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Logarithmic Enriques surfaces, II

By

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Introduction

This is a sequel of our paper [2]. Every thing will be defined over the complex number field **C**. Let $\bar{V}$ be a normal projective surface. A log Enriques surface can occur as the base space of a cy 3-fold with a fibration.

Definition 1. $\bar{V}$ is *a logarithmic* (*log*, for short) *Enriques surface* if the subsequent conditions are satisfied:

(1) $\bar{V}$ has at worst isolated quotient singularities;

(2) A multiple $NK_{\bar{V}}$ of a canonical divisor $K_{\bar{V}}$ of $\bar{V}$ is linearly equivalent to zero for some positive integer N;

(3) $H^1(\bar{V}, \mathcal{O}_{\bar{V}})$ vanishes.

The index of $\bar{V}$ is defined as:

$$I = \text{Index}(\bar{V}) = \text{Min}\{N \geq 1;\ NK_{\bar{V}} \sim 0\}.$$

A K3-surface (resp. an Enriques surface) is a log Enriques surface of index one (resp. two). It is known that $1 \leq I \leq 66$ (cf. Proposition 1.3 below). Furthermore, if I is a prime number then $I \leq 19$. Since $IK_{\bar{V}}$ is linearly equivalent to zero, there is a $\mathbf{Z}/I\mathbf{Z}$-covering $\pi\colon \bar{U} \to \bar{V}$ such that π is étale over the smooth part $\bar{V} - (\text{Sing}\,\bar{V})$ of $\bar{V}$ and that $\bar{U}$ is an abelian surface or a K3-surface possibly with isolated rational double singularities (cf. [2, Definition 2.1]). In particular, the canonical divisor $K_{\bar{U}}$ of $\bar{U}$ is linearly equivalent to zero.

Definition 2. $\pi\colon \bar{U} \to \bar{V}$ is *the canonical covering of* $\bar{V}$. Actually, $\bar{V}$ determines $\bar{U}$ uniquely up to isomorphisms.

A log Enriques surface of index one is a K3-surface possibly with rational double singularities. A log Enriques surface of index 2 is an Enriques surface possibly with rational double singularities or a rational surface (cf. [2, Proposition 1.3]). The latter surfaces are classified in [2, Theorem 3.6]. Log Enriques surfaces $\bar{V}$ of index I with smooth canonical coverings $\bar{U}$ are classified in [2, Theorems 4.1 and 5.1]. In particular, if $\bar{U}$ is an abelian surface then $I = 3$ or 5.

If $\bar{V}$ has rational double singular points, we denote by $\tilde{V}$ a minimal resolution of all rational double singularities of $\bar{V}$. Then $\tilde{V}$ is a log Enriques surface of the same index as $\bar{V}$. Instead of $\bar{V}$, we can treat $\tilde{V}$ without loss of generality.

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In view of the above arguments, we shall assume the following hypothesis in Theorem 2.11 below.

Hypothesis (A) (1) *The index I of $\bar{V}$ is greater than 2. Hence $\bar{V}$ is a rational surface (cf. [2, Proposition 1.3]) and $\bar{V}$ admits at least one singular point.*

(2) *The canonical covering $\bar{U}$ of $\bar{V}$ is not an abelian surface. Hence $\bar{U}$ is a $K3$-surface possibly with rational double singularities.*

(3) *Every singularity of $\bar{V}$ has multiplicity ≥ 3, i.e., $\bar{V}$ has no rational double singular points.*

If $I = pq$ for two positive integers p, q, we let $\bar{V}_1 := \bar{U}/(\mathbf{Z}/p\mathbf{Z})$. Then $\bar{V}_1$ is a log Enriques surface of index p (cf. [2, Lemma 2.2]) with $\bar{U}$ as its canonical covering. So, we shall mainly consider log Enriques surfaces of prime index (See Proposition 1.3, (2) below). The following theorem is a part of Theorem 2.11 in §2 and our starting point.

Theorem 2.11'. *Let $\bar{V}$ be a log Enriques surface satisfying the above Hypothesis (A). Assume that the index I of $\bar{V}$ is an odd prime number. Then we have:*

(1) *There is a composite $\bar{V}_n \xrightarrow{\bar{h}_n} \cdots \to \bar{V}_1 \xrightarrow{\bar{h}_1} \bar{V}_0 := \bar{V}$ $(n \geq 0)$ of combining morphisms (cf. Definition 2.1 and Proposition 2.8 below for the definition) between log Enriques surfaces of the same index I such that $\bar{U}_n$ is a $K3$-surface possibly with rational double singular points of Dynkin type A_1. Here $\pi_i\colon \bar{U}_i \to \bar{V}_i$ is the canonical covering of $\bar{V}_i$.*

(2) *For each singularity x of $\bar{U}_n$ the image $y := \pi_n(x) \in \bar{V}_n$ is a singularity isomorphic to $(\mathbf{C}^2/C_{2I,1}; 0)$, where $C_{2I,1} := \langle \sigma_{2I,1} \rangle \subseteq GL(2; \mathbf{C})$ is a cyclic subgroup of order $2I$ generated by*

$$\sigma_{2I,1} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix},$$

η being a primitive $2I$-th root of the unity.

(3) *Every $\bar{V}_i$ satisfies the Hypothesis (A).*

The above n, $\bar{U}_n$ and $\bar{V}_n$ are uniquely determined by the original surface $\bar{V}$ (cf. Theorem 2.11 in §2). We shall describe precisely $\bar{V}_n$ and $\bar{U}_n$ in Theorems 3.1–9.1. As consequences, we will have:

Main Theorem. *With the assumptions and notations of Theorem 2.11', we describe in Tables 1, 2, 3, 5, 7 all possible distributions of singular points on $\bar{V}_n$ and on $\bar{U}_n$ as well as the Picard number of $\bar{V}_n$.*

Corollary 1. (1) *If $I = 3$, then* $\#(\mathrm{Sing}\ \bar{U}_n) \leq 6$ *and* $\#(\mathrm{Sing}\ \bar{V}_n) \leq 15$.

(2) *If $I = 5$, then* $\#(\mathrm{Sing}\ \bar{U}_n) \leq 3$ *and* $\#(\mathrm{Sing}\ \bar{V}_n) \leq 16$.

(3) *If $I = 7$, then* $\#(\mathrm{Sing}\ \bar{U}_n) \leq 2$ *and* $\#(\mathrm{Sing}\ \bar{V}_n) \leq 15$.

(4) *If $I = 11$, then* $\#(\mathrm{Sing}\ \bar{U}_n) \leq 1$ *and* $\#(\mathrm{Sing}\ \bar{V}_n) = 2, 12, 13$.

(5) *If $I = 13$, then* $\#(\mathrm{Sing}\ \bar{U}_n) = 1$ *and* $\#(\mathrm{Sing}\ \bar{V}_n) = 10$.

(6) *If $I = 17$ or 19 then $\bar{U}_n$ is smooth.*

The upper bounds for $\#(\mathrm{Sing}\, \bar{U}_n)$ *and* $\#(\mathrm{Sing}\, \bar{V}_n)$ *in* (1), (2) *and* (3) *above are best ones* (*See* [2, *Examples* 6.11, 6.12 *and* 6.13]). *For* $I = 3, 5, 7, 11, 13$ *there are examples of* $\bar{V}$ *for which* $\bar{U}_n$ *admits at least one singular point* (*See Examples* 3.2, 4.3, 5.3, 6.3 *and* 7.3).

Corollary 2 (*cf. Lemmas* 1.2 *and* 2.3 *below*). *Let* $\bar{V}$ *be as in Theorem* 2.11'. *Let* $f: V \to \bar{V}$ *be a minimal resolution of singularities and set* $c := \#(\mathrm{Sing}\, \bar{V})$, $D := f^{-1}(\mathrm{Sing}\, \bar{V})$. *Then we have* $h^1(V, D + 2K_V) = c - 1 - (K_V^2) - (D, K_V) = 0$.

Remark. (1) If $\bar{V}$ is a log Enriques surface of index 13 then the canonincal covering of $\bar{V}$ admits at least one singular point (cf. [2, Theorems 4.1 and 5.1]).

(2) For each odd prime number I with $I \neq 13$ and $I \leq 19$ we gave examples in [2, §5] of log Enriques surfaces of index I with smooth canonical coverings.

When I is a prime number, the following result characterizes a combining morphism, which is indeed a crepant blowing-up (cf. Example 7.3 in §3).

Proposition 2.8. *Let* $\bar{V}$ *and* $\bar{V}_1$ *be two log Enriques surfaces of the same prime index* I. *Let* $\pi: \bar{U} \to \bar{V}$ *and* $\pi_1: \bar{U}_1 \to \bar{V}_1$ *be canonical coverings. Then the following conditions are equivalent*:

(1) *There is a combining morphism* $\bar{h}: \bar{V} \to \bar{V}_1$ *with exceptional curve* $\bar{E}$.

(2) *There is a point* y *of* $\bar{V}_1$ *which is not a rational double singular point and there is a birational morphism* $\bar{h}: \bar{V} \to \bar{V}_1$ *such that* $\bar{h}$ *is an isomorphism over* $\bar{V}_1 - \{y\}$, *the exceptional divisor* $\bar{h}^{-1}(y)$ *is an irreducible curve and* $\bar{h}^{-1}(y) \cap (\mathrm{Sing}\, \bar{V})$ *consists of two points* z_1, z_2.

(3) *There is a point* $x \in \bar{U}_1$ *and there is a* $\mathbf{Z}/I\mathbf{Z}$*-equivariant morphism* $\tilde{h}: \bar{U} \to \bar{U}_1$ *such that* $\pi_1(x)$ *is not a rational double singular point,* $\tilde{h}$ *is an isomorphism over* $\bar{U}_1 - \{x\}$, *the exceptional divisor* $\bar{F} := \tilde{h}^{-1}(x)$ *is an irreducible curve,* $\bar{F}$ *is* $\mathbf{Z}/I\mathbf{Z}$*-stable and* $\bar{F}$ *has exactly two* $\mathbf{Z}/I\mathbf{Z}$*-fixed points* $\{z'_1, z'_2\}$.

Under the above equivalent conditions, we have $\pi_1 \cdot \tilde{h} = \bar{h} \cdot \pi$. *Hence* $\bar{E} = \bar{h}^{-1}(y)$, $\bar{F} = \pi^{-1}(\bar{E})$, $x = \pi_1^{-1}(y)$ *and* $z'_i = \pi^{-1}(z_i)$ $(i = 1, 2)$ *after a suitable relabelling. Moreover,* $x \in \bar{U}_1$ *is a singular point, and* $y \in \bar{V}_1$ *and* $z_i \in \bar{V}(i = 1, 2)$ *are singularities of multiplicity* ≥ 3.

Terminology. A $(-n)$-curve on a nonsingular projective surface V is a nonsingular rational curve of self intersection number $-n$. A curve C on a surface V is called an m-section of a certain fibration from V onto a curve if $(C, \mathrm{F}) = m$ for a fiber F.

Notations. Let V be a nonsingular projective surface and let D, $H_1, H_2, \ldots,$ be divisors on V.

K_V: Canonical divisor of V

$\rho(V) :=$ rank $NS(V) \otimes_{\mathbf{Z}} \mathbf{Q}$, the Picard number of V, where $NS(V)$ is the Neron-Severi group of V

$H_1 \sim H_2$: linear equivalence

$H_1 \equiv H_2$: numerical equivalence
$f_*(D)$: the direct image of D by a morphism f
$f^*(D)$: the total transform of D by a morphism f
$f'(D)$: the proper transform of D by a birational morphism f
$\#(D)$: the number of irreducible components of Supp (D)
Sing $\bar{V}$: the singular locus of a variety $\bar{V}$

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§ 1. Preliminaries

Let $\bar{V}$ be a log Enriques surface of index I. Let $f: V \to \bar{V}$ be a minimal resolution of singularities. Denote by D the exceptional set $f^{-1}(\mathrm{Sing}\,\bar{V})$. Then D is a reduced effective divisor with only simple normal crossings and its dual graph is a disjoint union of trees. Moreover, every component of D is a nonsingular rational curve of self intersection number ≤ -2 and the intersection matrix of irreducible components of D is negative definite.

From now on, we shall confuse $\bar{V}$ with a triple (V, D, f) or a pair (V, D).

The following four results will be used in the sections below.

Lemma 1.1 (*cf.* [2, *Lemma* 1.2]). *Let $\bar{V}$ be a log Enriques surface and let $D = \sum_{i=1}^{n} D_i$ be the irreducible decomposition of D. Then we have*:

(1) $H^1(V, \mathcal{O}_V) = 0$.

(2) *There is a* **Q**-*divisor $D^* = \sum_{i=1}^{n} \alpha_i D_i$ such that $I\alpha_i$ is an integer with $0 \leq I\alpha_i \leq I - 1$ for each i and $f^*(IK_{\bar{V}}) \sim I(K_V + D^*) \sim 0$. Moreover, D^* is uniquely determined.*

(3) *$\alpha_i = 0$ if and only if the connected component of D containing D_i is contractible to a rational double singularity on $\bar{V}$.*

(4) $K_V \equiv -D^*$, $(K_{\bar{V}}^2) = (D^*)^2$.

Lemma 1.2. *Let $\bar{V}$ be a log Enriques surface of index I satisfying the Hypothesis (A) in the Introduction. Set $c := \#(\mathrm{Sing}\,\bar{V})$ which is also the number of connected components of D. Assume $I \geq 3$. Then we have $h^1(V, D + 2K_V) = c - 1 - (K_{\bar{V}}^2) - (D, K_V)$.*

Proof. By the proof of Proposition 1.6 in [2], we have

$$H^2(V, D + 2K_V) = H^0(V, D + 2K_V) = 0.$$

On the other hand, let $D = \sum_{i=1}^{n} D_i$ be the irreducible decomposition of D. Since the dual graph of D is a disjoint union of trees, we have the following computation:

$$(D, D + K_V) = \sum_{i=1}^{n} (D_i, D_i + K_V) + 2 \sum_{i<j} (D_i, D_j) = -2n + 2(n - c) = -2c.$$

Then Lemma 1.2 follows from the Riemann-Roch theorem.

Proposition 1.3 (*cf.* [2, *Lemmas* 2.3 *and* 2.4 *and Proposition* 6.6]). *Let* $\bar{V}$ *be a log Enriques surface of index* I *satisfying the Hypothesis* (A) *in the Introduction. Let* $\pi: \bar{U} \to \bar{V}$ *be the canonical covering. Let* $g: U \to \bar{U}$ *be a minimal resolution of singularities. Set* $c := \#(\mathrm{Sing}\, \bar{V})$. *Then we have*:

(1) *We have* $\varphi(I) \leq 22 - \rho(U) \leq 21$, *where* φ *is Euler's* φ*-function. Thence we have* $2 \leq I \leq 66$. *If* I *is a prime number then* $2 \leq I \leq 19$. *If* I *is not a prime number then* I *is divisible by* 2, 3 *or* 5.

(2) *If* I *is a prime number then we have*

$$c + \rho(U) - \rho(\bar{U}) + I(\rho(\bar{V}) - c + 2) = 24.$$

(3) *Assume* I *is an odd prime number and* $\bar{U}$ *admits at least one singular point. Then we have*

$$\rho(\bar{V}) \geq c - 1, \quad 2 \leq c \leq \mathrm{Min}\,\{16, 23 - I\}.$$

If $I = 3$ *then* $\rho(\bar{V}) \leq c + 4$. *If* $I = 5$ *then* $\rho(\bar{V}) \leq c + 2$. *If* $I = 7$ *then* $\rho(\bar{V}) \leq c + 1$. *If* $I \geq 11$ *then* $\rho(\bar{V}) = c - 1$. *If* $c = 16$ *then* $I = 5$.

Let η_n be a primitive n-th root of the unity and let k be an integer satisfying $1 \leq k \leq n - 1$ and $g.c.d.\,(n, k) = 1$. Then $C_{n,k}$ denotes a finite cyclic subgroup of order n in $GL(2, \mathbf{C})$ which is generated by

$$\sigma_{n,k} := \begin{pmatrix} \eta_n & 0 \\ 0 & \eta_n^k \end{pmatrix}.$$

Lemma 1.4. *Let* $\bar{V}$ *be a log Enriques surface of prime index* I *and let* $\pi: \bar{U} \to \bar{V}$ *be the canonical covering. Let* y *be a singularity of* $\bar{V}$ *of multiplicity* ≥ 3. *Then we have*:

(1) $x := \pi^{-1}(y)$ *consists of a single singular point of* $\bar{U}$. *Hence* x *is fixed by the natural* $\mathbf{Z}/I\mathbf{Z}$*-action on* $\bar{U}$. *The covering morphism* π *ramifies exactly over* $f(\mathrm{Supp}\, D^*)$ *which coincides with set of singularities of* $\bar{V}$ *of multiplicity* ≥ 3 (*cf. the notations of Lemma* 1.1).

(2) *Assume further* y *is a cyclic singularity. Then* x *is a rational double singularity of Dynkin type* A_{N-1} *for some* $N \geq 1$. *The case* $N = 1$ *corresponds to the case where* x *is smooth. Moreover, we have* $(\bar{V}, y) \cong (\mathbf{C}^2/C_{IN,k}, 0)$ *for an integer* k *which satisfies the conditions*:

$$(i)\ 1 \leq k \leq IN - 2, \quad (ii)\ N|(1 + k), \quad (iii)\ I \nmid k.$$

If $N = 1$, *we can list all possible cases of* k *as follows*:

(2–1) $I = 3$, $k = 1$.

(2–2) $I = 5$, $k = 1, 2$.

(2–3) $I = 7$, $k = 1, 2, 3$.

(2–4) $I = 11$, $k = 1, 2, 3, 5, 7$.

(2–5) $I = 13$, $k = 1, 2, 3, 4, 5, 6$.

(2–6) $I = 17$, $k = 1, 2, 3, 4, 5, 8, 10, 11$.

(2–7) $I = 19$, $k = 1, 2, 3, 4, 6, 7, 8, 9, 14$.

Proof. (1) Note that every singularity of $\bar{U}$ is a rational double singularity because $K_{\bar{U}} \sim 0$. Since the degree I of π is a prime number, $\pi^{-1}(y)$ consists of one or I points. If $\pi^{-1}(y)$ consists of I points x_i's then $(\bar{U}, x_i) \cong (\bar{V}, y)$ for each i. Hence y must be a rational double singularity. This contradicts the assumption. So, $\pi^{-1}(y)$ consists of one point x. The second assertion of (1) follows from $I(K_V + D^*) \sim 0$ (see the construction of $\bar{U}$ in [2, §2] and Lemma 1.1, (3)).

(2) Assume y is a cyclic singularity of multiplicity ≥ 3. Then $(\bar{V}, y) \cong (\mathbf{C}^2/G_y, 0)$ with a group G_y which is isomorphic to $C_{M,k}$ with $1 \leq k \leq M-2$ and $g.c.d.(M, k) = 1$ (cf. Brieskorn [1]). Moreover, x is a smooth point or a cyclic singularity. So, x has Dynkin type A_{N-1} for some $N \geq 1$. Namely, there is a subgroup $G_x \subseteq SL(2, \mathbf{C})$ of order N such that $(\bar{U}, x) \cong (\mathbf{C}^2/G_x, 0)$. Since G_x is a subgroup of G_y with index I we have $M = IN$. So, $G_x = \langle \sigma^I_{M,k} \rangle$ and $1 = \det(\sigma^I_{M,k}) = \eta_M^{I(1+k)}$. Hence $IN | I(1+k)$ and $N | (1+k)$. This is the condition (ii) of (2). The condition (iii) follows from $g.c.d.(IN, k) = 1$. The condition (i) follows from the choice of k.

