# LOGARITHMIC INTERTWINING OPERATORS AND GENUS-ONE CORRELATION FUNCTIONS

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## ABSTRACT OF THE DISSERTATION


# Logarithmic Intertwining Operators and Genus-One Correlation Functions

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We develop the theory of modular invariance for logarithmic intertwining operators. We construct and study genus-one correlation functions for logarithmic intertwining operators between generalized modules over a quasi-rational vertex operator algebra $V$. We consider generalized $V$-modules which admit a right action of some associative algebra $P$, and intertwining operators between modules in this class which commute with the action of $P$ ($P$-intertwining operators). We obtain duality properties, i.e., suitable associativity and commutativity properties, for $P$-intertwining operators. Using the concept of pseudotrace introduced by Miyamoto, we define formal $q$-traces of products of $P$-intertwining operators, and obtain certain identities for these formal series. This allows us to show that the formal $q$-traces satisfy a system of differential equations with regular singular points, and therefore are absolutely convergent in a suitable region and can be extended to yield multivalued analytic functions, called genus-one correlation functions. Furthermore, we show that the space of solutions of these differential equations is invariant under the action of the modular group. We obtain a characterization of symmetric functions on bimodules over associative algebras in terms of pseudotraces of certain "bimodule actions". We conclude by sketching the steps by which these results can be used to obtain a full modular invariance theorem for the genus-one correlation

functions at least when the central charge is not 0. This modular invariance generalizes the full modular invariance theorem by Huang in the rational case. Miyamoto was the first to obtain a partial result that does not involve logarithmic intertwining operators or even intertwining operators. This modular invariance has been a conjecture for many years.

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# Dedication

To Ping and Sofia, and to my parents, Sergio and Donatella.

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# Chapter 1

# Introduction

## 1.1 Background, motivations and description of results

The theory of vertex operator algebra arose independently in mathematics and physics and has been providing deep and remarkable connections between different fields. In mathematics, one of its most spectacular applications was the construction of the "moonshine module", a vertex operator algebra (usually denoted by $V^\natural$) whose group of automorphisms is the Monster group $\mathbb{M}$, the largest sporadic finite simple group. Noticing patterns relating the dimensions of irreducible modules for the Monster and the Fourier expansion of the modular function $J(q)$, McKay and Thompson conjectured the existence of a "natural" infinite dimensional graded module $V = \coprod_{n=-1}^{\infty} V_n$ for $\mathbb{M}$ whose graded dimension

$$\sum_{n=-1}^{\infty} \dim(V_n) q^n$$

is given exactly by $J(q)$. Additionally, Conway and Norton conjectured that for any element $g$ in the Monster, the series

$$\sum_{n\in\mathbb{Z}} \mathrm{tr}\ g|_{V_n} q^n$$

is the Fourier expansion of a generator of the field of modular functions for some genus zero subgroup of $SL_2(\mathbb{R})$. Frenkel, Lepowsky and Meurman constructed a module $V^\natural$ for the Monster group in [FLM], proving the McKay-Thompson conjecture and introducing the notion of vertex operator algebra, a variant of Borcherds' notion of vertex algebra ([B]). The full Conway-Norton conjecture for $V^\natural$ was later proved by Borcherds.

The connection between the theory of vertex operator algebras and the theory of modular functions has deep roots and the solution of the Moonshine conjecture is just a part of it. In his Ph.D. thesis [Z] Zhu obtained another general modular invariance result. Considering a class of "rational" vertex operator algebras satisfying a certain cofiniteness condition, Zhu studied traces of products of $n$ vertex operators associated to irreducible representations, and showed that these formal traces converge and the functions thus obtained (called $n$-point genus-one correlation functions) form a space invariant under the action of the modular group; as direct consequence, he established the modular invariance for the spaces of functions spanned by the graded dimension of the irreducible modules. Zhu's results were later extended by Dong, Li and Mason in [DLM2] to include twisted representations; and in [Miy1], Miyamoto considered traces of products of module maps and at most one intertwining operator.

All these results rely heavily on the use of the commutator formula to obtain recurrence relations for the $n$-point genus-one correlation function in terms of the $n-1$-point functions. Since this formula is not available for general intertwining operators, the methods do not generalize to product of more than one intertwining operator. In [H2], Huang overcame this difficulty and proved a full modular invariance theorem; he used commutativity and associativity for intertwining operators to obtain a system of "modular" differential equations, and to obtain genus-one commutativity and associativity properties. This modular invariance result is a crucial ingredient in other important works by Huang, including his proof of the Verlinde conjecture and the rigidity and modularity of the vertex tensor category (see [H4], [H5]) for "rational" vertex operator algebras.

In [M1] Milas considered a class of weak modules for non-rational vertex operator algebras, called "logarithmic modules". These are modules on which the operator $L(0)$ does not necessarily act semisimply, but can be expressed as direct sum of generalized eigenspaces for $L(0)$. Moreover, he introduced and studied "logarithmic intertwining operators" between these modules, that is, intertwining operators which involve (integral) powers of $\log x$ in addition to powers of the formal variable $x$. The theory of these kinds of modules and intertwining operators has since been extended (see [HLZ1]-[HLZ8]) and interesting classes of such modules have been constructed (see for instance [M2], [AM1]-[AM3]).

Huang conjectured that a full modular invariance result should hold for such classes of modules, and that it should play an important role in the study of the properties of logarithmic modules. Before this conjecture was explicitly formulated, a partial result generalizing Zhu's result in the context of logarithmic modules was obtained first by Miyamoto [Miy2], assuming only a cofiniteness condition for the vertex operator algebra $V$ and infinite dimensionality of all nonzero $V$-modules (an assumption which is used but not explicitly mentioned in Miyamoto's paper, as pointed out in [M2],[AN]). The main new idea is the use of a generalization of ordinary matrix traces, called "pseudotraces", to construct additional genus-one correlation functions; pseudotraces were successively studied by Arike in [Ar], who obtained a characterization in terms of projective bases (or coordinate systems) for projective modules over associative algebras. In [AM4], Adamović and Milas considered the graded dimensions of modules of certain non-rational vertex operator (super)algebras, and proved modularity of the differential equations these graded dimensions satisfy.

In this thesis, we obtain results that will lead us to a full modular invariance result for logarithmic intertwining operators in the sense of [H2]. We study pseudotraces of products of intertwining operators and genus-one correlation functions in an attempt to achieve such result. Our first concern is to construct genus-one correlation functions from products of intertwining operators; to do so, we are naturally led to consider pseudotraces of products of intertwining operators. In order for the pseudotrace to be well defined, we consider logarithmic modules which admit a right action of some associative algebra, and logarithmic intertwining operators whose products commute with this action. We then develop tools to study these "formal traces"; in particular, we formulate suitable associativity and commutativity statements for this kind of intertwining operator in Theorem 3.5.3 and Theorem 3.5.4.

Using these properties, we can verify that many identities for the formal traces in the semisimple case carry over to the logarithmic setting; however, we see that these traces satisfy a more complicated system of differential equations than the one in [H2] (Proposition 4.1.14). Nonetheless it is still possible to prove convergence of these formal series to multivalued analytic functions (the "genus-one correlation functions") and modular invariance of the space of solutions of the system of differential equations, in Proposition 4.2.5.

At this point the only step left to prove is modular invariance of the space spanned by the genus-one correlation function. Using duality properties, we reduce this problem to the case of the 1-point functions; we then study the "lower coefficients" of the formal series expansion. These coefficients yield a symmetric function on a certain bimodule over some associative algebra; therefore we need to obtain a characterization of such symmetric functions which would allows us to "construct" appropriate intertwining operators whose pseudotraces match the given coefficient.

We achieve this by considering pseudotraces of certain "bimodule actions" (Theorem 2.3.5) and using the results by Huang and Yang in [HY]. We conclude by outlining a proof of a modular invariance theorem for logarithmic intertwining operators, leaving details for a separate publication. The result applies to vertex algebras with nonzero central charge satisfying the $C_2$-cofiniteness condition, and in particular to the modules for the triplet vertex algebra $\mathcal{W}(p)$, and the vertex algebra $\mathcal{W}_{p,q}$ with $(p,q) \neq (2,3)$. We believe this result could have impact in the study of these algebras and their modules.

## 1.2 Summary of results

We study genus-one correlation functions for logarithmic intertwining operators. In Chapter 2 we introduce the pseudotraces, that is, symmetric functions on the ring $\mathrm{End}_P(U)$ for a finitely generated projective right $P$-module $U$ over some associative algebra $P$. These are generalizations of ordinary matrix traces, and have similar properties, including invariance under cyclic permutations. Specifically, given a right projective $P$-module $U$, we know that there exists a projective basis for $U$, that is, a set of elements $\{u_i\}_{i=1}^n$ in $U$ and a set of $P$-linear homomorphisms $\{\alpha_i\}_{i=1}^n \subseteq \mathrm{Hom}_P(U,P)$ such that for any element $u \in U$, $u = \sum_{i=1}^n u_i\alpha_i(u)$. If $\phi$ is a symmetric linear function on $P$, for $\alpha \in \mathrm{End}_P(U)$ we define the pseudotrace associated with $U$ and $\phi$ by

$$\phi_U(\alpha) = \phi\left(\sum_{i=1}^n \alpha_i \circ \alpha(u_i)\right).$$

Let $A$ be an associative algebra; it has been shown ([Ar]) that any symmetric function on $A$ can be expressed as sum of pseudotraces for appropriate symmetric (or Frobenius) algebras $P_i$ equipped with non-degenerate symmetric functions $\phi_i$, and $A$-$P_i$-bimodules $U_i$, projective as right $P$-modules. In order to obtain modular invariance of the genus-one correlation functions, we need a similar characterization of symmetric functions on bimodules over associative algebras. A symmetric function on an $A$-$A$-bimodule $M$ is a linear function $L : M \to \mathbb{C}$ such that

$$L(am) = L(ma)$$

for all $a \in A$ and $m \in M$. For any algebra $A$ and any $A$-$A$-bimodule $M$, we use pseudotraces to construct symmetric linear functions on the bimodule $M$ as follows: we consider an associative algebra $P$ equipped with a symmetric linear function $\phi$, and an $A$-$P$-bimodule $U$, projective as right module over $P$; then, given any homomorphism $f$ of $A$-$P$-bimodules from $W \otimes_A U$ in $U$, we have an $A$-$A$-bimodule homomorphism $T_f : M \to \mathrm{End}_P(U)$. Then the map $\phi_U^f : M \to \mathbb{C}$ defined by

$$\phi_U^f = \phi_U \circ T_f$$

is a symmetric linear function on the bimodule $M$. We show that any symmetric linear map on a bimodule $M$ can be expressed as sum of functions $\phi_{U_i}^{f_i}$ for appropriate symmetric algebras $(P_i, \phi_i)$, $A$-$P_i$-bimodules $U_i$, and homomorphisms $f_i$.

In Chapter 3 we recall concepts from the theory of vertex operator algebras and their modules. We will deal with grading restricted generalized $V$-modules, that is, weak modules $W$ which are direct sum of the generalized eigenspaces for the operator $L(0)$,

$$W = \coprod_{n \in \mathbb{C}} W_{[n]}$$

such that $W_{[n]}$ is finite dimensional for any $n \in \mathbb{C}$, and $W_{[n]} = 0$ for $\Re(n)$ sufficiently negative. A crucial condition for obtaining differential equations for genus-one correlation function is the $C_2$-cofiniteness condition, introduced first in [Z]: we shall say that a $V$-module $W$ satisfies the $C_2$-cofiniteness condition if the space $C_2(W)$ spanned by the set $\{v_{-2}w | v \in V,\ w \in W\}$ has finite codimension in $W$. We will consider vertex operator algebras whose grading restricted generalized modules all satisfy this condition.

We then define the formal $q$-traces of products of logarithmic intertwining operators; for a fixed vertex operator algebra $V$ and some associative algebra $P$, we consider $V$-modules $\tilde{W}_i$, $i = 1, \dots, n$ equipped with a right action of the algebra $P$ such that $\tilde{W}_n$ is projective as right $P$-module; if $\mathcal{Y}_i$ are intertwining operators of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, $i = 1, \dots, n$, (where we take $\tilde{W}_0 = \tilde{W}_n$) such that for all $i = 1, \dots, n$, $w_i \in W_i$, $\tilde{w}_i \in \tilde{W}_i$ and $p \in P$,

$$\mathcal{Y}_i(w, x)(\tilde{w}p) = (\mathcal{Y}_i(w, x)\tilde{w})p,$$

(we will call intertwining operators which satisfy this property $P$- intertwining operators) then the product $\mathcal{Y}_1(w_1, x_1) \dots \mathcal{Y}_n(w_n, x_n)$ is an element of

$$\mathrm{End}_P(\tilde{W}_n)\{x_1, \dots, x_n, \log x_1, \dots, \log x_n\}$$

and it is thus possible to evaluate the pseudotrace

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(w_1, x_1) \dots \mathcal{Y}_n(w_n, x_n) q^{L(0)}.$$

In particular we will consider a map $\mathcal{U}(1) : W \to W[x]$ and study properties of formal $q$-traces obtained by taking pseudotraces of products of *geometrically modified* logarithmic intertwining operators ([H2])

$$\mathcal{Y}(\mathcal{U}(x)w, x),$$

the first goal is to prove absolute convergence of such $q$-traces.

In Section 3.3 we recall some notions from the theory of elliptic functions and modular forms: in particular, we recall the Taylor expansion of the Weierstrass elliptic function $\wp$ and its derivatives, and the Fourier ($q$-expansion) of the Eisenstein series. These expansions, considered as formal power series, will appear as coefficients in identities for the formal $q$-traces and in the system of differential equations for the genus-one correlation function.

In Section 3.4 we derive identities for the formal $q$-traces; these identities have the same shape as the ones found in [H2], and the main tools used in this section are associativity and commutativity of intertwining operators. It is therefore necessary to obtain a formulation of these duality properties to use in the present context. In

Section 3.5, under the assumptions used in [HLZ7], we state and prove suitable commutativity and associativity properties for $P$-intertwining operators; we recall the required background from the theory of tensor product for modules of vertex operator algebras ([HLZ1]-[HLZ8]) in Section 3.2.

