}\\ K^f_{(3)}(0;s) &: -\frac{1}{3\pi^2 a^4}\\ K^f_{\rm ghost}(0;s) &: -\frac{11}{120\pi^2 a^4}\,.\end{aligned} \tag{6.32}$$

Adding up all the contributions we get the net contribution to $K(0;s)$ from the fermionic fields in the gravity multiplet:

$$K^f(0;s) = -\frac{31}{45\pi^2 a^4}\,. \tag{6.33}$$

We now need to remove from this the zero mode contribution. Analysis of the zero modes in the fermionic sector requires special care. Among the $l=0$ modes in the first line of (6.22) we have four vanishing $\widehat{\beta}_k$ giving a net contribution of $-1/\pi^2 a^4$. Thus naively we must remove this contribution from $K(0;s)$. However a detailed analysis shows that although the matrix $\widehat{\mathcal{M}}^2$ describing the square of the fermionic kinetic term has four eigenvectors with zero eigenvalues, the matrix $\widehat{\mathcal{M}}$ has only a pair of eigenvectors with zero eigenvalues. These eigenvectors are

$$\begin{aligned}\psi^{(1)}_m &= (i-\sigma_3\widehat{\Gamma}^4 - i\tau_3\widehat{\Gamma}^5 + \sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5)(a^{-1}\Gamma_m + 2\sigma_3 D_m)\chi\,,\\ \psi^{(2)}_m &= \sigma_3(i+\sigma_3\widehat{\Gamma}^4 + i\tau_3\widehat{\Gamma}^5 + \sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5)(a^{-1}\Gamma_m + 2\sigma_3 D_m)\chi\,.\end{aligned} \tag{6.34}$$

The other two eigenvectors of $\widehat{\mathcal{M}}^2$ which are not zero modes of $\widehat{\mathcal{M}}$ are

$$\begin{aligned}\xi^{(1)}_m &= (i+\sigma_3\widehat{\Gamma}^4 - i\tau_3\widehat{\Gamma}^5 - \sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5)(a^{-1}\Gamma_m + 2\sigma_3 D_m)\chi\,,\\ \xi^{(2)}_m &= \sigma_3(i-\sigma_3\widehat{\Gamma}^4 + i\tau_3\widehat{\Gamma}^5 - \sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5)(a^{-1}\Gamma_m + 2\sigma_3 D_m)\chi\,.\end{aligned} \tag{6.35}$$

The action of $\widehat{\mathcal{M}}$ on these modes are given by

$$\widehat{\mathcal{M}}\,\xi^{(1)}_m = -2\,i\,a^{-1}\,\psi^{(1)}_m,\quad \widehat{\mathcal{M}}\,\xi^{(2)}_m = 2\,i\,a^{-1}\,\psi^{(2)}_m,\quad \widehat{\mathcal{M}}\,\psi^{(1)}_m = 0,\quad \widehat{\mathcal{M}}\,\psi^{(2)}_m = 0\,. \tag{6.36}$$

From this we conclude that the contribution of only two of the four zero modes of $\widehat{\mathcal{M}}^2$ will have to be removed from the contribution to $K(0;s)$. This amounts to removing a factor of $-1/2\pi^2 a^4$ from $K(0;s)$. Subtracting this from (6.33) we get the net contribution to $K(0;s)$ from the non-zero modes of the gravity multiplet fermions:

$$-\frac{17}{90\pi^2 a^4}\,. \tag{6.37}$$

For later use it will be useful to find the physical interpretation of these zero modes. First we note that the zero modes satisfy the chirality projection condition:

$$\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\psi^{(k)}_m = i\psi^{(k)}_m,\quad \sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5\xi^{(k)}_m = -i\xi^{(k)}_m,\quad \text{for } k=1,2\,. \tag{6.38}$$

Choosing $\chi = \chi^+_k(i)$ in (6.34) and using (2.29) and the fact that χ has been chosen to satisfy $\widehat{\Gamma}^{45}\chi = i\chi$, we get

$$\begin{aligned}\psi^{(1)}_m &= (i-\sigma_3\widehat{\Gamma}^4 - \widehat{\Gamma}^4 - i\sigma_3)(\Gamma_m + 2\sigma_3 D_m)\chi\,,\\ \psi^{(2)}_m &= \sigma_3(i+\sigma_3\widehat{\Gamma}^4 + \widehat{\Gamma}^4 - i\sigma_3)(\Gamma_m + 2\sigma_3 D_m)\chi\,.\end{aligned} \tag{6.39}$$

Thus we have $\psi_m^{(1)} = -\psi_m^{(2)}$, i.e. they are not independent. A similar analysis for $\chi = \eta_k^+(i)$ will give $\psi_m^{(1)} = \psi_m^{(2)}$ again showing that they are not independent. This allows us to keep only one of these modes, — we shall take it to be $\psi^{(1)} - i\psi^{(2)}$ in both cases. Now one can show that

$$\begin{aligned}
\psi_\mu^{(1)} - i\psi_\mu^{(2)} &= D_\mu\epsilon + \frac{1}{\sqrt{2}}\,\Gamma^\sigma\,\left(\bar{F}^{(1)}_{\mu\sigma}\Gamma^4 + \bar{F}^{(2)}_{\mu\sigma}\Gamma^5\right)\epsilon\,, \\
&\quad \epsilon \equiv 2(i-\sigma_3\widehat{\Gamma}^4-i\tau_3\widehat{\Gamma}^5+\sigma_3\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5+\sigma_3-i\widehat{\Gamma}^4+\sigma_3\tau_3\widehat{\Gamma}^5-i\tau_3\widehat{\Gamma}^4\widehat{\Gamma}^5)\sigma_3\chi\,, \\
\Gamma^{\rho\sigma}\,\left(\bar{F}^{(1)}_{\rho\sigma}\Gamma^4+\bar{F}^{(2)}_{\rho\sigma}\Gamma^5\right)\epsilon &= 0
\end{aligned} \tag{6.40}$$

Comparing this with the supersymmetry transformations laws for the gravitino and the dilatino in the convention of [46]

$$\begin{aligned}
\delta\psi_\mu &= D_\mu\epsilon + \frac{1}{4\sqrt{2}}\,\left(4\delta^\rho_\mu\Gamma^\sigma - \Gamma_\mu\Gamma^{\rho\sigma}\right)\left(\bar{F}^{(1)}_{\rho\sigma}\Gamma^4 + \bar{F}^{(2)}_{\rho\sigma}\Gamma^5\right)\epsilon + \cdots \\
\delta\Lambda &= -\frac{1}{4}\,\Gamma^{\rho\sigma}\,\left(\bar{F}^{(1)}_{\rho\sigma}\Gamma^4 + \bar{F}^{(2)}_{\rho\sigma}\Gamma^5\right)\epsilon
\end{aligned} \tag{6.41}$$

where ϵ is the supersymmetry transformation parameter and $\cdots$ denotes terms which vanish in the near horizon background geometry, we see that $\psi_\mu^{(1)} - i\psi_\mu^{(2)}$ is associated with a supersymmetry transformation generated by the parameter ϵ. However since ϵ is obtained by the action of Γ matrices on $\chi^+(i)$ or $\eta^+(i)$, it is not normalizable.

7 Zero mode contribution

Adding (5.45) and (6.37) we get the net s-independent contribution to $K(0;s)$ from all the non-zero modes:

$$-\frac{101}{180\pi^2a^4} - \frac{17}{90\pi^2a^4} = -\frac{3}{4\pi^2a^4}\,. \tag{7.1}$$

To this we must add the result of carrying out the zero mode integration. This was described for the gauge fields in appendix A of [46]; we shall briefly review the argument since it can also be generalized to integration over the zero modes of the metric and the gravitino fields.