It remains to obtain the list for $N = 1$. First, we write down a list of integers (I, k) satisfying the conditions (i), (ii) and (iii). If $k' > k$ and $(\mathbf{C}^2/C_{IN,k'}, 0) \cong (\mathbf{C}^2/C_{IN,k}, 0)$, we can omit (I, k') from the list. A list, thus obtained, is the one given in (2).

§2. Proof of Theorem 2.11

Let $\bar{V}$ be a log Enriques surface of index I. We shall use the notaions (V, D, f) in §1. Assume that there is a (-1)-curve E on V; such a (-1)-curve always exists if $\bar{V}$ is a rational surface. Let $V = V_t \xrightarrow{h_t} V_{t-1} \xrightarrow{h_{t-1}} \cdots \xrightarrow{h_2} V_1$ be a composite of blowing-downs of (-1)-curves such that h_t is the blowing-down of $E_t := E$, $h_i (2 \leq i \leq t-1)$ is the blowing-down of a (-1)-curve $h^{(i+1)}(E_i)$ of $h^{(i+1)}_*(D)$ and $D_{(1)} := h_*(D)$ contains no (-1)-curves. Here we set $h^{(i+1)} := h_{i+1} \cdots h_t : V_t \to V_i$, $h^{(t+1)} = id$ and $h = h^{(2)}$.

Assume further that $D_{(1)}$ *is contractible to quotient singularities.* Let $f_1 : V_1 \to \bar{V}_1$ be the contraction of $D_{(1)}$, which makes V_1 a minimal resolution of $\bar{V}_1$. Set $\bar{E} := f(E)$ and denote by y the point $f_1 h(E)$ on $\bar{V}_1$. Then h induces a birational morphism $\bar{h} : \bar{V} \to \bar{V}_1$ such that $\bar{h} \cdot f = f_1 \cdot h$, $\bar{h}^{-1}(y) = \bar{E}$ and $\bar{h}$ is an isomorphism over $\bar{V}_1 - \{y\}$.

Definition 2.1 The morphism $\bar{h}$ is *a combining morphism with exceptional curve* $\bar{E}$.

Concerning $\bar{V}_1$, we have the following:

Lemma 2.2. *Let* $\bar{V}$ *be a log Enriques surface of index* I. *Let* $\bar{h} : \bar{V} \to \bar{V}_1$ *be a combining morphism. Then* $\bar{V}_1$ *is a log Enriques surface of the same index* I. *We have moreover* $D^*_{(1)} = h_*(D^*)$ *in the notations of Lemma* 1.1.

Proof. Note that $\bar{V}_1$ is birationally equivalent to $\bar{V}$ and $h^1(V_1, \mathcal{O}_{V_1}) = h^1(V, \mathcal{O}_V) = 0$. Note also that $\bar{V}_1$ has at worst quotient singularities by the definition of $\bar{h}$. So, $h^1(\bar{V}_1, \mathcal{O}_{\bar{V}_1}) = h^1(V_1, \mathcal{O}_{V_1}) = 0$. Let $\bar{E}$ be the exceptional curve of $\bar{h}$. Since $I(K_V + D^*) \sim 0$ (cf. Lemma 1.1), we have $I(K_{V_1} + h_* D^*) \sim 0$. Hence $f_1^*(IK_{\bar{V}_1}) \sim I(K_{V_1} + h_* D^*)$ and $IK_{\bar{V}_1} \sim 0$. So, $\bar{V}_1$ is a log Enriques surface and its index, say J, is a divisor of I. In view of Lemma 1.1, (2), we have only to show that $J = I$. We can write $0 \sim \bar{h}^*(JK_{\bar{V}_1}) \sim JK_{\bar{V}} + \alpha \bar{E}$ with a rational number α. Since $IK_{\bar{V}} \sim 0$, we have then $I\alpha\bar{E} \sim 0$. Hence $\alpha = 0$ and $JK_{\bar{V}} \sim 0$. So, we have $I | J$ by the definition of index. So, $J = I$.

In order to prove Proposition 2.8, we need the following Lemmas 2.3 $\sim$ 2.7. The assertion (4) in the following lemma will also be used in the proof of Corollary 2 which is stated in the Intoduction.

Lemma 2.3. *Let $\bar{h}: \bar{V} \to \bar{V}_1$ be a combinig morphism between two log Enriques surfaces of the same index I. We shall use the notations $(V_1, D_{(1)}, f_1)$, $E = E_t$, $y = \bar{h}(\bar{E})$, etc. in Definition 2.1. Then we have:*

(1) *For any i $(2 \le i \le t)$, $h^{(i+1)}(E_i)$ meets exactly two irreducible components $h^{(i+1)}(B_i')$ and $h^{(i+1)}(B_i'')$ of $h^{(i+1)}(D)$. For each $3 \le i \le t$, E_{i-1} is equal to one of B_i' and B_i''. Denoting by α_i'/I, α_i''/I the coefficient of B_i', B_i'' in D^*, respectively, we have $(h^{(i+1)}(E_i), h^{(i+1)}(B_i)) = (h^{(i+1)}(E_i), h^{(i+1)}(B_i'')) = 1$ and $\alpha_i' + \alpha_i'' \ge I$.*

(2) *Let Γ_1, Γ_2 be the connected components of D containing B_t', B_t'', respectively, and set $z_i := f(\Gamma_i)$ $(i = 1, 2)$. Then we have $z_1 \ne z_2$, $\bar{E} \cap (\mathrm{Sing}\, \bar{V}) = \{z_1, z_2\}$, $f^{-1}(z_i) = \Gamma_i$ and $f_1^{-1}(y) = h(E + \Gamma_1 + \Gamma_2)$. Moreover, $\bar{E}$ is a nonsingular rational curve.*

(3) *$y \in \bar{V}_1$ and $z_i \in \bar{V}$ $(i = 1, 2)$ are quotient singularities of multiplicity ≥ 3.*

(4) *$h^1(V, D + 2K_V) = h^1(V_1, D_{(1)} + 2K_{V_1})$.*

Proof. Since $D_{(1)} = h_*(D) = h_*(E + D)$ and $D_{(1)}$ is contractible to quotient singularities on V_1, the dual graph of $E + D$ is a disjoint union of trees and the (-1)-curve E meets at most two irreducible components of D. In particular, $\Gamma_1 \ne \Gamma_2$ and $z_1 \ne z_2$. If E meets two (one, none, resp.) irreducible components of D, denotes them by B_t' and B_t'' (B_t', ϕ, resp.). Accordingly, we have $0 = (E, K_V + D^*) = -1 + \alpha_t'/I + \alpha_t''/I (-1 + \alpha_t'/I, -1$, resp.). By Lemma 1.1, we have $\alpha_t'/I < 1$. Hence E meets exactly two components B_t' and B_t'' of D and we have $\alpha_t' + \alpha_t'' = I$. Let $h_t: V = V_t \to V_{t-1}$ be the blowing-down of $E = E_t$. Then we have $I(K_{V_{t-1}} + h_{t*} D^*) \sim 0$. If $h_{t*}(D)$ contains no (-1)-curves, then (1) is proved. If $h_{t*}(D)$ contains a (-1)-curve $h_t(E_{t-1})$, then E_{t-1} must be one of B_t' and B_t''. B_t' by the convention. Arguing similarly with $h_t(E_{t-1})$, we can conclude (1) and (2). Indeed, we have $0 = (E_{t-1}, K_{V_{t-1}} + h_{t*} D) \le -1 + \alpha_{t-1}'/I + \alpha_{t-1}''/I$.

(3) Note that $f_1 h(B_2'') = f_1 h(E + \Gamma_1 + \Gamma_2) = y$ and $\{f(B_2'), f(B_2'')\} = \{z_1, z_2\}$ as set. By Lemma 1.1, (2), the coefficients of B_2', B_2'' in D^* satisfy $\alpha_2'/I < 1$ and $\alpha_2''/I < 1$. Since $\alpha_2' + \alpha_2'' \ge I$, we have $\alpha_2' > 0$ and $\alpha_2'' > 0$. So, z_i $(i = 1, 2)$ is not a rational double singularity (cf. Lemma 1.1, (3)). Note that α_2'/I is also the coefficient of the irreducible component $h(B_2')$ in $D_{(1)}^* = h_*(D^*)$. So, $y = f_1 h(B_2')$

is not a rational double singularity.

(4) In view of Lemma 1.2, we have only to show that $f(t) = f(1)$. Here we set $c(i) := \#\{\text{connected component of } h_*^{(i+1)}D\}$ and

$$f(i) := c(i) - (K_{V_i}^2) - (h_*^{(i+1)}D, K_{V_i}).$$

We have $c(t) = c(i) + 1$ for $1 \le i \le t-1$, $(K_{V_{i-1}}^2) = (K_{V_i}^2) + 1$ and

$$(K_{V_{i-1}}, h_{i*}B) - (K_{V_i}, B) = \begin{cases} -1 & \text{if } B = B_i' \text{ or } B_i'' \\ 1 & \text{if } B = E_i \\ 0 & \text{otherwise.} \end{cases}$$

Note that $E = E_t$ is not contained in D and that E_i $(i \le t-1)$ is a component of D. We then obtain $f(t) = f(t-1) = \cdots = f(1)$.

Lemma 2.4. *Let $\bar{h}: \bar{V} \to \bar{V}_1$ be a birational morphism between two log Enriques surfaces of the same index I. Then the following two conditions are equivalent:*

(1) $\bar{h}$ is a combining morphism.

(2) There is a point y on $\bar{V}_1$ which is not a rational double singular point, such that $\bar{h}$ is an isomorphism over $\bar{V} - \{y\}$ and the exceptional divisor $\bar{E} := \bar{h}^{-1}(y)$ is an irreducible curve.

Assume the above equivalent conditions. Then y is a singularity of multiplicity ≥ 3 (cf. Lemma 2.3).

Proof. If $\bar{h}$ is a combining morphism, then the condition (2) follows from the definition of $\bar{h}$.

Now we assume the condition (2). We use the notations (V, D, f) for $\bar{V}$ and $(V_1, D_{(1)}, f_1)$ for $\bar{V}_1$. Set $E := f'(\bar{E})$. Note that E is not a component of D. Hence we have $(E, K_V) = (E, -D^*) \le 0$ (cf. Lemma 1.1, (4)). Moreover, we have $(E^2) < 0$ because E is contractible to the point y by the birational morphsim $\bar{h} \cdot f$. So, E is a (-1)-curve or a (-2)-curve.

Suppose $(E^2) = -2$. Then $E \cap D^* = \phi$. Let D_i $(1 \le i \le r)$ be all connected components of D with $(E, D_i) > 0$. Then D_i consists of (-2)-curves (cf. Lemma 1.1, (3)). Note that $\bar{h} \cdot f: V \to \bar{V}_1$ is a resolution of the singularity y on $\bar{V}_1$ with $(\bar{h}f)^{-1}(y) = E + \sum_i D_i$. This implies that y is a rational double singularity, a contradiction. So, we have $(E^2) = -1$.

Since $\bar{h} \cdot f: V \to \bar{V}_1$ is a resolution of singularities, there is a birational morphism $h: V \to V_1$ such that $f_1 \cdot h = \bar{h} \cdot f$. By the assumption on $\bar{h}$, the morphism h is a composite morphism of the blowing-down of E and the blowing-downs of several components of D. Moreover, $h_*(D) = D_{(1)}$. Hence $h_*(D)$ contains no (-1)-curves because f_1 is a minimal resolution. Since $\bar{V}_1$ is a log Enriques surface, $f_1 h_*(D) = \mathrm{Sing}(\bar{V}_1)$ consists of quotient singular points. So, $\bar{h}$ is a combining morphism by Definition 2.1.

Lemma 2.5. *Let $\bar{h}: \bar{V} \to \bar{V}_1$ be a combining morphism between two log Enriques surfaces of the same prime index I and with the exceptional curve $\bar{E} \subseteq \bar{V}$. Let $\pi: \bar{U} \to \bar{V}$, $\pi_1: \bar{U}_1 \to \bar{V}_1$ be canonical coverings. Set $y = \bar{h}(\bar{E})$, $\bar{E} \cap (\mathrm{Sing}\, \bar{V}) = \{z_1, z_2\}$*

(*cf.* *Lemma* 2.3), $\bar{F} := \pi^{-1}(\bar{E})$, $x := \pi^{-1}(y)$, $z_i' := \pi^{-1}(z_i)$. *Then we have*:

(1) *x and z_i' consist of a single point. $\bar{F}$ is a nonsingular irreducible rational curve.*

(2) *There is a birational morphism $\bar{h}: \bar{U} \to \bar{U}_1$ such that $\pi_1 \cdot \bar{h} = \bar{h} \cdot \pi$, the morphism $\bar{h}$ is an isomorphism over $\bar{U}_1 - \{x\}$ and $\bar{h}^{-1}(x) = \bar{F}$. Moreover, $\bar{F} \cap (\operatorname{Sing} \bar{U}) \subseteq \{z_1', z_2'\}$.*

(3) *z_1', z_2'' are all points on $\bar{F}$ fixed by the natural action of $\mathbf{Z}/I\mathbf{Z}$ on $\bar{U}$. The curve $\bar{F}$ is $\mathbf{Z}/I\mathbf{Z}$-stable. $\bar{h}$ is a $\mathbf{Z}/I\mathbf{Z}$-equivariant morphism.*

(4) *Let $\tilde{g}: \tilde{U} \to \bar{U}$ be a minimal resolution of singularities contained in $\{z_1', z_2'\}$. Then $\tilde{g}_1 := \bar{h} \cdot \tilde{g}: \tilde{U} \to \bar{U}_1$ is a minimal resolution of the singularity $x \in \bar{U}_1$ with $\tilde{g}^{-1}(\bar{F})$ as the exceptional set.*

(5) *Both $\bar{U}$ and $\bar{U}_1$ are $K3$-surfaces possibly with rational double singularities.*

Proof. (1) By Lemma 2.3, y and z_i are not rational double singular points. Then the first part of (1) follows from Lemma 1.4. Hence, $\bar{F}$ is connected. Since $E := f'(\bar{E})$ meets $\operatorname{Supp}(D^*)$ *transversally* in exactly two points (cf. Lemma 2.3), $\bar{F}$ is nonsingular and $\bar{F}$ is rational by the Hurwitz formula (cf. Lemma 1.4, (1)).

(2) Since π is etale over $\bar{V} - (\operatorname{Sing} \bar{V})$, we have $\operatorname{Sing}(\bar{U}) \subseteq \pi^{-1}(\operatorname{Sing}(\bar{V})$ and $\bar{F} \cap (\operatorname{Sing} \bar{U}) \subseteq \{z_1', z_2''\}$. Since $\bar{U}$, $\bar{U}_1$ are respectively normalizations of $\bar{V}$ and $\bar{V}_1$ in the function field $\mathbf{C}(\bar{U}) = \mathbf{C}(\bar{U}_1)$, (2) follows from properties of $\bar{h}$ before Definition 2.1.

(3) Since π ramifies exactly over {singularity of $\bar{V}$ of multiplicity ≥ 3} (cf. Lemma 1.4), the first assertion of (3) follows from Lemma 2.3, (3). By the same reasoning, x is fixed by the natural $\mathbf{Z}/I\mathbf{Z}$-action on $\bar{U}_1$. So, $\bar{U}_1 - \{x\}$ is $\mathbf{Z}/I\mathbf{Z}$-stable. Hence $\bar{U} - \bar{F}$ is $\mathbf{Z}/I\mathbf{Z}$-stable because the actions of $\mathbf{Z}/I\mathbf{Z}$ on $\bar{U} - \bar{F}$ and on $\bar{U}_1 - \{x\}$ are the same. The second and hence the third assertion of (3) follow.

(4) Note that $\tilde{g}_1 := \bar{h} \cdot \tilde{g}: \tilde{U} \to \bar{U}_1$ is a resolution of the singularity $x \in \bar{U}_1$. Since $\bar{U}$ has only rational double singular points, we have $K_{\tilde{U}} = \tilde{g}^*(K_{\bar{U}}) \sim 0$. Hence there are no (-1)-curves on $\tilde{U}$ and $\tilde{g}_1$ is a minimal resolution of the singularity $x \in \bar{U}_1$.

(5) We have only to show neither $\bar{U}$ nor $\bar{U}_1$ is an abelian surface. Since there is a rational curve $\bar{F}$ on $\bar{U}$, the surface $\bar{U}$ is not abelian. If $\bar{U}_1$ is an abelian surface, then $\bar{U}_1$ is especially smooth. However, the assertion (4) implies that $x \in \bar{U}_1$ is a singularity. This is a contradiction. So, $\bar{U}_1$ is not an abelian surface. This proves (5).

Lemma 2.6. *Let $\bar{V}_1$ be a log Enriques surface of prime index I and let $\pi_1: \bar{U}_1 \to \bar{V}_1$ be the canonical covering. Let $\bar{U}$ be a normal projective surface such that $K_{\bar{U}} \sim 0$, $\bar{U}$ has at worst rational double singularities and there is an action of $\mathbf{Z}/I\mathbf{Z}$ on $\bar{U}$. Assume that there is a point $x \in \bar{U}_1$ and there is a $\mathbf{Z}/I\mathbf{Z}$-equivariant morphism $\bar{h}: \bar{U} \to \bar{U}_1$ such that $\bar{h}$ is an isomorphism over $\bar{U}_1 - \{x\}$, the exceptional divisor $\bar{F} := \bar{h}^{-1}(x)$ is $\mathbf{Z}/I\mathbf{Z}$-stable and the action of $\mathbf{Z}/I\mathbf{Z}$ on $\bar{F}$ is non-trivial. Set*

$\bar{V} := \bar{U}/(\mathbf{Z}/I\mathbf{Z})$ and let $\pi: \bar{U} \to \bar{V}$ be the quotient morphism. Then $\bar{V}$ is a log Enriques surface of the same index I as $\bar{V}_1$ and $\bar{U}$ is the canonical covering.

Proof. Since $\tilde{h}: \bar{U} \to \bar{U}_1$ is a surjective $\mathbf{Z}/I\mathbf{Z}$-equivariant morphism and since $\bar{V} = \bar{U}/(\mathbf{Z}/I\mathbf{Z})$ and $\bar{V}_1 = \bar{U}_1/(\mathbf{Z}/I\mathbf{Z})$, there is a surjective morphism $\bar{h}: \bar{V} \to \bar{V}_1$ such that $\pi_1 \cdot \tilde{h} = \bar{h} \cdot \pi$. Since $\bar{F}$ is $\mathbf{Z}/I\mathbf{Z}$-stable, so does x. Therefore, $\pi_1^{-1}\pi_1(x) = x$ and $\pi^{-1}\pi(\bar{F}) = \bar{F}$. Set $y := \pi_1(x)$, $\bar{E} := \pi(\bar{F})$. By the properties of $\tilde{h}$, we see that $\bar{h}$ is an isomorphism over $\bar{V}_1 - \{y\}$ and that $\bar{E} = \bar{h}^{-1}(y)$. Hence every singularity of $\bar{V} - \bar{E}$ is an isolated quotient singularity. Let $f: V \to \bar{V}$ be a minimal resolution. Since the action of the group $\mathbf{Z}/I\mathbf{Z}$ of prime order on $\bar{F}$ is non-trivial, $\bar{F}$ contains only finitely many points with non-trivial isotropy group. So, every singularity of $\bar{V}$ contained in $\bar{E}$ is an isolated quotient singularity. Thus, $\bar{V}$ has at worst isolated quotient singularities and π is etale over $\bar{V} - \operatorname{Sing} \bar{V}$. Hence $h^1(\bar{V}, \mathcal{O}_{\bar{V}}) = h^1(V, \mathcal{O}_V) = h^1(V_1, \mathcal{O}_{V_1}) = 0$.

