Tôhoku Math. J.
47 (1995), 595–600

LOGARITHMIC DIVERGENCE OF HEAT KERNELS ON SOME QUANTUM SPACES

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(Received June 28, 1994, revised February 27, 1995)

Abstract. Asymptotic behaviour of the heat kernels on some explicitly known quantum spaces are studied. Then the heat kernels are shown to be logarithmically divergent. These results suggest to us that the "dimensions" of these quantum spaces would not be zero but less than one so that these quantum spaces look almost like "discrete spaces".

Introduction. In spectral geometry, there is a famous asymptotic formula of McKean and Singer [4] which relates the spectrum of the Laplacian and differential geometrical data (volume, the integration of the scalar curvature etc.) of a compact closed manifold. Namely, let $\{\lambda_1, \lambda_2, \dots\}$ be the spectrum of the Laplacian Δ (including the multiplicity) of a given compact closed manifold M of dimension n. Then the asymptotic behaviour of the heat kernel

$$H(t) := \mathrm{Trace}(e^{-t\Delta}) = \sum_{j=1}^{\infty} e^{-\lambda_j t}$$

for $t \downarrow 0$ is given by

$$\left(\frac{1}{4\pi t}\right)^{n/2} \{a_0 + a_1 t + a_2 t^2 + \cdots\} ,$$

where $a_0 = \mathrm{Volume}(M)$ is the volume of the manifold M and a_1 is the amount given in terms of the scalar curvature $\kappa(x)$ of the manifold M by

$$a_1 = \int_M \kappa(x) dv(x) .$$

The higher degree terms $a_2, a_3 \dots$ are also expressed in terms of differential geometrical data. Then this asymptotic exapansion formula tells us in particular that the dimension of a given manifold M is determined by the behaviour of the spectrum of the Laplacian on M.

This paper is devoted to the study of the asymptotic behaviour of the heat kernels associated with the quantum group $SU_q(2)$ and the quantum two-sphere $S_q^2(c, d)$ of Podleś. The motivation of our study comes from the fact that in the case of quantum

1991 *Mathematics Subject Classification.* Primary 58G25; Secondary 58B30, 33D80, 17B37.

groups or quantum homogeneous spaces, we have already observed several mysterious behaviours in contrast to the classical theory. The first observation is the discovery of the quantum two-sphere of Podleś which has extra parameters after being quantized, and we even have a discrete family of "quantum two-dimensional spheres". (See [6].)

The second observation is the vanishing of the third "deRham cohomology" of $SU_q(2)$ as well as the vanishing of the second "deRham cohomology" of $S_q^2(c, d)$. (For the precise statements, we fefer to [2] and [3].)

These "mysterious" behaviours of the "quantized objects" suggest to us that the process of "quantization" changes the "topological type" and "discretize" the manifold.

The main results of this paper are the explicit expressions of the asymptotic formulas of the heat kernels for the limit $t \downarrow 0$ associated with the quantum group $SU_q(2)$ and the quantum two sphere $S_q^2(c, d)$. Then we see that the formulas are logarithmically divergent in contrast to the classical case of having polynomial divergence with its degree determined by the dimension of the manifold. Our results suggests to us that the dimensions of these quantum spaces would not be zero but less than one, so that these "quantum spaces" look almost like "discrete spaces". This observation agrees with what we have already mentioned above.

Throughout this paper, we assume that the deformation parameter q is a real number satisfying the condition $0<q<1$.

ACKNOWLEDGMENT. The authors would like to thank the referee for pointing out some mistakes in the earlier version.

1. $SU_q(2)$ as a three dimensional quantum space. In this section, we study the case of $SU_q(2)$ viewed as a "three-dimensional quantum sphere". In the case of $SU_q(2)$ viewed as a "quantum manifold", the corresponding "Laplacian" is regarded to be the quantum Casimir operator C. (See [1], for example.) Then the n-th eigenvalue λ_n of the quantum Casimir operator C for $n=1, 2, \ldots$ is given by

$$\lambda_n = \left[\frac{n}{2}\right]_q^2 = \frac{q^n - 2 + q^{-n}}{(q-q^{-1})^2}$$

with its multiplicity given by $m_n = n^2$, where $[\alpha]_q$ is the homogeneous q-integer of $\alpha \in \boldsymbol{R}$ defined by

$$[\alpha]_q := \frac{q^\alpha - q^{-\alpha}}{q-q^{-1}}.$$

In the following discussions, we compute the asymptotic behaviour of the heat kernel

