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Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties

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Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

0. Introduction

In this paper we shall develop a theory of logarithmic deformations of normal crossing varieties, and prove that certain normal crossing varieties have flat deformations to smooth varieties.

A *normal crossing variety* is a reduced complex analytic space which is locally isomorphic to a normal crossing divisor on a smooth variety. Moreover, it is called a *simple normal crossing variety* if the irreducible components are smooth. According to Friedman [F], we can define the concept of *d-semistability* for simple normal crossing varieties (§1). For example, the central fiber of a semistable degeneration is a d-semistable simple normal crossing variety.

Let $f:\mathscr{X}\to\Delta$ be a semistable degeneration over a disk Δ such that the general fibers $f^{-1}(t)$ for $t\neq 0$ are smooth K3 surfaces and that the irreducible components of the central fiber $f^{-1}(0)$ are compact Kähler surfaces. Then by Kulikov's theorem on the minimal model of f([Ku] and [PP]), there exists another semistable degeneration $f':\mathscr{X}'\to\Delta$ which is bimeromorphically equivalent to f over Δ and such that the canonical divisor $K_{\mathscr{X}'}$ is trivial. In this case, the central fiber $X=f'^{-1}(0)$ can be easily classified.

Conversely, Friedman's theorem ([F]) states that for an arbitrary d-semistable simple normal crossing compact Kähler surface X with trivial K_X and $H^1(X,\mathscr{O}_X)$, there exists a semistable degeneration f' as above, i.e., the *d-semistable* K3 *surface* X has a *smoothing*.

The purpose of this paper is to give an alternative easy proof of Friedman's theorem, and generalize it to higher dimensional varieties.

The difficulty of the proof is [F] stems from the fact that the flat deformations of the above X is obstructed. The Kuranishi space of deformations of X has two components, one for the smoothings and the other for the locally trivial deformations which may destroy the d-semistability. In order to overcome this difficulty, we introduce the concept of the *logarithmic structures* by Deligne, Faltings, Fontaine and Illusie (cf. [Kat]) and the *logarithmic deformations* (§1 and §2). In general, a simple normal crossing variety X admits a logarithmic structure $\mathcal{U}$ if and only if it is d-semistable (Proposition 1.1).

The logarithmic deformations of the pair $(X, \mathcal{U})$ give us a covariant functor LD from the category of local Artin $\mathbb{C}[[t_1, \dots, t_m]]$-algebras with residue field $\mathbb{C}$ to the category of sets, where m is the number of connected components of the singular locus of X. LD has a hull in the sense of [Sch] (Theorem 2.3). LD has a similar property as the usual flat deformation functor F of X for the infinitesimal extension (Theorem 2.2). But we note that the tangent space of LD is bigger than that of F, since there are deformations of the logarithmic structure $\mathcal{U}$ which fix the underlying variety X.

The logarithmic deformations of the d-semistable K3 surface X correspond to the smoothing component of the Kuranishi space, and proved to be unobstructed since the obstruction group is trivial (Corollary 2.5). Here the unobstructedness means the smoothness of the hull of LD over $\mathbb{C}[[t_1, \dots, t_m]]$ instead of over the base field $\mathbb{C}$, and implies automatically the existence of the smoothing. We note also that our argument could be modified for algebraic schemes over a field of arbitrary characteristic.

In the higher dimensional case, we shall generalize the T^1-lifting principle ([R], [K]) slightly in order to prove the unobstructedness of logarithmic deformations (§3). Here the base field should be of characteristic zero. We shall prove that a compact Kähler normal crossing variety X of arbitrary dimension d with a logarithmic structure such that $K_X \sim 0$, $H^{d-1}(X, \mathcal{O}_X) = 0$, and $H^{d-2}(X^{[0]}, \mathcal{O}_{X^{[0]}}) = 0$ has unobstructed logarithmic deformations and is smoothable, where $X^{[0]}$ is the normalization of X (Theorem 4.2).

1. Logarithmic structures on normal crossing varieties

A reduced complex analytic space X is called a *normal crossing variety* (or *n.c. variety*) of dimension d if the local ring $\mathcal{O}_{X,p}$ at each point p is isomorphic to $\mathbb{C}\{x_0, \dots, x_d\}/(x_0 \dots x_r)$ for some $r = r(p)$ such that $0 \leqq r \leqq d$. In addition, if X has smooth irreducible components, then X is called a *simple normal crossing variety* (or *s.n.c. variety*).

A partial open covering of X with systems of holomorphic functions

$$\mathcal{U} = \{U_\lambda;\, z_0^{(\lambda)}, \dots, z_d^{(\lambda)}\}$$

is called a *logarithmic atlas* (or *log atlas*) if the following conditions are satisfied:

(a) $\bigcup_\lambda U_\lambda$ contains the singular locus D of X,

(b) there is an isomorphism φ from U_λ to an open neighborhood of $(0, \dots, 0)$ of the variety

$$V_r = \{(x_0, \dots, x_d) \in \mathbb{C}^{d+1};\, x_0 \dots x_r = 0\}$$

for some $r = r(\lambda)$ such that $\varphi^*(x_j) = z_j^{(\lambda)}$ for $0 \leqq j \leqq r$ and that $z_j^{(\lambda)}$ $(0 \leqq j \leqq d)$ are invertible,

(c) if $U_\lambda \cap U_\mu \neq \emptyset$ then there exist invertible holomorphic functions $u_j^{(\lambda\mu)}$ $(0 \leqq j \leqq d)$ on $U_\lambda \cap U_\mu$ and a permutation $\sigma = \sigma(\lambda, \mu) \in \mathfrak{S}_{d+1}$ such that

$$z_{\sigma(j)}^{(\lambda)} = u_j^{(\lambda\mu)} z_j^{(\mu)} \text{ and } u_0^{(\lambda\mu)} \dots u_d^{(\lambda\mu)} = 1 \text{ on } U_\lambda \cap U_\mu.$$

Two log atlases $\mathscr{U}$ and $\mathscr{U}'$ on a n.c. variety X are said to be *equivalent* if their union defines a long atlas on X. An equivalence class of log atlases is called a *logarithmic structure* (or *log structure*).

Let X be a s.n.c. variety, X_j $(0 \leqq j \leqq n)$ the irreducible components, and I_{X_j} (resp. I_D) the defining ideals of X_j (resp. D) in X. Then X is called *d-semistable* if

$$I_{X_0}/I_{X_0}I_D \otimes \dots \otimes I_{X_n}/I_{X_n}I_D \cong \mathcal{O}_D.$$

Proposition 1.1. *A s.n.c. variety X admits a logarithmic structure if and only if it is d-semistable.*

Proof. Let $\mathscr{U} = \{U_\lambda; z_0^{(\lambda)}, \dots, z_d^{(\lambda)}\}$ be a log atlas on X. Then for each λ there is an injection $\sigma_\lambda : \{0, \dots, r(\lambda)\} \to \{0, \dots, n\}$ such that $z_j^{(\lambda)}$ is a local equation of $X_{\sigma_\lambda(j)} \cap U_\lambda$. Define local sections $s_k^{(\lambda)}$ of $I_{X_k}/I_{X_k}I_D$ over U_λ as follows:

$$s_{\sigma_\lambda(0)}^{(\lambda)} \equiv z_0^{(\lambda)} z_{r(\lambda)+1}^{(\lambda)} \dots z_d^{(\lambda)} \,(\mathrm{mod}\ I_{X_{\sigma_\lambda(0)}} I_D)$$

$$s_k^{(\lambda)} \equiv z_j^{(\lambda)} (\mathrm{mod}\ I_{X_k} I_D) \quad \text{if} \quad k = \sigma_\lambda(j) \text{ for } j \neq 0$$

$$s_k^{(\lambda)} \equiv 1 \,(\mathrm{mod}\ I_{X_k} I_D) \quad \text{if} \quad k \neq \sigma_\lambda(j) \text{ for any } j$$