Since $\bar{V}$ is birational to $\bar{V}_1$ by a morphism $\bar{h}$ and since $\mathcal{O}(K_{\bar{V}_1})$ is not trivial, we can prove that $\mathcal{O}(K_{\bar{V}})$ is not trivial. On the other hand, the fact $K_{\bar{U}} \sim 0$ implies that $IK_{\bar{V}} \sim 0$ (cf. [2, Lemma 2.2]). Hence $\bar{V}$ is a log Enriques surface of index I. This proves Lemma 2.6.

Lemma 2.7. *Let $\bar{V}$ and $\bar{V}_1$ be two log Enriques surfaces of the same prime index I. Let $\pi: \bar{U} \to \bar{V}$, $\pi_1: \bar{U}_1 \to \bar{V}_1$ be canonical coverings. Then the following conditions are equivalent:*

(1) *There is a combining morphism $\bar{h}: \bar{V} \to \bar{V}_1$ with the exceptional curve $\bar{E}$. Set $y = \pi(\bar{E})$.*

(2) *There is a point $x \in \bar{U}_1$ and there is a $\mathbf{Z}/I\mathbf{Z}$-equivariant morphism $\tilde{h}: \bar{U} \to \bar{U}_1$ such that $\pi_1(x)$ is not a rational double singular point, $\tilde{h}$ is an isomorphism over $\bar{U}_1 - \{x\}$, the exceptional divisor $\bar{F} := \tilde{h}^{-1}(x)$ is an irreducible curve and $\bar{F}$ is $\mathbf{Z}/I\mathbf{Z}$-stable.*

Furthermore, suppose the equivalent conditions (1) *and* (2). *Then we have $\pi_1 \cdot \tilde{h} = \bar{h} \cdot \pi$. Hence $\bar{E} = \bar{h}^{-1}(y)$, $\bar{F} = \pi^{-1}(\bar{E})$, and $x = \pi_1^{-1}(y)$. In addition, $x \in \bar{U}_1$ is a singular point.*

Proof. Assume the condition (1). Set $x := \pi_1^{-1}(y)$ which is a single point and a singular point by Lemma 2.5. Let $\tilde{h}$ be the one given in Lemma 2.5. Then the condition (2) is satisfied (cf. Lemma 2.3, (3)).

Assume the condition (2). By the argument of Lemma 2.6, there is a birational morphism $\bar{h}: \bar{V} \to \bar{V}_1$ such that $\pi_1 \cdot \tilde{h} = \bar{h} \cdot \pi$ and $\bar{h}$ is an isomorphism over $\bar{V}_1 - \{\pi_1(x)\}$. Since $\bar{F}$ is an irreducible curve, so does $\bar{E} := \pi(\bar{F})$. Thus, $\bar{h}$ is a combining morphism with the exceptional curve $\bar{E}$ (cf. Lemma 2.4). The condition (1) is satisfied. The last assertion of Lemma 2.7 is proved in Lemma 2.5.

Now Proposition 2.8 in the Introduction follows from Lemmas 2.3, 2.4, 2.5 and 2.7.

We shall use the following two lemmas in the proof of Theorem 2.11.

Lemma 2.9. *Let $\bar{h}: \bar{V} \to \bar{V}_1$ be a combining morphism. Then $\bar{V}$ satisfies the*

Hypothesis (A) in the Introduction if and only if so does $\bar{V}_1$.

Proof. By Lemma 2.5, neither the canonical covering of $\bar{V}$ nor that of $\bar{V}_1$ is an abelian surface. Let $\bar{E}$ be the exceptional curve of $\bar{h}$ and set $y = \bar{h}(\bar{E})$. Note that $\bar{h}: \bar{V} \to \bar{V}_1$ is an isomorphism over $\bar{V}_1 - \{y\}$ and $\bar{E} \cap (\mathrm{Sing}\,\bar{V}) = \{z_1, z_2\}$ for two points z_1, z_2. Moreover, $y \in \bar{V}_1$, $z_i \in \bar{V} (i = 1, 2)$ are singularities of multiplicity ≥ 3 by Lemma 2.3. So, the assertion that every singularity has multiplicity ≥ 3 holds true for $\bar{V}$ if and only if so does for $\bar{V}_1$. By Lemma 2.2, $\bar{V}$ and $\bar{V}_1$ have the same index. This proves Lemma 2.9.

Lemma 2.10. *Let G be a group of odd order. Let Γ be a graph of Dynkin type $A_n (n \geq 1)$, $D_n (n \geq 5)$ or $E_n (n = 6, 7, 8)$. Assume G acts on Γ such that the action on edges is determined by the action on vertices in the following sense* (*). *Then the action of G on Γ is trivial.*

(*) *If e is an edge of Γ linking two vertices v_1, v_2, then for every element g of G, g(e) is a unique edge linking $g(v_1)$ and $g(v_2)$.*

Proof. Lemma 2.10 is clear in the case E_7 or E_8. Consider the case A_n. Note that the set of two tip vertices of the graph Γ is G-stable. Since the order of G is not divisible by 2, we see that each tip vertex of Γ is G-fixed. So G fixes every vertices by our assumption (*). Then we can deduce that G acts trivially on Γ by the same reasoning. The case $D_n (n \geq 5)$, E_6 can be proved similarly.

Now we can prove the following Theorem 2.11. We shall use the notations (V, D, f) of § 1 for $\bar{V}$.

Theorem 2.11. *Let $\bar{V}$ be a log Enriques surface satisfying the Hypothesis (A) in the Introduction. Assume that the index I of $\bar{V}$ is an odd prime number. Then we have:*

(1) *There is a composite $\bar{V}_n \xrightarrow{\bar{h}_n} \cdots \to \bar{V}_1 \xrightarrow{\bar{h}_1} \bar{V}_0 := \bar{V}$ $(n \geq 0)$ of combining morphisms (cf. Proposition 2.8) between log Enriques surfaces of the same index I such that $\bar{U}_n$ is a K3-surface possibly with rational double singularities of Dynkin type A_1. Here we let $\pi_i: \bar{U}_i \to \bar{V}_i$ be the canonical covering. Moreover, for each singularity x of $\bar{U}_n$ the image $y := \pi_n(x) \in \bar{V}_n$ is a singularity isomorphic to $(\mathbf{C}^2/C_{2I,1}; 0)$. Here $C_{2I,1} := \langle \sigma_{2I,1} \rangle \subseteq GL(2; \mathbf{C})$ is a cyclic subgroup of order 2I generated by*

$$\sigma_{2I,1} = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix},$$

η being a primitive 2I-th root of the unity. Finally, the hypothesis (A) is satisfied by every $\bar{V}_i$.

(2) *Let $g: U \to \bar{U}$ be a minimal resolution and denote by $\Gamma := g^{-1}(\mathrm{Sing}\,\bar{U})$ the exceptional divisor. Then there are natural $\mathbf{Z}/I\mathbf{Z}$-actions on U and $\bar{U}$ such that g is $\mathbf{Z}/I\mathbf{Z}$-equivariant and every irreducible component of Γ is $\mathbf{Z}/I\mathbf{Z}$-stable. Moreover, there are exactly n irreducible components $F_i (1 \leq i \leq n)$ of Γ*

on which $\mathbf{Z}/I\mathbf{Z}$ *does not trivially act. Finally, after relabelling subscripts of* F_i*'s, there is a contraction* $G_i: U \to \bar{U}_i$ *of* $\Gamma - (F_1 + \cdots + F_i)$ *and a contraction* $\tilde{h}_i: \bar{U}_i \to \bar{U}_{i-1}$ *of* $\bar{F}_i := G_i(F_i)$ *such that* $\tilde{h}_i \cdot G_i = G_{i-1}$ *and* $\pi_{i-1} \cdot \tilde{h}_i = \pi_i \cdot \tilde{h}_i$. *Here we set* $G_0 := g$, $\pi_0 := \pi$, $\bar{U}_0 := \bar{U}$.

Conversely, suppose $\bar{Y}_r \overset{\bar{p}_r}{\to} \cdots \to \bar{Y}_1 \overset{\bar{p}_1}{\to} \bar{Y}_0 := \bar{V}$ $(r \geq 0)$ *is a composite of combining morphisms with* $\varpi_i: \bar{X}_i \to \bar{Y}_i$ *a canonical covering and satisfying* $(\bar{Y}_r; \varpi_r(x)) \cong (\mathbf{C}^2/C_{2I,1}: 0)$ *for every singularity* x *of* $\bar{X}_r$. *Then we have* $r = n$. *Moreover, there is a strictly increasing sequence* $\{F_{k_1}\} \subset \{F_{k_1}, F_{k_2}\} \cdots \subset \{F_{k_1}, \ldots, F_{k_n}\}$ *and there is a contraction* $H_i: U \to \bar{X}_i$ *of* $\Gamma - (F_{k_1} + \cdots + F_{k_i})$. *In particular,* $\bar{X}_n = \bar{U}_n$ *and* $\bar{Y}_n = \bar{X}_n/(\mathbf{Z}/I\mathbf{Z}) = \bar{V}_n$.

(3) $n = \#\{$*exceptional curve of* $\tilde{h}_1 \cdots \tilde{h}_n: \bar{V}_n \to \bar{V}\} = \rho(U) - \rho(\bar{U}) - \#(\mathrm{Sing}\, \bar{U}_n) \leq \mathrm{Min}\,\{19, 22 - I\}$ (See also Proposition 1.3, (2)).

Proof. Let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Then $\bar{U}$ is a K3-surface possibly with rational double singularities by the Hypothesis (A). Let $g: U \to \bar{U}$ be a minimal resolution. The U is a K3-surface. Set $\Gamma := g^{-1}(\mathrm{Sing}\, \bar{U})$. Then Γ consists of (-2)-curves. Write $\Gamma = \sum_{i=1}^{m} \Gamma_i$ where Γ_i is a connected component of Γ. Then the dual graph of Γ_i has Dynkin type $A_{m_i} (m_i \geq 1)$, $D_{m_i} (m_i \geq 4)$ or $E_{m_i} (m_i = 6, 7, 8)$. By the Hypothesis (A) and by Lemma 1.4, every singular point of $\bar{U}$ is fixed by the $\mathbf{Z}/I\mathbf{Z}$-action. Hence there is a non-trivial $\mathbf{Z}/I\mathbf{Z}$-action on U such that g is a $\mathbf{Z}/I\mathbf{Z}$-equivariant birational morphism and every Γ_i is $\mathbf{Z}/I\mathbf{Z}$-stable. We prove first the following:

CLAIM. (1) Let $F_j (n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_i)$ be all irreducible components of Γ_i such that $\mathbf{Z}/I\mathbf{Z}$ *does not* act trivially on it. Set $n = \sum_{i=1}^{m} n_i$. Then every connected componet of $\Gamma - \sum_{j=1}^{n} F_j$ consists of a single (-2)-curve.

(2) Suppose $\mathbf{Z}/I\mathbf{Z}$ acts trivially on every irreducible component of Γ_i. Then Γ_i consists of a single (-2)-curve.

(3) Every irreducible component of Γ is $\mathbf{Z}/I\mathbf{Z}$-stable.

Proof of the claim. (1) Suppose there is a connected componet of $\Gamma - \sum_{j=1}^{n} F_j$ with at least two components. Then there are two componets L_1, L_2 of $\Gamma - \sum_{j=1}^{n} F_j$ with an intersection point P. Note that two tangents of L_1, L_2 at the point P are fixed by the $\mathbf{Z}/I\mathbf{Z}$-action. So, the action of $\mathbf{Z}/I\mathbf{Z}$ on U and $\bar{U}$ are trivial. This leads to that $\bar{V} = \bar{U}/(\mathbf{Z}/I\mathbf{Z}) = \bar{U}$ and the index I of $\bar{V}$ is equal to one. This is a contradiction. So, the assertion (1) of the claim is true. Then follows the assertion (2) of the claim.

(3) Suppose there is an irreducible component of Γ_i which is not $\mathbf{Z}/I\mathbf{Z}$-stable. We may assume $i = 1$. Set $x := g(\Gamma_1) \in \bar{U}$, $y := \pi(x) \in \bar{V}$ which are singular points. Set $\Delta := f^{-1}(y) \subseteq V$. Then the action of $\mathbf{Z}/I\mathbf{Z}$ on the dual graph of Γ_1 is not trivial. By Lemma 2.10, the dual graph of Γ_1 has Dynkin type D_4. Write $\Gamma_1 = \sum_{j=1}^{4} L_j$ with the central component L_1. We see that L_1 is $\mathbf{Z}/I\mathbf{Z}$-stable. Since I is not divisible by 2, we have $\eta(L_2) = L_3$, $\eta(L_3) = L_4$, $\eta(L_4) = L_2$ after relabelling subscripts. Here η is a generator of $\mathbf{Z}/I\mathbf{Z}$. So, $3|I$. Hence $I = 3$. Since $x = g(\Gamma_1)$ is a singularity of Dynkin type D_4, the dual

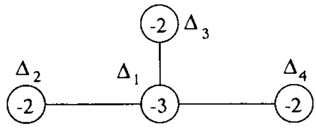

Figure 1

graph of Δ is given in Figure 1 (cf. [2, Proposition 6.1]).

In Figure (1), we have $\Delta = \sum_{i=1}^{4} \Delta_i$ with the central component Δ_1 and three irreducible components Δ_j $(j = 2, 3, 4)$ sprouting from Δ_1. Let $P_j := \Delta_1 \cap \Delta_j$ $(j = 2, 3, 4)$ be an intersection point. Let $h: V_1 \to V$ be the blowing-up of three points P_j's. Set $E_j := h^{-1}(P_j)$, $\Delta_i' := h'(\Delta_i)$. Note that the coefficients of Δ_i's in D^* for $i = 1, \dots, 4$ are respectively $\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. So, we have $0 \sim h^* 3(K_V + D^*) = 3(K_{V_1} + h'(D^*))$ (cf. Lemma 1.1). Let $f_1: V_1 \to \bar{V}_1$ be the contraction of $h'(D^*)$. Then we have $f_1^*(3K_{\bar{V}_1}) = 3(K_{V_1} + h'(D^*))$ and $3K_{\bar{V}_1} \sim 0$ (cf. [2, Lemma 1.2]). Set $\bar{E}_j := f_1(E_j)$, $z_1 := f_1(\Delta_1')$, $z_{2j} := f_1(\Delta_j')$. Then $\bar{E}_j \cap (\mathrm{Sing}\, \bar{V}_1) = \{z_1, z_{2j}\}$ and z_1, z_{2j}'s are quotient singular points. There is a birational morphism $\bar{h}: \bar{V}_1 \to \bar{V}$ such that $\bar{h} \cdot f_1 = f \cdot h$, the morphism $\bar{h}$ is an isomorphism over $\bar{V} - \{y\}$ and $\bar{h}^{-1}(y) = \bar{E}_2 + \bar{E}_3 + \bar{E}_4$. Thus, every singularity of $\bar{V}_1$ is an isolated quotient singularitiy. Hence $h^1(\bar{V}_1, \mathcal{O}_{\bar{V}_1}) = h^1(V_1, \mathcal{O}_{V_1}) = h^1(V, \mathcal{O}_V) = 0$. So, $\bar{V}_1$ is a log Enriques surface of index one or three. Since $\bar{V}$ and hence $\bar{V}_1$ are rational surfaces by the Hypothesis (A), $\bar{V}_1$ has index 3. By Definition 2.1, $\bar{h}$ is a composite morphism of three combining morphisms. Let $\pi_1: \bar{U}_1 \to \bar{V}_1$ be the canonical covering. Set $\bar{F}_i := \pi_1^{-1}(\bar{E}_i)$. Then $\bar{F}_i$ is an irreducible curve and is stable under the natural $\mathbf{Z}/I\mathbf{Z}$-action on $\bar{U}_1$ (see also Lemma 2.5). Note that $\pi_1^{-1}(z_{2j})$ is a smooth point and $Q_1 := \pi_1^{-1}(z_1)$ is a singular point of Dynkin type A_1. By the same argument of Lemma 2.5, we see that U is also a minimal resolution of $\bar{U}_1$. Let $g_1: U \to \bar{U}_1$ be the resolution which is, in fact, $\mathbf{Z}/I\mathbf{Z}$-equivariant. Then we have $\bar{h} \cdot \pi_1 \cdot g_1 = \pi \cdot g$. So, we have $L_1 = g_1^{-1}(Q_1)$ and $\{L_i | i = 2, 3, 4\} = \{g_1'(\bar{F}_j) | j = 2, 3, 4\}$. Hence L_i is also $\mathbf{Z}/I\mathbf{Z}$-stable for $i = 2, 3, 4$ (cf. Lemma 2.5, (3)). We reach a contradiction. Thus, the claim is proved.

Next we prove the assertion (1) of Theorem 2.11. We use the notations of the claim: n_i, $n = \sum_{i=1}^{m} n_i$, F_j $(1 \le j \le n)$.

Assume $n_i = 0$ for some i, say $i = 1$. Then Γ_1 is a single (-2)-curve on which $\mathbf{Z}/I\mathbf{Z}$ acts trivially by the claim. Set $x := g(\Gamma_1) \in \bar{U}$, $y := \pi(x) \in \bar{V}$ which are singular points. Note that $\pi^{-1}(y) = x$ (cf. Lemma 1.4). We can prove that the singularity y is isomorphic to $(\mathbf{C}^2/C_{2I,1}; 0)$.

Assume $n_j \ge 1$ for some j. Let $G_i (1 \le i \le n): U \to \bar{U}_i$ be the contraction of $\Gamma - (F_1 + \cdots + F_i)$. Set $\bar{F}_i := G_i(F_i)$. Let $\tilde{h}_i: \bar{U}_i \to \bar{U}_{i-1}$ be the contraction of $\bar{F}_i$. We set $\bar{U}_0 := \bar{U}$, $G_0 := g$ and $x_{i-1} := \tilde{h}_i(\bar{F}_i)$. Then $\tilde{h}_i \cdot G_i = G_{i-1}$. By (3) of the claim, there is a non-trivial $\mathbf{Z}/I\mathbf{Z}$-action on $\bar{U}_i$ such that $G_i, \tilde{h}_i$ are $\mathbf{Z}/I\mathbf{Z}$-equivariant and $\bar{F}_i$ is $\mathbf{Z}/I\mathbf{Z}$-stable. Since the action of $\mathbf{Z}/I\mathbf{Z}$ on F_i is non-trivial, so does on $\bar{F}_i$. Set $\bar{V}_i := \bar{U}_i/(\mathbf{Z}/I\mathbf{Z})$ and let $\pi_i: \bar{U}_i \to \bar{V}_i$ be the quotient morphism. Set $\bar{E}_i := \pi_i(\bar{F}_i)$. Applying Lemma 2.6 n-times, we see that every $\bar{V}_i$

is a log Enriques surface of index I and π_i is the canonical covering. Set $y_i = \pi_i(x_i)$. By Lemma 2.7, there is a combining morphism $\bar{h}_i\colon \bar{V}_i \to \bar{V}_{i-1}$ such that $\bar{h}_i \cdot \pi_i = \pi_{i-1} \cdot \tilde{h}_i$, $\bar{E}_i = \bar{h}_i^{-1}(y_{i-1})$ and $\bar{E}_i$ is the exceptional curve of $\bar{h}_i$ (cf. Lemma 2.9 and Hypothesis (A), (3)). Here we set $\bar{V}_0 := \bar{V}$, $\pi_0 := \pi$. We shall prove that $\bar{h}_i$'s satisfy the conditions in Theorem 2.11.