In Chapter 4 we use the identities obtained in Section 3.4 to obtain differential equation for the formal $q$-traces. We obtain a system of differential equations for which the series

$$\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})L(0)_n^{i_1}w_1, q_{z_1})\cdot\ldots\cdot\mathcal{Y}_n(\mathcal{U}(q_{z_n})L(0)_n^{i_n}w_n, q_{z_n})q^{L(0)-\frac{c}{24}},$$

$i_j\in\mathbb{N}$, $j=1,\ldots,n$ are solutions (here $L(0)_n$ denotes the locally nilpotent part of the operator $L(0)$). Due to the nonsemisimplicity of the operator $L(0)$, the system is not decoupled, but nonetheless the singular points in the variable $q$ are regular, and thus one can prove absolute convergence of the formal $q$-traces in a suitable domain. We then prove modular invariance for the solutions of the system of differential equations in Section 4.2. We consider a space of vector valued functions in the variables $z_1,\ldots,z_n$ and $\tau$ and we denote the components of these vector valued functions by $\phi_{i_1,\ldots,i_n}(z_1,\ldots,z_n;\tau)$ for $i_j\in\mathbb{N}$, $j=1,\ldots,n$. The solutions of the system of differential equations are naturally elements of this space. Then, for $g\in SL_2(\mathbb{Z})$,

$$g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

we define the action of $g$ on $\phi_{i_1,\ldots,i_n}$ by

$$\begin{aligned}&(g\phi_{i_1,\ldots,i_n})(z_1,\ldots,z_n;\tau)\\ &\quad=\left(\frac{1}{\gamma\tau+\delta}\right)^a\sum_{j_1=0}^{\infty}\cdots\sum_{j_n=0}^{\infty}\frac{\prod_{k=1}^n(\log(\gamma\tau+\delta))^{j_k}}{j_1!\cdots j_n!}\phi_{i_1+j_1,\ldots,i_n+j_n}\left(z_1',\ldots,z_n';\tau'\right)\end{aligned}$$

with $z'=\frac{z}{\gamma\tau+\delta}$ and $\tau'=\frac{\alpha\tau+\beta}{\gamma\tau+\delta}$, and where $a$ is the (rational) number $\mathrm{wt}\ w_1+\ldots+\mathrm{wt}\ w_n$. We then prove that the space of solutions of our differential equations is invariant under this action.

# Chapter 2

# Associative algebras and bimodules

## 2.1 Hattori-Stallings trace

In this section, we recall useful properties of projective modules over an associative algebra $A$ over $\mathbb{C}$. For additional background, see [AF], [Br]. We introduce the *Hattori-Stallings* trace, a function

$$\mathrm{tr}_M : \mathrm{End}_A(M) \to A/[A, A]$$

defined on the ring of $A$ endomorphisms of a right projective $A$-module $M$, with $[A, A]$ the linear span (over the complex field) of the set of commutators in $A$. The Hattori-Stallings trace will serve as the basic building block to construct symmetric linear functions on $\mathrm{End}_A(M)$.

Fix a finite dimensional associative unital algebra $A$ throughout this chapter.

Let $M$ be a right $A$-module; we equip $\mathrm{Hom}_A(M, A)$ with a left $A$-module structure with

$$(a\alpha)(m) = \alpha(ma)$$

for all $a \in A, \alpha \in \mathrm{Hom}_A(M, A)$ and $m \in M$. For a second right $A$-module $N$, we denote by $\tau_{N,M}$ the natural group homomorphism

$$\tau_{N,M} : M \otimes_A \mathrm{Hom}_A(N, A) \to \mathrm{Hom}_A(N, M)$$

given by $\tau_{N,M}(m_1 \otimes \alpha) : n \mapsto m_1\alpha(n)$, and let $\tau_M = \tau_{M,M}$. Moreover we have a "contraction" homomorphism

$$\begin{aligned} \pi : M \otimes_A \mathrm{Hom}_A(M, A) &\to A \\ m \otimes \alpha &\mapsto \alpha(m) \end{aligned}$$

**Definition 2.1.1.** A right $A$-module $M$ is called *projective* if every epimorphism of right modules $N \xrightarrow{\sigma} M \to 0$ splits, i.e., there exists a homomorphism $i : M \to N$ such that $\sigma \circ i = 1_M$.

**Proposition 2.1.2.** *Let $M$ be a right $A$-module; the following conditions are equivalent:*

*(i) $M$ is projective;*

*(ii) $M$ is direct summand of a free $A$-module;*

*(iii) For any epimorphism of right $A$-modules $N_1 \xrightarrow{g} N_2 \to 0$ and module map $\gamma : M \to N_2$, there exists a map $\bar{\gamma} : M \to N_1$ such that $g \circ \bar{\gamma} = \gamma$.*

*Proof.* $(i) \Rightarrow (ii)$: $M$ is a quotient of a free module: since $M$ is projective, the epimorphism splits.
$(ii) \Rightarrow (iii)$: Let $M$ be direct summand of the free module $F$ and consider the projection $p : F \to M$ and the inclusion map $i : P \to F$. Then there exists a map $f : F \to N_1$ such that $g \circ f = \gamma \circ p$. Now take $\bar{\gamma} = f \circ i$
$(iii) \Rightarrow (i)$ Let $g : N \to M$ an epimorphism. Then taking $\gamma = 1_M$, $g \circ \bar{\gamma} = 1_M$, so the sequence $N \to M \to 0$ splits. ∎

Let $M$ be a finitely generated right module. A *projective basis* for $M$ is a pair of sets $\{m_i\}_{i=1}^n \subseteq M$, $\{\alpha_i\}_{i=1}^n \subseteq \mathrm{Hom}_A(M, A)$ such that for all $m \in M$,

$$m = \sum_{i=1}^{n} m_i \alpha(m).$$

**Proposition 2.1.3.** *The module $M$ has a projective basis if and only if it is projective.*

*Proof.* $(\Rightarrow)$ Let $\{m_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$ be a projective basis for $M$. Consider the free module $F$ on the generators $x_1, \ldots, x_n$ and the surjective homomorphism $\pi : F \to M$ defined by $\pi(x_i) = m_i$, for $i = 1, \ldots, n$. Then $i : M \to F$, $i(m) = \sum_{i=1}^n x_i \alpha_i(m)$ is a section, and therefore $M$ is a direct summand of $F$.

$(\Leftarrow)$ Let $m_1, \ldots, m_n$ be a set of generators for $M$. Consider $F$ as defined above and let $i$ be a section from $M$ to $F$; we have maps $g_i \in \mathrm{Hom}_A(F, A)$ such that for $f \in F$,

$$f = \sum_{i=1}^{n} x_i g_i(f).$$

Define $\alpha_i(m) = g_i(i(m))$; then $\{m_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$ is a projective basis for $M$. ∎

**Proposition 2.1.4.** *Let $M$ be a finitely generated right $A$-module. The following are equivalent:*

*(i) $M$ is projective;*

*(ii) For any right $A$-module $N$, the map $\tau_{N,M} : M \otimes_A \operatorname{Hom}_A(N, A) \to \operatorname{Hom}_A(N, M)$ is an isomorphism.*

*(iii) The map $\tau_M : M \otimes_A \operatorname{Hom}_A(M, A) \to \operatorname{End}_A(M)$ is an isomorphism.*

*Proof.* $(i) \Rightarrow (ii)$: Let $\{m_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$ be a projective basis for $M$. If $f$ is an element of $\operatorname{Hom}_A(N, M)$, then for all $n \in N$,

$$\tau_M\left(\sum_{i=1}^n m_i \otimes \alpha_i \circ f\right)(n) = \sum_{i=1}^n m_i\alpha_i(f(n)) = f(n)$$

which shows that $\tau_{N,M}$ is onto.

Now suppose $\tau_M(\sum_{j=1}^k x_j \otimes f_j) = 0$, for $x_i \in M$ and $f_i \in \operatorname{Hom}_A(N, M)$; rewriting $x_i$ in terms of the basis elements, we obtain

$$\begin{aligned}\sum_{j=1}^k x_j \otimes f_j &= \sum_{j=1}^k \sum_{i=1}^n m_i\alpha_i(x_j) \otimes f_j \\ &= \sum_{i=1}^n m_i \otimes \sum_{j=1}^k \alpha_i(x_j) f_j\end{aligned}$$

Now for all elements $n \in N$,

$$\begin{aligned}\sum_{j=1}^k \alpha_i(x_j) f_j(n) &= \sum_{j=1}^k \alpha_i(x_j f_j(n)) \\ &= \alpha_i\left(\sum_{j=1}^k x_j f_j(n)\right) \\ &= \alpha_i\left(\tau_{N,M}\left(\sum_{j=1}^k x_j \otimes f_j\right)(n)\right) = 0\end{aligned}$$

which shows $\sum_{j=1}^k x_j \otimes f_j = 0$ and thus $\tau_{N,M}$ is injective.

$(ii) \Rightarrow (iii)$: Just take $N = M$.

$(iii) \Rightarrow (i)$: Consider $\sum_{i=1}^{n} m_i \otimes \alpha_i = \tau_M^{-1}(1_M)$; then for all $m \in M$,

$$\sum_{i=1}^{n} m_i \alpha_i(m) = m,$$

which shows that $\{m_i\}_{i=1}^{n}$, $\{\alpha_i\}_{i=1}^{n}$ is a projective basis for $M$, and so by Proposition 2.1.3, $M$ is projective. ∎

Let $[A, A]$ denote the subgroup of $A$ generated by commutators. For a finitely generated projective right module $M$, we will call the following the *Hattori-Stallings trace* of an endomorphism $\alpha \in \mathrm{End}_A(M)$ the element of $A/[A, A]$:

$$\mathrm{tr}_M(\alpha) = \pi(\tau_M^{-1}(\alpha)) \bmod [A, A]$$

**Proposition 2.1.5.** *Let $M$, $N$ be two projective right $A$-modules, and consider homomorphisms $f \in \mathrm{Hom}_A(M, N)$, $g \in \mathrm{Hom}_A(N, M)$. Then*

$$\mathrm{tr}_M g \circ f = \mathrm{tr}_N f \circ g.$$

*Proof.* By Proposition 2.1.4, $\tau_{M,N}^{-1}(f) = \sum_{i=1}^{k} x_i \otimes f_i$ for elements $x_i \in N$, $f_i \in \mathrm{Hom}_A(M, A)$ $i = 1, \ldots, k$, and $\tau_{N,M}^{-1}(g) = \sum_{j=1}^{l} y_j \otimes g_j$ for elements $y_j \in M$ and $g_j \in \mathrm{Hom}_A(N, A)$. Then $\tau_M^{-1}(g \circ f) = \sum_{i,j} y_j g_j(x_i) \otimes f_i$ and

$$\begin{aligned}
\mathrm{tr}_M g \circ f &= \mathrm{tr}_M \sum_{j=1}^{l} y_j \otimes g_j \sum_{i=1}^{k} x_i \otimes f_i \\
&= \mathrm{tr}_M \sum_{i,j} y_j g_j(x_i) \otimes f_i \\
&= \sum_{i,j} f_i(y_j) g_j(x_i) \bmod [A, A] \\
&= \sum_{i,j} g_j(x_i) f_i(y_j) \bmod [A, A] \\
&= \mathrm{tr}_N f \circ g.
\end{aligned}$$

which concludes the proof. ∎

## 2.2 Pseudotraces

In this section we recall the definition of pseudotraces for a right projective $A$-module $M$; pseudotraces are symmetric linear functions on $\mathrm{End}_A(U)$ which where introduced in [Miy2] and further studied in [Ar]; all symmetric linear functions on an algebra $A$ can be expressed as a sum of pseudotraces for suitable representations of $A$.

We will say that a linear function $\phi : A \to \mathbb{C}$ is *symmetric* if $\phi(ab) = \phi(ba)$ for all $a, b \in A$, and denote by $SLF(A)$ the vector space of all such functions; $SLF(A) \simeq (A/[A,A])^*$. Any symmetric linear function defines a symmetric bilinear form

$$\langle\ ,\ \rangle : A \times A \to \mathbb{C}$$

by $\langle a, b\rangle = \phi(ab)$, and $\phi$ is said to be *nondegenerate* if the corresponding bilinear form is nondegenerate. We denote by rad $\phi$ the two sided ideal of $A$

$$\text{rad } \phi = \big\{a \in A \mid \langle a, b\rangle = 0 \quad \text{for all } b \in A\big\}$$

(the radical of $\phi$); the symmetric function $\phi$ is nondegenerate if and only if rad $\phi = \{0\}$. Let $M$ be a finite dimensional projective left $A$-module, $\{m_i\}_{i=1}^n, \{\alpha_i\}_{i=1}^n$ a projective basis.

**Definition 2.2.1.** Let $\phi \in SLF(A)$. The *pseudtrace map* $\phi_M$ on $\text{End}_A(M)$ associated to $\phi$ is the function $\phi_M = \phi \circ \text{tr}_M$; we can express the pseudotrace of an endomorphism $\alpha$ in terms of the projective basis in the following way:

$$\begin{aligned} \phi_M : \text{End}_A(M) &\to \mathbb{C} \\ \alpha &\mapsto \phi\left(\sum_{i=1}^n \alpha_i(\alpha(m_i))\right). \end{aligned}$$

Pseudotrace maps are an extension of regular trace functions, and share similar properties; it is easy to see that the pseudotrace of a product is invariant under cyclic permutation of the factors [AN]. Moreover, it is clear from the definition that the pseudotrace map does not depend on the choice of projective basis for $M$.

**Proposition 2.2.2.** *Let $M_1$ and $M_2$ be two left projective $A$-modules, and consider homomorphisms $\alpha \in \text{Hom}_A(M_1, M_2)$ and $\beta \in \text{Hom}_A(M_2, M_1)$; then*

$$\phi_{M_1}(\beta \circ \alpha) = \phi_{M_2}(\alpha \circ \beta).$$

*Proof.* Follows from symmetry property for the Hattori-Stallings trace (Proposition 2.1.5). ∎

Importantly, for a given associative algebra $A$, the pseudotraces of representations of $A$ span the space $SLF(A)$ of symmetric linear functions on $A$.

**Definition 2.2.3.** Let $A$, $B$ be associative algebras; an *$A$-$B$-bimodule* is a space $M$ that is simultaneously a left $A$-module and a right $B$-module, such that the left action of $A$ commutes with the right action of $B$; i.e., for all $a \in A$, $b \in B$, $m \in M$,

$$a(mb) = (am)b.$$

**Definition 2.2.4.** A *symmetric algebra* (also called *Frobenius algebra*) is an associative algebra equipped with a nondegenerate symmetric function $\phi$.

**Definition 2.2.5.** A *basic algebra* is an algebra $A$ such that $A/J(A)$ is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$ (here $J(A)$ is the *Jacobson radical* of $A$, that is, the intersection of all maximal right ideals of $A$).