Then $\{s_0^{(\lambda)} \otimes \dots \otimes s_n^{(\lambda)}\}_\lambda$ defines an isomorphism

$$\mathcal{O}_D \cong I_{X_0}/I_{X_0}I_D \otimes \dots \otimes I_{X_n}/I_{X_n}I_D\,.$$

Conversely, suppose that X is d-semistable. Let $\{U_\lambda\}$ be a partial covering of X satisfying the condition (a) and such that $I_{X_j}/I_{X_j}I_D|_{U_\lambda} \cong \mathcal{O}_D|_{U_\lambda}$. We have a nowhere vanishing section s of $I_{X_0}/I_{X_0}I_D \otimes \dots \otimes I_{X_n}/I_{X_n}I_D$. We can write $s|_{U_\lambda} = s_0^{(\lambda)} \otimes \dots \otimes s_n^{(\lambda)}$ for some $s_j^{(\lambda)} \in H^0(I_{X_j}/I_{X_j}I_D|_{U_\lambda})$. We may assume that the $s_j^{(\lambda)} (0 \leqq j \leqq r(\lambda))$ are represented by some holomorphic functions $z_j^{(\lambda)} \in H^0(I_{X_j}|_{U_\lambda})$ and the other $s_j^{(\lambda)} (r(\lambda) < j \leqq n)$ are represented by $1 \in H^0(I_{X_j}|_{U_\lambda})$. If the U_λ are chosen small enough, then $\{U_\lambda; z_0^{(\lambda)}, \dots, z_d^{(\lambda)}\}$ satisfies the condition (b), and we can write $z_j^{(\lambda)} = u_j^{(\lambda\mu)} z_j^{(\mu)}$ on $U_\lambda \cap U_\mu$. The $u_j^{(\lambda\mu)}$ satisfy the equality $u_0^{(\lambda\mu)} \dots u_d^{(\lambda\mu)} = 1$ on $U_\lambda \cap U_\mu \cap D$, hence

$$u_0^{(\lambda\mu)} \dots u_d^{(\lambda\mu)} = 1 + \sum_{j=0}^{d} a_j z_0^{(\mu)} \dots z_{j-1}^{(\mu)} z_{j+1}^{(\mu)} \dots z_d^{(\mu)}$$

for some $a_j \in H^0(\mathcal{O}_{U_\lambda \cap U_\lambda})$. Since $z_0^{(\mu)} \dots z_d^{(\mu)} = 0$ on U_μ, one can add a multiple of $z_0^{(\mu)} \dots z_{j-1}^{(\mu)} z_{j+1}^{(\mu)} \dots z_d^{(\mu)}$ to $u_j^{(\lambda\mu)}$. Thus, if we replace each $u_j^{(\lambda\mu)}$ by

$$u_j^{(\lambda\mu)} - a_j z_0^{(\mu)} \dots z_{j-1}^{(\mu)} z_{j+1}^{(\mu)} \dots z_d^{(\mu)} (u_0^{(\lambda\mu)} \dots u_{j-1}^{(\lambda\mu)} u_{j+1}^{(\lambda\mu)} \dots u_d^{(\lambda\mu)})^{-1}$$

then we see that $\{U_\lambda; z_0^{(\lambda)}, \dots, z_d^{(\lambda)}\}$ satisfies the condition (c). Q.E.D.

2. Logarithmic deformations

Let $(X, \mathscr{U}_0)$ be a n.c. variety with a logarithmic structure as in §1, and $D_1, \dots, D_m$ the connected components of $D = \operatorname{Sing} X$. Let A be an Artin local $\mathbb{C}$-algebra with residue field $A/\mathfrak{m}_A = \mathbb{C}$, and $s_1, \dots, s_m$ elements of $\mathfrak{m}_A$. Then a *logarithmic deformation* $(\mathscr{X}, \mathscr{U})$ of $(X, \mathscr{U}_0)$ over $\mathscr{A} = (A; s_1, \dots, s_m)$ is a pair consisting of a flat deformation $\mathscr{X}$ of X over A and a *logarithmic atlas*

$$\mathscr{U} = \{U_\lambda; z_0^{(\lambda)}, \dots, z_d^{(\lambda)}\}$$

on $\mathscr{X}$ which is defined as follows: $\{U_\lambda\}$ is a partial open covering of $\mathscr{X}$ and the $z_j^{(\lambda)}$ are holomorphic functions on the U_λ such that

(a) $\{U_\lambda \cap X\}$ and the restrictions of the $z_j^{(\lambda)}$ define a logarithmic atlas on X which is equivalent to $\mathscr{U}_0$,

(b) $z_0^{(\lambda)} \dots z_d^{(\lambda)} = s_i$ if $U_\lambda \cap D_i \neq \emptyset$,

(c) if $U_\lambda \cap U_\mu \neq 0$, then there exist $u_j^{(\lambda\mu)} \in H^0(U_\lambda \cap U_\mu, \mathcal{O}_{\mathscr{X}}^*)$ for $0 \leqq j \leqq d$ such that

$$z_{\sigma(j)}^{(\lambda)} = u_j^{(\lambda\mu)} z_j^{(\mu)} \quad \text{and} \quad u_0^{(\lambda\mu)} \dots u_d^{(\lambda\mu)} = 1 \text{ on } U_\lambda \cap U_\mu.$$

where σ is a permutation.

Let $\Lambda_m = \mathbb{C}[[t_1, \dots, t_m]]$. We shall regard $\mathscr{A}$ as a Λ_m-algebra with the same underlying ring as A whose structure homomorphism $\Lambda_m \xrightarrow{\alpha} \mathscr{A}$ is given by $\alpha(t_i) = s_i$.

Two log deformations $(\mathscr{X}, \mathscr{U})$ and $(\mathscr{X}, \mathscr{U}')$ of $(X, \mathscr{U}_0)$ over $\mathscr{A}$ are said to be *equivalent* if they define the same log structure, i.e., if their union defines a log atlas on $\mathscr{X}$.

Let $\mathscr{A}' = (A'; s'_1, \dots, s'_m)$ be another Artin local Λ_m-algebra with a local homomorphism $\tilde{h}: \mathscr{A} \to \mathscr{A}'$ over Λ_m, i.e., $\tilde{h}$ is a $\mathbb{C}$-algebra local homomorphism $h: A \to A'$ such that $h(s_i) = s'_i$. Let $\mathscr{X}' = \mathscr{X} \times_A A'$ with the natural morphism $\varphi: \mathscr{X}' \to \mathscr{X}$. Then one can define a log structure $\mathscr{U}'$ on $\mathscr{X}'$ by pulling back $\mathscr{U}$ by φ. The log deformation over $\mathscr{A}'$ thus obtained is called the *pull-back* of $(\mathscr{X}, \mathscr{U})$ by h. In the case in which h is surjective, $(\mathscr{X}', \mathscr{U}')$ (resp. $(\mathscr{X}, \mathscr{U})$) is called the *restriction* (resp. *lifting*) of the other.