Note that G_n is a minimal resolution. A point x of $\bar{U}_n$ is a singular point if and only if $G_n^{-1}(x)$ is a connected component of $\Gamma - (F_1 + \cdots + F_n)$. By the claim, every connected component of $\Gamma - (F_1 + \cdots + F_n)$ is a single (-2)-curve on which $\mathbf{Z}/I\mathbf{Z}$ acts trivially. As in the case $n_i = 0$, we see that Sing $\bar{U}_n$ consists of singularities x such that $\pi_n(x) \in \bar{V}_n$ is a singularity isomorphic to $(\mathbf{C}^2/C_{2I,1}; 0)$. Since U is a $K3$-surface, every $\bar{U}_i$ is a $K3$-surface possibly with isolated rational double singularities. By Lemma 2.9, we see that $\bar{V}_i$'s satisfy the Hypothesis (A). Thus, $\bar{h}_i$'s satisfy the conditions in Theorem 2.11, (1). Hence (1) is proved. The first part of Theorem 2.11, (2) is also proved in the above arguments.

We now prove the converse part in Theorem 2.11, (2). By Lemma 2.5, U is a minimal resolution of each $\bar{X}_i$. Let $H_i\colon U \to \bar{X}_i$ be the resolution. Let $\bar{S}_i$ be the exceptional curve of $\bar{p}_i$ and set $\bar{T}_i := \varpi_i^{-1}(\bar{S}_i)$. By Lemma 2.5, $\bar{T}_i$ is a nonsingular irreducible rational curve and the natural $\mathbf{Z}/I\mathbf{Z}$-action on $\bar{T}_i$ is non-trivial. Moreover, there is a $\mathbf{Z}/I\mathbf{Z}$-equivariant birational morphism $\tilde{p}_i\colon \bar{X}_i \to \bar{X}_{i-1}$ such that $\varpi_{i-1} \cdot \tilde{p}_i = \bar{p}_i \cdot \varpi_i$, $\tilde{p}_i$ is the contraction of $\bar{T}_i$ and $\tilde{p}_i(\bar{T}_i) \in \bar{X}_{i-1}$ is a singular point. Set $T_i := H_i'(\bar{T}_i)$. Note that $\tilde{p}_1 \cdots \tilde{p}_r \cdot H_r\colon U \to \bar{X}_0 = \bar{U}$ is a minimal resolution and we may assume that it is equal to g. Denote by $\Gamma' := H_r^{-1}(\text{Sing } \bar{X}_r)$ the exceptional divisor of H_r. Then we have $\Gamma = g^{-1}(\text{Sing } \bar{U}) = \Gamma' + T_1 + \cdots + T_r$, $H_i^{-1}(\text{Sing } \bar{X}_i) = \Gamma - (T_1 + \cdots + T_i)$. Applying the above claim to $\bar{Y}_i$ (cf. Lemmas 2.9 and 2.2), we see that every component of $H_i^{-1}(\text{Sing } \bar{X}_i)$ is $\mathbf{Z}/I\mathbf{Z}$-stable. So, H_i is a $\mathbf{Z}/I\mathbf{Z}$-equivariant birational morphism. The action of $\mathbf{Z}/I\mathbf{Z}$ on T_i is non-trivial because so does on $\bar{T}_i$. Let x be a singular point of $\bar{X}_r$. Then $\varpi_r(x) \in \bar{Y}_r$ is a singularity isomorphic to $(\mathbf{C}^2/C_{2I,1}; 0)$. Since $\varpi_r(x)$ is a branch point of ϖ_r by Lemma 1.4, $H_r^{-1}(x)$ consists of a single (-2)-curve on which $\mathbf{Z}/I\mathbf{Z}$ acts trivially. Thus, we have $r = n$ and $\{T_i | 1 \le i \le n\} = \{F_j | 1 \le j \le n\}$. The converse part in Theorem 2.11, (2) is proved. Hence (2) is proved.

Finally, we shall prove (3). The first equality follows from the definition of $\bar{h}_i$'s (cf. Proposition 2.8 in the Introduction). In the notations of the statement of the present theorem, we have $n = \#\{\text{irreducible component of } \Gamma\} - \#\{\text{exceptional curve of } G_n\colon U_n \to \bar{U}_n\} = \rho(U) - \rho(\bar{U}) - \#(\text{Sing } \bar{U}_n)$ because $\bar{U}_n$ has only rational double singularities of Dynkin type A_1. For the last inequality, we have only to consider the case Sing $\bar{U} \ne \phi$. By virtue of Proposition 1.3, we have $\rho(U) - \rho(\bar{U}) = 24 - c - I(\rho(\bar{V}) - c + 2) \le 24 - c - I \le 22 - I \le 19$. This proves (3).

Thus, Theorem 2.11 is proved.

As a consequence, we have:

Corollary 2.12. *Let $\bar{V}$ be a log Enriques surface whose index I is an odd* *prime number. Let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Then the following two conditions are equivalent.*

(1) *$\bar{V}$ satisfies the Hypothesis (A) in the Introduction. For every singularity x of $\bar{U}$ the image $y := \pi(x) \in \bar{V}$ is a singularity isomorphic to $(\mathbf{C}^2/C_{2I,1}; 0)$.*

(2) *$\bar{V}$ satisfies the Hypothesis (A). We have $\bar{V} = \bar{V}_n$, i.e., $n = 0$ in the notations of Theorem* 2.11.

§3. Index 3 case

We shall prove the following Theorem 3.1 in the present section. In the Table 1 below, by $\mathrm{Sing}(\bar{U}) = mA_1$, we mean that $\bar{U}$ consists of exactly m singularities of Dynkin type A_1. By $\mathrm{Sing}(\bar{V}) = (3, 1)^i, (6, 1)^j$, we mean that $\bar{V}$ has exactly $i + j$ singularities, and i (resp. j) singularities of them are isomorphic to $(\mathbf{C}^2/C_{a,b}; 0)$ (cf. Lemma 1.4) with $(a, b) = (3, 1)$ (resp. $(6, 1)$). We also use the notations (V, D, f) in §1 for $\bar{V}$.

Theorem 3.1. *Let $\bar{V}$ be a log Enriques surface of index* 3 *and let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Assume $\bar{V}$ satisfies the condition* (1) *of Corollary* 2.12. *Then $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table* 1. *In particular, $H^1(V, D + 2K_V) = 0$.*

Table 1

No.	$\mathrm{Sing}(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	$\mathrm{Sing}(\bar{U})$
1	$(3, 1)^9, (6, 1)^6$	14	29	$6A_1$
2	$(3, 1)^8, (6, 1)^5$	13	26	$5A_1$
3	$(3, 1)^7, (6, 1)^4$	12	23	$4A_1$
4	$(3, 1)^6, (6, 1)^3$	11	20	$3A_1$
5	$(3, 1)^5, (6, 1)^2$	10	17	$2A_1$
6	$(3, 1)^4, (6, 1)$	9	14	A_1
7	$(3, 1)^3$	8	11	ϕ

Proof. If $\bar{U}$ is smooth, then $\bar{V}$ and $\bar{U}$ are described in the seventh row of the Table 1 by [2, Theorem 5.1]. So, we shall assume that $\bar{U}$ admits at least one singular point.

Let y_i for $1 \le i \le m_0$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{3,1}; 0)$. Let y_j for $m_0 + 1 \le j \le m_0 + m_1$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{6,1}; 0)$.

Note that $x_n := \pi^{-1}(y_n)$ $(1 \le n \le m_0 + m_1)$ consists of a single point (cf. Lemma 1.4). Moreover, x_i for $1 \le i \le m_0$ (resp. x_j for $m_0 + 1 \le j \le m_0 + m_1$) is a smooth point of $\bar{U}$ (resp. a singularity of Dynkin type A_1). By the condition (1) of Corollary 2.12, every singularity of $\bar{V}$ other than $y_j's$ is a cyclic singularity of order 3. So, by Lemma 1.4, we have $c := \#(\mathrm{Sing}\,\bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\,\bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \le i \le m_0)$ is a single (-3)-curve.

(2) $\Delta_j (m_0 + 1 \le j \le c)$ is a single (-6)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with

$$D^* = \frac{1}{3}\sum_i \Delta_i + \frac{2}{3}\sum_j \Delta_j.$$

Hence we have

$$-\frac{1}{3}m_0 - \frac{8}{3}m_1 = (D^*)^2 = (K_V^2) = 10 - \rho(V) =$$

$$10 - \rho(\bar{V}) - \#(D) = 10 - \rho(\bar{V}) - (m_0 + m_1), \quad \textit{and}$$

(3.1) $$\rho(\bar{V}) = 10 - \frac{2}{3}m_0 + \frac{5}{3}m_1.$$

This, together with Proposition 1.3, implies:

$$24 = c + \rho(U) - \rho(\bar{U}) + 3(\rho(\bar{V}) - c + 2) =$$

$$(m_0 + m_1) + m_1 + 3\left(10 - \frac{2}{3}m_0 + \frac{5}{3}m_1 - m_0 - m_1 + 2\right).$$

Hence we have:

(3.2) $$m_0 = 3 + m_1, \quad \textit{and}$$

(3.1′) $$\rho(\bar{V}) = 8 + m_1.$$

On the other hand, by Proposition 1.3, we have

$$-1 \le \rho(\bar{V}) - c = 8 + m_1 - (m_0 + m_1) \le 4.$$

Namely, $4 \le m_0 \le 9$. By noting that

$$\rho(V) = \rho(\bar{V}) + m_0 + m_1, \quad \mathrm{Sing}\,\bar{U} = m_1 A_1$$

and by equalities (3.1)′ and (3.2), we see that $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table 1. The second assertion of Theorem 3.1 follows from Lemma 1.2 and the Table 1. This proves Theorem 3.1.

The existence of the case No. 1 (resp. No. 6, or No. 7) in Table 1 of Theorem

3.1 was given in Example 6.11 (resp. Example 6.8 and Remark 6.7, or Example 5.3) of [2]. We shall give below examples of cases No. 2, No. 3, No. 4 and No. 5.

Examples 3.2. We can find a nonsingular rational surface V' and a $\mathbf{P}^1$-fibration $\Phi: V' \to \mathbf{P}^1$ such that the following two conditions are satisfied.

(1) All singular fibers of Φ are vertically shown in Figure m $(2 \leq m \leq 5)$. We set $F = F_1 + F_2$ in the case Figure 5. In particular, in the case Figure m for $m = 5$ (resp. $m = 2, 3, 4$), $F + D_1' + D_2'$ (resp. $F + D_1' + D_2' + D_3'$) is the support of a singular fiber of Φ. We have $\rho(V') = 11$.

(2) Denote by D' the reduced effective divisor consisting of all irreducible components in Figure m with self intersection number ≤ -2. Let $f_1: V' \to \bar{V}'$ be the contraction of D'. Then $\bar{V}'$ is a log Enriques surface of index 3.

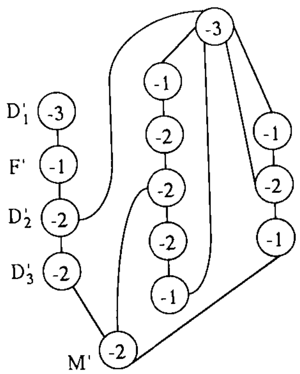

Figure 2

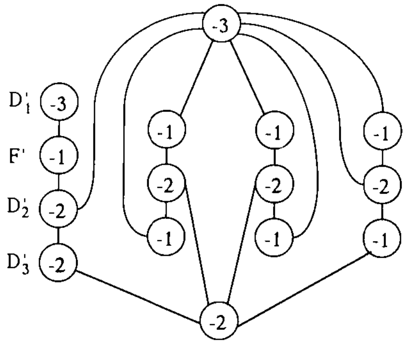

Figure 3

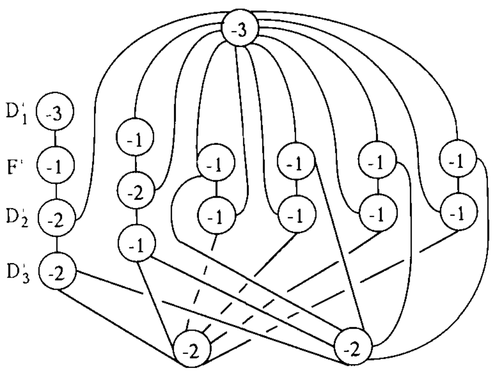

Figure 4

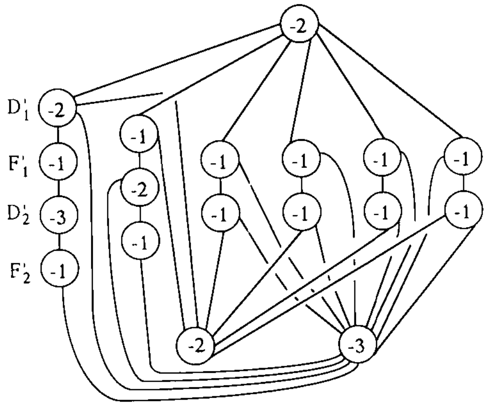

Figure 5

Let $\sigma: V' \to \Sigma_2$ be a composite morphism of blowing-downs onto a Hirzebruch surface Σ_2 such that $(\sigma(M')^2) = -2$. Then the existence of a pair (V', D') is equivelent to that of $(\Sigma_2, \sigma(D'))$. In addition, to meet the above condition (2), we just require that $3(K_{V'} + D'^*) \sim 0$ (cf. Lemma 1.1), or equivalently, $3(K_{\Sigma_2} + \sigma_*(D'^*)) \sim 0$.

Let $\pi_1: \bar{U}' \to \bar{V}'$ be the canonical covering. In the case Figure 2 (resp.

3, 4, 5), $\pi_1^{-1}(f_1(D'))$ consists of a smooth point and a singular point P of Dynkin type D_{16} (resp. D_{13}, D_{10}, D_7). We have $\mathrm{Sing}(\bar{U}') = \{P\}$.

Let $\bar{\tau}\colon \bar{V} \to \bar{V}'$ be a composite morphism of combining morphisms such that $\bar{V}$ satisfies the condition (1) of Corollary 2.12. In the notations of Theorem 2.11, we have $\bar{V} = \bar{V}_n'$ with $n = 11$ (resp. 9, 7, 5) for the case Figure 2 (resp. 3, 4, 5). Let $f\colon V \to \bar{V}$ be a minimal resolution of singularities. Then there is a composite morphism $\tau\colon V \to V'$ of blowing-ups of several intersection points of D' and their infinitely near points such that $\bar{\tau} \cdot f = f_1 \cdot \tau$ and that $\tau^{-1}(D') - D$ consists of exactly n disjoint (-1)-curves.

Finally, in the case Figure 2 (resp. 3, 4, 5), $\bar{V}$ is a log Enriques surface of index 3 fitting the case No. 2 (resp. 3, 4, 5) of the Table 1. For the concrete constructions of (V', D') and (V, D), we refer to Example 7.3.

§4. Index 5 case

We shall prove the following Theorem 4.1 in the present section. In the Table 2 below, by $\mathrm{Sing}(\bar{U}) = mA_1$, we mean that $\bar{U}$ consists of exactly m singularities of Dynkin type A_1. By $\mathrm{Sing}(\bar{V}) = (5, 1)^i, (5, 2)^j, (10, 1)^k$, we mean that $\bar{V}$ has exactly $i + j + k$ singularities, and i (resp. j, k) singularities of them are isomorphic to $(\mathbf{C}^2/C_{a,b}; 0)$ with $(a, b) = (5, 1)$ (resp. (5, 2), (10, 1)). We also use the notations (V, D, f) in §1 for $\bar{V}$.

Theorem 4.1. *Let $\bar{V}$ be a log Enriques surface of index 5 and let $\pi\colon \bar{U} \to \bar{V}$ be the canonical covering. Assume $\bar{V}$ satisfies the condition (1) of Corollary 2.12. Then $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table 2. In particular, $H^1(V, D + 2K_V) = 0$.*

Table 2

No.	$\mathrm{Sing}(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	$\mathrm{Sing}(\bar{U})$
1	$(5, 1)^4, (5, 2)^9, (10, 1)^3$	15	40	$3A_1$
2	$(5, 1)^3, (5, 2)^7, (10, 1)^2$	12	31	$2A_1$
3	$(5, 1)^2, (5, 2)^5, (10, 1)$	9	22	A_1
4	$(5, 1), (5, 2)^3$	6	13	ϕ

Proof. If $\bar{U}$ is smooth, then $\bar{V}$ and $\bar{U}$ are described in the fourth row of the Table 2 by [2, Theorem 5.1]. So, we shall assume that $\bar{U}$ admits at least one singular point.

Let y_i for $1 \le i \le m_0'$ and y_j for $m_0' + 1 \le j \le m_0' + m_0''$ be respectively all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{5,r}; 0)$ with $r = 1$ and $r = 2$. Set

$m_0 := m_0' + m_0''$. Let y_k for $m_0 + 1 \le k \le m_0 + m_1$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{10,1}; 0)$. As in Theorem 3.1, we have $c := \#(\mathrm{Sing}\, \bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\, \bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \le i \le m_0')$ is a single (-5)-curve.

(2) $\Delta_j (m_0' + 1 \le j \le m_0)$ is a chain of one (-2)-curve $\Delta_{j,1}$ and one (-3)-curve $\Delta_{j,2}$.