**Proposition 2.2.6** ([Ar], [Miy2])**.** *Let $A$ be an associative algebra and $\phi \in SLF(A)$. Then there exist basic symmetric algebras $P_i$ with symmetric linear functions $\phi_i$, and $A$-$P_i$-bimodules $M_i$, projective as right $P_i$-modules, such that*

$$\phi(a) = \sum_{i=1}^{n} \phi_{i\,M_i}(a)$$

*where we consider $a$ as an element of $\mathrm{End}_{P_i}(M_i)$ by left action. Furthermore, if $\nu$ is an element in $\mathrm{rad}\ \phi$, that is,*

$$\phi(\nu a) = 0$$

*for all $a \in A$, then $M_i$ can be chosen to satisfy $\nu M_i = 0$.*

## 2.3 Symmetric functions on bimodules

Here we introduce symmetric functions over $A$-$A$-bimodules. Let $A$ and $P$ be associative algebras; given an $A$-$A$-bimodule $M$, an $A$-$P$-bimodule $U$, and an $A$-$P$ endomorphism $f : M \otimes_A U \to U$, we use pseudotraces to construct symmetric functions over $M$. We prove that all symmetric functions on $M$ can be expressed as sums of such functions.

**Definition 2.3.1.** Let $A$ be an associative algebra, $M$ an $A$-$A$-bimodule. A linear function $\phi : M \to \mathbb{C}$ is called *symmetric* if for all $m \in M$ and $a \in A$,

$$\phi(am) = \phi(ma).$$

Let $U$ be an $A$-$P$-bimodule, projective as a right $P$-module. Let $\phi \in SLF(P)$. The endomorphism ring $\mathrm{End}(U_P)$ is an $A$-$A$-bimodule with actions given by

$$(a \cdot \tau)(u) = a(\tau(u))$$
$$(\tau \cdot a)(u) = \tau(au)$$

for all $a \in A$, $\tau \in \mathrm{End}(U_P)$, $u \in U$.

**Proposition 2.3.2.** *The pseudotrace $\phi_U$ on $\mathrm{End}_P(U)$ is a symmetric linear function of the $A$-$A$-bimodule $\mathrm{End}_P(U)$.*

*Proof.* This is just a consequence of the fact that $\phi_U$ is a symmetric linear function under composition. Note that for all $\tau \in \mathrm{End}_P(U)$ and $a \in A$, $a\tau = L_a \circ \tau$, $\tau a = \tau \circ\ L_a$. (Here $L_a$ is the $P$-endomorphism of $U$ given by the left action of the element $a \in A$.) Then

$$\begin{aligned}\phi_U(a\tau) &= \phi_U(L_a \circ \tau)\\ &= \phi_U(\tau \circ L_a)\\ &= \phi_U(\tau a)\end{aligned}$$

which proves our claim. ∎

Let $M$ be an $A$-$A$-bimodule, and let $\mathrm{Hom}_{A,P}(M \otimes_A U, U)$ be the set of all $A$-$P$-bimodule homomorphisms from $M \otimes_A U$ to $U$, and let $f$ be an element in this space. Then for any $m \in M$, the map

$$\begin{aligned}U &\to U\\ u &\mapsto f(m \otimes u)\end{aligned}$$

is a $P$-endomorphism of $U$; we can then define

$$\begin{aligned}T_f : M &\to \mathrm{End}(U_P)\\ m &\mapsto (u \mapsto f(m \otimes u)).\end{aligned} \tag{2.3.1}$$

**Proposition 2.3.3.** *The map $T_f$ is an $A$-$A$-bimodule homomorphism.*

*Proof.* Let $m \in M$, $a \in A$; then for all $u \in U$,

$$T_f(am)(u) = f(am \otimes u) = af(m \otimes u) = a(T_f(m)(u)) = (aT_f(m))(u)$$

and thus $T_f(am) = aT_f(m)$. Similarly,

$$T_f(ma)(u) = f(ma \otimes u) = f(m \otimes u) = T_f(m)(au) = (T_f(m)a)(u),$$

which proves $T_f(ma) = T_f(m)a$. ∎

We will consider linear maps $\phi_U^f$ defined by

$$\begin{aligned} \phi_U^f : M &\to \mathbb{C} \\ m &\mapsto \phi_U(T_f(m)). \end{aligned} \tag{2.3.2}$$

**Proposition 2.3.4.** *The linear map $\phi_U^f$ is symmetric, i.e., for all $a \in A$ and $m \in M$,*

$$\phi_U^f(am) = \phi_U^f(ma).$$

*Proof.* Using Propositions 2.3.2, 2.3.3,

$$\begin{aligned} \phi_U^f(am) &= \phi_U(T_f(am)) \\ &= \phi_U(aT_f(m)) \\ &= \phi_U(T_f(m)a) \\ &= \phi_U(T_f(ma)) \\ &= \phi_U^f(ma). \end{aligned}$$

∎

We have obtained a map $SLF(P) \otimes \operatorname{Hom}_{A,P}(M \otimes_A U, U) \to SLF(M)$. We now need to show that any elements $\psi$ in $SLF(M)$ can be obtained as a sum of functions $\phi_U^f$. We have the following:

**Theorem 2.3.5.** *Let $M$ be an $A$-$A$-bimodule, and let $\phi \in SLF(M)$. Then for $i = 1, \ldots, n$ there exist basic symmetric algebras $P_i$ equipped with symmetric linear functions $\phi_i$, $A$-$P_i$-bimodules $U_i$ (projective as right $P_i$-modules) and maps $f_i$ in the space* $\mathrm{Hom}_{A,P}(M \otimes_A U_i, U_i)$*, such that for any $m \in M$,*

$$\phi(m) = \sum_{i=1}^{n} (\phi_i)_{U_i}^{f_i}(m).$$

The proof of this Theorem is given in Section 2.4.

**Remark 2.3.6.** It is easy to see that it is enough to prove this for indecomposable bimodules: in fact, suppose $M = M_1 \oplus M_2$ and $\psi = \psi_1 + \psi_2$, where $\psi_i = \psi|_{M_i}$. Suppose there exist algebras $P_i$ with symmetric linear functions $\phi_i$, $A, P_i$-bimodules $U_i$, and functions $g_i \in \mathrm{Hom}_{A,P_i}(M_i \otimes U_i, U_i)$ such that $\psi_i(m_i) = \phi_{U_i}^{g_i}(m_i)$ for all $m_i \in M_i$. Then we can define functions $f_i \in \mathrm{Hom}_{A,P_i}(M \otimes_A U_i, U_i)$ by

$$f_i(m \otimes u) = g_i(m_i \otimes u) \qquad i = 1, 2$$

for all $m = m_1 + m_2$ with $m_i \in M_i$, $i = 1, 2$. Then

$$\phi_{U_i}^{f_i}(m) = \phi_{U_i}^{g_i}(m_i) = \psi_i(m_i) = \psi_i(m)$$

and

$$\psi(m) = \psi(m_1) + \psi(m_2) = \phi_{U_1}^{f_1}(m) + \phi_{U_2}^{f_2}(m).$$

**Remark 2.3.7.** In [Ar], the algebra $P$ is taken to be symmetric; although we will not need to, one can safely relax this hypothesis. In fact, suppose $U$ is a projective right $P$-module, $\phi \in SLF(P)$, and $\{u_i, \alpha_i\}$ is a coordinate system for $U_P$.
Let $I = \mathrm{rad}\phi$; consider the algebra $P/I$ and its symmetric linear function $\bar{\phi}$ defined by $\phi$ on the quotient. Then $U/UI$ is a projective $P/I$-module, and letting

$$\beta_i(u + UI) = \alpha_i(u) + UI,$$

$\{u_i + UI, \beta_i\}$ is a coordinate system for $U/UI$ as a $P/I$-module. Now if $f$ is an element of $\mathrm{Hom}_{A,P}(M \otimes_A U, U)$, we can define $g \in \mathrm{Hom}_{A,P/I}(M \otimes_A U/UI, U/UI)$ by

$$g(m \otimes (u + UI)) = f(m \otimes v) + UI$$

Then clearly $\phi_U^f(m) = \bar{\phi}_{U/UI}^g(m)$ for all $m \in M$.

## 2.4 A square zero extension

In this section we recall the construction of the split square zero extension of an associative algebra by a bimodule and use it to prove Theorem 2.3.5. Let $A$ be an associative algebra over the complex numbers and $M$ an $A$-$A$-bimodule. We consider the following product on the space $\bar{A} = A \oplus M$:

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2),$$

which makes $\bar{A}$ into an algebra which contains a copy of $A$ as a subalgebra and a copy of $W$ as a two sided ideal (this algebra is referred to as the *trivial square zero extension* of $A$ by $M$; see [W]).

*Proof of Theorem 2.3.5.* Let $\phi$ be a symmetric linear function on the bimodule $M$. Then one can extend $\phi$ to a linear function $\bar{\phi}$ on $\bar{A}$ by letting $\bar{\phi}(a, m) = \phi(m)$. The function $\bar{\phi}$ is a symmetric linear function on the algebra $\bar{A}$: in fact,

$$\bar{\phi}((a_1, m_1) \cdot (a_2, m_2)) = \phi(a_1 m_2 + m_1 a_2)$$
$$\bar{\phi}((a_2, m_2) \cdot (a_1, m_1)) = \phi(a_2 m_1 + m_2 a_1)$$

which are equal since $\phi$ is symmetric on the bimodule $M$.

By Proposition 2.2.6, there exist basic symmetric algebras $P_i$ equipped with symmetric linear functions $\phi_i$, and $\bar{A}$-$P_i$-bimodules $M_i$ such that on $\bar{A}$,

$$\bar{\phi} = \sum_{i=1}^{n} \mathrm{tr}_{U_i}^{\phi_i}.$$

In particular, for any $m \in M$,

$$\phi(m) = \bar{\phi}((0, m)) = \sum_{i=1}^{n} \mathrm{tr}_{U_i}^{\phi_i}((0, m)). \tag{2.4.3}$$

where $(0, m)$ on the right-hand side is seen as an element of $\mathrm{End}_{P_i}(U_i)$ by left action of $(0, m)$ on $U_i$, i.e., $\mathrm{tr}_{U_i}^{\phi_i}((0, m))$ is the pseudotrace of the $P_i$-module endomorphism of $U_i$ given by $u \to (0, m)u$.

Consider $U_i$ as a left $A$-module by $au = (a, 0)u$ and for $m \in M$, $u \in U_i$ define

$f_i(m \otimes u) = (0, m)u$. Since

$$\begin{aligned} f_i(ma \otimes u) &= (0, ma)u \\ &= ((0, m) \cdot (a, 0))u \\ &= (0, m)au \\ &= f_i(m \otimes au), \end{aligned}$$

$f_i \in \mathrm{Hom}_{A,P}(M \otimes_A U_i, U_i)$. Then $T_{f_i}(m)$ is the $P_i$ endomorphism given by the left action of $(0, m)$ on $U_i$; therefore, (2.4.3) can be rewritten as

$$\phi(m) = \sum_{i=1}^{n} (\phi_i)^{f_i}_{U_i}(m),$$

which concludes the proof. ∎

**Corollary 2.4.1.** *Suppose $\nu$ is an element in $A$ such that $\phi(\nu m) = 0$ for all $m \in M$. Then the modules $U_i$ can be chosen in such a way that $\nu U_i = 0$ for $i = 1, \ldots, n$.*

*Proof.* Let $\bar{A}$, $\bar{\phi}$ be defined as in the proof of Theorem 2.3.5. Since $\phi(\nu m) = 0$ for all $m \in M$, the element $(\nu, 0)$ belongs to rad $\bar{\phi}$ by definition of $\bar{\phi}$ and product on $\bar{A}$. The result then follows from Proposition 2.2.6. ∎

# Chapter 3

# Logarithmic intertwining operators

## 3.1 Generalized modules and logarithmic intertwining operators

After recalling some notions from logarithmic formal calculus, we recall the definitions of vertex operator algebras and classes of modules. In particular, we will deal with generalized modules (modules which decompose as direct sums of generalized eigenspaces for the operator $L(0)$), and we will consider logarithmic intertwining operators for this class of modules. For more details, see [FLM], [FHL], [LL], [HLZ1]-[HLZ7].

We will denote by $x$, $y$, $q$, $\log x$, $\log y$, $\log q$, $x_1$, $x_2$, $x_3 \ldots$, $\log x_1, \log x_2, \ldots$ independent commuting formal variables. For any set of commuting independent formal variables $X$ and for any vector space $\mathcal{W}$ which does not involve any element of $X$, we denote by $\mathcal{W}\{X\}$ the space of formal series in arbitrary complex powers of the formal variables in $X$. In particular we will consider the space $\mathcal{W}\{x, \log x\}$: an arbitrary element in this space can be written as

$$\sum_{m,n\in\mathbb{C}} w_{n,m} x^n (\log x)^m, \quad w_{n,m} \in \mathcal{W}. \tag{3.1.1}$$

The symbol $\frac{d}{dx}$ denotes the linear map (formal differentiation) defined on $W\{x, \log x\}$ by

$$\begin{aligned}\frac{d}{dx}&\left(\sum_{m,n\in\mathbb{C}} w_{n,m} x^n (\log x)^m\right)\\ &= \sum_{m,n\in\mathbb{C}} ((n+1)w_{n+1,m} + (m+1)w_{n+1,m+1}) x^n (\log x)^m\\ &\left(= \sum_{m,n\in\mathbb{C}} n w_{n,m} x^{n-1} (\log x)^m + \sum_{m,n\in\mathbb{C}} m w_{n,m} x^{n-1} (\log x)^{m-1}\right).\end{aligned} \tag{3.1.2}$$

We will make use of the notiation

$$\log(1-T) = -\sum_{n=1}^{\infty} \frac{T^n}{n}$$
$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!},$$

for any $T$ for which these expressions make sense. Also, for commuting independent formal variables $x$, $y$, we let

$$\log(x+y) = \log x + \log\left(1 + \frac{y}{x}\right) = \log x - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{y}{x}\right)^n.$$

For any formal series in $W\{x, \log x\}$ the following result holds:

**Theorem 3.1.1.** *For $f(x)$ as in (3.1.1), we have*

$$e^{y\frac{d}{dx}} f(x) = f(x+y) \tag{3.1.3}$$

*("Taylor's theorem" for logarithmic formal series) and*

$$e^{yx\frac{d}{dx}} f(x) = f(xe^y). \tag{3.1.4}$$

**Definition 3.1.2.** A *vertex operator algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

satisfying the two grading restrictions conditions $\dim V_{(n)} < \infty$ for all $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for $n$ sufficiently negative, equipped with a linear map

$$V \to (\mathrm{End}(V)[[x, x^{-1}]]$$
$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \tag{3.1.5}$$

and two distinguished vectors, $\mathbf{1} \in V_{(0)}$ (the vacuum vector) and $\omega \in V_{(2)}$ (the conformal vector) satisfying:

1. Lower truncation condition: for all $u, v \in V$,

$$v_n u = 0 \quad \text{for large enough } n$$

2. Vacuum property:

$$Y(\mathbf{1}, x) = 1_V$$

3. Creation property

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v$$

4. Jacobi identity: for all $u, v \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(v, x_1)Y(u, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(u, x_2)Y(v, x_1) =$$
$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y(Y(v, x_0)u, x_2)$$

5. Virasoro algebra relations: Let

$$Y(\omega, x) = \sum_{n \in Z} L(n)x^{-n-2};$$

then

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c;$$

where $c \in \mathbb{C}$ is called the *central charge* of $V$;

6. $L(0)$ grading: for $v \in V_{(n)}$,

$$L(0)v = nv;$$

(here $n$ is called the *weight* of $v$ and it is denoted by wt $v$)

7. $L(-1)$ derivative property

$$Y(L(-1)v, x) = \frac{d}{dx}Y(v, x).$$

This concludes the definition.