$$\begin{aligned} H_{SU_q(2)}(t) &:= \mathrm{Trace}(e^{-tC}) = \sum_{n=1}^{\infty} m_n e^{-\lambda_n t} \\ &= \sum_{n=1}^{\infty} n^2 \exp\left\{ \frac{-t}{(q-q^{-1})^2} (q^n - 2 + q^{-n}) \right\} \end{aligned}$$

for $t \downarrow 0$. Now, by putting $s := t/(q-q^{-1})^2$, the heat kernel $H_{SU_q(2)}(t)$ is expressed as $f(s) + A(s) + B(s)$, where

$$\begin{aligned} f(s) &:= \sum_{n=1}^{\infty} n^2 \exp(-sq^{-n}) , \\ A(s) &:= (e^{2s} - 1) f(s) , \\ B(s) &:= e^{2s} \sum_{k=1}^{\infty} \frac{(-1)^k s^k}{k!} \sum_{n=1}^{\infty} n^2 \exp(-sq^{-n}) q^{nk} . \end{aligned}$$

Here we have the estimate

$$\begin{aligned} |B(s)| &\leq e^{2s} \sum_{k=1}^{\infty} \frac{s^k}{k!} \sum_{n=1}^{\infty} n^2 \exp(-sq^{-n}) q^{nk} \leq e^{2s} \sum_{k=1}^{\infty} \frac{s^k}{k!} \sum_{n=1}^{\infty} n^2 q^n \\ &= e^{2s} \frac{q(1+q)}{(1-q)^3} \sum_{k=1}^{\infty} \frac{s^k}{k!} = e^{2s} \frac{q(1+q)}{(1-q)^3} (e^s - 1) = O(s) \quad \text{as } s \downarrow 0 . \end{aligned}$$

Therefore, the dominant part of the asymptotic exapansion of $H_{SU_q(2)}(t)$ is determined by $f(s)$. For simplicity, we put $\delta = \log 1/q$ and $\mu = \log 1/s$. Since $0 < q < 1$, we have $\delta > 0$. Furthermore, the limit $t \downarrow 0$ corresponds to $\mu \to +\infty$. We now use the inverse Mellin transformation of the Gamma function to obtain

$$f(s) = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \Gamma(z) \varphi(z) e^{\mu z} dz ,$$

where

$$\varphi(z) = \sum_{n=1}^{\infty} n^2 e^{-\delta z n} = \frac{e^{-\delta z}(1 + e^{-\delta z})}{(1 - e^{-\delta z})^3} .$$

By a change of the path of integration, we obtain

$$f(s) = \sum_{j=-\infty}^{+\infty} \operatorname*{Res}_{z = (2\pi i/\delta) j} [\Gamma(z) \varphi(z) e^{\mu z}] + \frac{1}{2\pi i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \Gamma(z) \varphi(z) e^{\mu z} dz . \tag{1.1}$$

Now, by simple computation, we have

$$\operatorname*{Res}_{z=0} [\Gamma(z) \varphi(z) e^{\mu z}] = \frac{1}{3\delta^3} \mu^3 - \frac{\gamma}{\delta^3} \mu^2 + \frac{\zeta(2) + \gamma^2}{\delta^3} \mu - \frac{2\zeta(3) + 3\zeta(2)\gamma + \gamma^3}{3\delta^3}$$

and for a non-zero integer k, we have

$$\operatorname*{Res}_{z=(2\pi i/\delta)k}[\Gamma(z)\varphi(z)e^{\mu z}]$$

$$=\left[\frac{\Gamma\left(\frac{2\pi i}{\delta}k\right)}{\delta^3}\mu^2+\frac{2\Gamma'\left(\frac{2\pi i}{\delta}k\right)}{\delta^3}\mu+\frac{\Gamma''\left(\frac{2\pi i}{\delta}k\right)}{\delta^3}\right]\exp\left(\frac{2\pi i}{\delta}k\mu\right),$$

where γ is the Euler constant and $\zeta(s)$ is the Riemann zeta function.

By making use of the asymptotic estimate

$$\Gamma\left(-\frac{1}{2}+iy\right)\sim\sqrt{2\pi}\,|y|^{-1}e^{-(\pi|y|)/2}\qquad\text{for}\quad |y|\to+\infty$$

and the boundedness of $|\varphi(-1/2+iy)|$ with respect to the variable y, the second term on the right hand side of (1.1) is bounded by $\text{Const.}e^{-\mu/2}=\text{Const.}\sqrt{t}$.