(3) $\Delta_k (m_0 + 1 \le k \le c)$ is a single (-10)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with

$$D^* = \frac{3}{5}\sum_i \Delta_i + \frac{1}{5}\sum_j (\Delta_{j,1} + 2\Delta_{j,2}) + \frac{4}{5}\sum_k \Delta_k.$$

As in Theorem 3.1, we have

$$-\frac{1}{5}(9m_0' + 2m_0'' + 32m_1) = (D^*)^2 = (K_{\bar{V}}^2) =$$

$$10 - \rho(\bar{V}) - (m_0' + 2m_0'' + m_1), \quad \textit{and}$$

$$5(\rho(\bar{V}) - 10) = 4m_0' - 8m_0'' + 27m_1. \tag{4.1}$$

This, together with Proposition 1.3, implies

$$24 = c + \rho(U) - \rho(\bar{U}) + 5(\rho(\bar{V}) - c + 2) =$$

$$(m_0' + m_0'' + m_1) + m_1 + (4m_0' - 8m_0'' + 27m_1)$$

$$+ 5(12 - m_0' - m_0'' - m_1).$$

Hence we have:

$$m_0'' = 3 + 2m_1. \tag{4.2}$$

By the same proposition, we can write

$$\rho(\bar{V}) = c - 1 + r = (m_0' + m_0'' + m_1) - 1 + r \qquad \text{for } r = 0, 1, 2 \text{ or } 3.$$

This, together with (4.2), makes (4.1) into the following form:

$$m_0' = 16 - 5r - 4m_1. \tag{4.1'}$$

On the other hand, by Lemma 1.2, we have:

$$h^1(V, D + 2K_V) = c - 1 - (K_{\bar{V}}^2) - (D, K_V) =$$

$$2c - 12 + r + \#(D) - (D, K_V) =$$

$$2(m_0' + m_0'' + m_1) - 12 + r$$

$$+ (m_0' + 2m_0'' + m_1) - (3m_0' + m_0'' + 8m_1) =$$

$$-12+r+3m_0''-5m_1=-3+r+m_1.$$

Hence we obtain

$$(4.3)\qquad 0\le h^1(V, D+2K_V)=-3+r+m_1.$$

Since $m_1\ge 1$, the equality (4.1′) implies that $5r=16-4m_1-m_0'\le 12$ and $r\le 2$. By making use of (4.1)′, (4.2) and (4.3), we shall show:

$$(r, m_0', m_0'', m_1)=(0,4,9,3),\ (0,0,11,4),\ (1,3,7,2)\ \textit{or}\ (2,2,5,1).$$

So, either the following case (5) occurs or $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table 2 (cf. the proof of Theorem 3.1).

Case (5) $\rho(\bar{V})=c-1=14$, $\rho(V)=40$, $\mathrm{Sing}(\bar{U})=4A_1$ and

$$D=\sum_{j=1}^{11}(\Delta_{j,1}+\Delta_{j,2})+\sum_{k=12}^{15}\Delta_k.$$

Here Δ_k is an isolated (-10)-curve of D. The curves $\Delta_{j,1}$ and $\Delta_{j,2}$ are respectively (-2)-curve and (-3)-curve, $\Delta_{j,1}+\Delta_{j,2}$ is a linear chain and $\Delta_{j,1}+\Delta_{j,2}$ is a connected component of D.

Actually, the above case (5) does not occur by the following Lemma 4.2. The second assertion of Theorem 4.1 follows from Lemma 1.2 and the Table 2. This proves Theorem 4.1.

Lemma 4.2. *The above case* (5) *does not occur.*

Proof. Assume, on the contrary, that $\bar{V}$ is a log Enriques surface satisfying the conditions of Theorem 4.1 and fitting the above case (5). We use the above notations for D. We can write

$$D^*=\frac{1}{5}\sum_{j=1}^{11}(\Delta_{j,1}+2\Delta_{j,2})+\frac{4}{5}\sum_{k=12}^{15}\Delta_k.$$

Set $V_1:=V$, $D_{(1)}:=D$. Suppose there is a (-1)-curve E_1 on V_1 such that E_1 meets a coefficient $\frac{4}{5}$ component of $D_{(1)}^*$, say Δ_{12}. Then E_1 meets a coefficient $\frac{1}{5}$ component of D^*, say $\Delta_{1,1}$, because $K_V\equiv -D^*$. Moreover, $(E_1, \Delta_{12})=(E_1, \Delta_{1,1})=1$ and E_1 meets no components of $D_{(1)}$ other than Δ_{12} and $\Delta_{1,1}$. Let σ_1: $V_1\to V_2$ be the smooth contraction of the (-1)-curve E_1 and the (-2)-curve $\Delta_{1,1}$. Set $D_{(2)}:=\sigma_{1*}(D_{(1)})$, $D_{(2)}^*:=\sigma_{1*}(D_{(1)}^*)$. Note that $5(K_{V_2}+D_{(2)}^*)\sim 0$. Continue this process. We obtain a composite of smooth contraction $V_1\xrightarrow{\sigma_1}V_2\xrightarrow{\sigma_2}\cdots\xrightarrow{\sigma_n}V_{n+1}$ such that the following claim holds, where $\sigma=\sigma_n\cdots\sigma_1$, $W:=V_{n+1}$, $B:=D_{(n+1)}=\sigma_*(D)$, $B^*:=D_{(n+1)}^*=\sigma_*(D^*)$.

Claim (1). No (-1)-curve on W meets any coefficient $\frac{4}{5}$ component of B^*.

Note that $5(K_W+B^*)\sim 0$. A connected component of B is either a chain $\Gamma_{j,1}+\Gamma_{j,2}$ $(1\le j\le 11-n)$ of one (-2)-curve $\Gamma_{j,1}$ and one (-3)-curve $\Gamma_{j,2}$, or a tree $\Gamma_{k,0} + \cdots + \Gamma_{k,r_k}$ $(12 \le k \le 15)$ of one $(2r_k - 10)$-curve $\Gamma_{k,0}$ as the central component and r_k $(r_k \ge 0)$ (-2)-curves $\Gamma_{k,1}, \dots, \Gamma_{k,r_k}$ as twigs. Let us write

$$B = \sum_j (\Gamma_{j,1} + \Gamma_{j,2}) + \sum_k (\Gamma_{k,0} + \cdots + \Gamma_{k,r_k}).$$

Then we have

$$B^* = \frac{1}{5}\sum_j (\Gamma_{j,1} + 2\Gamma_{j,2}) + \frac{2}{5}\sum_k (2\Gamma_{k,0} + \Gamma_{k,1} + \cdots + \Gamma_{k,r_k}).$$

By the construction of σ, we find that $\sum_{k=12}^{15} r_k = n \le 11$, $(K_W^2) = (K_V^2) + 2n \le -30 + 2 \times 11 = -8$.

The fact $K_W + B^* \equiv 0$ implies:

Claim (2). Suppose C_1 is a $(-m)$-curve on W with $m \ge 2$. Then either C_1 is a component of B or C_1 is a (-2)-curve disjoint from B.

Let $\Phi: W \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration. Since $(K_W^2) < 8$, there is at least one singular fiber S_1.

Claim (3). We can write $\operatorname{Supp} S_1 = \sum_i E_i + \sum_j C_j + \sum_k B_k$ such that E_i is a (-1)-curve not contained in B, B_k is a component of B and C_j is a (-2)-curve not contained in B. Moreover, $\sum_i E_i + \sum_k B_k$ is a connected tree.

Proof. The first assertion follows from the claim (2) and the fact $2r_k - 10 \ne -1$. For the second assertion, we use the negative semi-definiteness of the intersection matrix of S_1. This proves the claim (3).

Claim (4). There is a singular fiber of Φ, say S_1, such that S_1 contains a coefficient $\frac{4}{5}$ component of B^*.

Proof. Suppose the claim is false. Then all four coefficient $\frac{4}{5}$ components of B^* are transversal to the fibration Φ. This leads to $2 = (S_1, -K_W) = (S_1, B^*) \ge 4 \times \frac{4}{5}$. This is a contradiction. So, Claim (4) is true.

Claim (5). Let S_1 be a singular fiber containing a coefficient $\frac{4}{5}$ component $\Gamma_{k,0}$ of B^*. Let $\Gamma_{k,0} + \cdots + \Gamma_{k,r_k}$ be the connected component of B^* containing $\Gamma_{k,0}$ in the above notations. After relabelling the indices of $\Gamma_{k,s}$'s, we have one of the following cases:

Case (5–1) $r_k \le 5$ and there are (-1)-curves $E_s (1 \le s \le 10 - 2r_k)$ such that $(E_s, \Gamma_{k,s}) = 1$ and that $S_1 = \Gamma_{k,0} + \sum_{s=1}^{10-2r_k} (E_s + \Gamma_{k,s})$.

Case (5–2) $r_k \le 4$ and there are (-1)-curves $E_s (1 \le s \le 9 - 2r_k)$ such that $(E_s, \Gamma_{k,s}) = 1$ and that $S_1 = 2\Gamma_{k,0} + 2\sum_{s=1}^{9-2r_k} (E_s + \Gamma_{k,s}) + \Gamma_{k,10-2r_k} + \Gamma_{k,11-2r_k}$.

Proof. If S_1 contains no components of B except for some $\Gamma_{k,s}$'s, then the case (5–1) or (5–2) takes place by the claims (1) and (3). Suppose S_1 contains a component of B other than $\Gamma_{k,s}$'s. We shall show that this will leads to a contradiction and hence the claim is true.

By the claims (1) and (3), S_1 contains a (-1)-curve E_1, a component B_1 of B other than $\Gamma_{k,s}$'s and a component of B among $\Gamma_{k,s}$'s, say $\Gamma_{k,1}$, such that $(E_1, \Gamma_{k,1}) = (E_1, B_1) = 1$. Then $(B_1^2) \leq -3$. By the claim (1), B_1 is a (-3)-curve with coefficient $\frac{2}{5}$ in B^* and B_1, together with a (-2)-curve B_0, forms a connected component of B. The fact $(E_1, B^*) = 1$ implies that E_1 meets a coefficient $\frac{1}{5}$ component B_2 of B. Set $f_0 := 2E_1 + \Gamma_{k,1} + B_2$. Let $\Psi: W \to \mathbf{P}^1$ be the $\mathbf{P}^1$-fibration with f_0 as its singular fiber. Then $\Gamma_{k,0}$ is a cross-section of Ψ.

Case (5–3). $B_2 \neq B_0$. Then there is a (-3)-curve B_3 such that $B_2 + B_3$ is a chain and a connected component of B. We see that B_1 is a 2-section of Ψ and B_3 is a cross-section of Ψ. All components of $B - (B_1 + B_3 + \Gamma_{k,0})$ are contained in fibers. Let f_1 be the singular fiber of Ψ containing B_0. By the claims (1) and (2), f_1 contains a twig, say $\Gamma_{k,2}$ sprouting from the cross-section $\Gamma_{k,0}$. By the claim (3), f_1 contains a (-1)-curve E_2 and a component $B_4 (\neq B_0)$ of B such that $(E_2, B_0) = (E_2, B_4) = 1$. If $(B_4^2) = -2$ then $f_1 = 2E_2 + B_0 + B_4$. This leads to $(B_3, f_1) = (B_3, 2E_2) \neq 1$, a contradiction. So, $(B_4^2) = -3$ and B_4 has coefficient $\frac{2}{11}$ in B^* by the claim (1). This leads to that $(E_2, B_1) = 1$ or $(E_2, B_3) = 1$ because $(E_2, B^*) = 1$. Hence $(B_1, f_1) \geq 3$ or $(B_3, f_1) \geq 2$ because E_2 has multiplicity ≥ 2 in f_1. We reach a contradiction. So, the case (5–3) is impossible.

Case (5–4). $B_2 = B_0$. Then B_1 is a 3-section of Ψ. All components of $B - (B_1 + \Gamma_{k,0})$ are contained in fibers. Since $\rho(W) = 10 - (K_W^2) \geq 18 > 4$, there is an another singular fiber f_1 of Ψ. By the claims (1) and (2), f_1 contains some twig, say $\Gamma_{k,2}$ sprouting from the 3-section $\Gamma_{k,0}$. By the claim (3), f_1 contains a (-1)-curve E_2 such that $(E_2, \Gamma_{k,2}) = 1$. Since $(E_2, B^*) = 1$, the fiber f_1 contains a coefficient $\frac{1}{5}$ component B_3 of B^* such that $(E_2, B_3) = (E_2, B_1) = 1$. Then B_3 a (-2)-curve and $f_1 = 2E_2 + B_3 + \Gamma_{k,2}$. This leads to $(B_1, f_1) = (B_1, 2E_2) = 2$. We reach a contradiction. So, the case (5–4) is impossible.

This proves the claim (5).

Now we can finish the proof of Lemma 4.2. By making use of the claims (4) and (5), we can imply the assertion that all four coefficient $\frac{4}{5}$ components $\Gamma_{k,0}$ $(12 \leq k \leq 15)$ of B^* are contained in fibers of Φ. Indeed, if a coefficient $\frac{4}{5}$ component $\Gamma_{k',0}$ of B^* is transversal to the fibration, then $\Gamma_{k',0}$ meets a (-1)-curve of the fiber S_1 which is described in the claim (5). However, this contradicts the claim (1). Thus the assertion is proved. So, $\Gamma_{k,0}$ $(12 \leq k \leq 15)$ are contained in four distinct fibers, say S_k, and S_k, like S_1, fits the case (5–1) or (5–2) of the claim (5). By counting the number of twigs sprouting from the central component $\Gamma_{k,0}$, we see that $10 - 2r_k \leq r_k$, $2 + (9 - 2r_k) \leq r_k$ if S_k fits the case (5–1), (5–2), respectively. So, we obtain $r_k \geq 4$. This leads to $11 \geq n = \sum_{k=12}^{15} r_k \geq 4 \times 4$. We reach a contradiction. So, the case (5) shown in the proof of Theorem 4.1 is impossible. This proves Lemma 4.2.

The existence of the case No. 1 (resp. No. 4) in Table 2 of Theorem 4.1 was given in Example 6.12 (resp. Example 5.4) of [2]. We shall give below several examples of cases No. 1, No. 2 and No. 3.

Examples 4.3. We can find a nonsingular rational surface V' and a $\mathbf{P}^1$-fibration $\Phi: V' \to \mathbf{P}^1$ such that the following two conditions are satisfied.

(1) All singular fibers of Φ are vertically shown in Figure m $(6 \leq m \leq 15)$. We set $F = F_1 + F_2$ in the case Figure 14. In particular, in the case Figure m for $m = 15$ (resp. $m \neq 15$), $F + D_1' + D_2'$ (resp. $F + D_1' + D_2' + D_3'$) is the support of a singular fiber of Φ. For the case Figure m $(m = 6, \ldots, 15)$, we have respectively $\rho(V') = 12, 13, 12, 12, 14, 14, 12, 12, 13, 12$.

(2) Denote by D' the reduced effective divisor consisting of all irreducible components in Figure m with self intersection number ≤ -2. Let $f_1: V' \to \bar{V}'$ be the contraction of D'. Then $\bar{V}'$ is a log Enriques surface of index 5.

Let $\pi_1: \bar{U}' \to \bar{V}'$ be the canonical covering. Then π_1^{-1} (Sing $\bar{V}'$) consists of several smooth points and isolated singular points. We have $\mathrm{Sing}(\bar{U}') \subseteq \pi_1^{-1}(\mathrm{Sing}\,\bar{V}')$. More precisely, the Dynkin types of $\mathrm{Sing}(\bar{U}')$ for the cases Figure m $(6 \leq m \leq 15)$ are respectively given as follows:

$$A_1 + D_{16},\ A_{11} + D_6,\ A_{17},\ D_{10} + E_7,\ D_4 + E_6 + E_6,$$
$$D_5 + D_5 + E_6,\ A_1 + D_{11},\ A_{12},\ D_5 + E_6,\ E_6$$

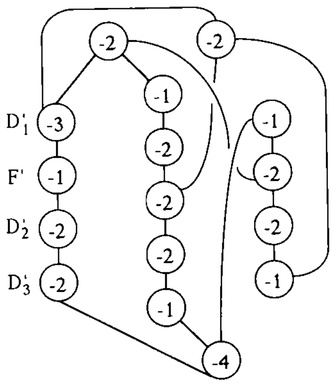

Figure 6

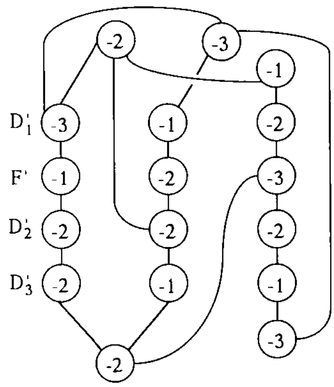

Figure 7

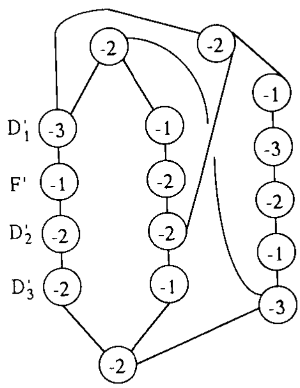

Figure 8

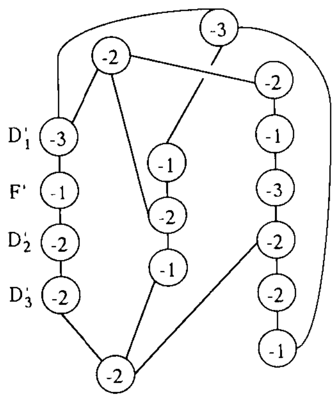

Figure 9

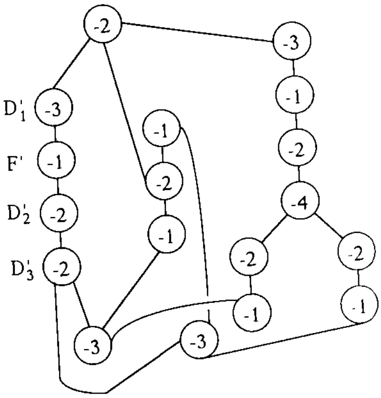

Figure 10

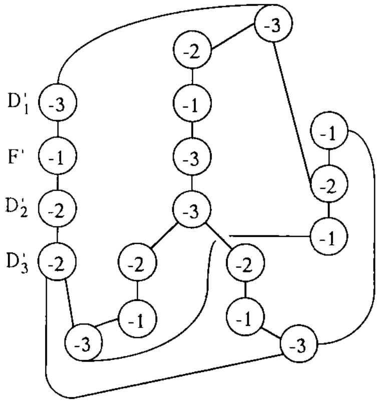

Figure 11

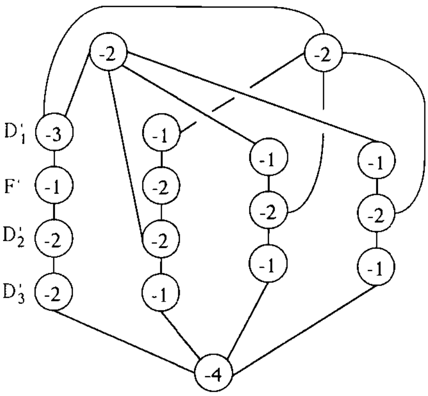

Figure 12

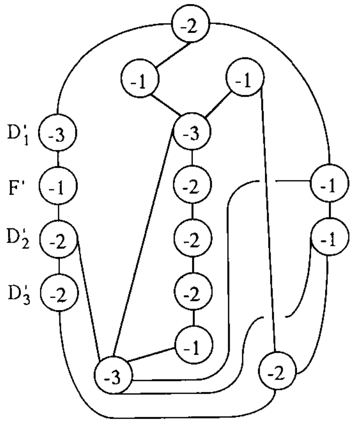

Figure 13

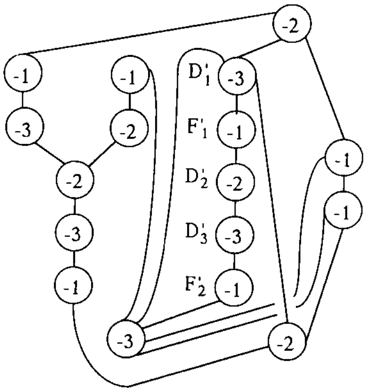

Figure 14

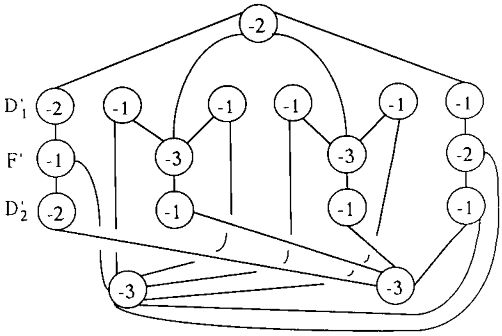

Figure 15

Let $\bar{\tau}: \bar{V} \to \bar{V}'$ be a composite morphism of combining morphisms such that $\bar{V}$ satisfies the condition (1) of Corollary 2.12. For the cases Figure m with $m = 6, \dots, 15$, we have, in the notations of Theorem 2.11, $\bar{V} = \bar{V}_n'$ with $n = 14, 14, 14, 14, 13, 13, 10, 10, 9, 5$, respectively.