The log deformations of $(X, \mathscr{U}_0)$ defines a covariant functor $\mathrm{LD} = \mathrm{LD}(X, \mathscr{U}_0)$ from the category of Artin local Λ_m-algebras with residue field $\mathbb{C}$ to the category of sets:

$$\mathrm{LD}: (\mathrm{Art}_{\Lambda_m}) \to (\mathrm{Set.})$$

Let $(\mathscr{X}, \mathscr{U})$ be a log deformation of a n.c. variety $(X, \mathscr{U}_0)$ over $\mathscr{A}$. Then the sheaf of relative logarithmic differentials $\Omega^1_{\mathscr{X}/\mathscr{A}}(\log)$ is a locally free $\mathcal{O}_{\mathscr{X}}$-module of rank d defined as follows:

(α) on each U_λ, it is a quotient sheaf of the direct sum of $\Omega^1_{U_\lambda/A}$ and a free module $\bigoplus_{j=0}^d e_j^{(\lambda)} \mathcal{O}_{U_\lambda}$ by the submodule generated by

$$dz_j^{(\lambda)} - z_j^{(\lambda)} e_j^{(\lambda)} (0 \leqq j \leqq d) \quad \text{and} \quad \sum_{j=0}^{d} e_j^{(\lambda)},$$

(β) if $U_\lambda \cap U_\mu \neq 0$, then the gluing on the overlap is given by

$$e_{\sigma(j)}^{(\lambda)} = e_j^{(\mu)} + du_j^{(\lambda\mu)}/u_j^{(\lambda\mu)}$$

as well as the identification of $\Omega^1_{U_\lambda/A}$ and $\Omega^1_{U_\mu/A}$ there.

The transitivity of the above gluing can be checked by using the Lemma 2.1 below, and the sheaf $\Omega^1_{\mathscr{X}/\mathscr{A}}(\log)$ is well defined. We shall denote $e_j^{(\lambda)} = dz_j^{(\lambda)}/z_j^{(\lambda)}$. We also define

$$\Omega^p_{\mathscr{X}/\mathscr{A}}(\log) = \bigwedge^p \Omega^1_{\mathscr{X}/\mathscr{A}}(\log)$$

$$T_{\mathscr{X}/\mathscr{A}}(\log) = \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{\mathscr{X}/\mathscr{A}}(\log), \mathscr{O}_X)$$

We note that $\Omega^d_{\mathscr{X}/\mathbb{C}}(\log)\partial \quad \omega_X$.

Lemma 2.1. *Let $(\mathscr{X}, \mathscr{U})$ be a logarithmic deformation of $(\mathscr{X}, \mathscr{U}_0)$ over $\mathscr{A}$, and $(U;\ z_0, \ldots, z_d)$ a member of $\mathscr{U}$. Let u_j $(0 \leqq j \leqq d)$ be invertible holomorphic functions on U such that*

$$z_j = u_j z_j \quad \text{and} \quad u_0 \ldots u_d = 1.$$

Then $u_0 = \cdots = u_d = 1$.

Proof. We proceed by induction on the length of A. We assume that the z_j are not invertible for $0 \leqq j \leqq r$ and invertible for $r < j \leqq d$. It is clear that $u_j = 1$ for $r < j \leqq d$. We shall consider the u_j for $0 \leqq j \leqq r$. If $A = \mathbb{C}$, the U has an equation $z_0 \ldots z_r = 0$ in an open set of $\mathbb{C}^{d+1}$. Since $z_j = u_j z_j$, we have

$$u_j - 1 = a_j z_0 \ \ldots \ z_{j-1} z_{j+1} \cdots z_r$$

for some holomorphic functions a_j. Then

$$1 = u_0 \ldots u_d = 1 + \sum_{j=0}^{d} a_j z_0 \ \ldots \ z_{j-1} z_{j+1} \ldots z_r.$$

Hence $u_j = 1$ for all j.

If $A \neq \mathbb{C}$, we take a principal ideal $J = aA \subset A$ such that $\mathfrak{m}_A J = 0$, and let $\bar{A} = A/J$. Let $\bar{U}$ denote the restriction of U over Spec $\bar{A}$. By the induction hypothesis applied to $\bar{U}$, we have

$$u_j = 1 + a a_j z_0 \ \ldots \ z_{j-1} z_{j+1} \cdots z_r$$

for some holomorphic functions a_j. Then as before, from $u_0 \ \ldots \ u_d = 1$ follows that $u_j = 1$. Q.E.D.

Theorem 2.2. *Let $(X, \mathscr{U}_0)$ be a normal crossing variety with a logarithmic structure whose singular locus has m connected components, and $(\mathscr{X}, \mathscr{U})$ a logartithmic deformation of $(\mathscr{X}, \mathscr{U}_0)$ over a Λ_m-algebra $\mathscr{A}$. Let*

$$0 \to J \to \mathscr{A}' \to \mathscr{A} \to 0$$

be an extension of Λ_m-algebras an ideal J such that $J^2 = 0$.

(1) *Assume that* $(\mathscr{X}, \mathscr{U})$ *lifts to a logarithmic deformation* $(\mathscr{X}', \mathscr{U}')$ *over* $\mathscr{A}'$. *Then the set of automorphisms of* $\mathscr{X}'$ *over* $\mathscr{A}'$ *which fix the log structure* $\mathscr{U}'$ *and induce the identity on* $\mathscr{X}$ *is bijective to*

$$H^0(\mathscr{X}, T_{\mathscr{X}/\mathscr{A}}(\log) \otimes_{\mathscr{A}} J).$$

(2) *Under the same assumption as in* (1), *the set of equivalence classes of log deformations over* $\mathscr{A}'$ *which are liftings of* $(\mathscr{X}, \mathscr{U})$ *is a torsor on*

$$H^1(\mathscr{X}, T_{\mathscr{X}/\mathscr{A}}(\log) \otimes_{\mathscr{A}} J).$$

(3) *The obstruction to the existence of the lifting of* $(\mathscr{X}, \mathscr{U})$ *over* $\mathscr{A}'$ *is in*

$$H^2(\mathscr{X}, T_{\mathscr{X}/\mathscr{A}}(\log) \otimes_{\mathscr{A}} J)$$

Proof. Let $(U; z_0, \ldots, z_d)$ be a logarithmic chart belonging to the atlas $\mathscr{U}$. If U is small enough, then a lifting $(U'; z'_0, \ldots, z'_d)$ over $\mathscr{A}'$ always exists uniquely up to the equivalence. The set of automorphisms φ of U' over $\mathscr{A}'$ which fix the log structure of U' and induce the identity on U correspond bijectively to the set of ring isomorphisms φ^* of $\mathscr{O}_{U'}$ which induce the identity on $\mathscr{O}_U$ with invertible holomorphic functions u'_j for $0 \leqq j \leqq d$ such that