Let A_μ be a vector field on $AdS_2 \times S^2$ and $g_{\mu\nu}$ be the background metric of the form $a^2\,g^{(0)}_{\mu\nu}$ where a is radius of curvature of S^2 and AdS_2 and $g^{(0)}_{\mu\nu}$ is independent of a. The path integral over A_μ is normalized such that

$$\int [DA_\mu]\exp\left[-\int d^4x\,\sqrt{\det g}\,g^{\mu\nu}A_\mu A_\nu\right] = 1\,, \tag{7.2}$$

i.e.

$$\int [DA_\mu]\exp\left[-a^2\int d^4x\,\sqrt{\det g^{(0)}}\,g^{(0)\mu\nu}A_\mu A_\nu\right] = 1\,. \tag{7.3}$$

From this we see that up to an a independent normalization constant, $[DA_\mu]$ actually corresponds to integration with measure $\prod_{\mu,x} d(aA_\mu(x))$. On the other hand the gauge field zero modes are associated with deformations produced by the gauge transformations with non-normalizable parameters: $\delta A_\mu \propto \partial_\mu\Lambda(x)$ for some functions $\Lambda(x)$ with a-independent integration range. Thus the result of integration over the gauge field zero modes can be found by first changing the integration over the zero modes of (aA_μ) to integration over Λ and then picking up the contribution from the Jacobian in this change of variables. This gives a factor of a from integration over each zero mode of A_μ. Thus if there are N zero modes then we shall get a factor of a^N. Of course N is infinite, but it needs to be regularized by subtracting from it a term proportional to the length of the boundary of AdS_2. We shall now describe two equivalent ways of computing N: one is a somewhat indirect but useful method and the other is a more direct method, but involves a little bit of additional computation.

First let us describe the indirect method. For a non-zero mode, the path integral weighted by the exponential of the action produces a factor of $\kappa_n^{-1/2}$ where κ_n is the eigenvalue of the kinetic operator. Since κ_n has the form b_n/a^2 where b_n is an a independent constant, integration over a non-zero mode produces a factor proportional to a. Including the zero mode contribution to $K(0,s)$ is equivalent to counting the same factor of a from integration over the zero modes as well. Thus when we remove from the determinant the contribution due to the zero modes, we remove a factor of a for each zero mode. However the analysis of the previous paragraph showed that integration over the zero modes gives us back a factor of a. Thus the net effect of integrating over the gauge field zero modes is to cancel the effect of the subtraction of the zero mode contribution from $K(0;s)$. In other words the net effect of integration over the six gauge field zero modes amounts to effecively adding a contribution of $6/8\pi^2a^4$ to $K(0;s)$.[17] Using (3.26) we see that this corresponds to a contribution of $-6\ln a$ in the entropy, i.e. $-\ln a$ for each gauge field.

Next we shall describe a direct method for evaluating the zero mode contribution from the gauge fields which does not make any reference to the result on integration over the non-zero modes. Let $f_m^{(\ell)}$ denote the normalized zero mode wave functions of gauge fields on AdS_2 given in (2.6). Then the total number of zero modes may be written as

$$\sum_{\ell\in\mathcal{F},\ell\neq 0}\int d\theta d\eta\sqrt{\det g_{AdS_2}}\, g^{mn}_{AdS_2} f_m^{(\ell)*} f_n^{(\ell)}\,, \tag{7.4}$$

where g_{AdS_2} is the metric on AdS_2. We now use the fact that $\sum_\ell g^{mn}_{AdS_2} f_m^{(\ell)*} f_n^{(\ell)}$ must be independent of the AdS_2 coordinates (η,θ) since AdS_2 is a homogeneous space. Thus we can evaluate this at $\eta=0$. In this case only $\ell=\pm 1$ modes contribute, leading to the result $1/(2\pi a^2)$. Integrating this over AdS_2 with a cut-off $\eta\leq\eta_0$, we get the result

$$\frac{1}{2\pi a^2}\,2\pi a^2\,(\cosh\eta_0-1)\,. \tag{7.5}$$

The term proportional to $\cosh\eta_0$ can be interpreted as a shift in the ground state energy. Thus we are left with an effective contribution of -1. From this we conclude that for every gauge field the integration over the zero modes gives a factor of a^{-1} to $e^{S_{BH}}$, i.e. $-\ln a$ to the black hole entropy.

The effect of integration over the zero modes of the fluctuations $h_{\mu\nu}$ of the metric (including those of the SU(2) gauge fields arising from the dimensional reduction of the

[17]Note that this is not the actual modification of the heat kernel, but represents the effective contribution to be added to $K(0;s)$ that reproduces, via (3.26), the net contribution to the one loop determinant due to the zero modes.

metric on S^2) can be found in a similar way, with (7.2), (7.3) replaced by

$$\int [Dh_{\mu\nu}] \exp\left[-\int d^4x \sqrt{\det g}\, g^{\mu\nu} g^{\rho\sigma} h_{\mu\rho} h_{\nu\sigma}\right] = 1\,, \tag{7.6}$$

i.e.

$$\int [Dh_{\mu\nu}] \exp\left[-\int d^4x \sqrt{\det g^{(0)}}\, g^{(0)\mu\nu} g^{(0)\rho\sigma} h_{\mu\rho} h_{\nu\sigma}\right] = 1\,. \tag{7.7}$$

Thus the correctly normalized integration measure, up to an a independent constant, is $\prod_{x,(\mu\nu)} dh_{\mu\nu}(x)$. We now note that the zero modes are associated with diffeomorphisms with non-normalizable parameters: $h_{\mu\nu} \propto D_\mu \xi_\nu + D_\nu \xi_\mu$, with the diffeomorphism parameter $\xi^\mu(x)$ having a independent integration range. Thus the a dependence of the integral over the metric zero modes can be found by finding the Jacobian from the change of variables from $h_{\mu\nu}$ to ξ^μ. Lowering of the index of ξ^μ gives a factor of a^2, leading to a factor of a^2 per zero mode. On the other hand following the same logic as in the case of gauge fields we find that the removal of the integration over the metric zero modes from the heat kernel removes a factor of a per zero mode from the integrand. Thus the effect of integration over the metric zero modes will be to add the double of the contribution that one removes. Since we had removed from $K(0;s)$ a contribution of $3/8\pi^2 a^4 + 3/8\pi^2 a^4 = 6/8\pi^2 a^4$ (see eq. (5.44)) we need to add a factor of $12/8\pi^2 a^4$.

Finally we turn to the fermion zero modes.[18] The normalization of the zero modes is determined from

$$\int [D\psi_\mu][D\bar{\psi}_\mu] \exp\left[-\int d^4x \sqrt{\det g}\, g^{\mu\nu} \bar{\psi}_\mu \psi_\nu\right] = 1\,, \tag{7.8}$$

i.e.

$$\int [D\psi_\mu][D\bar{\psi}_\mu] \exp\left[-a^2 \int d^4x \sqrt{\det g^{(0)}}\, g^{(0)\mu\nu} \bar{\psi}_\mu \psi_\nu\right] = 1\,, \tag{7.9}$$

indicating that $a\psi_\mu$ and $a\bar{\psi}_\mu$ are the correctly normalized integration variables. As discussed in the previous section, the fermion zero modes are associated with the asymptotic supersymmetry transformations, with the anti-commutator of a pair of supersymmetry transformations generating a diffeomorphism with parameter $\bar{\epsilon}\Gamma^\mu\epsilon$. Since $\Gamma^\mu \sim a^{-1}$, and since $\bar{\epsilon}\Gamma^\mu\epsilon$ has a-independent integration range, we see that the correctly normalized ϵ is $\epsilon_0 = a^{-1/2}\epsilon$ for which the supersymmetry algebra generated by ϵ_0 and ξ^μ does not involve any a dependence. Thus integration over each ψ_μ zero mode is equivalent to integration over $a\psi_\mu \sim a^{3/2}\,\epsilon_0$, producing a factor of $a^{-3/2}$. On the other hand a non-zero mode of the fermion will produce a factor of $a^{-1/2}$ after integration since the kinetic operator of the fermion is of order a^{-1}. Thus removing a fermion zero mode contribution from the heat kernel removes a factor of $a^{-1/2}$ for each zero mode. Thus the effect of integration over the fermion zero modes is to add back three times the amount we remove from the heat kernel while removing the fermion zero mode contribution. This gives a net contribution of $-3/2\pi^2 a^4$ to the effective heat kernel.