In what follows, unless otherwise mentioned, we will fix a vertex operator algebra $V$ such that $V_{(n)} = 0$ whenever $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$. For a vertex operator algebra $V$, we set

$$C_n(V) = \text{span}\{v_{-n}u \mid v \in \coprod_{n>0} V_{(n)}, u \in V\}.$$

**Definition 3.1.3.** We say that $V$ is $C_n$-cofinite if

$$\dim V/C_n(V) < \infty.$$

**Definition 3.1.4** ([H2])**.** A *weak $V$-module* is a vector space $W$ equipped with a linear map

$$\begin{aligned} V &\to (\mathrm{End}(W)[[x, x^{-1}]] \\ v &\mapsto Y_W(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

such that the following properties hold:

1. Lower truncation condition: for all $u, v \in V$,

$$v_n u = 0 \;\text{ for large enough } n$$

2. Vacuum property:

$$Y_W(\mathbf{1}, x) = 1_W$$

3. Jacobi identity: for all $u, v \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(v, x_1) Y_W(u, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(u, x_2) Y_W(v, x_1) =$$
$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) Y_W(Y(v, x_0)u, x_2)$$

**Definition 3.1.5** ([Z])**.** An $\mathbb{N}$-*gradable weak module* is a weak $V$-module equipped with an $\mathbb{N}$-grading $W = \coprod_{n\in N} W_{(n)}$ such that for all homogeneous $v \in V$, $w \in W$ and $n \in \mathbb{Z}$,

$$v_n w \in W_{(m+\mathrm{wt}\ v-n-1)}$$

if $w$ belongs to the homogeneous subspace $W_{(m)}$.

**Definition 3.1.6** ([M1], [HLZ1])**.** A $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ equipped with a linear map

$$\begin{aligned} Y_W : V \otimes W &\to W((x)) \\ v \otimes w &\mapsto Y_W(v, x)w \end{aligned}$$

is called a *generalized $V$-module* if for $n \in \mathbb{C}$, the homogeneous subspaces $W_{[n]}$ are the generalized eigenspaces of $L(0) = \mathrm{Res}_x x Y_W(\omega, x)$ with eigenvalues $n$, that is, for $n \in \mathbb{C}$, $w \in W_{[n]}$, there exists $K \in \mathbb{Z}_+$, depending on $w$, such that $(L(0) - n)^K w = 0$. For $w \in W_{[n]}$, we denote the generalized eigenvalue $n$ by $\mathrm{wt}\, w$.

We define *homomorphisms* (or *module maps*) and *isomorphisms* between generalized $V$-modules, *generalized $V$-submodules*, and *quotient generalized $V$-modules* in the obvious ways.

**Definition 3.1.7.** An *irreducible generalized $V$-module* is a generalized $V$-module $W$ such that there is no generalized $V$-submodule of $W$ that is neither 0 nor $W$ itself. A *lower bounded generalized $V$-module* is a generalized $V$-module $W$ such that $W_{[n]} = 0$ when $\Re(n)$ is sufficiently negative. We say that lower-bounded generalized $V$-module $W$ *has a lowest conformal weight*, or for simplicity, $W$ *has a lowest weight* if there exists $n_0 \in \mathbb{C}$ such that $W_{[n_0]} \neq 0$ but $W_{[n]} = 0$ when $\Re(n) < \Re(n_0)$ or $\Re(n) = \Re(n_0)$ but $\Im(n) \neq \Im(n_0)$. In this case, we call $n_0$, $W_{[n_0]}$ and elements of $W_{[n_0]}$ the *lowest conformal weight* or *lowest weight* of $W$, the *lowest weight space* or *lowest weight space* of $W$ and *lowest conformal weight vectors* or *lowest weight vectors* of $W$, respectively. A *grading restricted generalized $V$-module* is a generalized $V$-module $W$ such that $W$ is lower bounded and $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$. An *(ordinary) $V$-module* is a generalized $V$-module $W$ such that $W$ is grading restricted and $W_{[n]} = W_{(n)}$ for $n \in \mathbb{C}$, where for $n \in \mathbb{C}$, $W_{(n)}$ are the eigenspaces of $L(0)$ with eigenvalues $n$. An *irreducible $V$-module* is a $V$-module such that it is irreducible as a generalized $V$-module. A *generalized $V$-module of length $l$* is a generalized $V$-module $W$ such that there exist generalized $V$-submodules $W = W_1 \supset \cdots \supset W_{l+1} = 0$ such that $W_i/W_{i+1}$ for $i = 1, \ldots, l$ are irreducible $V$-modules. A *finite length generalized $V$-module* is a generalized $V$-module of length $l$ for some $l \in \mathbb{Z}_+$. *Homomorphisms* and *isomorphisms* between lower-bounded, grading-restricted or finite length generalized $V$-modules are homomorphisms and isomorphisms between the underlying generalized $V$-modules.

For a generalized module $W$, the *formal completion* of $W$ is the space

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]},$$

and for $n \in \mathbb{C}$ we denote the projection from $\overline{W}$ to $W_{[n]}$ by $\pi_n$.

**Definition 3.1.8.** Let $(W_1, Y_1)$, $(W_2, Y_2)$ and $(W_3, Y_3)$ be generalized modules for a vertex operator algebra $V$. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1\, W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3[\log x]\{x\},$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1}(\log x)^k \in W_3[\log x]\{x\}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$${w_{(1)}}^{\mathcal{Y}}_{n+m;\,k} w_{(2)} = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large, independently of } k;$$

the *Jacobi identity*:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) & Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ & -x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ = x_2^{-1}\delta & \left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned} \tag{3.1.6}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the first term on the left-hand side is meaningful because of the lower truncation condition) and the $L(-1)$-*derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x).$$

We will denote the space of all logarithmic intertwining operators of type $\binom{W_3}{W_1 W_2}$ by $\mathcal{V}^{W_3}_{W_1 W_2}$.

Note that if the three modules $W_1, W_2, W_3$ are ordinary modules, then all logarithmic intertwining operators are in fact ordinary intertwining operator, i.e., there are no logarithmic terms.

For a generalized $V$-module $W$, we consider the semisimple part of the operator $L(0)$, denoted by $L(0)_s$; also denote by $L(0)_n$ the locally nilpotent part $L(0) - L(0)_s$.

Then $L(0)_n$ is a module endomorphism of $W$ (i.e., it commutes with the action of $V$ on $W$):

**Lemma 3.1.9.** *Let $\mathcal{Y}(x,w)$ be an intertwining operator; then*

$$L(0)_n\mathcal{Y}(x,w) - \mathcal{Y}(x,w)L(0)_n = 0$$

**Definition 3.1.10.** Let $W$ be a generalized module for a vertex operator algebra. We define

$$x^{\pm L(0)} : W \to W\{x\}[\log x] \subset W[\log x]\{x\}$$

by

$$x^{\pm L(0)} = x^{\pm L(0)_s} e^{\pm \log x (L(0)-L(0)_s)}$$

**Lemma 3.1.11.** *Using the same notation as above, we have*

$$\frac{d}{dx}x^{L(0)} = L(0)x^{L(0)-1}.$$

**Proposition 3.1.12.** *Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$ and let $w \in W_1$. Then*

*(a)*

$$e^{yL(-1)}\mathcal{Y}(w,x)e^{-yL(-1)} = \mathcal{Y}(e^{yL(-1)}w,x) = \mathcal{Y}(w,x+y)$$

*(b)*

$$y^{L(0)}\mathcal{Y}(w,x)y^{-L(0)} = \mathcal{Y}(y^{L(0)}w,xy)$$

*(c)*

$$e^{yL(1)}\mathcal{Y}(w,x)e^{-yL(1)} = \mathcal{Y}(e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w, x(1-yx)^{-1}).$$

## 3.2 Tensor product of modules and associativity of intertwining operators

In this section, we recall the notion of $P(z)$-tensor product for modules of vertex operator algebras introduced in [HL1]-[HL3], [H3] (and then generalized to the case of logarithmic modules in [HLZ1]-[HLZ8]), and some related results that we will need later; in particular, we will state associativity for intertwining operators.

Related to the concept of intertwining operator is that of *intertwining map*: a $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$ is a linear function

$$I_z : W_1 \otimes W_2 \to \overline{W}_3$$

satisfying the following conditions: the *lower truncation condition*: for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large;}$$

and the *Jacobi identity*: for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) & Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)}) \\ &= z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) I(Y_1(v, x_0) w_{(1)} \otimes w_{(2)}) \\ &\quad + x_0^{-1}\delta\left(\frac{-z + x_1}{x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}). \end{aligned}$$

Let $\mathcal{Y} \in \binom{W_3}{W_1 W_2}$, and let $p \in \mathbb{Z}$; then the map

$$\begin{aligned} I_{\mathcal{Y},p} &: W_1 \otimes W_2 \to W_3 \\ & w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x) w_{(2)} \Big|_{x^n = e^{n(\log z + 2\pi i p)}, (\log(x))^m = (\log z + 2\pi i p)^m} \end{aligned}$$

is a well defined $P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$; for any $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto I_{\mathcal{Y},p}$ is a bijection with inverse denoted by $I \mapsto \mathcal{Y}_{I,p}$ ([HLZ3]).

**Definition 3.2.1** ([HLZ1]-[HLZ7])**.** Given two generalized $V$-modules $W_1$, $W_2$, their $P(z)$-tensor product is a third module, denoted by $W_1 \boxtimes_{P(z)} W_2$, equipped with an intertwining map

$$\boxtimes_{P(z)} : W_1 \otimes W_2 \to \overline{W_1 \boxtimes_{P(z)} W_2}$$

such that for any generalized $V$-module $W_3$ and intertwining map $I : W_1 \otimes W_2 \to \overline{W}_3$, there exists a unique $V$-module morphism $\eta : W_1 \boxtimes_{P(z)} W_2 \to W_3$ such that

$$I = \overline{\eta} \circ \boxtimes_{P(z)},$$

where $\overline{\eta}$ is the unique map $\overline{\eta} : \overline{W_1 \boxtimes_{P(z)} W_2} \to \overline{W}_3$ extending $\eta$.

From the definition, one can see that given two $V$-modules, if their $P(z)$-tensor product exists then it is unique; moreover we have the following:

**Proposition 3.2.2.** *Let $W_1$, $W_2$, $W_3$ and $W_4$ be generalized $V$-modules and $\varphi : W_1 \to W_3$ and $\psi : W_2 \to W_4$ be $V$-module homomorphisms. Suppose that the $P(z)$-tensor products $W_1 \boxtimes_{P(z)} W_2$ and $W_3 \boxtimes_{P(z)} W_4$ exist and denote their intertwining maps by $I_1$, $I_2$ respectively. Then there exists a unique $V$ homomorphism*

$$\varphi \boxtimes_{P(z)} \psi : W_1 \boxtimes W_2 \to W_3 \boxtimes W_4$$

*such that for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,*

$$I_2(\varphi(w_{(1)}) \otimes \psi(w_{(2)})) = \overline{\varphi \boxtimes_{P(z)} \psi} \circ I_1(w_{(1)} \otimes w_{(2)}).$$

We recall some results from [HLZ5] concerning associativity of logarithmic intertwining operators.

**Theorem 3.2.3** ([HLZ5])**.** *Let $V$ be a vertex operator algebra whose category of generalized modules is closed under $P(z)$-tensor product, and such that every generalized module satisfies the $C_1$-cofiniteness condition and the quasi-finite dimensionality condition; let $W_1$, $W_2$, $W_3$, $W_4$, $M$ be generalized $V$-modules.*

1. *Consider intertwining operators $\mathcal{Y}_1 \in \binom{W_4}{W_1 M}$, $\mathcal{Y}_2 \in \binom{M}{W_2 W_3}$. Then there exists a unique intertwining operator $\mathcal{Y}^1 \in \binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2 \ W_3}$ such that*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\ x_2=z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0=z_0,\ x_2=z_2}$$

   *whenever $z_0 = z_1 - z_2$ and $|z_1| > |z_2| > |z_0| > 0$, for all $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.*

2. *Let $\mathcal{Y}^1 \in \binom{W_4}{M W_3}$, $\mathcal{Y}^2 \in \binom{M}{W_1 W_2}$. Then there exists a unique intertwining operator $\mathcal{Y}_1$ of type $\binom{W_4}{W_1 W_2 \boxtimes_{P(z_2)} W_3}$ such that*

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0=z_1-z_2,\ x_2=z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\ x_2=z_2}$$

*whenever* $|z_1| > |z_2| > |z_1 - z_2| > 0$ *for all* $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3$ *and* $w'_{(4)} \in W'_4$.