$$\varphi^*(z'_j) = u'_j z'_j \text{ and } u'_0 \ldots u'_d = 1,$$

since the u'_j are uniquely determined by φ^* by Lemma 2.1. The ring isomorphism φ^* corresponds to a derivation $\varphi^* - \mathrm{id}$ from $\mathscr{O}_U$ to $J\mathscr{O}_{U'} \cong J \otimes_{\mathscr{A}} \mathscr{O}_U$. The corresponding $\mathscr{O}_U$-homomorphism $\delta_0 : \Omega^1_{U/\mathscr{A}} \to J \otimes_{\mathscr{A}} \mathscr{O}_U$ can be extended uniquely to $\delta \in \mathrm{Hom}_{\mathscr{O}_U}(\Omega^1_{U/\mathscr{A}}(\log), J \otimes_{\mathscr{A}} \mathscr{O}_U)$ if we set $\delta(dz_j/z_j) = u'_j - 1$, since $\sum_{j=0}^d (u'_j - 1) = 0$. Conversely, δ determines an automorphism φ of U' inducing the identity on U given by $\varphi^*(w) = \omega + \delta(dw)$ for $w \in \mathscr{O}_{U'}$. Then $\varphi^*(z'_j) = (1 + \delta(dz'_j/z'_j))z'_j$. Since $\sum_{j=0}^d dz'_j/z'_j = 0$, φ preserves the logarithmic structure. Therefore, we obtain (1). The rest of the theorem follows from the general theory in [Gro]. Q.E.D.

Theorem 2.3. *Let* $(X, \mathscr{U}_0)$ *be a compact normal crossing variety with a logarithmic structure. Then the logarithmic deformation functor* $\mathrm{LD}(X, \mathscr{U}_0)$ *has a hull* $R(X, \mathscr{U}_0)$ *in the sense of* [Sch].

Proof. Similar to the case of the usual flat deformations. Q.E.D.

Corollary 2.4. *Let* $(X, \mathscr{U}_0)$ *be a compact normal crossing variety with a logarithmic structure, and m the number of connected components of the singular locus of* X. *Assume that* $H^2(X, T_{X/\mathbb{C}}(\log)) = 0$. *Then* $\mathrm{LD}(X, \mathscr{U}_0)$ *is unobstructed, i.e,* $R(X, \mathscr{U}_0)$ *is formally smooth over* $A_m = \mathbb{C}[[t_1, \ldots, t_m]]$. *Moreover,* X *is smoothable by a flat deformation.*

Proof. The first assertion follows immediately from Theorem 2.2. By [Gra], there exists a semi-universal family of flat deformations $\mathscr{X} \to S$. Let R be the formal completion of the local ring $\mathscr{O}_{S,s}$, where s is the base point of S.

By forgetting the log structure, we obtain a natural homomorphism $\alpha: R \to R(X, \mathscr{U})$. By assumption, there exists a formal arc $\beta: R(X, \mathscr{U}) \to \mathbb{C}[[t]]$, where the target has a Λ_m-algebra structure given by $t_i \to t$ for all i. By Artin's approximation theorem ([A]), there exists a homomorphism $\mathcal{O}_{S,s} \to \mathbb{C}\{t\}$ which coincides with $\beta \circ \alpha$ up to the first order. Therefore, X has a smoothing. Q.E.D.

Corollary 2.5. ([F]) *Let X be a compact Kähler normal crossing variety of dimension 2 which admits a logarithmic structure. Assume that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. Then X is smoothable by a flat deformation.*

Proof. By [F, 5.7], $h^2(X, T_{X/\mathbb{C}}(\log)) = h^0(X, \Omega^1_{X/\mathbb{C}}(\log) = 0$. Q.E.D.

It is possible to develop a theory of logarithmic deformations of *algebraic* normal crossing varieties over a field of arbitrary characteristic, and extend Corollary 2.5. to positive characteristic (cf. [Kat]).

3. T^1-lifting property

Let $(X, \mathscr{U}_0)$ be a n.c. variety with a log structure, m the number of connected components of the singular locus of X, and LD $=$ LD$(X, \mathscr{U}_0)$ the log deformation functor of $(X, \mathscr{U}_0)$. Let

$$A_k = \mathbb{C}[t]/(t^{k+1}) \quad \text{and} \quad A_k[\varepsilon] = \mathbb{C}[t, \varepsilon]/(t^{k+1}, \varepsilon) \quad \text{for } k \geqq -1.$$

We note that they are zero rings if $k = -1$. Set $\Lambda_m = \mathbb{C}[[t_1, \ldots, t_m]]$ as before. Let $\mathscr{A}_k$ and $\mathscr{A}_k[\varepsilon]$ be Λ_m-algebras whose underlying $\mathbb{C}$-algebras are A_k and $A_k[\varepsilon]$, respectively. We set LD$(0) =$ LD$(\mathbb{C})$. We always assume that the images of the t_i are in the maximal ideals for $k \geqq 0$, and the natural homomorphisms $\mathscr{A}_k \to \mathscr{A}_{k-1}$, $\mathscr{A}_k[\varepsilon] \to \mathscr{A}_{k-1}[\varepsilon]$ and $\mathscr{A}_k[\varepsilon] \to \mathscr{A}_k$ are over Λ_m for all k.

Then LD is said to have the *T^1-lifting property* if the natural maps

$$\mathrm{LD}(\mathscr{A}_k[\varepsilon]) \to \mathrm{LD}(\mathscr{A}_k) \times_{\mathrm{LD}(\mathscr{A}_{k-1})} \mathrm{LD}(\mathscr{A}_{k-1}[\varepsilon])$$

are surjective for all $k \geqq 0$ and all Λ_m-algebra structures on the A_k, etc. as above. We note that the T^1-lifting in the case $k = 0$ is also a non-trivial condition. We extend the T^1-lifting principle on unobstructed deformations in [R], [K] to our situation in the following.

Theorem 3.1. *Let $(X, \mathscr{U}_0)$ be a n.c. variety, m the number of connected components of Sing X, and LD its logarithmic deformation functor. Assume that LD is pro-representable and has the T^1-lifting property. Then LD is unobstructed, i.e., its hull R is formally smooth over Λ_m.*

Proof. We fix a Λ_m-algebra structure $\mathscr{A}_{k+1}$ on A_{k+1} for some $k \geqq 0$. Let $C_k = \mathbb{C}[t, \varepsilon]/(t^{k+1}, t^k\varepsilon, \varepsilon^2) = A_k \times_{A_{k-1}} A_{k-1}[\varepsilon]$. We define homomorphisms $\alpha: A_{k+1} \to A_k[\varepsilon]$ and $\alpha': A_k \to C_k$ by setting $\alpha(t) = \alpha'(t) = t + \varepsilon$. Then we consider

Λ_m-algebra structures $\mathscr{A}_k$, $\mathscr{A}_k[\varepsilon]$ and $\mathscr{C}_k$ on A_k, $A_k[\varepsilon]$, and C_k, respectively, such that the following commutative diagram

$$\begin{array}{ccc} \mathscr{A}_{k+1} & \longrightarrow & \mathscr{A}_k \\ \alpha\downarrow & & \downarrow\alpha' \\ \mathscr{A}_k[\varepsilon] & \longrightarrow & \mathscr{C}_k \end{array}$$

consists of Λ_m-homomorphisms, where horizontal arrows are natural homomorphisms. We note that $\mathscr{C}_k$ coincides with the fiber product $\mathscr{A}_k \times_{\mathscr{A}_{k-1}} \mathscr{A}_{k-1}[\varepsilon]$ for the induced Λ_m-algebras $\mathscr{A}_{k-1}$ and $\mathscr{A}_{k-1}[\varepsilon]$. Then by Theorem 2.2(3), we have the following commutative diagram of obstruction sequences:

$$\begin{array}{ccccc} \mathrm{LD}(\mathscr{A}_{k+1}) & \longrightarrow & \mathrm{LD}(\mathscr{A}_k) & \xrightarrow{\delta_1} & (t^{k+1}) \otimes H^2(X, T_{X/\mathbb{C}}(\log)) \\ \downarrow & & \downarrow & & \alpha''\downarrow \\ \mathrm{LD}(\mathscr{A}_k[\varepsilon]) & \longrightarrow & \mathrm{LD}(\mathscr{C}_k) & \xrightarrow[\delta_2]{} & (t^k\varepsilon) \otimes H^2(X, T_{X/\mathbb{C}}(\log)) \end{array}$$

where α'' is induced by α. By the T^1-lifting property, we have $\delta_2 = 0$. Since α'' is an isomorphism, we also have $\delta_1 = 0$, hence the hull R of LD is formally smooth over Λ_m. Q.E.D.