Finally, $\bar{V}$ is a log Enriques surface of index 5 fitting respectively the cases No. 1, 1, 1, 1, 1, 1, 2, 2, 2, 3 of the Table 2. For the concrete constructions of (V', D') and (V, D), we refer to Examples 7.3 and 3.2.

§5. Index 7 case

We shall prove the following Theorem 5.1 in the present section. In the Tables 3 and 4 below, by $\mathrm{Sing}(\bar{U}) = mA_1$, we mean that $\bar{U}$ consists of exactly m singularities of Dynkin type A_1. By $\mathrm{Sing}(\bar{V}) = (7, 1)^i, (7, 2)^j, (7, 3)^k, (14, 1)^r$, we mean that $\bar{V}$ has exactly $i + j + k + r$ singularities, and i (resp. j, k, r) singularities of them are isomorphic to $(\mathbf{C}^2/C_{a,b}; 0)$ with $(a, b) = (7, 1)$ (resp. $(7, 2), (7, 3), (14, 1)$). We also use the notations (V, D, f) in §1 for $\bar{V}$.

Theorem 5.1. *Let $\bar{V}$ be a log Enriques surface of index 7 and let $\pi: \bar{U} \to \bar{V}$*

Table 3

No.	$\mathrm{Sing}(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	$\mathrm{Sing}(\bar{U})$
1	$(7, 1)^2, (7, 2)^5, (7, 3)^6, (14, 1)^2$	14	46	$2A_1$
2	$(7, 1), (7, 2)^3, (7, 3)^4, (14, 1)$	9	29	A_1
3	$(7, 2), (7, 3)^2$	4	12	ϕ
4	$(7, 1)^3, (7, 2)^2, (7, 3)^8, (14, 1)^2$	14	47	$2A_1$
5	$(7, 1), (7, 2)^8, (7, 3)^4, (14, 1)^2$	14	45	$2A_1$

Table 4

No.	$\mathrm{Sing}(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	$\mathrm{Sing}(\bar{U})$
6	$(7, 2)^2, (7, 3)^9, (14, 1)^3$	13	47	$3A_1$
7	$(7, 2)^{11}, (7, 3)^2, (14, 1)^2$	14	44	$2A_1$
8	$(7, 2)^2, (7, 3)^6, (14, 1)$	9	30	A_1
9	$(7, 2)^6, (7, 3)^2, (14, 1)$	9	28	A_1

be the canonical covering. Assume $\bar{V}$ satisfies the condition (1) *of Corollary* 2.12. *Then $\bar{V}$ and $\bar{U}$ are described in one of five rows of the Table* 3. *In particular*, $H^1(V, D + 2K_V) = 0$.

Proof. If $\bar{U}$ is smooth, then $\bar{V}$ and $\bar{U}$ are described in the third row of the Table 3 by [2, Theorem 5.1]. So, we shall assume that $\bar{U}$ admits at least one singular point.

Let y_i for $1 \le i \le n_1$, y_j for $n_1 + 1 \le j \le n_1 + n_2$ and y_k for $n_1 + n_2 + 1 \le k \le n_1 + n_2 + n_3$ be respectively all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{7,s}; 0)$ with $s = 1, 2$ and 3. Set $m_0 := n_1 + n_2 + n_3$. Let y_r for $m_0 + 1 \le r \le m_0 + m_1$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{14,1}; 0)$. As in Theorem 3.1, we have $c := \#(\mathrm{Sing}\,\bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\,\bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \le i \le n_1)$ is a single (-7)-curve.

(2) $\Delta_j (n_1 + 1 \le j \le n_1 + n_2)$ is a chain of one (-2)-curve Δ_{j1} and one (-4)-curve Δ_{j2}.

(3) $\Delta_k (n_1 + n_2 + 1 \le k \le m_0)$ is a chain of two (-2)-curves Δ_{k1}, Δ_{k2}, and one (-3)-curve Δ_{k3} with $(\Delta_{ka}, \Delta_{k,a+1}) = 1\ (a = 1, 2)$.

(4) $\Delta_r (m_0 + 1 \le r \le c)$ is a single (-14)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with $D^* =$

$$\frac{5}{7}\sum_i \Delta_i + \frac{2}{7}\sum_j (\Delta_{j1} + 2\Delta_{j2}) + \frac{1}{7}\sum_k (\Delta_{k1} + 2\Delta_{k2} + 3\Delta_{k3}) + \frac{6}{7}\sum_r \Delta_r.$$

As in Theorem 3.1, we have

$$-\frac{1}{7}(25n_1 + 8n_2 + 3n_3 + 72m_1) = (D^*)^2 =$$

$$(K_{\bar{V}}^2) = 10 - \rho(\bar{V}) - (n_1 + 2n_2 + 3n_3 + m_1), \quad \textit{and}$$

$$7(\rho(\bar{V}) - 10) - 18n_1 + 6n_2 + 18n_3 - 65m_1 = 0. \tag{5.1}$$

This, together with Proposition 1.3, implies

$$24 = c + \rho(U) - \rho(\bar{U}) + 7(\rho(\bar{V}) - c + 2) =$$

$$(n_1 + n_2 + n_3 + m_1) + m_1 + (18n_1 - 6n_2 - 18n_3 + 65m_1)$$

$$+ 7(12 - n_1 - n_2 - n_3 - m_1).$$

Hence we have:

$$n_1 = n_2 + 2n_3 - 5m_1 - 5. \tag{5.2}$$

By the same proposition, we can write

$$\rho(\bar{V}) = c - 1 + r = (n_1 + n_2 + n_3 + m_1) - 1 + r \qquad \text{for } r = 0, 1 \text{ or } 2.$$

This, together with (5.2), makes (5.1) into the following form:

(5.1′) $$2n_2 = 22 - 7r - 3n_3 + 3m_1 .$$

Using (5.1′), we make (5.2) into the following

(5.2′) $$2n_1 = 12 - 7r + n_3 - 7m_1 .$$

By (5.2′), we eliminate n_3 in (5.1′) and obtain:

(5.3) $$0 \leq 2n_2 = 58 - 28r - 18m_1 - 6n_1 .$$

This and the fact $m_1 \geq 1$ imply $r \leq 1$.

By making use of (5.1′), (5.2′) and (5.3), we can show that $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table 3 or 4 (cf. the proof of Theorem 4.1). Then Theorem 5.1 follows from Proposition 5.2 below (cf. the proof of Theorem 3.1).

Proposition 5.2. *The cases of Table 4 are impossible.*

Proof. This can be proved by the same fashion as in the proof of Lemma 4.2.

The existence of the case No. 1 (resp. No. 3) in Table 3 of Theorem 5.1 was given in Example 6.13 (resp. Example 5.5) of [2]. We shall give below an example of the case No. 2. We do not know yet whether or not the cases No. 4 and No. 5 exist.

Example 5.3. We can find a nonsingular rational surface V' and a $\mathbf{P}^1$-fibration $\Phi: V' \to \mathbf{P}^1$ such that the following two conditions are satisfied.

(1) All singular fibers of Φ are vertically shown in Figure 16. In particular, $F + D_1' + \cdots + D_5'$ is the support of a singular fiber of Φ. We have $\rho(V') = 13$.

(2) Denote by D' the reduced effective divisor consisting of all irreducible components in Figure 16 with self intersection number ≤ -2. Let $f_1 : V' \to \bar{V}'$ be the contraction of D'. Then $\bar{V}'$ is a log Enriques surface of index 7.

Let $\pi_1 : \bar{U}' \to \bar{V}'$ be the canonical covering. Then $\pi_1^{-1}(f_1(D'))$ consists of a smooth point and a singular point P of Dynkin type A_8. We have $\mathrm{Sing}(\bar{U}') = \{P\}$.

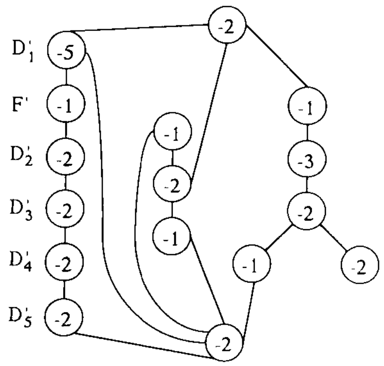

Figure 16

Let $\bar{\tau}: \bar{V} \to \bar{V}'$ be a composite morphism of combining morphisms such that $\bar{V}$ satisfies the condition (1) of Corollary 2.12. In the notations of Theorem 2.11, we have, $\bar{V} = \bar{V}_n'$ with $n = 7$.

Finally, $\bar{V}$ is a log Enriques surface of index 7 fitting the case No. 2 of the Table 3. For the concrete constructions of (V', D') and (V, D), we refer to Example 7.3 and 3.2.

Table 5

No.	Sing $(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	Sing $(\bar{U})$
1	$(11, 1), (11, 2)^2, (11, 3)^2, (11, 5)^3, (11, 7)^3, (22, 1)$	11	48	A_1
2	$(11, 1), (11, 2), (11, 3)^3, (11, 5)^2, (11, 7)^4, (22, 1)$	11	47	A_1
3	$(11, 2)^3, (11, 3)^3, (11, 5), (11, 7)^4, (22, 1)$	11	45	A_1
4	$(11, 1)^2, (11, 3)^2, (11, 5)^4, (11, 7)^3, (22, 1)$	11	50	A_1
5	$(11, 2)^4, (11, 3)^2, (11, 5)^2, (11, 7)^3, (22, 1)$	11	46	A_1
6	$(11, 1)^2, (11, 2), (11, 3), (11, 5)^5, (11, 7)^2, (22, 1)$	11	51	A_1
7	$(11, 1), (11, 2)^3, (11, 3), (11, 5)^4, (11, 7)^2, (22, 1)$	11	49	A_1
8	$(11, 2)^5, (11, 3), (11, 5)^3, (11, 7)^2, (22, 1)$	11	47	A_1
9	$(11, 1)^3, (11, 5)^7, (11, 7), (22, 1)$	11	54	A_1
10	$(11, 1)^2, (11, 2)^2, (11, 5)^6, (11, 7), (22, 1)$	11	52	A_1
11	$(11, 1), (11, 2)^4, (11, 5)^5, (11, 7), (22, 1)$	11	50	A_1
12	$(11, 1)^3, (11, 2)^4, (11, 7)^6$	12	47	ϕ
13	$(11, 1)^4, (11, 2), (11, 3), (11, 7)^7$	12	48	ϕ
14	$(11, 1)^4, (11, 2)^2, (11, 5), (11, 7)^6$	12	49	ϕ
15	$(11, 1)^5, (11, 5)^2, (11, 7)^6$	12	51	ϕ
16	$(11, 5), (11, 7)$	2	11	ϕ

§ 6. Index 11 case

We shall prove the following Theorem 6.1 in the present section. In the Tables 5 and 6 below, by $\mathrm{Sing}(\bar{U}) = mA_1$, we mean that $\bar{U}$ consists of exactly m singularities of Dynkin type A_1. By $\mathrm{Sing}(\bar{V}) = (11, 1)^i, (11, 2)^j, (11, 3)^k, (11, 5)^r, (11, 7)^s, (22, 1)^t$, we mean that $\bar{V}$ has exactly $i + j + k + r + s + t$ singularities, and i (resp. j, k, r, s, t) singularities of them are isomorphic to $(\mathbf{C}^2/C_{a,b}; 0)$ with $(a, b) = (11, 1)$ (resp. $(11, 2), (11, 3), (11, 5), (11, 7), (22, 1)$). We also use the notations (V, D, f) in § 1 for $\bar{V}$.

Theorem 6.1. *Let $\bar{V}$ be a log Enriques surface of index* 11 *and let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Assume $\bar{V}$ satisfies the condition* (1) *of Corollary* 2.12. *Then $\bar{V}$ and $\bar{U}$ are described in one of* 16 *rows of the Table* 5. *In particular,* $H^1(V, D + 2K_V) = 0$.

Proof. If $\bar{U}$ is smooth, then $\bar{V}$ and $\bar{U}$ are described in n-th row ($n = 12, \ldots, 16$) of the Table 5 by [2, Theorem 5.1]. So, we shall assume that $\bar{U}$ admits at least one singular point.

Let y_i for $1 \le i \le n_1$, y_j for $n_1 + 1 \le j \le n_1 + n_2$, y_k for $n_1 + n_2 + 1 \le k \le n_1 + n_2 + n_3$, y_r for $n_1 + n_2 + n_3 + 1 \le r \le n_1 + \cdots + n_4$ and y_s for $n_1 + \cdots + n_4 + 1$

Table 6

No.	$\mathrm{Sing}(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	$\mathrm{Sing}(\bar{U})$
17	$(11, 1), (11, 3)^4, (11, 5), (11, 7)^5, (22, 1)$	11	46	A_1
18	$(11, 2)^2, (11, 3)^4, (11, 7)^5, (22, 1)$	11	44	A_1
19	$(11, 2)^6, (11, 5)^4, (11, 7), (22, 1)$	11	48	A_1
20	$(11, 1)^2, (11, 3), (11, 5), (11, 7)^7, (22, 1)$	11	49	A_1
21	$(11, 1), (11, 2)^2, (11, 3), (11, 7)^7, (22, 1)$	11	47	A_1
22	$(11, 1)^2, (11, 2), (11, 5)^2, (11, 7)^6, (22, 1)$	11	50	A_1
23	$(11, 1), (11, 2)^3, (11, 5), (11, 7)^6, (22, 1)$	11	48	A_1
24	$(11, 2)^5, (11, 7)^6, (22, 1)$	11	46	A_1
25	$(11, 3), (11, 5)^6, (11, 7)^2, (22, 1)^2$	10	52	$2A_1$
26	$(11, 5)^3, (11, 7)^6, (22, 1)^2$	10	51	$2A_1$
27	$(11, 2), (11, 5)^7, (11, 7), (22, 1)^2$	10	53	$2A_1$

$\leq s \leq n_1 + \cdots + n_5$ be respectively all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{11,v}; 0)$ with $v = 1, 2, 3, 5$ and 7. Set $m_0 := n_1 + \cdots + n_5$. Let y_t for $m_0 + 1 \leq t \leq m_0 + m_1$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{22,1}; 0)$. As in Theorem 3.1, we have $c := \#(\mathrm{Sing}\,\bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\,\bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \leq i \leq n_1)$ is a single (-11)-curve.

(2) $\Delta_j (n_1 + 1 \leq j \leq n_1 + n_2)$ is a chain of one (-2)-curve Δ_{j1} and one (-6)-curve Δ_{j2}.

(3) $\Delta_k (n_1 + n_2 + 1 \leq k \leq n_1 + n_2 + n_3)$ is a chain of one (-3)-curve Δ_{k1} and one (-4)-curve Δ_{k2}.

(4) $\Delta_r (n_1 + n_2 + n_3 + 1 \leq r \leq n_1 + \cdots + n_4)$ is a chain of four (-2)-curves $\Delta_{r1}, \ldots, \Delta_{r4}$ and one (-3)-curve Δ_{r5} with $(\Delta_{ra}, \Delta_{r,a+1}) = 1\ (1 \leq a \leq 4)$.

(5) $\Delta_s (n_1 + \cdots + n_4 + 1 \leq s \leq m_0)$ is a chain of three (-2)-curves $\Delta_{s1}, \Delta_{s2}, \Delta_{s4}$ and one (-3)-curve Δ_{s3} with $(\Delta_{sa}, \Delta_{s,a+1}) = 1\ (1 \leq a \leq 3)$.

(6) $\Delta_t (m_0 + 1 \leq t \leq c)$ is a single (-22)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with $D^* =$

$$\frac{9}{11}\sum_i \Delta_i + \frac{4}{11}\sum_j (\Delta_{j1} + 2\Delta_{j2}) + \frac{1}{11}\sum_k (6\Delta_{k1} + 7\Delta_{k2}) +$$

$$\frac{1}{11}\sum_r (\Delta_{r1} + 2\Delta_{r2} + 3\Delta_{r3} + 4\Delta_{r4} + 5\Delta_{r5}) +$$

$$\frac{1}{11}\sum_s (2\Delta_{s1} + 4\Delta_{s2} + 6\Delta_{s3} + 3\Delta_{s4}) + \frac{10}{11}\sum_t \Delta_t.$$

By Proposition 1.3, we have $\rho(\bar{V}) = c - 1$. As in Theorem 3.1, we have

$$-\frac{1}{11}(81n_1 + 32n_2 + 20n_3 + 5n_4 + 6n_5 + 200m_1) =$$

$$(D^*)^2 = (K_V^2) = 11 - c - \#(D) =$$

$$11 - (n_1 + n_2 + n_3 + n_4 + n_5 + m_1)$$

$$-(n_1 + 2n_2 + 2n_3 + 5n_4 + 4n_5 + m_1), \quad \textit{and}$$

$$(6.1) \qquad 121 + 59n_1 - n_2 - 13n_3 - 61n_4 - 49n_5 + 178m_1 = 0.$$

By Proposition 1.3, we have

$$24 = c + \rho(U) - \rho(\bar{U}) + 11 \times 1 = c + m_1 + 11.$$

Hence we have:

$$(6.2) \qquad c = \sum_{i=1}^{5} n_i + m_1 = 13 - m_1.$$

Eliminating n_1 by (6.2), we deduce the following (6.1′) from (6.1).

$$(6.1') \qquad 74 - 5n_2 - 6n_3 - 10n_4 - 9n_5 + 5m_1 = 0.$$

On the other hand, as in Theorem 4.1, we have

$$h^1(V, D + 2K_V) = 2c - 12 + \#(D) - (D, K_V) =$$

$$2\sum_{i=1}^{5} n_i + 2m_1 - 12 + (n_1 + 2n_2 + 2n_3 + 5n_4 + 4n_5 + m_1)$$

$$- (9n_1 + 4n_2 + 3n_3 + n_4 + n_5 + 20m_1), \quad \textit{and}$$

$$- h^1(V, D + 2K_V) = 12 + 6n_1 - n_3 - 6n_4 - 5n_5 + 17m_1.$$

This and the above equalities (6.1′) and (6.2) imply:

$$- 5h^1(V, D + 2K_V) =$$

$$5(12 + 6n_1 - n_3 - 6n_4 - 5n_5 + 17m_1)$$

$$- 30(n_1 + n_2 + n_3 + n_4 + n_5 + 2m_1 - 13)$$

$$- 6(74 - 5n_2 - 6n_3 - 10n_4 - 9n_5 + 5m_1), \quad \textit{and}$$

$$(6.3) \qquad 0 \leq 5h^1(V, D + 2K_V) = n_5 - n_3 - 1 - 5(1 - m_1).$$

Note that (6.1′) and (6.2) imply

$$0 \geq 74 + 5m_1 - 10(n_2 + n_3 + n_4 + n_5) =$$

$$74 + 5m_1 - 10(13 - 2m_1) + 10n_1 = - 56 + 25m_1 + 10n_1.$$

Hence $m_1 \leq 2$. So, $m_1 = 1, 2$.