## 3.3 Elliptic functions and Eisenstein series

In this section we recall some basic properties of the Weierstrass $\wp$ function and Eisenstein series; in particular, we will be using the Taylor and Fourier $q$-expansions of such functions. For additional background, see [L, Z]. For $z \in \mathbb{C}$, we will use the notation $q_z = e^{2\pi i z}$. We first introduce a formal power series related to the expansion of the Weierstrass $\wp$ function: for $m \geq 0$,

$$P_{m+1}(x;q) = (2\pi i)^{m+1} \sum_{l>0} \left( \frac{l^m}{m!} \frac{x^l}{1-q^l} - \frac{(-1)^m l^m}{m!} \frac{q^l x^{-l}}{1-q^l} \right)$$

where $(1-q^l)^{-1}$ is the power series $\sum_{k \geq 0} q^{lk}$ in the formal variable $q$. For $\tau, z \in \mathbb{C}$ satisfying $|q_\tau| < |q_z| < 1$, the series $P_{m+1}(q_z; q_\tau)$ is absolutely convergent, and for $|q_z| < 1$ the $q$-coefficients of $P_{m+1}(q_z; q)$ are absolutely convergent. Let

$$\wp_1(z;\tau) = \frac{1}{z} + \sum_{(k,l) \neq (0,0)} \left( \frac{1}{z - (k\tau + l)} + \frac{1}{k\tau + l} + \frac{z}{(k\tau + l)^2} \right)$$

$$\wp_2(z;\tau) = \frac{1}{z^2} + \sum_{(k,l) \neq (0,0)} \left( \frac{1}{(z - (k\tau + l))^2} - \frac{1}{(k\tau + l)^2} \right);$$

and for $m \geq 2$, let

$$\wp_{m+1}(z;\tau) = -\frac{1}{m} \frac{\partial}{\partial z} \wp_m(z;\tau).$$

These functions have Laurent expansion

$$\wp_m(z;\tau) = \frac{1}{z^m} + (-1)^m \sum_{k \geq 1} \binom{2k+1}{m-1} G_{2k+2}(\tau) z^{2k+2-m}$$

in the region $0 < |z| < \min(1, |\tau|)$, where $G_{2k+2}(\tau)$ are the Eisenstein series defined by

$$G_{2k+2}(\tau) = \sum_{(m,l) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m\tau + l)^{2k+2}}$$

for $k \geq 1$. Moreover, let

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \sum_{l \in \mathbb{Z}} \frac{1}{(m\tau + l)^2}.$$

It is known that the Eisenstein series have $q$-expansion

$$G_{2k+2}(\tau) = 2\zeta(2k+2) + \frac{2(2\pi i)^{2k+2}}{(2k+1)!}\sum_{l=1}^{\infty}\frac{l^{2k+1}q_\tau^l}{1-q_\tau^l}$$

We use the following notation to denote these as formal power series in the variable $q$:

$$\tilde{G}_{2k+2}(q) = 2\zeta(2k+2) + \frac{2(2\pi i)^{2k+2}}{(2k+1)!}\sum_{l=1}^{\infty}\frac{l^{2k+1}q^l}{1-q^l}, \qquad k \in \mathbb{N}$$

and similarly for the expansion of the elliptic functions

$$\tilde{\wp}_m(x;q) = \frac{1}{x^m} + (-1)^m \sum_{k=1}^{\infty}\binom{2k+1}{m-1}\tilde{G}_{2k+2}(q)x^{2k+2-m}, \qquad m = 1, 2, \ldots$$

For $z \in \mathbb{C}$ such that $0 < |z| < 1$,

$$\tilde{\wp}_m(z;q) = (-1)^m\left(P_m(q_z;q) - \frac{\partial^{m-1}}{\partial z^{m-1}}(\tilde{G}_2(q)z - \pi i)\right)$$

and $\tilde{\wp}_m(z;q_\tau) = \wp_m(z;\tau)$. The following is well known:

**Proposition 3.3.1.** *For any element* $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$, *and* $m = 1, 2, \ldots$

$$\wp_m|_g(z;\tau) := (\gamma\tau+\delta)^{-m}\wp_m\left(\frac{z}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \wp_m(z;\tau)$$

*if* $m > 1$,

$$\wp_m(z+\tau;\tau) = \wp_m(z+1;\tau) = \wp_m(z;\tau)$$

*and*

$$\wp_1(z+1;\tau) = \wp_1(z;\tau) + G_2(q)$$
$$\wp_1(z+\tau;\tau) = \wp_1(z;\tau) + G_2(\tau)\tau - 2\pi i$$

**Proposition 3.3.2.** *Let* $f(q)$ *be a modular form of weight* $k$. *Then the function* $\vartheta_k(f)$ *defined by*

$$(2\pi i)^2 q\frac{d}{dq}f(q) + kG_2(q)f(q)$$

*is a modular form of weight* $k+2$.

## 3.4 Formal $q$-traces of logarithmic intertwining operators

In this section we consider modules for a vertex operator algebra which admit a right action (by module endomorphisms) of an associative algebra $P$. We then consider products of intertwining operators which commute with this action; in particular, we are able to define the formal $q$-trace of such products by using pseudotraces on the $P$-endomorphism ring of the $L(0)$ generalized eigenspaces in the $V$-modules. We use properties of pseudotraces and intertwining operators to derive identities for the formal $q$-traces.

Let $P$ be an associative algebra equipped with a symmetric linear function $\phi$, and fix a vertex operator algebra $V$. We say that a generalized $V$-module $W$ is a *$V$-$P$-bimodule* if $W$ is a right $P$ module and $P$ acts on $W$ by $V$-module endomorphisms, that is, for any $v \in V$, $w \in W$, and $p \in P$,

$$Y(v,x)(wp) = (Y(v,x)w)p.$$

**Proposition 3.4.1.** *Let $W_{[n]}$ be an $L(0)$-generalized eigenspace of $W$ for the eigenvalue $n$. Then $W_{[n]}$ is a $P$-submodule of $W$; if $W$ is a projective right $P$-module, so is $W_{[n]}$.*

*Proof.* This is clear since the action of $P$ commutes with $L(0)$: let $p \in P$ and $w \in W_{[n]}$; then there exists $k \in \mathbb{N}$ such that $(L(0)-n)^k w = 0$. Then

$$(L(0)-n)^k (wp) = ((L(0)-n)^k w)p = 0$$

which proves the first part of the claim; the second part follows since $W_{[n]}$ is a direct summand of $W$. ∎

As a consequence, the action of $P$ commutes with $L(0)_s$ and $L(0)_n$. Suppose now $W$ is a grading restricted generalized $V$-module, projective as right $P$-module. Then for any generalized eigenspace $W_{[n]}$, we can define the pseudotrace

$$\phi_{W_{[n]}} : \mathrm{End}_P(W_{[n]}) \to \mathbb{C};$$

and for a given element $a(x_1,\ldots,x_k) \in \mathrm{End}_P(W)\{x_1,\ldots,x_k,\log x_1,\ldots,\log x_k\}$, we define

$$\mathrm{tr}^{\phi}_{W} a(x_1,\ldots,x_n)q^{L(0)} = \sum_{n\in\mathbb{C}} \phi_{W_{[n]}}\left(\pi_n a(x_1,\ldots,x_k)\sum_{i=0}^{\infty}\frac{(L(0)_n)^i}{i!}(\log q)^i\right)\Bigg|_{W_{[n]}} q^n$$

where $\pi_n : W \to W_{[n]}$ is the projection on the generalized eigenspace $W_{[n]}$. Note that since $L(0)_n$ is locally nilpotent, the summation over $i$ is finite for any value of $n \in \mathbb{C}$. If $W$ has finite length $l$, then the powers of $\log q$ are globally bounded by $l$.

**Remark 3.4.2.** Suppose $T \in \mathrm{End}_P\left(W_{[n]}\right)$, and let $\{w_i\}_{i=1}^s$, $\{\alpha_i\}_{i=1}^s$ be a projective basis for $W_{[n]}$. Let $w_i' \in W_{[n]}'$ be the linear function defined by $\langle w_i', w\rangle = \phi(\alpha_i(w))$ for all $w \in W_{[n]}$. Then one can express the pseudotrace of $T$ as

$$\phi_{W_{[n]}}(T) = \sum_{i=1}^{s}\langle w_i', Tw_i\rangle.$$

In particular, for any $n \in \mathbb{C}$ let $\{w_{n,i}\}_{i=1}^{s_n}$, $\{\alpha_{n,i}\}_{i=1}^{s_n}$ be a projective basis of $W_{[n]}$ and let $a(x_1,\ldots,x_k)$ as above, one can express the $q$-trace of $a(x_1,\ldots,x_n)$ as

$$\mathrm{tr}^{\phi}_{W} a(x_1,\ldots,x_n) = \sum_{n\in\mathbb{C}}\sum_{i=1}^{s_n}\left(\sum_{j=1}^{\infty}\left\langle w_{n,i}', a(x_1,\ldots,a_n)\frac{(L(0)_n)^j}{j!}w_{n,i}\right\rangle(\log q)^j\right)q^n$$

where $w_{n,i}'$ is defined as above and extended to an element of $W'$ by letting it map to 0 the generalized subspaces for eigenvalues different from $n$.

Now let $W_i$, $\tilde{W}_i$, $i = 1,\ldots,n$, be grading restricted generalized modules for $V$, and suppose $\tilde{W}_0$ is a $V$-$P$-bimodules, projective as right $P$-module. Moreover, consider logarithmic intertwining operators $\mathcal{Y}_i \in \binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, $i = 1,\ldots,n$, where we use the convention $\tilde{W}_0 = \tilde{W}_n$. If the action of $P$ commutes with the product of the intertwining operators, i.e., for all $w_i \in W_i$, $\tilde{w}_n \in \tilde{W}_n$, and $p$ in $P$,

$$\mathcal{Y}_1(w_1,x_1)\cdots\mathcal{Y}_n(w_n,x_n)(\tilde{w}_n p) = (\mathcal{Y}_1(w_1,x_1)\cdots\mathcal{Y}_n(w_n,x_n)\tilde{w}_n)p,$$

then one can define the *formal $q$-trace*

$$\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(w_1,x_1)\cdots\mathcal{Y}_n(w_n,x_n)q^{L(0)}.$$

**Remark 3.4.3.** Using the same notation as above, suppose the modules $\tilde{W}_i$ for $i = 1,\ldots,n$ are $V$-$P$-bimodules, with $W_0$ projective as right $P$-module, and the action of

$P$ commutes with all the intertwining operators, i.e., for all $i = 1, \dots, n$ for all $w_i \in W_i$, $\tilde{w}_i \in \tilde{W}_i$, and $p$ in $P$,

$$\mathcal{Y}(w_i, x)(\tilde{w}_i p) = (\mathcal{Y}(w_i, x)\tilde{w}_i)p.$$

Then the product of the intertwining operators commute with the action of $P$ and

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n) q^{L(0)}$$

is well defined.

**Definition 3.4.4.** Let $W$, be a $V$-module and $W_1, W_2$ be $V$-$P$-bimodules; we will say that a logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_2}{W W_1}$ is a *$P$-intertwining operator* if

$$\mathcal{Y}(w, x)(w_1 p) = (\mathcal{Y}(w, x)w_1)p$$

for all $p \in P$, $w \in W$, $W_1 \in W_1$.

**Remark 3.4.5.** Note that for any set of intertwining operators $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ of the above types, one can always consider $\tilde{W}_i$ as a projective right $P$-module with $P = \mathbb{C}$ and $\phi = 1_{\mathbb{C}}$. In this case, $\mathrm{tr}^{\phi}$ corresponds to the ordinary matrix trace of the action of the intertwining operators on the vector space $\tilde{W}_n$.

**Remark 3.4.6.** Using the same notation as in Remark 3.4.2, we can express the formal $q$-trace as

$$\begin{aligned}
&\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n) q^{L(0)} \\
&\quad = \sum_{n \in \mathbb{C}} \sum_{i=1}^{s_n} \left( \sum_{j=1}^{\infty} \left\langle w'_{n,i}, \mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n) \frac{(L(0)_n)^j}{j!} w_{n,i} \right\rangle (\log q)^j \right) q^n.
\end{aligned}$$

In particular, as a formal series in the variables $q$, $\log q$, its coefficients are finite sums of genus-one correlation functions; one will be able to use properties of these correlation functions to obtain properties for formal $q$ traces.

Following [H2], we use the concept of *geometrically modified intertwining operator*. Let $A_j$, $j \in \mathbb{Z}_+$ be the numbers defined by the formal relation

$$\frac{1}{2\pi i} \log(1 + 2\pi i y) = \left( \exp\left( \sum_{j \in \mathbb{Z}_+} A_j y^{j+1} \frac{\partial}{\partial y} \right) \right) y$$

and let $L_+(A) = \sum_{j \in Z_+} A_j L(j)$; then the operator $\mathcal{U}(1)$ is defined by

$$\mathcal{U}(1) = (2\pi i)^{L(0)} e^{-L_+(A)}.$$

Also let $\mathcal{U}(x) = x^{L(0)}\mathcal{U}(1)$, for any element $x$ for which the expression makes sense: it follows that

$$y^{L(0)}\mathcal{U}(x) = \mathcal{U}(yx).$$

**Definition 3.4.7** ([H2])**.** Let $W_1$, $W_2$, $W_3$ be $V$-modules, and $\mathcal{Y}$ a logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. The operator $\mathcal{Y}(\mathcal{U}(x)w, x)$ is called a *geometrically modified intertwining operator.*

Here we recall some of the properties of the operator $\mathcal{U}(1)$ and of the geometrically modified intertwining operators; see [H2] for these results.

**Lemma 3.4.8.** *Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1, W_2}$ for grading restricted generalized $V$-modules $W_1$, $W_2$, $W_3$. Then for $u \in V$ and $w \in W_1$,*

$$\begin{aligned} &[Y(\mathcal{U}(x_1)u, x_1), \mathcal{Y}(\mathcal{U}(x_2)w, x_2)] \\ &\qquad = 2\pi i \mathrm{Res}_y \delta\left(\frac{x_1}{e^{2\pi i y} x_2}\right) \mathcal{Y}(\mathcal{U}(x_2)Y(u, y)w, x_2) \end{aligned} \tag{3.4.7}$$

**Lemma 3.4.9.** *Let $W_1$, $W_2$, $W_3$ be grading restricted generalized $V$-modules, and $\mathcal{Y}$ and logarithmic intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then for any $w_1 \in W_1$,*

$$\mathcal{Y}(\mathcal{U}(x)L(-1)w_1, x) = 2\pi i x \frac{d}{dx} \mathcal{Y}(\mathcal{U}(x)w_1, x). \tag{3.4.8}$$

**Lemma 3.4.10.** *For any generalized $V$-module $W$ and $u \in V$, we have the $\mathcal{U}(x)$ conjugation property*

$$\mathcal{U}(x)Y(u, y) = Y(\mathcal{U}(xe^{2\pi i y})u, x(e^{2\pi i y} - 1))U(x). \tag{3.4.9}$$

We will consider formal $q$-traces of products of geometrically modified intertwining operators

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}$$

for intertwining operators whose product commutes with the action of $P$. Many of the properties of regular traces which hold in the completely reducible case carry over to the logarithmic setting.