[18]Naively integration over the fermion zero modes will make the integral vanish. However it was shown in [54] using localization techniques that the zeroes due to the fermionic zero mode integrals cancel the infinities coming from integration over the bosonic zero modes of the metric.

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Adding up the contribution from all the zero modes we see that the net effect of integration over the zero modes is to effectively add a factor of

$$\frac{6}{8\pi^2 a^4} + \frac{12}{8\pi^2 a^4} - \frac{3}{2\pi^2 a^4} = \frac{3}{4\pi^2 a^4}\,, \tag{7.10}$$

to $K(0;s)$. Note that the contribution from the graviton and the gravitino zero modes cancel — the final result $3/4\pi^2 a^4$ is the contribution of the six gauge fields in the gravity multiplet.

Adding (7.10) to the contribution (7.1) due to the non-zero modes we get the net contribution to the effective heat kernel to be

$$-\frac{3}{4\pi^2 a^4} + \frac{3}{4\pi^2 a^4} = 0\,. \tag{7.11}$$

This is perfectly consistent with the microscopic result (1.1).

8 $\mathcal{N} = 8$ black holes

In this section we shall briefly describe the analysis of logarithmic corrections to the entropy of 1/8 BPS black holes in $\mathcal{N} = 8$ supersymmetric string theories obtained by compactifying type IIB string theory on T^6. For this we first note that there is a consistent truncation of $\mathcal{N} = 8$ supergravity to $\mathcal{N} = 4$ supergravity by projecting on to the $(-1)^{F_L}$ even states in which we set all the RR and R-NS sector fields to zero. Using this embedding of the $\mathcal{N} = 4$ supergravity into $\mathcal{N} = 8$ supergravity, the quarter BPS black hole in $\mathcal{N} = 4$ supergravity that we have already analyzed can now be regarded as the 1/8 BPS black hole in the $\mathcal{N} = 8$ supergravity. Since the projection on to the $\mathcal{N} = 4$ supergravity is a consistent truncation it is guaranteed that at the quadratic level, the fluctuations of the additional fields in the $(-1)^{F_L}$ odd sector does not mix with the fluctuations of the fields in the $(-1)^{F_L}$ even sector. Thus the one loop effective action of full $\mathcal{N} = 8$ supergravity receives the contribution already computed for the $\mathcal{N} = 4$ black holes plus an additional contribution from the determinant of the $(-1)^{F_L}$ odd fields.

We begin with the contribution due to the extra bosons. There are sixteen gauge bosons, — one from the ten dimensional gauge field A_μ and fifteen from the components $C_{mn\mu}$ of the 3-form field with m, n along T^6 and μ along $AdS_2 \times S^2$. There are also thirty two scalars, — six from the components A_m of the ten dimensional gauge field along T^6, twenty from the components C_{mnp} of the 3-form field along T^6 and six from dualizing the components $C_{m\mu\nu}$ of the 3-form field. These fields can be labelled as $\mathcal{A}^{(r)}_\mu$, ϕ_{1r} and ϕ_{2r} with $1 \le r \le 16$, and, in the Feynman gauge, the quadratic terms in the action in the near horizon background geometry takes the form:

$$\int d^4x\,\sqrt{\det g}\,\Bigg[\frac{1}{2}\sum_{r=1}^{16} \mathcal{A}^{(r)}_\mu (g^{\mu\nu}\Box - R^{\mu\nu})\mathcal{A}^{(r)}_\nu + \frac{1}{2}\sum_{r=1}^{16}(\phi_{1r}\,\Box\,\phi_{1r} + \phi_{2r}\,\Box\,\phi_{2r})$$
$$+\sum_{r=1}^{8}\Big(2a^{-1}\phi_{2r}\,\varepsilon^{\gamma\beta}\partial_\gamma \mathcal{A}^{(r)}_\beta - a^{-2}\,\phi_{2r}\,\phi_{2r}\Big)$$

$$+\sum_{r=9}^{16}\left(-2i\,a^{-1}\,\phi_{1r}\,\varepsilon^{mn}\partial_m\mathcal{A}_n^{(r)}+a^{-2}\,\phi_{1r}\,\phi_{1r}\right)\Bigg]\,. \tag{8.1}$$

This has the same structure as the bosonic part of the matter multiplet fields analyzed in [46] except that for $1\le r\le 8$ only the components of the gauge fields along S^2 and the scalar fields ϕ_{2r} are affected by the background flux, while for $9\le r\le 16$ only the components of the gauge fields along AdS_2 and the scalar fields ϕ_{1r} are affected by the flux.[19] Thus the analysis proceeds as in [46] and we find, after including the contribution due to the ghosts, that the net contribution to the heat kernel is given by

$$8\left[8K^s_{AdS_2}(0;s)K^s_{S^2}(0;s)+\frac{1}{2\pi a^2}\left\{K^s_{S^2}(0;s)-K^s_{AdS_2}(0;s)\right\}\right]. \tag{8.2}$$

The small s expansion of this can be found by standard methods described earlier and we get the s independent contribution to (8.2) to be

$$\frac{34}{45\pi^2a^4}\,. \tag{8.3}$$

Next we consider the contribution from the extra fermion fields. These fields can be labelled by ψ'_μ $(0\le\mu\le 3)$, Λ' and φ'_r $(4\le r\le 9)$ where for each μ and r, ψ'_μ and φ'_r are 16 component right handed Majorana-Weyl spinor of the ten dimensional Lorentz group, and Λ' is a 16 component left-handed Majorana-Weyl spinor of the ten dimensional Lorentz group. Physically ψ'_μ and φ'_r are the four dimensional and internal components of the ten dimensional gravitino arising in the R-NS sector. In the presence of the background field, the quadratic action of these fermionic fields can be obtained by the dimensional reduction of the ten dimensional action of type IIA supergarvity. The result is:

$$-\frac{1}{2}\Bigg[\bar\psi'_\mu\Gamma^{\mu\nu\rho}D_\nu\psi'_\rho+\bar\Lambda'\Gamma^\mu D_\mu\Lambda'+\sum_{r=4}^{9}\bar\varphi'_r\Gamma^\mu D_\mu\varphi'_r-\frac{1}{2}\bar\psi'_\mu\Gamma^\mu\Gamma^\nu D_\nu\Gamma^\rho\psi'_\rho$$
$$-\frac{1}{2\sqrt{2}}\Bigg\{\left(-\bar\psi'_\rho\Gamma^{\mu\nu}\Gamma^\rho+\sqrt{2}\bar\Lambda'\Gamma^{\mu\nu}\right)(\varphi'_4\bar F^1_{\mu\nu}+\varphi'_5\bar F^2_{\mu\nu})$$

[19] This is not an accident but follows from the following considerations. We could have gotten an $\mathcal{N}=4$ supergravity theory from the original $\mathcal{N}=8$ supergravity by projecting out all fields which are odd under $\mathcal{I}_4$ where $\mathcal{I}_4$ represents the transformation that changes the sign of the coordinates $x^6,\cdots x^9$. The eight vectors $C_{mn\mu}$, $C_{45\mu}$ and A_μ with $6\le m,n\le 9$, $0\le\mu\le 3$ and the sixteen scalars C_{mn4}, C_{mn5}, A_4, A_5 and the duals of $C_{4\mu\nu}$, $C_{5\mu\nu}$ survive the projection. Four of the gauge fields and the 16 scalars will form part of the bosonic sector the four matter multiplets. For completing the matter multiplets we need eight more scalars which will come from the components of the NSNS 2-form field and the metric along 6789 directions, but from the analysis of [46] we know that these describe free scalars in $AdS_2\times S^2$ background. Thus the net contribution from the eight vector and sixteen RR scalar fields to the heat kernel will be given by that of four matter multiplets of $\mathcal{N}=4$ supergravity minus eight free scalar fields. The four remaining gauge fields will describe the four non-interacting vector fields of the gravity multiplet, and their contribution to the heat kernel will be given by that of four vector fields in $AdS_2\times S^2$ as given in (5.4). For the RR fields which are odd under $\mathcal{I}_4$ we can repeat the argument by using projection by the operator $(-1)^{F_L}\times\mathcal{I}_4$, — this will pick the complementary set. Thus in total the contribution to the heat kernel will be given by that of the bosonic sector of eight matter multiplets of the $\mathcal{N}=4$ theory, plus that of eight free vector fields minus that of sixteen free scalar fields on $AdS_2\times S^2$. This is precisely (8.2).