Theorem 3.2. *Let $(X, \mathscr{U}_0)$ be a n.c. variety, m the number of connected components of Sing X, and LD its log deformation functor. Assume the following conditions:*

(i) *for an arbitrary nonnegative integer k, an arbitrary Λ_m-algebra structure on $\mathscr{A}_k$, and for an arbitrary log deformation $(\mathscr{X}_k, \mathscr{U}_k)$* of *$(X, \mathscr{U}_0)$ over $\mathscr{A}_k$, the natural homomorphism*

$$H^2(\mathscr{X}_k, T_{\mathscr{X}_k/\mathscr{A}_k}(\log)) \to \mathrm{Ext}^2_{\mathscr{O}_X}(\Omega^1_{\mathscr{X}_k/\mathscr{A}_k}, \mathscr{O}_{X_k})$$

is injective,

(ii) *for an arbitrary positive integer k, and arbitrary $\mathscr{A}_k$* and *$(\mathscr{X}_k, \mathscr{U}_k)$ as in (i) if $\mathscr{A}_{k-1}$ is the induced Λ_m-algebra structure on A_{k-1}, and $(\mathscr{X}_{k-1}, \mathscr{U}_{k-1})$ is the restriction over $\mathscr{A}_{k-1}$, then the natural homomorphism*

$$H^1(\mathscr{X}_k, T_{\mathscr{X}_k/\mathscr{A}_k}(\log)) \to H^1(\mathscr{X}_{k-1}, T_{\mathscr{X}_{k-1}/\mathscr{A}_{k-1}}(\log))$$

is surjective.
Then the hull of LD is formally smooth over Λ_m if LD is pro-representable.

Proof. Let $\mathscr{A}_k[\varepsilon]$ be an arbitrary Λ_m-algebra structure on $A_k[\varepsilon]$ for $k \geqq 0$, and $(\mathscr{X}_k, \mathscr{U}_k) \in \mathrm{LD}(\mathscr{A}_k)$ for the induced Λ_m-algebra $\mathscr{A}_k$. Since $\mathscr{X}_k$ can always be liftable over $A_k[\varepsilon]$ as a flat deformation, the condition (i) implies that it is liftable over $\mathscr{A}_k[\varepsilon]$ as a log deformation by Theorem 2.2(3). In particular, (i) for the case $k = 0$ implies the T^1-lifting for $k = 0$. Then by Theorem 2.2(2), the condition (ii) implies the T^1-lifting in general. Q.E.D.

4. Smoothing of degenerate Calabi-Yau varieties

Lemma 4.1. *Let $(X, \mathscr{U}_0)$ be a compact Kähler n.c. variety with a log structure, m the number of connected components of Sing X, and $(\mathscr{X}, \mathscr{U})$ a log deformation of $(X, \mathscr{U}_0)$ over a Λ_m-algebra $\mathscr{A}$. Then*

(1) *$H^q(\mathscr{X}, \Omega^p_{\mathscr{X}/\mathscr{A}}(\log))$ is a free $\mathscr{A}$-module and commutes with base change of $\mathscr{A}$ for any p and q,*

(2) *the Hodge spectral sequence*

$$E_1^{p,q} = H^q(\mathscr{X}, \Omega^p_{\mathscr{X}/\mathscr{A}}(\log) \Rightarrow \mathbb{H}^{p+q}(\mathscr{X}, \Omega^\bullet_{\mathscr{X}/\mathscr{A}}(\log))$$

degenerates at E_1.

Proof. First, we consider the case in which $\mathscr{A} = \mathscr{A}_k$ is an Λ_m-algebra structure on A_k. We can define a locally free $\mathcal{O}_{\mathscr{X}_k}$-module $\Omega^1_{\mathscr{X}_k}(\log)$ as follows:

(α) on each log chart U_λ where $z_0^{(\lambda)} \dots z_d^{(\lambda)} = ct^{ni}$ with $c \in \mathbb{C}^*$, it is the quotient of the direct sum of $\Omega^1_{U_\lambda}$ and a free module $\bigoplus_{j=0}^d e_j^{(\lambda)} \mathcal{O}_{U_\lambda}$ by the submodule generated by

$$dz_j^{(\lambda)} - z_j^{(\lambda)} e_j^{(\lambda)}\ (0 \leqq j \leqq d) \quad \text{and} \quad n_i dt - t \sum_{j=0}^{d} e_j^{(\lambda)} ,$$

(β) if $U_\lambda \cap U_\mu \neq 0$, then we identify on the overlap by

$$e_{\sigma(j)}^{(\lambda)} = z_j^{(\mu)} + du_j^{(\lambda\mu)}/u_j^{(\lambda\mu)}$$

as well as $\Omega^1_{U_\lambda}$ and $\Omega^1_{U_\mu}$.

We also define $\Omega^p_{\mathscr{X}_k}(\log) = \Lambda^p \Omega^1_{\mathscr{X}_k}(\log)$. Then following [St, 2.6], we define a complex of $\mathcal{O}_{X_k}$-modules

$$L^\bullet = \bigoplus_{s=0}^{\infty} \Omega^\bullet_{\mathscr{X}_k}(\log)(\log t)^s$$

with a differential given by the rule

$$d\left(\sum_{s=0}^{N} \omega_s(\log t)^s\right) = \sum_{s=0}^{N} (d\omega_s(\log t)^s + (s\omega_s \Lambda dt/t)(log\ t)^{s-1}) .$$

Then the homomorphism $\varphi: L^\bullet \to \Omega^\bullet_{X/\mathbb{C}}(\log)$ which assigns $\sum_{s=0}^N \omega_s(\log t)^s$ to the image of ω_0 in $\Omega^\bullet_{X/\mathbb{C}}(\log)$ is a quasi-isomorphism as in [St, §2]. Since ψ can be factored as $L^\bullet \to \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log) \to \Omega^\bullet_{X/\mathbb{C}}(\log)$, the homomorphisms

$$\mathbb{H}^n(\mathscr{X}_k, \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log)) \to \mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}(\log))$$

are surjective. Then the long exact sequence of hypercohomologies associated to the exact sequence of complexes

$$0 \to \Omega^\bullet_{\mathscr{X}_{k-1}/\mathscr{A}_{k-1}}(\log)) \xrightarrow{t} \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log)) \to \Omega^\bullet_{X/\mathbb{C}}(\log) \to 0$$

splits into short exact sequences. By counting the dimension as vector spaces over $\mathbb{C}$, we infer the splitting of those associated to

$$0 \to \Omega^\bullet_{X/\mathbb{C}}(\log) \xrightarrow{t^k} \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log) \to \Omega^\bullet_{\mathscr{X}_{k-1}/\mathscr{A}_{k-1}}(\log) \to 0$$

Hence

$$\mathbb{H}^n(\mathscr{X}_k, \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log))/t\mathbb{H}^n(\mathscr{X}_k, \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log))\partial \quad \mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}(\log))$$

so $\mathbb{H}^n(\mathscr{X}_k, \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log))$ is a free $\mathscr{A}_k$-module.