In the following, for any $v \in V$, we denote by $o(v)$ the constant term of the operator $Y(x^{L(0)}v, x)$ acting on a generalized $V$-module; that is,

$$o(v) = \mathrm{Res}_x x^{-1} Y(x^{L(0)}v, x) = v_{\mathrm{wt}\ v-1}.$$

**Lemma 3.4.11.** *Consider grading restricted generalized $V$-modules $W_i$, $\tilde{W}_i$ for $i = 1, \dots, n$, with $\tilde{W}_0 = \tilde{W}_n$, and logarithmic intertwining operators $\mathcal{Y}_i$ of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ for $i = 1, \dots, n$. Moreover, suppose $\tilde{W}_0$ is a $V$-$P$-bimodule projective as a right $P$-module for some algebra $P$ equipped with a symmetric linear function $\phi$, and that the product of the intertwining operators $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ commutes with the action of $P$. Then for any $v \in V$, $w_i \in W_i$, we have*

$$\begin{aligned}
&\mathrm{tr}^{\phi}_{\tilde{W}_n} Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)} \\
&\quad = \sum_{i=1}^{n} \sum_{m \geq 0} P_{m+1}\left(\frac{x_i}{x}; q\right) \mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdot \\
&\qquad\qquad \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i) \cdot \\
&\qquad\qquad \cdot \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)} \\
&\quad\quad + \mathrm{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)u) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}
\end{aligned} \tag{3.4.10}$$

*and*

$$\begin{aligned}
&\sum_{i=1}^{n} \mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\qquad \cdot \mathcal{Y}_i(\mathcal{U}(x_i)u_0 w_i, x_i) \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)} \\
&\qquad = 0
\end{aligned} \tag{3.4.11}$$

*Proof.* By the commutator formula,

$$\begin{aligned}
&Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) \\
&\quad = \sum_{i=1}^{n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\qquad\quad \cdot [Y(\mathcal{U}(x)u, x), \mathcal{Y}_i(\mathcal{U}(x_i)w_i, x_i)] \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot \\
&\qquad\quad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) \\
&\qquad + \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) Y(\mathcal{U}(x)u, x)
\end{aligned}$$

$$= \sum_{i=1}^{n} 2\pi i \mathrm{Res}_y \delta\left(\frac{x}{e^{2\pi i y} x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$
$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$
$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)$$
$$+ \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) Y(\mathcal{U}(x)u, x).$$

Since $P$ acts on $\tilde{W}_n$ as $V$-module endomorphisms, and the product of the intertwining operators $\mathcal{Y}_1 \ldots \mathcal{Y}_n$ commutes with $P$, the pseudotrace of each term in this expression is well defined. Therefore, by linearity of pseudotraces, $q^{L(0)}$ conjugation property and cyclic property of pseudotraces,

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}$$
$$= \sum_{i=1}^{n} \mathrm{tr}^{\phi}_{\tilde{W}_n} 2\pi i \mathrm{Res}_y \delta\left(\frac{x}{e^{2\pi i y} x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$
$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$
$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}$$
$$+ \mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) Y(\mathcal{U}(x)u, x) q^{L(0)}$$
$$= \sum_{i=1}^{n} \mathrm{tr}^{\phi}_{\tilde{W}_n} 2\pi i \mathrm{Res}_y \delta\left(\frac{x}{e^{2\pi i y} x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$
$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$
$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}$$
$$+ \mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)} Y\left(\mathcal{U}\left(\frac{x}{q}\right)u, \frac{x}{q}\right)$$
$$= \sum_{i=1}^{n} \mathrm{tr}^{\phi}_{\tilde{W}_n} 2\pi i \mathrm{Res}_y \delta\left(\frac{x}{e^{2\pi i y} x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$
$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$
$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}$$
$$+ \mathrm{tr}^{\phi}_{\tilde{W}_n} Y\left(\mathcal{U}\left(\frac{x}{q}\right)u, \frac{x}{q}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}$$

$$= \sum_{i=1}^{n} \mathrm{tr}_{\tilde{W}_n}^{\phi} 2\pi i \mathrm{Res}_y \delta\left(\frac{x}{e^{2\pi i y}x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$

$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$+ \left(q^{-x\frac{\partial}{\partial x}}\right) \mathrm{tr}_{\tilde{W}_n}^{\phi} Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

and thus

$$\left(1 - q^{-x\frac{\partial}{\partial x}}\right) \mathrm{tr}_{\tilde{W}_n}^{\phi} Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$= \sum_{i=1}^{n} \mathrm{tr}_{\tilde{W}_n}^{\phi} 2\pi i \mathrm{Res}_y \delta\left(\frac{x}{e^{2\pi i y}x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$

$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$= \sum_{i=1}^{n} \mathrm{tr}_{\tilde{W}_n}^{\phi} 2\pi i \mathrm{Res}_y e^{2\pi i y x_i \frac{\partial}{\partial x_i}} \delta\left(\frac{x}{x_i}\right) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$

$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$= \sum_{i=1}^{n} \sum_{m=0}^{\infty} \mathrm{tr}_{\tilde{W}_n}^{\phi} \frac{(2\pi i)^{m+1}}{m!} \left(x_i \frac{\partial}{\partial x_i}\right)^m \delta\left(\frac{x}{x_i}\right) \cdot$$

$$\cdot \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$

$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$= \sum_{i=1}^{n} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} \frac{(2\pi i)^{m+1}}{m!} \left(x_i \frac{\partial}{\partial x_i}\right)^m \left(\frac{x^l}{x_i^l} + \frac{x^{-l}}{x_i^{-l}}\right)$$

$$\mathrm{tr}_{\tilde{W}_n}^{\phi} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot$$

$$\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$+ 2\pi i \sum_{i=1}^{n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_1)u_0 w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

Since the operator $(1 - q^{-x\frac{\partial}{\partial x}})$ kills constant expressions in the variable $x$, the left hand side has no constant term as a series in $x$. This implies

$$\sum_{i=1}^{n} \mathrm{tr}_{\tilde{W}_n}^{\phi} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot$$

$$\cdot \mathcal{Y}_i(\mathcal{U}(x_i)u_0 w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

$$= 0;$$

therefore,

$$\begin{aligned}
&\left(1 - q^{-x\frac{\partial}{\partial x}}\right) \mathrm{tr}_{\tilde{W}_n}^{\phi} Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \sum_{i=1}^{n} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} \frac{(2\pi i)^{m+1}}{m!} \left( (-l)^m \frac{x^l}{x_i^l} + l^m \frac{x^{-l}}{x_i^{-l}} \right) \\
&\qquad\quad \mathrm{tr}_{\tilde{W}_n}^{\phi} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_i(\mathcal{U}(x_1)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot \\
&\qquad\quad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}.
\end{aligned}$$

Then invert the operator $(1 - q^{-x\frac{\partial}{\partial x}})$, with the appropriate constant term:

$$\begin{aligned}
&\mathrm{tr}_{\tilde{W}_n}^{\phi} Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \mathrm{tr}_{\tilde{W}_n}^{\phi} o(U(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad + \left(1 - q^{-x\frac{\partial}{\partial x}}\right)^{-1} \sum_{i=1}^{n} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} \frac{(2\pi i)^{m+1}}{m!} \left( (-l)^m \frac{x^l}{x_i^l} + l^m \frac{x^{-l}}{x_i^{-l}} \right) \\
&\qquad\quad \mathrm{tr}_{\tilde{W}_n}^{\phi} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_i(\mathcal{U}(x_1)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot \\
&\qquad\quad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \mathrm{tr}_{\tilde{W}_n}^{\phi} o(U(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad + \sum_{i=1}^{n} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} \frac{(2\pi i)^{m+1}}{m!} \left( -(-l)^m \frac{q^l x^l}{(1-q^l)x_i^l} + l^m \frac{x^{-l}}{(1-q^l)x_i^{-l}} \right) \\
&\qquad\quad \mathrm{tr}_{\tilde{W}_n}^{\phi} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_i(\mathcal{U}(x_1)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdot \\
&\qquad\quad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{tr}^{\phi}_{\tilde{W}_n} o(U(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&\quad + \sum_{i=1}^{n}\sum_{m=0}^{\infty}\sum_{l=1}^{\infty}\frac{(2\pi i)^{m+1}}{m!}\left(-(-l)^m\frac{q^l\left(\frac{x_i}{x}\right)^{-l}}{1-q^l}+l^m\frac{\left(\frac{x_i}{x}\right)^{l}}{(1-q^l)}\right)\\
&\quad \mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot\\
&\quad \cdot\mathcal{Y}_i(\mathcal{U}(x_1)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdot\\
&\quad \cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&= \mathrm{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&\quad + \sum_{i=1}^{n}\sum_{m\geq 0}P_{m+1}\left(\frac{x_i}{x};q\right)\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdot\\
&\qquad \cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i)\cdot\\
&\qquad \cdot\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}
\end{aligned}
$$

∎

We will now consider finite length module whose weights are real. We choose the branch of $\log z$ such that $0 \leq \Im(z) < 2\pi$, and for any $z, n \in \mathbb{C}$, we define $z^n = e^{n\log z}$. Moreover, we will make the following assumptions:

- (Convergence of genus zero correlation functions) For any generalized $V$-modules $W_i$, $\tilde{W}_i$, $i = 0 \ldots n$ with $\tilde{W}_n = \tilde{W}_0$, for any elements $w_i \in W_i$, $\tilde{w}_n \in \tilde{W}_n$, $\tilde{w}'_n \in \tilde{W}'_n$, and for logarithmic intertwining operators $\mathcal{Y}_i \in \binom{\tilde{W}_{i-1}}{W_i\tilde{W}_i}$, $i = 0 \ldots n$, the genus zero correlation function

$$\langle \tilde{w}'_n, \mathcal{Y}_1(w_1, z_1)\cdots\mathcal{Y}_n(w_n, z_n)\tilde{w}_n\rangle$$

  is absolutely convergent in the region $|z_1| > |z_2| > \ldots > |z_n| > 0$.

- (Associativity for $P$-intertwining operators) For any generalized $V$-modules $\tilde{W}_1$, $\tilde{W}_2$, $V$-$P$-bimodules $W_1$, $W_2$, $W_3$, logarithmic $P$-intertwining operators $\mathcal{Y}_1 \in \binom{W_2}{\tilde{W}_2, W_3}$, $\mathcal{Y}_2 \in \binom{W_1}{\tilde{W}_1, W_2}$ there exists a $V$-$P$-bimodule $M$, a logarithmic intertwining operator $Y_3 \in \binom{M}{\tilde{W}_1, \tilde{W}_2}$ and a $P$-intertwining operator $\mathcal{Y}_4 \in \binom{W_1}{M, W_3}$ such that for any complex numbers $z_1, z_2$ with $|z_1| > |z_2| > |z_1 - z_2| > 0$ and $\tilde{w}_1 \in \tilde{W}_1$,

$\tilde{w}_2 \in W_2$, $w_3 \in W_3$, $w_1' \in W_1'$

$$\begin{aligned}&\langle w_1', \mathcal{Y}_1(\tilde{w}_1, z_1)\mathcal{Y}_2(\tilde{w}_2, z_2)w_3\rangle\\ &\quad = \langle w_1', \mathcal{Y}_4(\mathcal{Y}_3(\tilde{w}_1, z_1 - z_2)\tilde{w}_2, z_2)w_3\rangle\end{aligned}$$

- (Commutativity for $P$-intertwining operators) For any generalized $V$-modules $\tilde{W}_1$, $\tilde{W}_2$, $W_1$, $W_2$, $W_3$, logarithmic $P$-intertwining operators $\mathcal{Y}_1 \in \binom{W_2}{\tilde{W}_2, W_3}$, $\mathcal{Y}_2 \in \binom{W_1}{\tilde{W}_1, W_2}$there exists a $V$-$P$-bimodule $M$ and logarithmic $P$-intertwining operators $Y_3 \in \binom{W_1}{\tilde{W}_2, M}$ and $\mathcal{Y}_4 \in \binom{M}{\tilde{W}_1, W_3}$ such that for any $\tilde{w}_1 \in \tilde{W}_1$, $\tilde{w}_2 \in W_2$, $w_3 \in W_3$, $w_1' \in W_1'$, the multivalued analytic function

$$\langle w_1', \mathcal{Y}_1(\tilde{w}_1, z_1)\mathcal{Y}_2(\tilde{w}_2, z_2)w_3\rangle$$

in the region $|z_1| > |z_2| > 0$ is an analytic continuation of the multivalued analytic function

$$\langle w_1', \mathcal{Y}_3(\tilde{w}_2, z_2)\mathcal{Y}_4(\tilde{w}_1, z_1)w_3\rangle$$

in the region $|z_2| > |z_1| > 0$.

Our goal is to obtain differential equations for the genus-one correlation functions. In order to do that, we need to derive formulae for

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_i(\mathcal{U}(x_i)L(-1)w_i, x_i) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)},$$

which is related to the derivative of the formal $q$-trace with respect to the variable $x_i$; hence, we consider the expression

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_i(\mathcal{U}(x_i)Y(u, y)w_i, x_i) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

Using the $\mathcal{U}(x)$ conjugation property, one can rewrite this as

$$\begin{aligned}&\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdot\\ &\qquad \cdots \mathcal{Y}_i(Y(\mathcal{U}(x_i e^{2\pi i y}u, x_i(e^{2\pi i y} - 1))\mathcal{U}(x_i)w_i, x_i)\cdot\\ &\qquad\quad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\end{aligned}$$

then, in order to use formula 3.4.10, one rewrites the iterate as a product using associativity for intertwining operators.