By making use of (6.1′), (6.2) and (6.3), we can show that $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table 5 or 6 (cf. the proof of Theorem 4.1). Then Theorem 6.1 follows from Proposition 6.2 below (cf. the proof of Theorem 3.1).

Proposition 6.2. *The cases of Table* 6 *are impossible.*

Proof. This can be proved by the same fashion as in the proof of Lemma 4.2.

The existence of the case No. 16 in Table 5 of Theorem 6.1 was given in [2, Example 5.6]. We shall give below an example of the case No. 1 in Table 5. We do not know yet whether or not the other cases of Table 5 occur.

Example 6.3. We can find a nonsingular rational surface V' and a $\mathbf{P}^1$-fibration $\Phi: V' \to \mathbf{P}^1$ such that the following two conditions are satisfied.

(1) All singular fibers of Φ are vertically shown in Figure 17. In particular, $F + D'_1 + \cdots + D'_5$ is the support of a singular fiber of Φ. We have $\rho(V') = 14$.

(2) Denote by D' the reduced effective divisor consisting of all irreducible components in Figure m with self intersection number ≤ -2. Let $f_1: V' \to \bar{V}'$ be the contraction of D'. Then $\bar{V}'$ is a log Enriques surface of index 11.

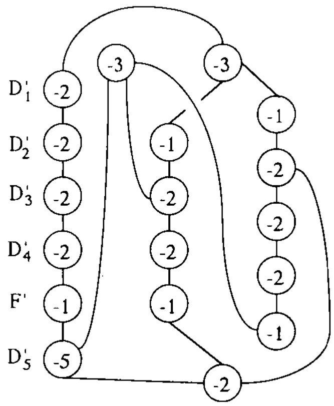

Figure 17

Let $\pi_1: \bar{U}' \to \bar{V}'$ be the canonical covering. Then $\pi_1^{-1}(f_1(D'))$ consists of a smooth point and a singular point P of Dynkin type A_{11}. We have $\mathrm{Sing}(\bar{U}') = \{P\}$.

Let $\bar{\tau}: \bar{V} \to \bar{V}'$ be a composite morphism of combining morphisms such that $\bar{V}$ satisfies the condition (1) of Corollary 2.12. In the notations of Theorem 2.11, we have, $\bar{V} = \bar{V}_n'$ with $n = 10$.

Finally, $\bar{V}$ is a log Enriques surface of index 11 fitting the case No. 1 of the Table 5. For the concrete constructions of (V', D') and (V, D), we refer to Examples 7.3 and 3.2.

§7. Index 13 case

We shall prove the following Theorem 7.1 in the present section. In the Tables 7 and 8 below, by Sing $(\bar{U}) = mA_1$, we mean that $\bar{U}$ consists of exactly m singularities of Dynkin type A_1. By $\mathrm{Sing}(\bar{V}) = (13, 1)^i, (13, 2)^j, (13, 3)^k, (13, 4)^r, (13, 5)^s, (13, 6)^t, (26, 1)^u$ we mean that $\bar{V}$ has exactly $i+j+k+r+s+t+u$ singularities, and i (resp. j, k, r, s, t, u) singularities of them are isomorphic to $(\mathbf{C}^2/C_{a,b}; 0)$ with $(a, b) = (13, 1)$ (resp. $(13, 2), (13, 3), (13, 4), (13, 5), (13, 6), (26, 1)$). We also use the notations (V, D, f) in §1 for $\bar{V}$.

Theorem 7.1. *Let $\bar{V}$ be a log Enriques surface of index* 13 *and let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Assume $\bar{V}$ satisfies the condition* (1) *of Corollary* 2.12. *Then $\bar{V}$ and $\bar{U}$ are described in one of nine rows of the Table* 7. *In particular*, $H^1(V, D + 2K_V) = 0$.

Proof. By [2, Theorem 5.1], we know that $\bar{U}$ admits at least one singular point. Let y_i for $1 \le i \le n_1$, y_j for $n_1 + 1 \le j \le n_1 + n_2$, y_k for $n_1 + n_2 + 1 \le k \le n_1 + n_2 + n_3$, y_r for $n_1 + n_2 + n_3 + 1 \le r \le n_1 + \cdots + n_4$, y_s for $n_1 + \cdots + n_4 + 1 \le s \le n_1 + \cdots + n_5$ and y_t for $n_1 + \cdots + n_5 + 1 \le s \le n_1 + \cdots + n_6$ be respectively all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{13,v}; 0)$ with $v = 1, 2, 3, 4, 5$ and 6. Set $m_0 := n_1 + \cdots + n_6$. Let y_u for $m_0 + 1 \le u \le m_0 + m_1$ be all

Table 7

No.	Sing ($\bar{V}$)	$\rho(\bar{V})$	$\rho(V)$	Sing ($\bar{U}$)
1	$(13, 2), (13, 3)^2, (13, 5)^3, (13, 6)^3, (26, 1)$	9	45	A_1
2	$(13, 1)^2, (13, 6)^7, (26, 1)$	9	54	A_1
3	$(13, 1), (13, 4), (13, 5)^3, (13, 6)^4, (26, 1)$	9	48	A_1
4	$(13, 1), (13, 3), (13, 4)^2, (13, 5), (13, 6)^4, (26, 1)$	9	49	A_1
5	$(13, 1), (13, 2), (13, 4), (13, 5), (13, 6)^5, (26, 1)$	9	50	A_1
6	$(13, 2), (13, 3), (13, 4)^3, (13, 5)^2, (13, 6)^2, (26, 1)$	9	45	A_1
7	$(13, 2), (13, 3)^3, (13, 4), (13, 5), (13, 6)^3, (26, 1)$	9	46	A_1
8	$(13, 2)^2, (13, 4)^2, (13, 5)^2, (13, 6)^3, (26, 1)$	9	46	A_1
9	$(13, 2)^2, (13, 3)^2, (13, 5), (13, 6)^4, (26, 1)$	9	47	A_1

singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{26,1}; 0)$. As in Theorem 3.1, we have $c := \#(\mathrm{Sing}\,\bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\,\bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \le i \le n_1)$ is a single (-13)-curve.

(2) $\Delta_j (n_1 + 1 \le j \le n_1 + n_2)$ is a chain of one (-2)-curve Δ_{j1} and one (-7)-curve Δ_{j2}.

(3) $\Delta_k (n_1 + n_2 + 1 \le k \le n_1 + n_2 + n_3)$ is a chain of two (-2)-curves Δ_{k1}, Δ_{k2} and one (-5)-curve Δ_{k3} with $(\Delta_{ka}, \Delta_{k,a+1}) = 1\,(a = 1, 2)$.

(4) $\Delta_r (n_1 + n_2 + n_3 + 1 \le r \le n_1 + \cdots + n_4)$ is a chain of three (-2)-curves $\Delta_{r1}, \Delta_{r2}, \Delta_{r3}$ and one (-4)-curve Δ_{r4} with $(\Delta_{ra}, \Delta_{r,a+1}) = 1\,(1 \le a \le 3)$.

(5) $\Delta_s (n_1 + \cdots + n_4 + 1 \le s \le n_1 + \cdots + n_5)$ is a chain of one (-2)-curve Δ_{s1} and two (-3)-curves Δ_{s2}, Δ_{s3} with $(\Delta_{sa}, \Delta_{s,a+1}) = 1\,(a = 1, 2)$.

(6) $\Delta_t (n_1 + \cdots + n_5 + 1 \le t \le m_0)$ is a chain of five (-2)-curves $\Delta_{t1}, \ldots, \Delta_{t5}$ and one (-3)-curve Δ_{t6} with $(\Delta_{ta}, \Delta_{t,a+1}) = 1\,(1 \le a \le 5)$.

(7) $\Delta_u (m_0 + 1 \le u \le c)$ is a single (-26)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with $D^* =$

$$\frac{11}{13}\sum_i \Delta_i + \frac{5}{13}\sum_j (\Delta_{j1} + 2\Delta_{j2}) + \frac{3}{13}\sum_k (\Delta_{k1} + 2\Delta_{k2} + 3\Delta_{k3}) +$$

$$\frac{2}{13}\sum_r (\Delta_{r1} + 2\Delta_{r2} + 3\Delta_{r3} + 4\Delta_{r4}) + \frac{1}{13}\sum_s (4\Delta_{s1} + 8\Delta_{s2} + 7\Delta_{s3}) +$$

Table 8

No.	Sing $(\bar{V})$	$\rho(\bar{V})$	$\rho(V)$	Sing $(\bar{U})$
10	$(13, 1), (13, 4)^5, (13, 6)^3, (26, 1)$	9	49	A_1
11	$(13, 1), (13, 3)^3, (13, 6)^5, (26, 1)$	9	50	A_1
12	$(13, 4)^2, (13, 5)^6, (13, 6), (26, 1)$	9	42	A_1
13	$(13, 4)^6, (13, 5)^3, (26, 1)$	9	43	A_1
14	$(13, 3), (13, 4)^3, (13, 5)^4, (13, 6), (26, 1)$	9	43	A_1
15	$(13, 3), (13, 4)^7, (13, 5), (26, 1)$	9	44	A_1
16	$(13, 3)^2, (13, 5)^5, (13, 6)^2, (26, 1)$	9	43	A_1
17	$(13, 3)^2, (13, 4)^4, (13, 5)^2, (13, 6), (26, 1)$	9	44	A_1
18	$(13, 3)^3, (13, 4), (13, 5)^3, (13, 6)^2, (26, 1)$	9	44	A_1
19	$(13, 3)^3, (13, 4)^5, (13, 6), (26, 1)$	9	45	A_1
20	$(13, 3)^4, (13, 4)^2, (13, 5), (13, 6)^2, (26, 1)$	9	45	A_1
21	$(13, 3)^6, (13, 6)^3, (26, 1)$	9	46	A_1
22	$(13, 2), (13, 4)^2, (13, 5)^4, (13, 6)^2, (26, 1)$	9	44	A_1
23	$(13, 2), (13, 4)^6, (13, 5), (13, 6), (26, 1)$	9	45	A_1
24	$(13, 2), (13, 3)^2, (13, 4)^4, (13, 6)^2, (26, 1)$	9	46	A_1
25	$(13, 2)^2, (13, 3), (13, 4)^3, (13, 6)^3, (26, 1)$	9	47	A_1
26	$(13, 2)^3, (13, 4)^2, (13, 6)^4, (26, 1)$	9	48	A_1

$$\frac{1}{13}\sum_t(\Delta_{t1} + 2\Delta_{t2} + 3\Delta_{t3} + 4\Delta_{t4} + 5\Delta_{t5} + 6\Delta_{t6}) + \frac{12}{13}\sum_u \Delta_u.$$

By Proposition 1.3, we have $\rho(\bar{V}) = c - 1$ and

$24 = c + \rho(U) - \rho(\bar{U}) + 13 \times 1 = c + m_1 + 13.$

So, we obtain:

$$(7.1) \qquad 11 = c + m_1 = 2m_1 + \sum_{i=1}^{6} n_i.$$

On the other hand, as in Theorem 3.1, we can compute as follows:

$$-\frac{1}{13}(121n_1 + 50n_2 + 27n_3 + 16n_4 + 15n_5 + 6n_6 + 288m_1) =$$

$$(D^*)^2 = (K_{\bar{V}}^2) =$$

$$11 - c - (n_1 + 2n_2 + 3n_3 + 4n_4 + 3n_5 + 6n_6 + m_1),$$

$$\frac{1}{12}(13(11 - c) + 275m_1) + 9n_1 + 2n_2 - n_3 - 3n_4 - 2n_5 - 6n_6 = 0, \quad \textit{and}$$

$$\frac{1}{12}(13(11 - c) + 287m_1) - c + 10n_1 + 3n_2 - 2n_4 - n_5 - 5n_6 = 0.$$

The latter equality and the equality (7.1) imply:

$$(7.2) \qquad -11 + 26m_1 + 10n_1 + 3n_2 - 2n_4 - n_5 - 5n_6 = 0.$$

By (7.1), we eliminate m_1 in (7.2) and obtain $0 = -11 + 13 \times 11 - 3n_1 - 10n_2 - 13n_3 - 15n_4 - 14n_5 - 18n_6 \geq 12 \times 11 - 18\sum_{i=1}^{6} n_i$. Hence $\sum_{i=1}^{6} n_i \geq 8$. On the other hand, (7.1) implies that $\sum_{i=1}^{6} n_i$ is an odd integer satisfying $\sum_{i=1}^{6} n_i = 11 - 2m_1 \leq 9$. So, $\sum_{i=1}^{6} n_i = 9$. Thus, we have proved:

$$(7.1') \qquad m_1 = 1, \quad \sum_{i=1}^{6} n_i = 9, \quad c = 10.$$

In particular, we have $\rho(\bar{V}) = c - 1 = 9$ and $\mathrm{Sing}\, \bar{U} = A_1$. Using (7.2) again, we obtain $0 = -11 + 26 \times 1 + 10\sum_{i=1}^{6} n_i - 7n_2 - 10n_3 - 12n_4 - 11n_5 - 15n_6$ and

$$(7.2') \qquad 7n_2 + 10n_3 + 12n_4 + 11n_5 + 15n_6 = 105.$$

As in Theorem 4.1, by (7.2'), we have:

$$h^1(V, D + 2K_V) = 8 + \#(D) - (D, K_V) =$$

$$8 + (n_1 + 2n_2 + 3n_3 + 4n_4 + 3n_5 + 6n_6 + m_1)$$

$$-(11n_1 + 5n_2 + 3n_3 + 2n_4 + 2n_5 + n_6 + 24m_1) =$$

$$-15 - 10n_1 - 3n_2 + 2n_4 + n_5 + 5n_6 = 0.$$

Note that (7.2') implies $105 \leq 15\sum_{i=2}^{6} n_i$ and $\sum_{i=2}^{6} n_i \geq 7$. So, $\sum_{i=2}^{6} n_i = 7, 8, 9$ and $n_1 = 2, 1, 0$, respectively.

By making use of (7.1') and (7.2'), we can show that $\bar{V}$ and $\bar{U}$ are described in one of the rows of the Table 7 or 8 (cf. the proof of Theorem 4.1). Then Theorem 7.1 follows from Proposition 7.2 below (cf. the proof of Theorem 3.1).

Proposition 7.2. *The cases of Table* 8 *are impossible.*

Proof. This can be proved by the same fashion as in the proof of Lemma 4.2.

We shall give below an example of the case No. 1 in Table 7 of Theorem 7.1. We do not know yet whether or not the other cases of Table 7 occus.

Example 7.3 (Case No. 1 of Table 7). Let $\pi: \Sigma_2 \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration on the Hirzebruch surface Σ_2 and let M be the (-2)-curve on Σ_2. Let L be a fiber of π. Take two nonsingular irreducible members $C_1, C_2 \in |M + 2L|$ such that C_1 and C_2 share exactly one comon point, say P_1. Let L_3 be the fiber of π containing P_1. Let L_1 and L_2 be two fibers of π other than L_3. Denote by P_2 (resp. P_3, P_4) the unique intersection point of L_1 (resp. L_2, L_2) with C_2 (resp. C_1, C_2). Let $\sigma: V' \to \Sigma_2$ be the blowing-up of four points P_i's and ten infinitely near points of them such that $\sigma^*(L_i)$ $(i = 1, 2, 3)$ is vertically given in Figure 18. Here we denote by M', $C_i'(i = 1, 2)$, $L_j'(j = 1, 2, 3)$ the proper transforms on V' of the curves M, C_i, L_j, respectively. To be precise, $\sigma^*(L_1) = L_1' + F' + E_{10}' + E_9' + E_8' + E_7' + E_5' + E_2'$. Set $D' := M' + C_1' + C_2' + L_1' + L_2' + E_1' + \cdots + E_{10}'$.

We shall show that (V', D') is a log Enriques surface of index 13. Set $\Delta := 3E_3' + 6L_2' + 9M' + 12L_1' + 10C_1' + 5E_6' + E_{10}' + 2E_9' + 3E_8' + 4E_7' + 5E_5' + 6E_2' + 7C_2' + 8E_4' + 4E_1'$. Note that $10C_1 + 7C_2 + 9M + 12L_1 + 6L_2 + 13K_{\Sigma_2} \sim 0$. We can check easily that $\Delta + 13K_{V'} \sim 0$. Let $f_1: V' \to \bar{V}'$ be the contraction of D'. Then $13K_{\bar{V}'} \sim 0$. Hence $\bar{V}'$ is a log Enriques surface of index 13. Moreover, $D'^* = \frac{1}{13}\Delta$ in the notations of Lemma 1.1. Let $\pi_1: \bar{U}' \to \bar{V}'$ be the canonical covering. Then $\pi_1^{-1}(f_1(\Gamma))$ for $\Gamma := E_3' + L_2' + M' + L_1' + C_1' + E_6'$ (resp. $\Gamma := E_{10}' + E_9' + E_8' + E_7' + E_5' + E_2' + C_2' + E_4' + E_1'$) is a singularity of Dynkin type A_8 (resp. A_1) and there are no other singular points on $\bar{U}'$ (cf. Lemma 1.4).

Let $\tau: V \to V'$ be the blowing-up of several intersection points of D' and their infinitely near points such that $\tau^{-1}(D')$ has Figure 19 as its weighted dual graph.

In Figure 19, $\tilde{M}$, $\tilde{C}_i$ $(i = 1, 2)$, $\tilde{L}_j(j = 1, 2)$, $\tilde{E}_k(k = 1, \ldots, 10)$ are the proper transforms on V of the curves M', C_i', L_j', E_k', respectively. We denote by D the reduced effective divisor consisting of all components of $\tau^{-1}(D')$ of self intersection

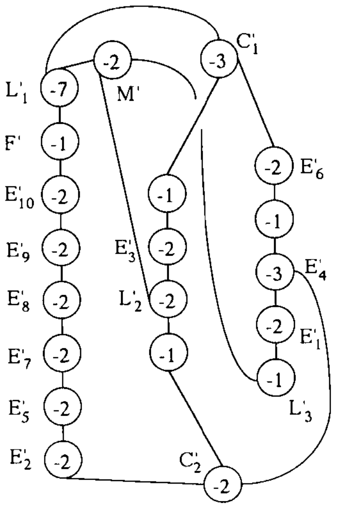

Figure 18

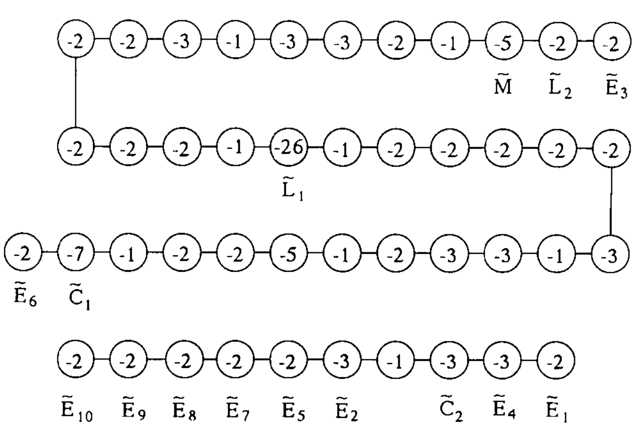

Figure 19

number ≤ -2. Then $\tau^{-1}(D') - D$ consists of 8 disjoint (-1)-curves. Let $f: V \to \bar{V}$ be the contraction of D. We see that $\bar{V}$ is a log Enriques surface of index 13. Indeed, by the fact that $13(K_{V'} + D'^*) \sim 0$, we can check that $13(K_V + D^*) \sim 0$ in the notations of Lemma 1.1. We have also $D := f^{-1}(\mathrm{Sing}\,\bar{V})$. Note that $\rho(V') = 2 + 14 = 16$ and $\rho(V) = 16 + 29 = 45$. So, $\rho(\bar{V}) = \rho(V) - \#\{\text{irreducible component of } D\} = 9$. Let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Then $\pi^{-1}(f(\tilde{L}_1))$ is a rational double singularity of Dynkin type A_1 and there are no other singular points on $\bar{U}$. Since the weighted dual graph of $f^{-1}(\mathrm{Sing}\,\bar{V})$ is precisely given as a subgraph of $\tau^{-1}(D')$, the singular locus of $\bar{V}$ is as described in the first row of Table 7 (cf. Brieskorn [1]). So, $\mathrm{Sing}(\bar{V})$, $\rho(\bar{V})$, $\rho(V)$ and $\mathrm{Sing}(\bar{U})$ are as described in the first row of Table 7. Since $\mathrm{Sing}(\bar{U}) \neq \phi$, the surface $\bar{U}$ is not an abelian surface. Thus, we see that $\bar{V}$ satisfies the condition (1) of Corollary 2.12 and fits the case No. 1 of Table 7.