Next, we consider the general case. Let $\mathfrak{m}$ be the maximal ideal of $\mathscr{A}$. Assume that $\mathfrak{m}^{k+1} = 0$ for a positive integer k, and set $\mathscr{A}' = \mathscr{A}/\mathfrak{m}^k$ and $\mathscr{X}' = \mathscr{X} \times_{\mathscr{A}} \mathscr{A}'$. We shall prove that the homomorphism

$$\mathbb{H}^n(\mathscr{X}, \Omega^\bullet_{\mathscr{X}/\mathscr{A}}(\log)) \to \mathbb{H}^n(\mathscr{X}', \Omega^\bullet_{\mathscr{X}'/\mathscr{A}'}(\log))$$

is surjective. Suppose the contrary. Then the obstruction homomorphism

$$\delta: \mathbb{H}^n(\mathscr{X}', \Omega^\bullet_{\mathscr{X}'/\mathscr{A}'}(\log)) \to \mathbb{H}^{n+1}(X', \Omega^\bullet_{X/\mathbb{C}}(\log) \otimes \mathfrak{m}^k)$$

is not zero. For a Λ_m-homomorphism $\alpha: \mathscr{A} \to \mathscr{A}_k$, we define $\mathscr{X}_k = \mathscr{X} \times_{\mathscr{A}} \mathscr{A}_k$, etc. Then there is a commutative diagram of obstruction homomorphisms

$$\begin{array}{ccc}
\mathbb{H}^n(\mathscr{X}, \Omega^\bullet_{\mathscr{X}/\mathscr{A}}(\log)) & \longrightarrow & \mathbb{H}^n(\mathscr{X}_k, \Omega^\bullet_{\mathscr{X}_k/\mathscr{A}_k}(\log)) \\
\downarrow & & \downarrow \\
\mathbb{H}^n(\mathscr{X}', \Omega^\bullet_{\mathscr{X}'/\mathscr{A}'}(\log)) & \longrightarrow & \mathbb{H}^n(\mathscr{X}_{k-1}, \Omega^\bullet_{\mathscr{X}_{k-1}/\mathscr{A}_{k-1}}(\log)) \\
\delta\downarrow & & \downarrow\delta' \\
\mathbb{H}^{n+1}(X, \Omega^\bullet_{X/\mathbb{C}}(\log) \otimes m^k) & \underset{\alpha*}{\longrightarrow} & \mathbb{H}^{n+1}(X, \Omega^\bullet_{X/\mathbb{C}}(\log))
\end{array}$$

If α is chosen to be general, the $\alpha_* \circ \delta \neq 0$, but $\delta' = 0$ by the previous argument, a contradiction.

In the rest of the proof, we shall show that the complex $\Omega^\bullet_{X/\mathbb{C}}(\log))$ allows a structure of a cohomological mixed Hodge complex on X. Then by [D, 8.19], the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}(\log)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}(\log))$$

degenerates at E_1, and the homomorphisms

$$H^q(\mathscr{X}, \Omega^p_{\mathscr{X}/\mathscr{A}}(\log)) \to H^q(X, \Omega^p_{X/\mathbb{C}}(\log))$$

are surjective, hence (1) and (2).

In order to define a $\mathbb{Z}$-structure on our complex, we shall construct a morphism of semi-analytic spaces $\tilde{\rho}: \tilde{X} \to X$. It is called *real blow-up* in [P] (see also [C]), but our $\tilde{X}$ is a real analytic manifold with corner instead of a C^∞-manifold.

As in §1, we let $V = V_d = \{x_0, \ldots, x_d) \in \mathbb{C}^{d+1};\ x_0 \ldots x_d = 0\}$, and set $x_j = s_j e^{\sqrt{-1}\theta_j}$. Correspondingly, we consider a real analytic space

$$\tilde{W} = \left\{(s_0, \theta_0, \ldots, s_d, \theta_d) \in (\mathbb{R} \times S^1)^{d+1};\ s_0 \ldots s_d = 0,\ \sum_{j=0}^{d} \theta_j = 0\right\}$$

and its semi-analytic subspace $\tilde{V}$ defined by equations $s_j \geqq 0$ for all j. If $\mathcal{O}^{\mathrm{re}}_{\tilde{W}}$ is the structure sheaf of a real analytic space $\tilde{W}$, then $\mathcal{O}^{\mathrm{re}}_{\tilde{V}} = i^{-1}\,\mathcal{O}^{\mathrm{re}}_{\tilde{W}}$ is that of a semi-analytic space $\tilde{V}$ for $i:\tilde{V} \subset \tilde{W}$. The natural projection $\tilde{\rho}:\tilde{V} \to V$ is a morphism of semi-analytic spaces.

We define the sheaf of logarithmic differential 1-forms $\Omega^{1,\mathrm{re}}_{\tilde{V}}(\log)$ as a locally free $\mathcal{O}^{re}_{\tilde{V}}$-module of rank $2d$ generated by

$$ds_0/s_0,\, d\theta_0,\, \ldots, ds_d/s_d,\, d\theta_d$$

with relations $\sum_j ds_j/s_j = \sum_j d\theta_j = 0$. We also define

$$\Omega^{p,\mathrm{re}}_{\tilde{V}}(\log)_{\mathbb{C}} = \bigwedge^{p} \Omega^{1,\mathrm{re}}_{\tilde{V}}(\log) \otimes \mathbb{C}.$$

Then by the same argument as in [St, 1.13], we have

$$\mathscr{H}^p(\Omega^{\bullet,\mathrm{re}}_{\tilde{V}}(\log)_{\mathbb{C}})_Q \cong \bigwedge^{p}\left(\mathbb{C}\langle ds_0/s_0, \ldots, ds_r/s_r\rangle \Big/ \left(\sum_{j=0}^{r} ds_j/s_j\right)\right)$$

if $s_j = 0$ for $0 \leqq j \leqq r$ and $s_j \neq 0$ for $r < j \leqq d$ at $Q \in \tilde{V}$.

Let $\tilde{V}^0 = \tilde{V} \backslash \tilde{\rho}^{-1}(\mathrm{Sing}\,\tilde{V})$ and $\imath:\tilde{V}^0 \subset \tilde{V}$. Let $M^\bullet$ be the subcomplex of $\imath_*\Omega^{\bullet,\mathrm{re}}_{\tilde{V}}(\log)_{\mathbb{C}}$ generated by $\Omega^{\bullet,\mathrm{re}}_{\tilde{V}}(\log)_{\mathbb{C}}$ and the functions $(\log s_0)^{i_0} \cdots (\log s_d)^{i_d}$ for $i_0, \ldots, i_d \in \mathbb{Z}_{\geqq 0}$, where we define $\log s_j = -\sum_{k \neq j} \log s_k$ on the locus on which $s_j = 0$. Then we have

$$M^p_Q = \bigoplus_{i_0, \ldots, i_r \geqq 0} \Omega^{p,\mathrm{re}}_{\tilde{V}}(\log)_{\mathbb{C},Q}\, (\log s_0)^{i_0} \cdots (\log s_r)^{i_r} \Big/ \left(\sum_{j=0}^{d} \log s_j\right)$$

for the above Q.