$$+(\bar{\varphi}'_4\bar{F}^1_{\mu\nu}+\bar{\varphi}'_5\bar{F}^2_{\mu\nu})\left(-\Gamma^\rho\Gamma^{\mu\nu}\psi'_\rho-\sqrt{2}\Gamma^{\mu\nu}\Lambda'\right)\Bigg\}\Bigg], \tag{8.4}$$

where the last term in the first line is the gauge fixing term. This also leads to ghosts which have the same action as given in the last two terms in (4.13), except that the new ghost fields $\tilde{b}'$, $\tilde{c}'$ and $\tilde{e}'$ have opposite ten dimensional chirality compared to the superghosts $\tilde{b}$, $\tilde{c}$ and $\tilde{e}$ respectively of $\mathcal{N}=4$ supergravity. Using (4.6) we can express (8.4) as

$$\begin{aligned}
&-\frac{1}{2}\Bigg[\bar{\psi}'_\mu\Gamma^{\mu\nu\rho}D_\nu\psi'_\rho-\frac{1}{2}\bar{\psi}'_\mu\Gamma^\mu\Gamma^\nu D_\nu\Gamma^\rho\psi'_\rho+\bar{\Lambda}'\Gamma^\mu D_\mu\Lambda'+\sum_{r=4}^{9}\bar{\varphi}'_r\Gamma^\mu D_\mu\varphi'_r\\
&+\frac{1}{2a}\Bigg\{\left(\bar{\psi}'_m\Gamma^m-\bar{\psi}'_\alpha\Gamma^\alpha+\sqrt{2}\bar{\Lambda}'\right)\tau_3\varphi'_4+i\left(-\bar{\psi}'_m\Gamma^m+\bar{\psi}'_\alpha\Gamma^\alpha+\sqrt{2}\bar{\Lambda}'\right)\sigma_3\varphi'_5\\
&\qquad+\bar{\varphi}'_4\tau_3\left(\Gamma^m\psi'_m-\Gamma^\alpha\psi'_\alpha-\sqrt{2}\Lambda'\right)+i\bar{\varphi}'_5\sigma_3\left(-\Gamma^m\psi'_m+\Gamma^\alpha\psi'_\alpha-\sqrt{2}\Lambda'\right)\Bigg\}\Bigg].
\end{aligned} \tag{8.5}$$

As in (6.1) we shall express this as

$$-\frac{1}{2}\left[\sum_{r=6}^{9}\bar{\varphi}'_r\Gamma^\mu D_\mu\varphi'_r+\left(\bar{\Lambda}'\mathcal{K}^{(1)}+\bar{\psi}'^\alpha\mathcal{K}^{(2)}_\alpha+\bar{\psi}'^m\mathcal{K}^{(3)}_m+\bar{\varphi}'_4\mathcal{K}^{(4)}+\bar{\varphi}'_5\mathcal{K}^{(5)}\right)\right], \tag{8.6}$$

where

$$\begin{aligned}
\mathcal{K}^{(1)} &= (\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2})\Lambda'+\frac{1}{\sqrt{2}\,a}(\tau_3\varphi'_4+i\sigma_3\varphi'_5),\\
\mathcal{K}^{(2)}_\alpha &= -\frac{1}{2}\Gamma^n(\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2})\Gamma_\alpha\psi'_n-\frac{1}{2}\Gamma^\beta\left(\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2}\right)\Gamma_\alpha\psi'_\beta-\frac{1}{2a}\Gamma_\alpha(\tau_3\varphi'_4-i\sigma_3\varphi'_5)\\
\mathcal{K}^{(3)}_m &= -\frac{1}{2}\Gamma^\beta(\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2})\Gamma_m\psi'_\beta-\frac{1}{2}\Gamma^n\left(\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2}\right)\Gamma_m\psi'_n+\frac{1}{2a}\Gamma_m(\tau_3\varphi'_4-i\sigma_3\varphi'_5)\\
\mathcal{K}^{(4)} &= (\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2})\varphi'_4+\frac{1}{2a}\tau_3\left(\Gamma^m\psi'_m-\Gamma^\alpha\psi'_\alpha-\sqrt{2}\Lambda'\right)\\
\mathcal{K}^{(5)} &= (\not{D}_{S^2}+\sigma_3\not{D}_{AdS_2})\varphi'_5+\frac{i}{2a}\sigma_3\left(-\Gamma^m\psi'_m+\Gamma^\alpha\psi'_\alpha-\sqrt{2}\Lambda'\right).
\end{aligned} \tag{8.7}$$

The fields $\varphi'_6,\cdots\varphi'_9$ represent free fermions in $AdS_2\times S^2$ background, and the net contribution from these fields to the heat kernel is given by [46]

$$\begin{aligned}
&-\frac{4}{\pi^2a^4}\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}\lambda^2}\sum_{l=0}^{\infty}(2l+2)e^{-\bar{s}(l+1)^2}\\
&=-\frac{2}{\pi^2a^4\bar{s}^2}\left(1-\frac{11}{180}\bar{s}^2+\mathcal{O}(\bar{s}^3)\right).
\end{aligned} \tag{8.8}$$

The overall normalization is fixed by noting that each of the φ'_r's represent four Majorana fermions in four dimensions. Thus altogether we have 16 Majorana or equivalently eight Dirac fermions. The overall minus sign is a reflection of the fact that the path integral over the fermions gives the determinant of the kinetic operator instead of the inverse of the determinant.

For computing the contribution from the other fields we expand them as in (6.6)

$$\begin{aligned}
\Lambda' &= a_1\chi + a_2\sigma_3\chi \\
\psi'_\alpha &= b_1\Gamma_\alpha\chi + b_2\sigma_3\Gamma_\alpha\chi + b_3 D_\alpha\chi + b_4\sigma_3 D_\alpha\chi \\
\psi'_m &= c_1\Gamma_m\chi + c_2\sigma_3\Gamma_m\chi + c_3\sigma_3 D_m\chi + c_4 D_m\chi \\
\varphi'_4 &= \tau_3(h_1\chi + h_2\sigma_3\chi) \qquad (8.9)\\
\varphi'_5 &= (g_1\chi + g_2\sigma_3\chi) \qquad (8.10)
\end{aligned}$$

where a_i, b_i, c_i, h_i and g_i are grassman parameters and χ is the product of an arbitrary spinor of the SO(6) Clifford algebra generated by $\widetilde{\Gamma}^4, \cdots \widehat{\Gamma}^9$, χ^+_{lm} (or η^+_{lm}) defined in (2.16) and $\chi^\pm_k(\lambda)$ (or $\eta^\pm_k(\lambda)$) defined in (2.20). χ satisfies

$$\not{D}_{S^2}\chi = i\zeta_1\,\chi, \quad \not{D}_{AdS_2}\chi = i\zeta_2\,\chi, \quad \zeta_1 > 0\,. \qquad (8.11)$$

As in (6.7), we expand $\mathcal{K}^{(1)}, \cdots \mathcal{K}^{(5)}$ as

$$\begin{aligned}
\mathcal{K}^{(1)} &= A_1\chi + A_2\sigma_3\chi \\
\mathcal{K}^{(2)}_\alpha &= B_1\Gamma_\alpha\chi + B_2\sigma_3\Gamma_\alpha\chi + B_3 D_\alpha\chi + B_4\sigma_3 D_\alpha\chi \\
\mathcal{K}^{(3)}_m &= C_1\Gamma_m\chi + C_2\sigma_3\Gamma_m\chi + C_3\sigma_3 D_m\chi + C_4 D_m\chi \\
\mathcal{K}^{(4)} &= \tau_3(H_1\chi + H_2\sigma_3\chi) \\
\mathcal{K}^{(5)} &= (G_1\chi + G_2\sigma_3\chi) \qquad (8.12)
\end{aligned}$$