There is a composite morphism $\bar{\tau}: \bar{V} \to \bar{V}'$ of 8 combining morphisms such that $\bar{\tau} \cdot f = f_1 \cdot \tau$. In the notations of Theorem 2.11, we have $\bar{V} = \bar{V}_n'$, $\bar{U} = \bar{U}_n'$ and $\bar{\tau} = \bar{h}_1 \cdots \bar{h}_n$ with $n = 8$.

§8. Index 17 case

We shall prove the following Theorem 8.1 in the present section.

Theorem 8.1. *Let $\bar{V}$ or synonymously (V, D, f) be a log Enriques surface of index 17 and let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Assume $\bar{V}$ satisfies the condition (1) of Corollary 2.12. Then $\bar{U}$ is nonsingular. Hence possible distributions of singular points of $\bar{V}$ are given in [2, Theorem 5.1]. (See also [ibid., Example 5.7].) In particular, $H^1(V, D + 2K_V) = 0$.*

Proof. Suppose, on the contrary, that $\bar{U}$ admit at least one singular point. Let y_i for $1 \leq i \leq n_1$, y_j for $n_1 + 1 \leq j \leq n_1 + n_2$, y_k for $n_1 + n_2 + 1 \leq k \leq n_1 + n_2 + n_3$, y_r for $n_1 + n_2 + n_3 + 1 \leq r \leq n_1 + \cdots + n_4$, y_s for $n_1 + \cdots + n_4 + 1 \leq s \leq n_1 + \cdots + n_5$, y_t for $n_1 + \cdots + n_5 + 1 \leq t \leq n_1 + \cdots + n_6$, y_u for $n_1 + \cdots + n_6 + 1 \leq u \leq n_1 + \cdots + n_7$ and y_v for $n_1 + \cdots + n_7 + 1 \leq v \leq n_1 + \cdots + n_8$ be respectively all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{17,z}; 0)$ with $z = 1, 2, 3, 4, 5, 8, 10$ and 11. Set $m_0 := n_1 + \cdots + n_8$. Let y_w for $m_0 + 1 \leq w \leq m_0 + m_1$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{34,1}: 0)$. As in Theorem 3.1, we have $c := \#(\mathrm{Sing}\,\bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\,\bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \leq i \leq n_1)$ is a single (-17)-curve.

(2) $\Delta_j (n_1 + 1 \leq j \leq n_1 + n_2)$ is a chain of one (-2)-curve Δ_{j1} and one (-9)-curve Δ_{j2}.

(3) $\Delta_k (n_1 + n_2 + 1 \leq k \leq n_1 + n_2 + n_3)$ is a chain of one (-3)-curve Δ_{k1} and one (-6)-curve Δ_{k2}.

(4) $\Delta_r (n_1 + n_2 + n_3 + 1 \leq r \leq n_1 + \cdots + n_4)$ is a chain of three (-2)-curves

$\Delta_{r1}, \Delta_{r2}, \Delta_{r3}$ and one (-5)-curve Δ_{r4} with $(\Delta_{ra}, \Delta_{r,a+1}) = 1\ (1 \leq a \leq 3)$.

(5) $\Delta_s(n_1 + \cdots + n_4 + 1 \leq s \leq n_1 + \cdots + n_5)$ is a chain of one (-3)-curve Δ_{s1}, one (-2)-curve Δ_{s2} and one (-4)-curve Δ_{s3} with $(\Delta_{sa}, \Delta_{s,a+1}) = 1\ (a = 1, 2)$.

(6) $\Delta_t(n_1 + \cdots + n_5 + 1 \leq t \leq n_1 + \cdots + n_6)$ is a chain of seven (-2)-curves $\Delta_{t1}, \ldots, \Delta_{t7}$ and one (-3)-curve Δ_{t8} with $(\Delta_{ta}, \Delta_{t,a+1}) = 1\ (1 \leq a \leq 7)$.

(7) $\Delta_u(n_1 + \cdots + n_6 + 1 \leq u \leq n_1 + \cdots + n_7)$ is a chain of three (-2)-curves $\Delta_{u1}, \Delta_{u2}, \Delta_{u4}$ and one (-4)-curve Δ_{u3} with $(\Delta_{ua}, \Delta_{u,a+1}) = 1\ (1 \leq a \leq 3)$.

(8) $\Delta_v(n_1 + \cdots + n_7 + 1 \leq v \leq m_0)$ is a chain of five (-2)-curves $\Delta_{v1}, \ldots, \Delta_{v4}$, Δ_{v6} and one (-3)-curve Δ_{v5} with $(\Delta_{va}, \Delta_{v,a+1}) = 1\ (1 \leq a \leq 5)$.

(9) $\Delta_w(m_0 + 1 \leq w \leq c)$ is a single (-34)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with $D^* =$

$$\frac{15}{17}\sum_i \Delta_i + \frac{7}{17}\sum_j (\Delta_{j1} + 2\Delta_{j2}) + \frac{1}{17}\sum_k (10\Delta_{k1} + 13\Delta_{k2}) +$$

$$\frac{3}{17}\sum_r (\Delta_{r1} + 2\Delta_{r2} + 3\Delta_{r3} + 4\Delta_{r4}) + \frac{1}{17}\sum_s (9\Delta_{s1} + 10\Delta_{s2} + 11\Delta_{s3}) +$$

$$\frac{1}{17}\sum_t (\Delta_{t1} + 2\Delta_{t2} + 3\Delta_{t3} + 4\Delta_{t4} + 5\Delta_{t5} + 6\Delta_{t6} + 7\Delta_{t7} + 8\Delta_{t8}) +$$

$$\frac{2}{17}\sum_u (2\Delta_{u1} + 4\Delta_{u2} + 6\Delta_{u3} + 3\Delta_{u4}) +$$

$$\frac{1}{17}\sum_v (2\Delta_{v1} + 4\Delta_{v2} + 6\Delta_{v3} + 8\Delta_{v4} + 10\Delta_{v5} + 5\Delta_{v6}) + \frac{16}{17}\sum_w \Delta_w.$$

By Proposition 1.3, we have $\rho(\bar{V}) = c - 1$ and

$$24 = c + \rho(U) - \rho(\bar{U}) + 17 \times 1 = c + m_1 + 17.$$

So, we obtain:

$$7 = c + m_1 = 2m_1 + \sum_{i=1}^{8} n_i. \tag{8.1}$$

On the other hand, as in Theorem 3.1, we can compute as follows:

$$-\frac{1}{17}(225n_1 + 98n_2 + 62n_3 + 36n_4 + 31n_5 + 8n_6 + 24n_7 + 10n_8 + 512m_1) =$$

$$(D^*)^2 = (K_V^2) = 11 - c$$

$$-(n_1 + 2n_2 + 2n_3 + 4n_4 + 3n_5 + 8n_6 + 4n_7 + 6n_8 + m_1), \quad \textit{and}$$

$$0 = \frac{1}{4}(17(11 - c) + 495m_1)$$

$$+ 52n_1 + 16n_2 + 7n_3 - 8n_4 - 5n_5 - 32n_6 - 11n_7 - 23n_8 =$$

$$\frac{1}{4}(17(11 - c) + 515m_1) - 5c + 57n_1 + 21n_2 + 12n_3 - 3n_4 - 27n_6 - 6n_7 - 18n_8.$$

Dividing the latter equality by 3 and using (8.1), the following equality can be obtained:

$$-6+46m_1+19n_1+7n_2+4n_3-n_4-9n_6-2n_7-6n_8=0. \tag{8.2}$$

The equalities (8.1) and (8.2) imply

$$0 \geq -6+46m_1-9\sum_{i=1}^{8} n_i = -6+46m_1-9(7-2m_1) = -69+64m_1.$$

Hence $m_1 = 1$. So, $c = 6$ and $\sum_i n_i = 5$ by (8.1). Using (8.2) again, we obtain

$$0 = 8\sum_i n_i + 19n_1 + 7n_2 + 4n_3 - n_4 - 9n_6 - 2n_7 - 6n_8 \quad \textit{and}$$

$$n_6 = 27n_1 + 15n_2 + 12n_3 + 7n_4 + 8n_5 + 6n_7 + 2n_8. \tag{8.3}$$

This equality implies

$$n_6 \geq 2\left(\sum_i n_i - n_6\right) = 2(5-n_6).$$

So, $n_6 = 4, 5$ and $\sum_{i \neq 6} n_i = 5 - n_6 = 1, 0$, respectively. This contradicts (8.3).

Therefore $\bar{U}$ is nonsingular. Then $\bar{V}$ is described in [2, Theorem 5.1]. With the help of Lemma 1.2, the second assertion of Theorem 8.1 is proved there. This proves Theorem 8.1.

In [2, Example 5.7], we gave an example of log Enriques surface (V', D') of index 17 whose canonical covering admits at least one singularity of multiplicity ≥ 3.

§9. Index 19 case

We shall prove the following Theorem 9.1 in the present section.

Theorem 9.1. *Let $\bar{V}$ or synonymously (V, D, f) be a log Enriques surface of index 19 and let $\pi: \bar{U} \to \bar{V}$ be the canonical covering. Assume $\bar{V}$ satisfies the condition (1) of Corollary 2.12. Then $\bar{U}$ is nonsingular. Hence possible distributions of singular points of $\bar{V}$ are given in [2, Theorem 5.1]. (See also [ibid., Example 5.8].) In particular, $H^1(V, D+2K_V) = 0$.*

Proof. Suppose, on the contrary, that $\bar{U}$ admit at least one singular point. Let y_i for $1 \leq i \leq n_1$, y_j for $n_1+1 \leq j \leq n_1+n_2$, y_k for $n_1+n_2+1 \leq k \leq n_1+n_2+n_3$, y_r for $n_1+n_2+n_3+1 \leq r \leq n_1+\cdots+n_4$, y_s for $n_1+\cdots+n_4+1 \leq s \leq n_1+\cdots+n_5$, y_t for $n_1+\cdots+n_5+1 \leq t \leq n_1+\cdots+n_6$, y_u for $n_1+\cdots+n_6+1 \leq u \leq n_1+\cdots+n_7$, y_v for $n_1+\cdots+n_7+1 \leq v \leq n_1+\cdots+n_8$ and y_w for $n_1+\cdots+n_8+1 \leq w \leq n_1+\cdots+n_9$ be respectively all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{19,z}; 0)$ with $z = 1, 2, 3, 4, 6, 7, 8, 9$ and 14. Set $m_0 := n_1+\cdots+n_9$. Let y_h for $m_0+1 \leq h \leq m_0+m_1$ be all singularities of $\bar{V}$ isomorphic to $(\mathbf{C}^2/C_{38,1}; 0)$.

As in Theorem 3.1, we have $c := \#(\mathrm{Sing}\,\bar{V}) = m_0 + m_1$. We have also $\rho(U) - \rho(\bar{U}) = m_1$ and $\mathrm{Sing}\,\bar{U} = m_1 A_1$. Here U is a minimal resolution of $\bar{U}$. Set $\Delta_n := f^{-1}(y_n) \subseteq V$ and $D := \sum_{n=1}^{c} \Delta_n$. Then we have:

(1) $\Delta_i (1 \le i \le n_1)$ is a single (-19)-curve.

(2) $\Delta_j (n_1 + 1 \le j \le n_1 + n_2)$ is a chain of one (-2)-curve Δ_{j1} and one (-10)-curve Δ_{j2}.

(3) $\Delta_k (n_1 + n_2 + 1 \le k \le n_1 + n_2 + n_3)$ is a chain of two (-2)-curves Δ_{k1}, Δ_{k2} and one (-7)-curve Δ_{k3} with $(\Delta_{ka}, \Delta_{k,a+1}) = 1\,(a = 1, 2)$.

(4) $\Delta_r (n_1 + n_2 + n_3 + 1 \le r \le n_1 + \cdots + n_4)$ is a chain of one (-4)-curve Δ_{r1} and one (-5)-curve Δ_{r2}.

(5) $\Delta_s (n_1 + \cdots + n_4 + 1 \le s \le n_1 + \cdots + n_5)$ is a chain of five (-2)-curves $\Delta_{s1}, \dots \Delta_{s5}$ and one (-4)-curve Δ_{s6} with $(\Delta_{sa}, \Delta_{s,a+1}) = 1\,(1 \le a \le 5)$.

(6) $\Delta_t (n_1 + \cdots + n_5 + 1 \le t \le n_1 + \cdots + n_6)$ is a chain of one (-2)-curve Δ_{t1}, one (-4)-curve Δ_{t2} and one (-3)-curve Δ_{t3} with $(\Delta_{ta}, \Delta_{t,a+1}) = 1\,(a = 1, 2)$.

(7) $\Delta_u (n_1 + \cdots + n_6 + 1 \le u \le n_1 + \cdots + n_7)$ is a chain of two (-2)-curves Δ_{u1}, Δ_{u3} and two (-3)-curves Δ_{u2}, Δ_{u4} with $(\Delta_{ua}, \Delta_{u,a+1}) = 1\,(1 \le a \le 3)$.

(8) $\Delta_v (n_1 + \cdots + n_7 + 1 \le v \le n_1 + \cdots + n_8)$ is a chain of eight (-2)-curves $\Delta_{v1}, \dots, \Delta_{v8}$ and one (-3)-curve Δ_{v9} with $(\Delta_{va}, \Delta_{v,a+1}) = 1\,(1 \le a \le 8)$.

(9) $\Delta_w (n_1 + \cdots + n_8 + 1 \le w \le m_0)$ is a chain of five (-2)-curves $\Delta_{w1}, \Delta_{w2}, \Delta_{w3}, \Delta_{w5}, \Delta_{w6}$ and one (-3)-curve Δ_{w4} with $(\Delta_{wa}, \Delta_{w,a+1}) = 1\,(1 \le a \le 5)$.

(10) $\Delta_b (m_0 + 1 \le b \le c)$ is a single (-38)-curve.

We can check that $f^*(K_{\bar{V}}) \equiv K_V + D^*$ with $D^* =$

$$\frac{17}{19}\sum_i \Delta_i + \frac{8}{19}\sum_j (\Delta_{j1} + 2\Delta_{j2}) + \frac{5}{19}\sum_k (\Delta_{k1} + 2\Delta_{k2} + 3\Delta_{k3}) +$$

$$\frac{1}{19}\sum_r (13\Delta_{r1} + 14\Delta_{r2}) + \frac{2}{19}\sum_s (\Delta_{s1} + 2\Delta_{s2} + 3\Delta_{s3} + 4\Delta_{s4} + 5\Delta_{s5} + 6\Delta_{s6}) +$$

$$\frac{1}{19}\sum_t (7\Delta_{t1} + 14\Delta_{t2} + 11\Delta_{t3}) + \frac{1}{19}\sum_u (6\Delta_{u1} + 12\Delta_{u2} + 11\Delta_{u3} + 10\Delta_{u4}) +$$

$$\frac{1}{19}\sum_v (\Delta_{v1} + 2\Delta_{v2} + 3\Delta_{v3} + 4\Delta_{v4} + 5\Delta_{v5} + 6\Delta_{v6} + 7\Delta_{v7} + 8\Delta_{v8} + 9\Delta_{v9}) +$$

$$\frac{1}{19}\sum_w (3\Delta_{w1} + 6\Delta_{w2} + 9\Delta_{w3} + 12\Delta_{w4} + 8\Delta_{w5} + 4\Delta_{w6}) + \frac{18}{19}\sum_b \Delta_b.$$

By Proposition 1.3, we have $\rho(\bar{V}) = c - 1$ and

$$24 = c + \rho(U) - \rho(\bar{U}) + 19 \times 1 = c + m_1 + 19.$$

So, we obtain:

(9.1) $$5 = c + m_1 = 2m_1 + \sum_{i=1}^{9} n_i.$$

On the other hand, as in Theorem 3.1, we can compute as follows:

$$-\frac{1}{19}(289n_1 + 128n_2 + 75n_3 + 68n_4 + 24n_5$$
$$+ 39n_6 + 22n_7 + 9n_8 + 12n_9 + 648m_1) =$$
$$(D^*)^2 = (K_V^2) = 11 - c$$
$$- (n_1 + 2n_2 + 3n_3 + 2n_4 + 6n_5 + 3n_6 + 4n_7 + 9n_8 + 6n_9 + m_1) \quad \textit{and}$$
$$0 = \frac{1}{6}(19(11 - c) + 629m_1)$$
$$+ 45n_1 + 15n_2 + 3n_3 + 5n_4 - 15n_5 - 3n_6 - 9n_7 - 27n_8 - 17n_9 =$$
$$\frac{1}{6}(19(11 - c) + 647m_1) - 3c$$
$$+ 48n_1 + 18n_2 + 6n_3 + 8n_4 - 12n_5 - 6n_7 - 24n_8 - 14n_9.$$

Dividing the latter equality by 2 and using (9.1), the following equality can be obtained:

(9.2) $$2 + 57m_1 + 24n_1 + 9n_2 + 3n_3 + 4n_4 - 6n_5 - 3n_7 - 12n_8 - 7n_9 = 0.$$

This equality and the equality (9.1) imply

$$0 \geq 2 + 57m_1 - 12\sum_{i=1}^{9} n_i = 2 + 57m_1 - 12(5 - 2m_1) = -58 + 81m_1 > 0.$$

This is a contradiction.

Therefore, $\bar{U}$ is nonsingular. Then $\bar{V}$ is described in [2, Theorem 5.1]. With the help of Lemma 1.2, the second assertion of Theorem 9.1 is proved there. This proves Theorem 9.1.

In [2, Example 5.8], we gave an example of log Enriques surface (V', D') of index 19 whose canonical covering admits at least one singularity of multiplicity ≥ 3.

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