We claim that $M^\bullet$ gives a resolution of the constant sheaf $\mathbb{C}_{\tilde{V}}$. In fact, since the problem is local, we have only to check it at $Q \in \tilde{V}$ as above, and it is similar to [St, 2.15].

Let $\mathscr{U}$ be the log atlas of X as in §1. For a log chart U_λ, we define the real blow-up $\tilde{\rho}_\lambda:\tilde{U}_\lambda \to U_\lambda$ and a complex $M^\bullet_\lambda$ on $\tilde{U}_\lambda$ as the pull-back of $\tilde{\rho}$ and $M^\bullet$ by a morphism $\psi_\lambda:U_\lambda \to V$ given by $\psi^*_\lambda(x_j) = z^{(\lambda)}_j$ for all j. We can check that the $\tilde{\rho}_\lambda$ and the $M^\bullet_\lambda$ for the λ can be glued to give the real blow-up $\tilde{X}$ of X with a semi-analytic morphism $\tilde{\rho}:\tilde{X} \to X$ and a complex $M^\bullet_{\tilde{X}}$ on $\tilde{X}$. In fact, for the real variables s and θ such that $z^{(\lambda)}_j = s^{(\lambda)}_j e^{\sqrt{-1}\theta^{(\lambda)}_j}$, $z^{(\mu)}_j = s^{(\mu)}_j e^{\sqrt{-1}\theta^{(\mu)}_j}$ and $u^{(\lambda\mu)}_j = s^{(\lambda\mu)}_j e^{\sqrt{-1}\theta^{(\lambda\mu)}_j}$, if we put $s^{(\lambda)}_{\sigma(j)} = s^{(\lambda\mu)}_j s^{(\mu)}_j$, $\theta^{(\lambda)}_{\sigma(j)} = \theta^{(\lambda\mu)}_j + \theta^{(\mu)}_j$ and $\sum\theta^{(\lambda)}_j = \sum\theta^{(\mu)}_j$, then $z^{(\lambda)}_{\sigma(j)} = u^{(\lambda\mu)}_j z^{(\mu)}_j$ and $u^{(\lambda\mu)}_0 \ldots u^{(\lambda\mu)}_d = 1$ on the overlap $U_\lambda \cap U_\mu$. Moreover, $M^\bullet_{\tilde{X}}$ gives a resolution of $\mathbb{C}_{\tilde{X}}$.

Now we have a natural inclusion of complexes $\tilde{\rho}^*:\Omega^\bullet_{X/\mathbb{C}}(\log) \to \tilde{\rho}_* M^\bullet_{\tilde{X}}$ and we can check directly that the combination of the homomorphisms

$$\Omega^\bullet_{X/\mathbb{C}}(\log) \to \mathbf{R}\tilde{\rho}_* M^\bullet_{\tilde{X}} \to \mathbf{R}\tilde{\rho}_* \mathbb{C}_{\tilde{X}}$$

is a quasi-isomorphism. Therefore, we define a complex of $\mathbb{Z}$-modules on X by

$$A^\bullet_{\mathbb{Z}} = \mathbf{R}\tilde{\rho}_* \mathbb{Z}_{\tilde{X}}\,.$$

Next, we define a weight filtration defined over $\mathbb{Q}$. For this purpose, we define a semi-analytic morphism $\rho: X^* \to X$. Instead of $\tilde{W}$ and $\tilde{V}$, we define W^* and V^* by the equations $s_0 \ldots s_d = 0$ and $s_j \geqq 0$ for all j. Thus we have an additional real variable $\theta = \sum_j \theta_j$.

The sheaf of logarithmic differential 1-forms $\Omega_{V^*}^{1,\mathrm{re}}(\log)$ is defined as a locally free $\mathcal{O}_{V^*}^{\mathrm{re}}$-module of rank $2d+1$. A complex N^* on V^* is defined as before so that

$$N_Q^p = \bigoplus_{i_0, \ldots, i_r \geqq 0} \Omega_{V^*}^{p,\mathrm{re}}(\log)_{\mathbb{C},Q} (\log s_0)^{i_0} \cdots (\log s_r)^{i_r} \Big/ \left(\sum_{j=0}^{d} \log s_j \right)$$

if $s_j = 0$ for $0 \leqq j \leqq r$ and $s_j \neq 0$ for $r < j < d$ at $Q \in V^*$. Thus $N^\bullet$ gives a resolution of the constant sheaf $\mathbb{C}_{V^*}$. By gluing, we define X^*, etc. We can check again that the combination of the homomorphisms

$$\Omega_X^\bullet(\log) \to \mathbf{R}\rho_* N_{X^*}^\bullet \to \mathbf{R}\rho_* \mathbb{C}_{X^*}$$

is a quasi-isomorphism.

Now the rest of the proof is similar to [S, §4]. We only explain how to modify it. Let

$$H_{\mathbb{Q}}^k = \mathbf{R}\rho_* \mathbb{Q}_{X^*}(k+1)[k+1]/(W_k \mathbf{R}\rho_* \mathbb{Q}_{X^*})(k+1)[k+1]$$

where $\{W_k\}$ is the canonical filtration given by the truncations. The 1-form $d\theta$ gives an element of $H^1(X^*, \mathbb{Q}(1))$, and by using it, we can define a complex $A_{\mathbb{Q}}^\bullet$ with a weight filtration W on it such that

$$A_{\mathbb{Z}}^\bullet \otimes \mathbb{Q} \partial \quad A_{\mathbb{Q}}^\bullet .$$

We also define

$$A^{p,q} = \Omega_X^{p+q+1}(\log)/W_q \Omega_X^{p+q+1}(\log)$$

where W is defined by the multiplicity of log poles. Then the associated single complex $A_{\mathbb{C}}^\bullet$ has a weight filtrarion W and a Hodge filtration F such that

$$(\Omega_{X/\mathbb{C}}^\bullet(\log), F) \partial \quad (A_{\mathbb{C}}^\bullet, F)$$

$$(A_{\mathbb{Q}}^\bullet \otimes \mathbb{C}, W_{\mathbb{C}}) \partial \quad (A_{\mathbb{C}}^\bullet, W).$$

Moreover, the $\mathrm{Gr}_r^W(A_{\mathbb{Q}}^\bullet)$ are described by the constant sheaves of some compact Kähler manifolds, and the filtrations on the $\mathrm{Gr}_r^W(A_{\mathbb{C}}^\bullet)$ induced by F coincide with the Hodge filtrations. So the data

$$A_{\mathbb{Z}}^\bullet, (A_{\mathbb{Q}}^\bullet, W), (A_{\mathbb{C}}^\bullet, F, W)$$

becomes a cohomological mixed Hodge complex on X. Q.E.D.