With these discussions, we have the following asymptotic expansion of the heat kernel $H_{SU_q(2)}(t)$:

THEOREM 1. *In the asymptotic expansion of $t\downarrow 0$, the heat kernel $H_{SU_q(2)}(t)$ has an expression*

$$H_{SU_q(2)}(t)=\frac{1}{3\delta^3}\left\{\log\frac{(q-q^{-1})^2}{t}\right\}^3$$

$$+\sum_{j=0}^{2}h_j\left(\log\frac{(q-q^{-1})^2}{t}\right)\left\{\log\frac{(q-q^{-1})^2}{t}\right\}^j+O(\sqrt{t}),$$

where $h_j(\mu)$ for $j=1, 2, 3$ are smooth and δ-periodic functions of μ given by

$$h_0(\mu):=-\frac{2\zeta(3)+3\zeta(2)\gamma+\gamma^3}{3\delta^3}+\sum_{k\in\boldsymbol{Z}\setminus\{0\}}\frac{\Gamma''\left(\frac{2\pi i}{\delta}k\right)}{\delta^3}\exp\left(\frac{2\pi i}{\delta}k\mu\right),$$

$$h_1(\mu):=\frac{\zeta(2)+\gamma^2}{\delta^3}+\sum_{k\in\boldsymbol{Z}\setminus\{0\}}\frac{2\Gamma'\left(\frac{2\pi i}{\delta}k\right)}{\delta^3}\exp\left(\frac{2\pi i}{\delta}k\mu\right),$$

$$h_2(\mu):=-\frac{\gamma}{\delta^3}+\sum_{k\in\boldsymbol{Z}\setminus\{0\}}\frac{\Gamma\left(\frac{2\pi i}{\delta}k\right)}{\delta^3}\exp\left(\frac{2\pi i}{\delta}k\mu\right).$$

REMARK. The coefficients of

$$\mu^j = \left\{ \log \frac{(q-q^{-1})^2}{t} \right\}^j, \qquad j=0,1,2$$

in the above asymptotic expansion formula are not constants but periodic functions of μ. However, in the classical McKean-Singer asymptotic expansion formula, the coefficients of the expansion are all constants. Therefore, in combination with the fact that the order of divergence is logarithmic, this type of formula in the quantum case is very different from that in the classical situation.

2. The case of quantum two-spheres $S_q^2(c, d)$ of Podleś. In this section, we deal with the quantum two spheres $S_q^2(c, d)$ of Podleś viewed as a "two-dimensional quantum sphere". In this case, the corresponding "Laplacian" is again regarded to be the action of the same quantum Casimir operator C. (See [5], for example.) Then the n-th eigenvalue λ_n of the quantum Casimir operator C for $n=0, 1, \ldots$ is given by

$$\lambda_n = \left[n + \frac{1}{2} \right]_q^2 = \frac{q^{2n+1} - 2 + q^{-2n-1}}{(q-q^{-1})^2}$$

with its multiplicity given by $m_n = 2n+1$. Then, by the same type of discussions as in the case of $SU_q(2)$, we obtain the asymptotic behaviour of the heat kernel

$$\begin{aligned} H_{S_q^2(c,d)}(t) &:= \operatorname{Trace}(e^{-tC}) = \sum_{n=0}^{\infty} m_n e^{-\lambda_n t} \\ &= \sum_{n=0}^{\infty} (2n+1) \exp\left\{ \frac{-t}{(q-q^{-1})^2} (q^{2n+1} - 2 + q^{-2n-1}) \right\} \end{aligned}$$

for $t \downarrow 0$.

THEOREM 2. *In the asymptotic expansion of $t \downarrow 0$, the heat kernel $H_{S_q^2(c,d)}(t)$ has an expression*:

$$\begin{aligned} H_{S_q^2(c,d)}(t) &= \frac{1}{4\delta^2} \left\{ \log \frac{(q-q^{-1})^2}{t} \right\}^2 \\ &\quad + c_1\left(\log \frac{(q-q^{-1})^2}{t} \right) \left\{ \log \frac{(q-q^{-1})^2}{t} \right\} + c_0\left(\log \frac{(q-q^{-1})^2}{t} \right) + O(\sqrt{t}), \end{aligned}$$

where $c_j(\mu)$ for $j=0, 1$ are smooth and δ-periodic functions of μ given by

$$c_0(\mu) := \frac{\zeta(2) + \gamma^2}{4\delta^2} + \frac{1}{12} + \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{(-1)^k}{2\delta^2} \Gamma'\left(\frac{\pi i}{\delta} k \right) \exp\left(\frac{\pi i}{\delta} k\mu \right),$$

$$c_1(\mu) := -\frac{\gamma}{2\delta^2} + \sum_{k \in \mathbf{Z} \setminus \{0\}} \frac{(-1)^k}{2\delta^2} \Gamma\left(\frac{\pi i}{\delta} k \right) \exp\left(\frac{\pi i}{\delta} k\mu \right).$$

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