**Lemma 3.4.12.** *Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_0}{W, W_1}$ for grading restricted generalized $V$-modules $W, W_0, W_1$. Then for any $w_0' \in W_0', w_1 \in W_1, w \in W$, and for any complex number $z$ satisfying $|q_z| > 1 > |q_z - 1| > 0$,*

$$\begin{aligned}
&\langle w_0', \mathcal{Y}(Y(\mathcal{U}(xq_z)u, x(q_z-1))\mathcal{U}(x)w, x)w_1\rangle \\
&\quad = \langle w_0', Y(\mathcal{U}(xq_z)u, xq_z)\mathcal{Y}(\mathcal{U}(x)w, x)w_1\rangle
\end{aligned} \tag{3.4.12}$$

*Proof.* Using associativity for intertwining operators, we see that

$$\begin{aligned}
&\langle w_0', \mathcal{Y}(Y(\mathcal{U}(z_1q_z)u, z_1(q_z-1))\mathcal{U}(z_1)w, z_1)w_1\rangle \\
&\quad = \langle w_0', Y(\mathcal{U}(z_1q_z)u, z_1q_z)\mathcal{Y}(\mathcal{U}(z_1)w, z_1)w_1\rangle
\end{aligned} \tag{3.4.13}$$

holds for any complex numbers $z, z_1$ in the region $|z_1q_z| > |z_1| > |z_1(q_z - 1)| > 0$, or whenever $z_1 \neq 0$ and $|q_z| > 1 > |q_z - 1| > 0$. Then for a fixed $z$ in the above region, we have two series in powers of $z_1$ (not necessarily integral), and $\log z_1$; by our assumption on the modules, these powers must form a unique expansion set, and thus the coefficients of the two series must be equal. Then, we can replace the complex variable $z_1$ in (3.4.13) with the formal variable $x$, which concludes the proof. ∎

**Proposition 3.4.13.** *Consider grading restricted generalized $V$-modules $W_i$ and $V$-$P$-bimodules $\tilde{W}_i$ for $i = 1, \ldots, n$, with $\tilde{W}_0 = \tilde{W}_n$, and $P$-intertwining operators $\mathcal{Y}_i$ of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ for $i = 1, \ldots, n$. Moreover, suppose $\tilde{W}_0$ is projective as right $P$-module. Then for any $v \in V$, $w_i \in W_i$, and any integer $j$, $1 \leq j \leq n$,*

$$\begin{aligned}
&\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{j-1}(\mathcal{U}(x_{j-1})w_{j-1}, x_{j-1})\mathcal{Y}_j(\mathcal{U}(x_j)Y(v, y)w_j, x_j) \\
&\qquad \cdot \mathcal{Y}_{j+1}(\mathcal{U}(x_{j+1})w_{j+1}, x_{j+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \sum_{m \geq 0} (-1)^{m+1} \left( \tilde{\wp}_{m+1}(-y; q) + \frac{\partial^m}{\partial y^m}(\tilde{G}_2(q)y + \pi i) \right) \cdot \\
&\qquad \mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{j-1}(\mathcal{U}(x_{j-1})w_{j-1}, x_{j-1}) \cdot \\
&\qquad \cdot \mathcal{Y}_j(\mathcal{U}(x_j)v_m w_j, x_j)\mathcal{Y}_{j+1}(\mathcal{U}(x_{j+1})w_{j+1}, x_{j+1}) \cdot \\
&\qquad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad + \sum_{i \neq j} \sum_{m \geq 0} P_{m+1}\left( \frac{x_i}{x_j e^{2\pi i y}}; q \right) \cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\cdot \mathcal{Y}_i(\mathcal{U}(x_i)v_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&+ \operatorname{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)v)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}.
\end{aligned} \tag{3.4.14}
$$

*Proof.* By induction on $j$; when $j = 1$, by Lemma 3.4.9,

$$
\begin{aligned}
&\operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)Y(v, y)w_1, x_1)\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(Y(\mathcal{U}(x_1 e^{2\pi i y})v, x_1(e^{2\pi i y} - 1))\mathcal{U}(x_1)w_1, x_1) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}
\end{aligned}
$$

and using (3.4.12), for any complex number $z$ such that $|q_z| > 1 > |q_z - 1| > 0$,

$$
\begin{aligned}
&\operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)Y(v, z)w_1, x_1)\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(Y(\mathcal{U}(x_1 q_z)v, x_1(q_z - 1))\mathcal{U}(x_1)w_1, x_1) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \operatorname{tr}^{\phi}_{\tilde{W}_n} Y(\mathcal{U}(x_1 q_z)v, x_1 q_z)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}
\end{aligned}
$$

Now by 3.4.10 with $x = x_1 q_z$, we get

$$
\begin{aligned}
&\operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)Y(v, z)w_1, x_1)\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \sum_{m \geq 0} P_{m+1}\left(\frac{1}{q_z}; q\right) \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)v_m w_1, x_1) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad + \sum_{i=2}^{m} \sum_{m \geq 0} P_{m+1}\left(\frac{x_i}{x_1 q_z}; q\right) \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_i(\mathcal{U}(x_1)v_m w_i, x_i) \cdot \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad + \operatorname{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)v)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \sum_{m \geq 0} (-1)^{m+1} \left(\tilde{\wp}_{m+1}(-z; q) + \frac{\partial^m}{\partial z^m}(\tilde{G}_2(q)z + \pi i)\right) \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)v_m w_1, x_1) \cdot \\
&\qquad\quad \cdot \mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad + \sum_{i=2}^{m} \sum_{m \geq 0} P_{m+1}\left(\frac{x_i}{x_1 q_z}; q\right) \operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot
\end{aligned}
$$

$$\cdot\, \mathcal{Y}_i(\mathcal{U}(x_1)v_m w_i, x_i) \cdot \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$
$$+ \operatorname{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)v)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}$$

which proves the base case. Now suppose the result holds for $j \geq 1$, and use commutativity for intertwining operators. ∎

**Proposition 3.4.14.** *Using the same notation as in the previous Proposition, for all $v \in V$ and $l \geq 1$, we have*

$$\begin{aligned}
&\operatorname{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})v_{-l}w_j, q_{z_j})\cdot \\
&\qquad \cdot \mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)} \\
&= \sum_{k=1}^{\infty}(-1)^{l+1}\binom{2k+1}{l-1}\tilde{G}_{2k+2}(q)\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdot \\
&\qquad \cdots \mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})v_{2k+2-l}w_j, q_{z_j})\cdot \\
&\qquad \cdot \mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)} \\
&\quad + \sum_{i\neq j}\sum_{m=0}^{\infty}(-1)^{m+l}\binom{-m-1}{l-1}\tilde{\wp}_{m+l}(z_i - z_j; q)\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdot \\
&\qquad \cdots \mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\mathcal{Y}_i(\mathcal{U}(q_{z_i})v_m w_i, q_{z_i})\cdot \\
&\qquad \cdot \mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)} \\
&\quad + \delta_{l,1}\tilde{G}_2(q)\sum_{i=1}^{n}\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot \\
&\qquad \cdot \mathcal{Y}_i(\mathcal{U}(q_{z_i})(v_1 + v_0 z_i)w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdot \\
&\qquad \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)} \\
&\quad + \delta_{l,1}\operatorname{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)v)\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)} \qquad (3.4.15)
\end{aligned}$$

*Proof.* Notice that the coefficients of (3.4.14) as a series in the formal variables $q$, $\log q$ are absolutely convergent in the region $|q_{z_1}| > \ldots > |q_{z_n}| > 0$. Since the $q$ coefficients of $\tilde{\wp}_m(z, q)$ are absolutely convergent when $|z| < 1$, we can substitute $y = z$, $x_i = q_{z_i}$, for $i = 1, \ldots, n$ in (3.4.14),

$$\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})Y(v, z)w_j, q_{z_j})$$
$$\cdot\, \mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})w_{j+1}, q_{z_{j+1}}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}$$

$$
\begin{aligned}
&= \sum_{m\geq 0}(-1)^{m+1}\left(\tilde{\wp}_{m+1}(-z;q)+\frac{\partial^m}{\partial z^m}(\tilde{G}_2(q)z+\pi i)\right)\cdot \\
&\qquad \mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1},q_{z_{j-1}})\cdot \\
&\qquad \cdot\mathcal{Y}_j(\mathcal{U}(q_{z_j})v_m w_j,q_{z_j})\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})w_{j+1},q_{z_{j+1}})\cdot \\
&\qquad \cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)} \\
&\quad +\sum_{i\neq j}\sum_{m\geq 0}\tilde{(}-1)^{m+1}\Bigg(\wp_{m+1}(z_i-z_j-z;q) \\
&\qquad\qquad +(-1)^{m+1}\frac{\partial^m}{\partial z_i^m}(\tilde{G}_2(q)(z_i-z_j-z)-\pi i)\Bigg)\cdot \\
&\qquad \cdot\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1},q_{z_{i-1}})\cdot \\
&\qquad \cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})v_m w_i,q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1},q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)} \\
&\quad +\mathrm{tr}^{\phi}_{\tilde{W}_n}o(\mathcal{U}(1)v)\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)}.
\end{aligned}
$$

The result follows by taking the coefficient of $z^{l-1}$ and using the $q$ expansion of $\tilde{\wp}(z;q)$ and (3.4.11). ∎

Taking $v=\omega$ and $l=1$ in (3.4.15), one obtains

$$
\begin{aligned}
&\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1},q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})\omega_{-1}w_j,q_{z_j})\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1},q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)} \\
&\qquad =\sum_{k=1}^{\infty}\tilde{G}_{2k+2}(q)\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdot \\
&\qquad\qquad \cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1},q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})\omega_{2k+1}w_j,q_{z_j})\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1},q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)} \\
&\qquad\quad +\sum_{i\neq j}\sum_{m=0}^{\infty}(-1)^{m+1}\tilde{\wp}_{m+1}(z_i-z_j;q)\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdot \\
&\qquad\qquad \cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1},q_{z_{i-1}})\mathcal{Y}_i(\mathcal{U}(q_{z_i})\omega_m w_i,q_{z_i})\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1},q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)} \\
&\qquad\quad +\tilde{G}_2(q)\sum_{i=1}^{n}\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1},q_{z_{i-1}})\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})(\omega_1+\omega_0 z_i)w_i,q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1},q_{z_{i+1}})\cdot \\
&\qquad\qquad \cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)}
\end{aligned}
$$

$$+\operatorname{tr}^{\phi}_{\tilde{W}_n} o(\mathcal{U}(1)\omega)\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}$$

now since $w_k = L(k-1)$, and since $\mathcal{U}(1)\omega = (2\pi i)^2\left(\omega - \frac{c}{24}\mathbf{1}\right)$, we obtain

$$\begin{aligned}
&\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})L(-2)w_j, q_{z_j})\cdot\\
&\qquad\cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\quad=\sum_{k=1}^{\infty}\tilde{G}_{2k+2}(q)\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdot\\
&\qquad\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})L(2k)w_j, q_{z_j})\cdot\\
&\qquad\cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\quad+\sum_{i\neq j}\sum_{m=0}^{\infty}(-1)^{m+1}\tilde{\wp}_{m+1}(z_i - z_j; q)\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdot\\
&\qquad\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\mathcal{Y}_i(\mathcal{U}(q_{z_i})L(m-1)w_i, q_{z_i})\cdot\\
&\qquad\cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\quad+\tilde{G}_2(q)\sum_{i=1}^{n}\operatorname{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})(L(0)+L(-1)z_i)w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\quad+(2\pi i)^2\operatorname{tr}^{\phi}_{\tilde{W}_n}\left(L(0)-\frac{c}{24}\right)\cdot\\
&\qquad\cdot\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}.
\end{aligned}\tag{3.4.16}$$

## 3.5 Duality properties for $P$-intertwining operators

In this section we derive associativity and commutativity properties for $P$-intertwining operators, under the same assumptions as Proposition 3.2.3: we show that in the statement of these two properties, all the modules can be taken to be $V$-$P$-bimodules and all intertwining operators to be $P$-intertwining operators. In particular, under those assumptions all identities in the previous sections hold.

Let $W_1$ be a generalized $V$-module and $W_2$ be a $V$-$P$-bimodule. We can define a right action of $P$ on $W_1 \boxtimes_{P(z)} W_2$ the following way: for any $p \in P$, consider the map

$$\begin{aligned} \mathrm{Id}_{W_1} \otimes p : W_1 \otimes W_2 &\to W_1 \otimes W_2 \\ w_1 \otimes w_2 &\mapsto w_1 \otimes (w_2 p); \end{aligned}$$

then, by Proposition 3.2.2, there exists a unique $V$-homomorphism $\mathrm{Id}_{W_1} \boxtimes_{P(z)} p$ of $W_1 \boxtimes_{P(z)} W_2$ such that

$$\boxtimes_{P(z)} \circ (\mathrm{Id}_{W_1} \otimes p) = \overline{(\mathrm{Id}_{W_1} \boxtimes_{P(z)} p)} \circ \boxtimes_{P(z)}. \tag{3.5.17}$$

so we let $p$ act on $W_1 \boxtimes_{P(z)} W_2$ by

$$wp = (\mathrm{Id}_{W_1} \boxtimes_{P(z)} p)(w)$$

for any $w \in W_1 \boxtimes_{P(z)} W_2$, and extend it to the formal completion $\overline{W_1 \boxtimes_{P(z)} W_2}$; thus (3.5.17) becomes

$$\boxtimes_{P(z)}(w_1 \otimes (w_2 p)) = \boxtimes_{P(z)}(w_1 \otimes w_2)p \tag{3.5.18}$$

for all $w_1 \in W_1$ and $w_2 \in W_2$. Thus, $W_1 \boxtimes_{P(z)} W_2$ can be seen naturally as a $V$-$P$-bimodule; moreover, the intertwining map $\boxtimes_{P(z)}$ commutes with the action of $P$. In particular, $P$ commutes with the intertwining operator $\mathcal{Y}_{\boxtimes_{P(z)},0}$.

**Proposition 3.5.1.** *Let $\mathcal{Y}_1 \in \binom{W_4}{W_1 M}$ and $\mathcal{Y}_2 \in \binom{M}{W_2 W_3}$ be two logarithmic intertwining operators, where $W_1, W_2, W_3$ are $V$-$P$-bimodules, $M$ is a generalized $V$-module, and $P$ commutes with $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Let $\mathcal{Y}^1 \in \binom{W_4}{W W_3}$ be the logarithmic intertwining operator as in Proposition 3.2.3 (1.) (here $W = W_1 \boxtimes_{P(z_0)} W_2$). Then $P$ commutes with $\mathcal{Y}^1$.*

*Proof.* Let $p$ be any element in $P$; for any two $V$-$P$-bimodules $W^2, W^3$, generalized $V$-module $W^1$, and intertwining operator $\mathcal{Y} \in \binom{W^3}{W^1 W^2}$, consider the logarithmic intertwining operators $\mathcal{Y}_p, \mathcal{Y}^p \in \binom{W^3}{W^1 W^2}$ defined by

$$\begin{aligned} \mathcal{Y}_p(w^{(1)}, x)w^{(2)} &= \mathcal{Y}(w^{(1)}, x)(w^{(2)}p) \\ \mathcal{Y}^p(w^{(1)}, x)w^{(2)} &= (\mathcal{Y}(w^{(1)}, x)w^{(2)})p \end{aligned}$$

for all $w^{(1)} \in W^1, w^{(2)} \in W^2$. It is clear that $\mathcal{Y}_p$ (resp. $\mathcal{Y}^p$) is an intertwining operator since $p$ acts on $W^2$ (resp. $W^3$) as a $V$-module homomorphism. Then $P$ commutes with $\mathcal{Y}$ if and only if $\mathcal{Y}_p = \mathcal{Y}^p$ for all $p \in P$. Consider $(\mathcal{Y}^1)_p$ and $(\mathcal{Y}^1)^p$: then for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, and for all $z_0, z_1, z_2$ such that $z_0 = z_1 - z_2$, $|z_1| > |z_2| > |z_0| > 0$,

$$\begin{aligned}
&\langle w'_{(4)}, (\mathcal{Y}^1)^p(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle w'_{(4)}, (\mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)})p\rangle \\
&\quad = \langle pw'_{(4)}, \mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle pw'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle w'_{(4)}, (\mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)})p\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)(\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)})p\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)(\mathcal{Y}_2)^p(w_{(2)}, z_2)w_{(3)}\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\langle w'_{(4)}, (\mathcal{Y}^1)_p(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_0)},0}(w_{(1)}, z_0)w_{(2)}, z_2)(w_{(3)}p)\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)(w_{(3)}p)\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)(\mathcal{Y}_2)_p(w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)(\mathcal{Y}_2)^p(w_{(2)}, z_2)w_{(3)}\rangle.
\end{aligned}$$