Explicit computation yields

$$\begin{aligned}
A_1 &= i\zeta_1 a_1 + i\zeta_2 a_2 + \frac{1}{\sqrt{2}a}h_1 + \frac{i}{\sqrt{2}a}g_2 \\
A_2 &= i\zeta_2 a_1 - i\zeta_1 a_2 + \frac{1}{\sqrt{2}a}h_2 + \frac{i}{\sqrt{2}a}g_1 \\
B_1 &= -i\zeta_1 b_1 + \frac{1}{2}\zeta_1^2 b_3 + \frac{1}{2}\zeta_1\zeta_2 b_4 + i\zeta_1 c_1 - \frac{1}{2}\zeta_1\zeta_2 c_3 + \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right)c_4 - \frac{1}{2a}h_1 + \frac{i}{2a}g_2 \\
B_2 &= i\zeta_1 b_2 + \frac{1}{2}\zeta_1\zeta_2 b_3 - \frac{1}{2}\zeta_1^2 b_4 + i\zeta_1 c_2 - \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right)c_3 - \frac{1}{2}\zeta_1\zeta_2 c_4 + \frac{1}{2a}h_2 - \frac{i}{2a}g_1 \\
B_3 &= i\zeta_2 b_4 - 2c_1 - i\zeta_2 c_3 \\
B_4 &= i\zeta_2 b_3 - 2c_2 - i\zeta_2 c_4 \\
C_1 &= -i\zeta_2 b_2 + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right)b_3 + \frac{1}{2}\zeta_1\zeta_2 b_4 - i\zeta_2 c_2 - \frac{1}{2}\zeta_1\zeta_2 c_3 + \frac{1}{2}\zeta_2^2 c_4 + \frac{1}{2a}h_1 - \frac{i}{2a}g_2 \\
C_2 &= i\zeta_2 b_1 - \frac{1}{2}\zeta_1\zeta_2 b_3 + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right)b_4 - i\zeta_2 c_1 + \frac{1}{2}\zeta_2^2 c_3 + \frac{1}{2}\zeta_1\zeta_2 c_4 + \frac{1}{2a}h_2 - \frac{i}{2a}g_1 \\
C_3 &= 2b_2 + i\zeta_1 b_4 - i\zeta_1 c_3 \\
C_4 &= -2b_1 - i\zeta_1 b_3 + i\zeta_1 c_4 \\
H_1 &= i\zeta_1 h_1 - i\zeta_2 h_2 - \frac{1}{\sqrt{2}a}a_1 - \frac{1}{a}b_1 - \frac{i}{2a}\zeta_1 b_3 + \frac{1}{a}c_1 + \frac{i}{2a}\zeta_2 c_3 \\
H_2 &= -i\zeta_2 h_1 - i\zeta_1 h_2 - \frac{1}{\sqrt{2}a}a_2 + \frac{1}{a}b_2 + \frac{i}{2a}\zeta_1 b_4 + \frac{1}{a}c_2 + \frac{i}{2a}\zeta_2 c_4
\end{aligned}$$

$$G_1 = i\zeta_1 g_1 + i\zeta_2 g_2 - \frac{i}{\sqrt{2}a} a_2 - \frac{i}{a} b_2 + \frac{1}{2a}\zeta_1 b_4 - \frac{i}{a} c_2 + \frac{1}{2a}\zeta_2 c_4$$
$$G_2 = i\zeta_2 g_1 - i\zeta_1 g_2 - \frac{i}{\sqrt{2}a} a_1 + \frac{i}{a} b_1 - \frac{1}{2a}\zeta_1 b_3 - \frac{i}{a} c_1 + \frac{1}{2a}\zeta_2 c_3 \,. \tag{8.13}$$

We can express this as

$$\begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \\ \vec{H} \\ \vec{G} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{h} \\ \vec{g} \end{pmatrix}, \tag{8.14}$$

$\mathcal{M}$ being a 14×14 matrix. Let us also introduce a matrix $\mathcal{M}_1$ through

$$\mathcal{M}^2 = -(\zeta_1^2 + \zeta_2^2) I_{14} + a^{-2}\mathcal{M}_1 \,, \tag{8.15}$$

where I_{14} denotes the 14×14 identity matrix, and denote by β_k for $1 \le k \le 14$ the 14 eigenvalues of the matrix $\mathcal{M}_1$. Then following the logic leading to (6.14) one can show that the contribution to the heat kernel from the fermionic modes for $|\zeta_1| > 1$, i.e. $l > 0$, will be given by

$$K^f_{(1)}(0;s) = -\frac{1}{2\pi^2 a^4} \sum_{l=1}^{\infty} (2l+2) \int_0^\infty d\lambda \lambda \coth(\pi\lambda) e^{-\bar{s}(l+1)^2 - \bar{s}\lambda^2} \sum_{k=1}^{14} e^{\bar{s}\beta_k} \,, \tag{8.16}$$

where l and λ are related to ζ_1 and ζ_2 via

$$|\zeta_1| = (l+1)/a, \qquad \zeta_2 = \lambda/a \,. \tag{8.17}$$

The overall normalization is fixed by noting that ψ'_μ, φ'_r for $r = 4,5$ and Λ' altogether has degrees of freedom equal to that of $(4+2+1) \times 4 = 28$ Majorana fermions or equivalently 14 Dirac fermions.

The contribution from the $|\zeta_1| = 1/a$, i.e. $l = 0$ term has to be evaluated separately following the same logic that lead to (6.16). We choose the coefficients b_3 and b_4 to be zero and replace in (8.13) the expressions for B_k by that of $B_k + \frac{i}{2a} B_{k+2}$ for $k = 1,2$. This leads to a 12×12 matrix $\widetilde{\mathcal{M}}$. We now define a matrix $\widetilde{\mathcal{M}}_1$ through

$$\widetilde{\mathcal{M}}^2 = -(a^{-2} + \zeta_2^2) I_{12} + a^{-2}\widetilde{\mathcal{M}}_1 \,, \tag{8.18}$$

where I_{12} denotes the 12×12 identity matrix. If $\widetilde{\beta}_k$'s are the eigenvalues of $\widetilde{\mathcal{M}}_1$ then the contribution from the $l = 0$ modes to the heat kernel may be expressed as

$$K^f_{(2)}(0;s) = -\frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda \coth(\pi\lambda) e^{-\bar{s} - \bar{s}\lambda^2} \sum_{k=1}^{12} e^{\bar{s}\widetilde{\beta}_k} \,. \tag{8.19}$$

We can combine (8.16) and (8.19) to write

$$K^f_{(1)}(0;s) + K^f_{(2)}(0;s) = \widetilde{K}^f_{(1)}(0;s) + \widetilde{K}^f_{(2)}(0;s) \,, \tag{8.20}$$

where

$$\widetilde{K}^f_{(1)}(0;s) = -\frac{1}{2\pi^2 a^4}\sum_{l=0}^{\infty}(2l+2)\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}(l+1)^2-\bar{s}\lambda^2}\sum_{k=1}^{14}e^{\bar{s}\beta_k}$$

$$= -\frac{1}{\pi^2 a^4}\,Im\int_0^{e^{i\kappa}\times\infty}d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}\widetilde{\lambda}^2-\bar{s}\lambda^2}\sum_{k=1}^{14}e^{\bar{s}\beta_k|_{l+1\to\widetilde{\lambda}}} \tag{8.21}$$

$$\widetilde{K}^f_{(2)}(0;s) = -\frac{1}{\pi^2 a^4}\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}-\bar{s}\lambda^2}\left[\sum_{k=1}^{12}e^{\bar{s}\widetilde{\beta}_k}-\sum_{k=1}^{14}e^{\bar{s}\beta_k|_{l=0}}\right]. \tag{8.22}$$