Theorem 4.2. *Let $(X, \mathscr{U}_0)$ be a compact Kähler normal crossing variety with a logarithmic structure of dimension $d \geqq 3$, m the number of connected components of the singular locus of X, $v: X^{[0]} \to X$ the normalization, and LD the* *logarithmic deformation functor of* $(X, \mathcal{U}_0)$. *Assume the following conditions*:

(a) $\omega_X \partial \quad \mathcal{O}_X$,

(b) $H^{d-1}(X, \mathcal{O}_X) = 0$,

(c) $H^{d-2}(X^{[0]}, \mathcal{O}_{X^{[0]}}) = 0$.

Then the hull of LD is formally smooth over Λ_m, *and* X *is smoothable by a flat deformation.*

Proof. We shall check the conditions of Theorem 3.2. The condition (ii) follows immediately from Lemma 4.1. Let $(\mathcal{X}, \mathcal{U})$ be a log deformation of $(X, \mathcal{U}_0)$ over a Λ_m-algebra $\mathcal{A}$. By the Serre duality, (i) follows from the surjectivity of the homomorphism

$$H^{d-2}(\mathcal{X}_k, \Omega^1_{\mathcal{X}_k/\mathcal{A}_k}) \to H^{d-2}(\mathcal{X}_k, \Omega^1_{\mathcal{X}_k/\mathcal{A}_k}(\log)).$$

Since $d\log/2\pi i$ defines homomorphisms of sheaves

$$\mathcal{O}^*_{\mathcal{X}_k} \to \Omega^1_{\mathcal{X}_k/\mathcal{A}_k} \to \Omega^1_{\mathcal{X}_k/\mathcal{A}_k}(\log),$$

it is sufficient to prove that the image of

$$d\log/2\pi i : H^{d-2}(\mathcal{X}_k, \mathcal{O}^*_{\mathcal{X}_k}) \to H^{d-2}(\mathcal{X}_k, \Omega^1_{\mathcal{X}_k/\mathcal{A}_k}(\log))$$

generates $H^{d-2}(\mathcal{X}_k, \Omega^1_{\mathcal{X}_k/\mathcal{A}_k}(\log))$ as an $\mathcal{A}_k$-model.

We proceed by induction on k. First, we treat the case $k = 0$. The homomorphism $d\log/2\pi i : H^{d-2}(X, \mathcal{O}^*_X) \to H^{d-1}(X, \mathbb{Z})$ is surjective by (b). By [F, 1.5], if τ^p_X denotes the torsion part of $\Omega^p_{X/\mathbb{C}}$, then we have a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}/\tau^p_X) \Rightarrow H^{p+q}(X, \mathbb{C})$$

which degenerates at E_1. So the homomorphism

$$H^{d-1}(X, \mathbb{C}) \to H^{d-2}(X, \Omega^1_{X/\mathbb{C}}/\tau^1_X)$$

is surjective by (b) again.

Let $\mathcal{F}$ and $\mathcal{F}'$ be the cokernels of the natural homomorphisms $\mathcal{O}_X \to \nu_*\mathcal{O}_{X^{[0]}}$ and $\Omega^1_{X/\mathbb{C}}/\tau^1_X \to \Omega^1_{X/\mathbb{C}}(\log)$, respectively. Then we claim that $\mathcal{F} \partial \quad \mathcal{F}'$. In fact, locally on a log chart $(U; z_0, \ldots, z_d)$, $\mathcal{F}'$ is generated by $e_j = dz_j/z_j$ for $0 \leqq j \leqq r$. So we have

$$0 \to \mathcal{O}_U \to \bigoplus_{j=0}^{r} e_j(\mathcal{O}_U/(z_j)) \to \mathcal{F}'|_U \to 0.$$

The residue gives a natural identification of e_j to $1 \in \mathcal{O}_U/(z_j)$. Thus we have $\mathcal{F}' \partial \quad \mathcal{F}$.

Then by (b) and (c), $H^{d-2}(X, \mathcal{F}) = H^{d-2}(X, \mathcal{F}') = 0$, hence

$$H^{d-2}(X, \Omega^1_{X/\mathbb{C}}/\tau^1_X) \to H^{d-2}(X, \Omega'_{X/\mathbb{C}}(\log))$$

is surjective, and we finish the proof in the case $k = 0$.

Now we treat the case $k > 0$. We have an exact sequence

$$0 \to t^k\mathcal{O}_X \to \mathcal{O}^*_{\mathcal{X}_k} \to \mathcal{O}^*_{\mathcal{X}_{k-1}} \to 0.$$

So by (b), the homomorphism

$$H^{d-2}(\mathscr{X}_k, \mathcal{O}^*_{\mathscr{X}_{k-1}}) \to H^{d-2}(\mathscr{X}_{k-1}, \mathcal{O}^*_{\mathscr{X}_{k-1}})$$

is surjective. By the induction hypothesis and the Nakayama lemma, we complete the proof of (i) using (ii). Since X is Kähler and $W_X \partial \;\; \mathcal{O}_X$, a log automorphism is always liftable by Theorem 4.4 and [Wa, §3]. Thus LD is pro-representable. Q.E.D.

We shall give a simple example of producing Calabi-Yau 3-folds by applying the above theorem. Let $X = X_1 \cup X_2$ and $X_1 \cap X_2 = D$, where D is a smooth quartic in $\mathbb{P}^3$, and X_1 (resp. X_2) are obtained by blowing up $\mathbb{P}^3$ successively with smooth centers $C_1, \ldots, C_s$ (resp. $C'_1, \ldots, C'_t$) contained in the strict transforms of D in this order. In order that X has a logarithmic structure, we should have $C + C' \in |\mathcal{O}_D(8)|$ for $C = \sum_i C_i$ and$C' = \sum_j C'_j$, so that $N_{D/X_1} \otimes N_{D/X_2} \partial \;\; \mathcal{O}_D$. If there exists a polarization on X, then it is easy to see that the remaining conditions of Theorem 4.2 are satisfied and X has a smoothing. For example, if $s = 1$ and $t = 0$, then there exists a smoothing with Euler number $e = -296$.

We take a D defined by an equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$. For a primitive 8-th root of unity ζ, the divisors $\Gamma_{i,j,k}$ on D defined by $\Gamma_{i,j,k} = \{(x_0, \ldots, x_3) \in D; x_i = \zeta^k x_j\}$ consist of each 4 lines. We can choose 4 different $\Gamma_{i,j,k}$, denoted by $\Gamma_1, \ldots, \Gamma_4$, which have no common lines. Let a be a non-negative integer such that $a \leqq 4$. Then we take $C = C_1 + \cdots + C_{s'} + \Gamma_1 + \cdots + \Gamma_a$, and $C' = C'_1 + \cdots + C'_{t'} + \Gamma_1 + \cdots + \Gamma_a$, where C_i (resp. C'_j) for $1 \leqq i \leqq s'$ (resp. $1 \leqq j \leqq t'$) are members of $|\mathcal{O}_D(a_i)|$ (resp. $|\mathcal{O}_D(b_j)|$) such that $2a + \sum a_i + \sum b_j = 8$. Then we can prove that there exists a polarization on X. In this way, we can produce Calabi Yau 3-folds with the following Euler numbers: -296, -240, -200, -192, -176, -168, -160, -152, -144, -136, -128, -120, -112, -104, -96, -88, -80, -72, -64, -56, -48, -40, -32, -24, 0, 24.

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