We also need to compute the contribution due to the discrete modes described in (2.28). For this we set the fields Λ', ψ'_α, φ'_4 and φ'_5 to 0, and expand ψ'_m as in (8.9) with $c_{k+2}=2c_k a$ for $k=1,2$, with $\zeta_2=i/a$, $|\zeta_1|\geq 1/a$, i.e. $l\geq 0$. It can be seen that with this choice A_i, B_i, H_i, G_i computed from (8.13) vanish and we have $C_{k+2}=2C_k a$ for $1\leq k\leq 2$. Thus we can express these relations as

$$\begin{pmatrix}C_1\\C_2\end{pmatrix} = \widehat{\mathcal{M}}\begin{pmatrix}c_1\\c_2\end{pmatrix}, \tag{8.23}$$

for some 2×2 matrix $\widehat{\mathcal{M}}$. If $\widehat{\beta}_k$ denote the eigenvalues of $a^2\{\widehat{\mathcal{M}}^2+(\zeta_1^2-a^{-2})I_2\}$ then the contribution to $K(0;s)$ from these modes is given by

$$K^f_{(3)}(0;s) = -\frac{1}{2\pi^2 a^4}\sum_{l=0}^{\infty}(2l+2)e^{\bar{s}-\bar{s}(l+1)^2}\sum_{k=1}^{2}e^{\bar{s}\widehat{\beta}_k}$$

$$= -\frac{1}{\pi^2 a^4}\,Im\int_0^{e^{i\kappa}\times\infty}d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\,e^{\bar{s}-\bar{s}\widetilde{\lambda}^2}\sum_{k=1}^{2}e^{\bar{s}\widehat{\beta}_k|_{l+1\to\widetilde{\lambda}}}, \tag{8.24}$$

Explicit computation using (8.13) gives $\widehat{\beta}_1=\widehat{\beta}_2=-1$. Hence we have

$$K^f_{(3)}(0;s) = -\frac{2}{\pi^2 a^4}\,Im\int_0^{e^{i\kappa}\times\infty}d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\,e^{-\bar{s}\widetilde{\lambda}^2}. \tag{8.25}$$

Finally the three sets of bosonic ghosts $\widetilde{b}'$, $\widetilde{c}'$ and $\widetilde{e}'$ associated with gauge fixing of local supersymmetry, each of which gives rise to four Majorana fermions in four dimensions, contributes

$$K^f_{\text{ghost}} = \frac{3}{\pi^2 a^4}\sum_{l=0}^{\infty}(2l+2)\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}(l+1)^2-\bar{s}\lambda^2}$$

$$= \frac{6}{\pi^2 a^4}\,Im\int_0^{e^{i\kappa}\times\infty}d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-\bar{s}\widetilde{\lambda}^2-\bar{s}\lambda^2}, \tag{8.26}$$

to $K(0;s)$.

To evaluate the right hand sides of (8.21) and (8.22) we use the relations

$$\sum_k e^{\bar{s}\beta_k} = \sum_{n=0}^{\infty}\frac{1}{n!}\bar{s}^n\sum_k\beta_k^n = \sum_{n=0}^{\infty}\frac{1}{n!}\bar{s}^n Tr(\mathcal{M}_1^n), \tag{8.27}$$

$$\sum_k e^{\bar{s}\widetilde{\beta}_k} = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{s}^n \sum_k \widetilde{\beta}_k^n = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{s}^n Tr(\widetilde{\mathcal{M}}_1^n)\,. \tag{8.28}$$

Explicit computation gives

$$\begin{aligned}
Tr(\mathcal{M}_1) &= 0 \\
Tr(\mathcal{M}_1^2) &= -8 + 16(l+1)^2 - 16\lambda^2 \\
Tr(\mathcal{M}_1^3) &= -6(l+1)^2 - 6\lambda^2 \\
Tr(\mathcal{M}_1^4) &= 8 - 16(l+1)^2 + 40(l+1)^4 + 16\lambda^2 - 48(l+1)^2\lambda^2 + 40\lambda^4\,.
\end{aligned} \tag{8.29}$$

$$\begin{aligned}
Tr(\widetilde{\mathcal{M}}_1) &= -2 \\
Tr(\widetilde{\mathcal{M}}_1^2) &= 6 - 16\lambda^2 \\
Tr(\widetilde{\mathcal{M}}_1^3) &= -8 - 6\lambda^2 \\
Tr(\widetilde{\mathcal{M}}_1^4) &= 30 - 32\lambda^2 + 40\lambda^4\,.
\end{aligned} \tag{8.30}$$

Following the procedure of section 6 we can now carry out the small s expansion of the heat kernels. We get the following contribution to the order s^0 term in the small s expansion of various terms:

$$\begin{aligned}
\widetilde{K}^f_{(1)}(0;s) &: -\frac{43}{360\pi^2 a^4} \\
\widetilde{K}^f_{(2)}(0;s) &: \frac{1}{6\pi^2 a^4} \\
K^f_{(3)}(0;s) &: \frac{1}{6\pi^2 a^4} \\
K^f_{\text{ghost}}(0;s) &: -\frac{11}{120\pi^2 a^4}\,.
\end{aligned} \tag{8.31}$$

Adding up all the contributions in eq. (8.31) and the contribution from (8.8) we get the net contribution to $K(0;s)$ from the extra fermionic fields of $\mathcal{N}=8$ supergravity:

$$K^f(0;s) = \frac{11}{45\pi^2 a^4}\,. \tag{8.32}$$

Adding (8.32) to the bosonic contribution (8.3) we get the net contribution to the order s^0 terms in the heat kernel from all the extra fields appearing in $\mathcal{N}=8$ supergravity:

$$\frac{1}{\pi^2 a^4}\,. \tag{8.33}$$

It is also easy to see that the only zero modes among these extra fields arise from the gauge fields. In particular there are no fermion zero modes since both the $\widehat{\beta}_k$'s in (8.24) take the value -1 for $l=0$. Now we have already seen that for the gauge fields the integration over the zero modes gives us back the same result that we remove from the heat kernel. Thus removing the zero mode contribution of the sixteen gauge fields from the heat kernel and then including the contribution due to the zero mode integrals does not give any net contribution, and (8.33) represents the net extra contribution to the heat kernel from the extra fields of $\mathcal{N}=8$ supergravity. Since for the $\mathcal{N}=4$ supergravity the net s independent

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contribution to the effective heat kernel vanished, (8.33) represents the net contribution in $\mathcal{N}=8$ supergravity. According to (3.26) this gives a logarithmic correction to the black hole entropy of the form:

$$-4\ln a^2 = -2\ln\Delta\,. \tag{8.34}$$

This is in perfect agreement with the microscopic answer (1.2).

For identifying separately the contributions from the zero modes and the non-zero modes we note that the $\mathcal{N}=8$ supergravity has 28 gauge fields whose zero mode contribution to the entropy is $-28\ln a = -7\ln\Delta$. This represents the net zero mode contribution since the contribution from the graviton and the gravitino zero modes cancel. The rest of the contribution $5\ln\Delta$ comes from non-zero modes.

9 Half BPS black holes in STU model

Our analysis also gives the result for logarithmic corrections to the entropy of half BPS black holes in the STU model [57, 58] which has been studied recently in [59, 73, 74] in the context of black hole entropy. The STU model is constructed by beginning with type IIA string theory on $T^4\times T^2$ and taking an orbifold of this theory with a $\mathbb{Z}_2\times\mathbb{Z}_2$ group. The first $\mathbb{Z}_2$ acts as $(-1)^{F_L}$ times half a unit of shift along one of the circles of T^2 and the second $\mathbb{Z}_2$ acts as $\mathcal{I}_4$ times a shift along the second circle of T^2 where $\mathcal{I}_4$ denotes changing the sign of all the coordinates of T^4. If we label the two circles of T^2 by x^4 and x^5 then the black hole solution described at the beginning of section 4 survives the orbifold projection and hence continues to describe a black hole solution in this theory. The first $\mathbb{Z}_2$ projection removes from the spectrum all the masless RR and R-NS sector states and hence the low energy theory is an $\mathcal{N}=4$ supergravity theory, — with a structure identical to that of heterotic string theory on T^6 except that the sixteen matter multiplet fields associated with the dimensional reduction of ten dimensional $E_8\times E_8$ gauge fields are absent. The action of the second $\mathbb{Z}_2$ orbifold projection breaks the $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=2$. Under this a matter multiplet of $\mathcal{N}=4$ supergravity decomposes into a vector multiplet and a hypermultiplet, and we need to examine which components of the fields survive the projection. Similarly the gravity multiplet fields of the $\mathcal{N}=4$ supergravity decompose into different supermultiplets of $\mathcal{N}=2$ supergravity, and only some of these survive the orbifold projection.

For later use it is useful to note that in the fermionic sector the orbifold operation projects onto modes which are even under the action of Γ^{6789} accompanied by $(x^6,\cdots x^9)\to(-x^6,\cdots -x^9)$. Using the ten dimensional chirality of Λ and the ten dimensional gravitino field ψ_M $(0\le M\le 9)$, this condition translates to

$$\sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\psi_\mu = i\psi_\mu\,,\quad \sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\psi_{4,5} = i\psi_{4,5}\,,\quad \sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\psi_{6,7,8,9} = -i\psi_{6,7,8,9},\quad \sigma_3\tau_3\widehat\Gamma^4\widehat\Gamma^5\Lambda = -i\Lambda\,, \tag{9.1}$$

together with similar projection on the ghost fields.

Let G_{MN} and B_{MN} be the ten dimensional metric and NSNS 2-form fields. We begin with the two matter multiplet fields of $\mathcal{N}=4$ supergravity whose vector fields come from $G_{4\mu}-B_{4\mu}$ and $G_{5\mu}-B_{5\mu}$. Their scalar partners are G_{44}, G_{45}, G_{55}, B_{45}, $G_{4m}-B_{4m}$ and $G_{5m}-B_{5m}$ for $6\le m\le 9$. Under the orbifold projection the two vector fields as well as the scalars G_{44}, G_{45}, G_{55}, B_{45} survive, but the rest of the scalars are projected out. The surviving fields belong to two vector multiplets of $\mathcal{N}=2$ supersymmetry. The contribution to the heat kernel from these scalar and vector fields and the ghosts associated with the vector fields can be read out from the results of [46]. The vector couples to the two scalars due to the presence of the background flux and the net contribution to the heat kernel from the bosonic fields (including the ghosts) is given by $4K^s(0;s)$. There are zero modes of the gauge fields whose contribution needs to be removed from this and then added separately, but as we have seen before, this does not change the result.

The fermionic components of these two matter multiplets come from the components ψ_4 and ψ_5 of the ten dimensional gravitino. As was shown in [46], acting on these fermions, the kinetic operator takes the form:

$$\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2} - \frac{i}{2} a^{-1} \widehat{\Gamma}^5 \tau_3 - \frac{1}{2} a^{-1} \sigma_3 \widehat{\Gamma}^4 . \tag{9.2}$$

It follows from (9.1) that acting on the fields $\psi_{4,5}$ the last two terms in (9.2) cancel and the kinetic operator reduces to $\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2}$, i.e. that of a free fermion in $AdS_2 \times S^2$. The heat kernel of this is given by 1/8 of the contribution shown in (8.8). Adding this to the bosonic contribution $4K^s(0;s)$ given in (3.20) we get the net contribution to the heat kernel from each of the vector multiplets to be:

$$K^{vector}(0;s) = \frac{1}{180\pi^2 a^4} + \frac{11}{720\pi^2 a^4} + \cdots = \frac{1}{48\pi^2 a^4} + \cdots . \tag{9.3}$$

This corresponds to a correction of $-\frac{1}{24} \ln \Delta$ per vector multiplet, i.e. a total of $-\frac{1}{12} \ln \Delta$ to the black hole entropy from the two vector multiplets coming from the two matter multiplets of $\mathcal{N}=4$ supergravity.

Next we turn to the four matter multiplet fields of $\mathcal{N}=4$ supergravity whose vector fields come from $G_{m\mu} - B_{m\mu}$ where m is along T^4 and μ is along the non-compact direction. Their scalar components are G_{mn}, B_{mn}, $G_{m4} - B_{m4}$ and $G_{m5} - B_{m5}$. Under the orbifold projection the scalars G_{mn}, B_{mn} survive but the vector fields as well as the scalars $G_{m4} - B_{m4}$ and $G_{m5} - B_{m5}$ are projected out. This corresponds to removing the vector multiplets and keeping the hypermltiplet fields. Since the net contribution to $K(0;s)$ from a hypermultiplet and a vector multiplet vanishes, we could directly conclude that the hypermultiplet contribution to $K(0;s)$ will be negative of the contribution (9.3) from the vector multiplet. However it is instructive to carry out the computation directly. For each hypermultiplet we have four scalars without any coupling to the background gauge fields, and their contribution to the heat kernel is given by $4K^s(0;s)$. In the fermionic sector we have the fields $\psi_6, \cdots \psi_9$ subject to the orbifold projection (9.1). This makes the contribution from the last two terms in (9.2) identical, and we can express the operator as $\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2} - i\, a^{-1} \widehat{\Gamma}^5 \tau_3$. We need to compute its determinant on the subspace of states subject to the projection (9.1). Now note that since $\sigma_3 \tau_3 \widehat{\Gamma}^4$ anti-commutes with the projection operator it takes a state satisfying the orbifold projection to a state satisfying the opposite projection and vice versa. Since it also anti-commutes with the kinetic operator $\not{D}_{S^2} + \sigma_3 \not{D}_{AdS_2} - i\, a^{-1} \widehat{\Gamma}^5 \tau_3$, the action of $\sigma_3 \tau_3 \widehat{\Gamma}^4$ changes the eigenvalue of the kinetic operator. Thus we see that the matrix representing the kinetic operator in the subspace satisfying opposite projection is just the negative of the kinetic operator acting on the subspace satisfying the correct projection. Thus we could evaluate the determinant ignoring the projection condition and then take the square root of the modulus of the determinant. We now note that in the unprojected space the operator $\not{D}_{S^2} - i\,a^{-1}\,\widehat{\Gamma}^5\,\tau_3$ anti-commutes with the operator $\sigma_3\not{D}_{AdS_2}$. Thus the squares of the eigenvalues of $\not{D}_{S^2} + \sigma_3\not{D}_{AdS_2} - i\,a^{-1}\,\widehat{\Gamma}^5\,\tau_3$ will be given by the sum of the squares of the eigenvalues of $\not{D}_{S^2} - i\,a^{-1}\,\widehat{\Gamma}^5\,\tau_3$ and $\not{D}_{AdS_2}$. Of these $\not{D}_{AdS_2}$ has eigenvalues $\pm i a^{-1}\lambda$. On the other hand since $\not{D}_{S^2}$ has eigenvalues $\pm i a^{-1}(l+1)$ and $\widehat{\Gamma}^5\,\tau_3$ has eigenvalues ± 1, and they act on independent spaces, the eigenvalues of $\not{D}_{S^2} - i\,a^{-1}\,\widehat{\Gamma}^5\,\tau_3$ are given by $\pm i a^{-1}(l+1\pm 1)$ with $l = 0, 1, \cdots \infty$. This gives the net contribution to $K(0;s)$ from the fermionic components of the hypermultiplet to be

$$-\frac{1}{2\pi^2 a^4} Im \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\coth\pi\widetilde{\lambda} \int_0^{\infty} d\lambda\,\lambda\,\coth\pi\lambda\, e^{-s\lambda^2 - s\widetilde{\lambda}^2} \left[e^{-s-2s\widetilde{\lambda}} + e^{-s+2s\widetilde{\lambda}}\right]. \tag{9.4}$$

We can evaluate this by expanding the term in the square bracket in a power series in s, or by shifting the sum over l as in [46]. Both ways give the same result and adding this to the scalar contribution $4K^s(0;s)$ we get the contribution to the heat kernel from each hypermultiplet fields to be

$$K^{hyper}(0;s) = -\frac{1}{48\pi^2 a^4} + \cdots . \tag{9.5}$$

This corresponds to a correction of $\frac{1}{24}\ln\Delta$ per hypermultiplet, i.e. a total of $\frac{1}{6}\ln\Delta$ to the black hole entropy from the four hypermultiplets coming from the matter multiplets of $\mathcal{N}=4$ supergravity.

Finally we have to compute the contribution from the fields which survive from the gravity multiplet of $\mathcal{N}=4$ supergravity. In the bosonic sector the four gauge fields $G_{m\mu} + B_{m\mu}$ for m along T^4 are projected out but all other fields survive. Thus we need to remove the contribution given by (5.4) together with a contribution of $-8K^s_{AdS_2\times S^2}(0;s)$ representing the contribution of the eight ghost fields associated with these four gauge fields. The small s expansion of this is given by [46]:

$$\frac{1}{\pi^2 a^4 \bar{s}^2}\left(1 + \frac{16}{45}\bar{s}^2 - \frac{1}{2} - \frac{1}{90}\bar{s}^2 + \cdots\right) = \frac{1}{\pi^2 a^4 \bar{s}^2}\left(\frac{1}{2} + \frac{31}{90}\bar{s}^2 + \cdots\right), \tag{9.6}$$

where the $-1/2 - \bar{s}^2/90$ is the contribution due to the ghosts. Of these $4/8\pi^2 a^4$ can be identified as the contribution due to the gauge field zero modes. Thus the net non-zero mode contribution from these four gauge fields is $31/90\pi^2 a^4 - 1/2\pi^2 a^4 = -7/45\pi^2 a^4$, — this needs to be removed from the non-zero mode contribution (5.45) from the gravity multiplet of full $\mathcal{N}=4$ supergravity. Thus the net s-independent contribution to the heat kernel from the bosonic non-zero modes of $\mathcal{N}=4$ gravity multiplet which survive the orbifold projection is given by:

$$-\frac{101}{180\pi^2 a^4} + \frac{7}{45\pi^2 a^4} = -\frac{73}{180\pi^2 a^4}. \tag{9.7}$$

The fermionic components of the $\mathcal{N}=4$ gravity multiplet are given by ψ_μ and Λ, but we need to work in the subspace of these fermions which satisfy the conditions (9.1).

This requires us to impose the following restriction on the various coefficients appearing in section 6:

$$
\begin{array}{llll}
a_4 = i\, a_1, & a_3 = -i\, a_2, & a'_4 = i\, a'_1, & a'_3 = -i\, a'_2, \\
b_4 = -i\, b_1, & b_3 = i\, b_2, & b_8 = -i\, b_5, & b_7 = i\, b_6 \\
b'_4 = -i\, b'_1, & b'_3 = i\, b'_2, & b'_8 = -i\, b'_5, & b'_7 = i\, b'_6 \\
c_4 = -i\, c_1, & c_3 = i\, c_2, & c_8 = -i\, c_5, & c_7 = i\, c_6 \\
c'_4 = -i\, c'_1, & c'_3 = i\, c'_2, & c'_8 = -i\, c'_5, & c'_7 = i\, c'_6\,.
\end{array} \tag{9.8}
$$

Furthermore after the action of the kinetic operator on the fields the result will be a fermion of opposite chirality and hence the coefficients $A_i, A'_i, B_i, B'_i, C_i, C'_i$ are no longer all independent. This allows us to remove half of these coefficients and keep A_i, A'_i for $i = 1,2$ and B_i, B'_i, C_i, C'_i for $i = 1,2,5,6$ as the independent constants labelling the state obtained by the action the kinetic operator on the fields. This essentially halves the dimensions of all the matrices $\mathcal{M}_1$, $\widetilde{\mathcal{M}}_1$ and $\widehat{\mathcal{M}}_1$ appearing in section 6. The rest of the analysis proceeds exactly as in section 6, and we find the following results for the traces of the various matrices:

$$
\begin{aligned}
Tr(\mathcal{M}_1) &= 16 \\
Tr(\mathcal{M}_1^2) &= 64 - 32(l+1)^2 - 32\lambda^2 \\
Tr(\mathcal{M}_1^3) &= 256 - 192(l+1)^2 - 192\lambda^2 \\
Tr(\mathcal{M}_1^4) &= 1024 - 1024(l+1)^2 + 128(l+1)^4 - 1024\lambda^2 + 256(l+1)^2\lambda^2 + 128\lambda^4\,.
\end{aligned} \tag{9.9}
$$

$$
\begin{aligned}
Tr(\widetilde{\mathcal{M}}_1) &= 8 \\
Tr(\widetilde{\mathcal{M}}_1^2) &= 16 - 16\lambda^2 \\
Tr(\widetilde{\mathcal{M}}_1^3) &= 32 - 96\lambda^2 \\
Tr(\widetilde{\mathcal{M}}_1^4) &= 64 - 384\lambda^2 + 64\lambda^4\,.
\end{aligned} \tag{9.10}
$$

$$
\begin{aligned}
Tr(\widehat{\mathcal{M}}_1) &= 0 \\
Tr(\widehat{\mathcal{M}}_1^2) &= 0 \\
Tr(\widehat{\mathcal{M}}_1^3) &= 0 \\
Tr(\widehat{\mathcal{M}}_1^4) &= 0\,.
\end{aligned} \tag{9.11}
$$

This in turn gives the following order $\bar{s}^0$ terms in the small $\bar{s}$ expansion of various parts of the fermionic heat kernel:

$$
\begin{aligned}
\widetilde{K}^f_{(1)}(0;s) &: \frac{11}{144\pi^2 a^4} \\
\widetilde{K}^f_{(2)}(0;s) &: -\frac{5}{12\pi^2 a^4} \\
K^f_{(3)}(0;s) &: -\frac{5}{12\pi^2 a^4} \\
K^f_{\text{ghost}}(0;s) &: -\frac{11}{240\pi^2 a^4}\,.
\end{aligned} \tag{9.12}
$$

Adding up all the contributions and subtracting the zero mode contribution $-1/2\pi^2 a^4$ we get the net contribution to $K(0;s)$ from the non-zero modes of the surviving fermionic fields in the $\mathcal{N}=4$ gravity multiplet after the orbifold projection:

$$K^f(0;s) = -\frac{109}{360\pi^2 a^4} + \cdots . \tag{9.13}$$

Adding this to (9.7) we get a net contribution of $-17/24\pi^2 a^4$ from the non-zero modes. On the other hand the zero modes of two gauge fields, the metric and the gravitino gives a net contribution of

$$\frac{2}{8\pi^2 a^4} + \frac{12}{8\pi^2 a^4} - \frac{3}{2\pi^2 a^4} = \frac{1}{4\pi^2 a^4}\,, \tag{9.14}$$

to the effective heat kernel. Adding this to the sum of (9.7) and (9.13) we get

$$-\frac{11}{24\pi^2 a^4}\,, \tag{9.15}$$

leading to a correction of $\frac{11}{12}\ln\Delta$ to the entropy. Adding this to the contribution of $\frac{1}{6}\ln\Delta$ from the four hypermultiplets and $-\frac{1}{12}\ln\Delta$ from the two vector multiplets we arrive at a net correction of

$$\ln\Delta\,, \tag{9.16}$$

to the entropy of a half BPS black hole in the STU model.

We can identify separately the zero mode and the non-zero mode contributions by noting that the four gauge field zero modes give a contribution of $-\ln\Delta$ and the contributions from the metric and the gravitino zero modes cancel. The rest of the contribution $2\ln\Delta$ comes from the non-zero modes.

Finally we note that the analysis of this section can be extended to any $\mathcal{N}=2$ supergravity theory whose low energy effective action can be obtained by a consistent truncation of the $\mathcal{N}=4$ supergravity action in which two of the six vector fields of the gravity multiplet survive. In that case we can consider a black hole solution whose electric and magnetic charges are carried by these vector fields and the analysis of logarithmic corrections proceed in an identical manner. The FHSV model of [75] is another example of such a model.

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