Thus, by uniqueness in Proposition 3.2.3 (2.) applied to the intertwining operators $\mathcal{Y}_1$ and $(\mathcal{Y}_2)^p$ we see that $(\mathcal{Y}^1)^p = (\mathcal{Y}^1)_p$ and therefore $P$ commutes with $\mathcal{Y}^1$. ∎

**Proposition 3.5.2.** *Using the notation of Proposition 3.2.3 (2.), suppose the modules $W_3$ and $W_4$ are $V$-$P$-bimodules, and the intertwining operator $\mathcal{Y}^1$ commutes with $P$. Then the logarithmic intertwining operator $\mathcal{Y}_1$ also commutes with $P$.*

*Proof.* Note that since $W_3$ is a $V$-$P$-module, the right action of $P$ on $W_3$ defines a right action of $P$ on $W_2 \boxtimes W_3$, and $\mathcal{Y}_{\boxtimes_{P(z_2)},0}$ commutes with $P$. Now for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, and for complex numbers $z_0, z_1, z_2$ such that $|z_1| > |z_2| > |z_0| > 0$, $z_0 = z_1 - z_2$, for any $p \in P$

$$\langle w'_{(4)}, (\mathcal{Y}_1)^p(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)w_{(3)}\rangle$$

$$\begin{aligned}
&= \langle w'_{(4)}, (\mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)w_{(3)})p\rangle \\
&= \langle pw'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)w_{(3)}\rangle \\
&= \langle pw'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle \\
&= \langle w'_{(4)}, (\mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)})p\rangle \\
&= \langle w'_{(4)}, (\mathcal{Y}^1)^p(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle.
\end{aligned}$$

Similarly, since $\mathcal{Y}_{\boxtimes_{P(z_2)},0}$ commutes with $P$,

$$\begin{aligned}
&\langle w'_{(4)}, (\mathcal{Y}_1)_p(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)(\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)w_{(3)})p\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, z_2)(w_{(3)}p)\rangle \\
&\quad = \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)(w_{(3)}p)\rangle \\
&\quad = \langle w'_{(4)}, (\mathcal{Y}^1)_p(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle \\
&\quad = \langle w'_{(4)}, (\mathcal{Y}^1)^p(\mathcal{Y}^2(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle.
\end{aligned}$$

The conclusion thus follows from uniqueness in Proposition 3.2.3 part (2.) applied to the intertwining operators $(\mathcal{Y}^1)^p$ and $\mathcal{Y}^2$. ∎

We summarize these results in the following.

**Theorem 3.5.3** (Associativity for $P$-intertwining operators)**.** *Let $W_1, W_2, W_3$ be $V$-$P$-bimodules.*

*(i) Let $M$ be a $V$-$P$-bimodule and $\mathcal{Y}_1$, $\mathcal{Y}_2$ be two $P$-logarithmic intertwining operators of types $\binom{W_4}{W_1 M}$, $\binom{M}{W_2 W_3}$, respectively. Then there exist a generalized module $W$, a $P$-intertwining operator $\mathcal{Y}^1$ of type $\binom{W}{W_1 W_2}$ and a logarithmic intertwining operator $\mathcal{Y}^2$ of type $\binom{W_4}{W W_3}$ such that for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w_{(4)} \in W'_4$,*

$$\begin{aligned}
&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle \\
&\qquad = \langle w'_{(4)}, \mathcal{Y}^2(\mathcal{Y}^1(w_{(1)}, z_1 - z_2)w_{(2)}, z_2)w_{(3)}\rangle
\end{aligned}$$

*for all $z_1, z_2$ such that $|z_1| > |z_2| > |z_1 - z_2|$.*

*(ii) Let $W$ be a generalized $V$-module, $\mathcal{Y}^1$ a logarithmic intertwining operator of type*

$\binom{W_4}{WW_3}$, *and* $\mathcal{Y}^2$ *a* $P$*-intertwining operator of type* $\binom{W}{W_1W_2}$. *Then there exists a* $V$*-*$P$*-bimodule* $M$ *and* $P$*-intertwining operators* $\mathcal{Y}_1$, $\mathcal{Y}_2$ *of types* $\binom{W_4}{W_1M}$, $\binom{M}{W_2W_3}$, *respectively such that the same conclusion as in (i) holds.*

We now state the commutativity property.

**Theorem 3.5.4** (Commutativity for $P$-intertwining operators)**.** *Let* $\mathcal{Y}_1$, $\mathcal{Y}_2$ *be* $P$*-logarithmic intertwining operators of types* $\binom{W_4}{W_1M}$, $\binom{M}{W_2W_3}$, *respectively, for generalized* $V$*-modules* $W_1, W_2$ *and* $V$*-*$P$*-bimodules* $W_3, W_4, M$. *Then there exist a* $V$*-*$P$*-module* $M_1$ *and logarithmic intertwining operators* $\mathcal{Y}_3, \mathcal{Y}_4$ *of type* $\binom{W_4}{W_2M_1}$, $\binom{M_1}{W_1W_3}$ *respectively, commuting with the action of* $P$, *such that for any* $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, *the multivalued analytic function*

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle$$

*on the region* $|z_1| > |z_2| > 0$ *and the multivalued analytic function*

$$\langle w'_{(4)}, \mathcal{Y}_3(w_{(2)}, z_2)\mathcal{Y}_4(w_{(1)}, z_1)w_{(3)}\rangle$$

*on the region* $|z_2| > |z_1| > 0$ *are analytic extensions of each other.*

*Proof.* Using associativity twice, by the previous propositions one sees that all the outer intertwining operators commute with $P$. Recall ([HLZ2]) the operator $\Omega_r : \mathcal{V}^{W^3}_{W^1W^2} \to \mathcal{V}^{W^3}_{W^2W^1}$ defined by

$$\Omega_r(\mathcal{Y})(w_{(2)}, x)w_{(1)} = e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)}$$

for any $r \in \mathbb{Z}$; then $\Omega_{-r-1}(\Omega_r(\mathcal{Y})) = \Omega_r(\Omega_{-r-1}(\mathcal{Y})) = \mathcal{Y}$ for any $\mathcal{Y} \in \mathcal{V}^{W^3}_{W^1W^2}$.

We know by Theorem 3.5.3 that there exist a module $M$, an intertwining operator $\mathcal{Y}^1$ and a $P$-intertwining operator $\mathcal{Y}^2$ such that

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle\\ &\qquad = \langle w'_{(4)}, \mathcal{Y}^2(\mathcal{Y}^1(w_{(1)}, z_0)w_{(2)}, z_2)w_{(3)}\rangle;\end{aligned}$$

now substituting

$$\begin{aligned}\mathcal{Y}^1(w_{(1)}, x_0)w_{(2)} &= \Omega_0(\Omega_{-1}(\mathcal{Y}^1))(w_{(1)}, x_0)w_{(2)}\\ &= e^{x_0L(-1)}\Omega_{-1}(\mathcal{Y}^1)(w_{(2)}, e^{\pi i}x_0)w_{(1)}\end{aligned}$$

we obtain (in the region $|z_1| > |z_2| > |z_1 - z_2|$)

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2)w_{(3)}\rangle\\&\quad = \langle w'_{(4)}, \mathcal{Y}^2(\Omega_1(\mathcal{Y}^1)(w_{(2)}, e^{\pi i}z_0)w_{(1)}, z_2 + z_0)w_{(3)}\rangle;\end{aligned}$$

which is an extension of $\langle w'_{(4)}, \mathcal{Y}^2(\Omega_1(\mathcal{Y}^1)(w_{(2)}, z_2 - z_1)w_{(1)}, z_1)w_{(3)}\rangle$ defined on the region $|z_1| > |z_2 - z_1| > 0$. Now by Theorem 3.5.3 (ii), we know that there exist a $V$-$P$-bimodule $M_1$ and $P$ intertwining operators $\mathcal{Y}_3$, $\mathcal{Y}_4$ of type $\binom{W_4}{W_2 M_1}$, $\binom{M_1}{W_1 W_3}$ respectively, such that

$$\begin{aligned}&\langle w'_{(4)}, \mathcal{Y}^2(\Omega_1(\mathcal{Y}^1)(w_{(2)}, z_2 - z_1)w_{(1)}, z_1)w_{(3)}\rangle\\&\quad = \langle w'_{(4)}, \mathcal{Y}_3(w_{(2)}, z_2)\mathcal{Y}_4(w_{(1)}, z_1)w_{(3)}\rangle\end{aligned}$$

This concludes the proof. ∎

# Chapter 4

# Genus one correlation functions

## 4.1 Differential Equations

In this section we derive a system of differential equations satisfied by the formal $q$-traces, using the identities obtained in the previous chapter. The main technical assumption used in this chapter is the $C_2$-cofiniteness of the generalized $V$-modules, which is used to prove that a particular module over a ring of functions is finitely generated.

We denote by $G$ the space of all multivalued analytic functions defined on the region $|z_1| > |z_2| > \ldots > |z_n| > 0$ with preferred branches on $|z_1| > |z_2| > \ldots > |z_n| > 0$, $0 \leq \arg z_i < 2\pi$ for $i = 1 \ldots n$. For any such function $f(z_1, \ldots, z_n)$, the function $f(q_{z_1}, \ldots, q_{z_n})$ is a multivalued analytic function defined when $|q_{z_1}| > \ldots > |q_{z_n}|$. We denote by $G_q$ the space of such functions. Let

$$R = \mathbb{C}[\tilde{G}_4(q), \tilde{G}_6(q), \tilde{\wp}_2(z_i - z_j; q), \tilde{\wp}_3(z_i - z_j; q)]$$

let $V$ be a vertex operator algebra with central charge $c$, and assume all lower bounded generalized $V$- modules are $\mathbb{R}$-graded and satisfy the $C_2$ cofiniteness condition. For $V$-modules $W_1, W_1, \ldots, W_n$ we denote by $T$ the graded $R$-module

$$R \otimes W_1 \otimes \ldots \otimes W_n,$$

with grading induced by the grading by generalized eigenvalues on $W_1, \ldots, W_n$. Denote by $T_r$ the homogeneous subspace of degree $r \in \mathbb{R}$; moreover, define a filtration on $T$ by

$F(T)_r = \coprod_{s \leq r} T_s$. Let $J$ be the $R$-submodule of $T$ generated by the elements

$$\begin{aligned}
&A_j(v; w_1, \ldots, w_n) \\
&\quad = 1 \otimes w_1 \otimes \ldots \otimes w_{j-1} \otimes v_{-2} w_j \otimes w_{j+1} \otimes \ldots \otimes w_n \\
&\quad + \sum_{k=1}^{\infty} (2k+1) \tilde{G}_{2k+2}(q) \otimes w_1 \otimes \ldots \otimes w_{j-1} \otimes v_{2k} w_j \otimes w_{j+1} \otimes \ldots \otimes w_n \\
&\quad + \sum_{i \neq j} \sum_{m=0}^{\infty} (-1)^m (m+1) \tilde{\wp}_{m+2}(z_i - z_j) \otimes \\
&\qquad w_1 \otimes \ldots \otimes w_{j-1} \otimes v_m w_j \otimes w_{j+1} \otimes \ldots \otimes w_n
\end{aligned}$$

for $v \in V$, $w_i \in W_i$, $i = 1 \ldots, n$, and $1 \leq j \leq n$. The filtration on $T$ induces one on $J$, and we will denote by $F(J)_r$ the $r$-th subspace in the filtration.

**Proposition 4.1.1.** *There exists $N \in \mathbb{R}$ such that for any $r \in \mathbb{R}$, $F(T)_r = F(T)_N + F(J)_r$.*

*Proof.* Let $N$ such that

$$\coprod_{r > N} T_r \subseteq \sum_{i=1}^{n} R \otimes W_1 \otimes \ldots W_{i-1} \otimes C_2(W_i) \otimes W_{i+1} \otimes \ldots W_n;$$

then clearly if $r \leq N$, $F(T)_r \subseteq F(T)_N = F(T)_N + F(J)_r$. We now prove on induction on $k \in \mathbb{N}$ that if $r = N + k$, then $F(T)_r = F(T)_n + F(J)_r$. Let $r = N + k + 1$ and let $t \in F(T)_r$. By definition of $N$, we can assume

$$t = 1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes v_{-2} w_i \otimes w_{i+1} \otimes \ldots \otimes w_n$$

for some $v \in V$ and $w_j \in W_j$, $j = 1 \ldots, n$. Observe that the element $A(v; w_1, \ldots, w_n)$ belongs to $F(J)_r$ and

$$\begin{aligned}
S = &\sum_{k=1}^{\infty} (2k+1) \tilde{G}_{2k+2}(q) \otimes w_1 \otimes \ldots \otimes w_{j-1} \otimes v_{2k} w_j \otimes w_{j+1} \otimes \ldots \otimes w_n \\
&\quad + \sum_{i \neq j} \sum_{m=0}^{\infty} (-1)^m (m+1) \tilde{\wp}_{m+2}(z_i - z_j) \otimes \\
&\qquad w_1 \otimes \ldots \otimes w_{j-1} \otimes v_m w_j \otimes w_{j+1} \otimes \ldots \otimes w_n
\end{aligned}$$

belongs to $F(T)_{r-1}$. By induction hypothesis, $S \in F(T)_N + F(J)_{r-1}$. Then $t = A(v, w_1, \ldots, w_n) - s \in F(J)_r + F(T)_N + F(J)_{r-1} = F(T)_N + F(J)_r$. This concludes the proof. ∎

**Corollary 4.1.2.** *We have $T = F(T)_N + J$ and $T/J$ is a finitely generated $R$-module.*

*Proof.* The first assertion follows immediately from Proposition 4.1.1; the second follows since $F(T)_N$ is finitely generated. ∎

In the following we will fix an associative algebra $P$ and a symmetric linear function $\phi$ on $P$. For $V$-$P$-bimodules $\tilde{W}_0 \ldots \tilde{W}_n$ with $\tilde{W}_0 = \tilde{W}_n$ and $P$-intertwining operators $\mathcal{Y}_i$ of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, $i = 1 \ldots n$, and for $w_i \in W_i$, $i = 1, \ldots, n$, we consider the map

$$\begin{aligned} &F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\ &\qquad = \mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q^{L(0)-\frac{c}{24}} \end{aligned}$$

and we extend it to an $R$-module map $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n} : T \to G_q((q))[\log q]$ defined by

$$f \otimes w_1 \otimes \ldots \otimes w_n \mapsto f \cdot F(w_1, \ldots, w_1; z_1, \ldots, z_n; q).$$

We will also use the notation $F^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$ and $\psi^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$ when we need to specify the dependence on the symmetric function $\phi$.

**Proposition 4.1.3.** *The submodule $J$ is contained in the kernel of $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$; hence $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$ induces a map, also denoted by $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$, from $T/J$ to $G_q((q))[\log q]$.*

*Proof.* This follows from applying $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$ to (3.4.15) with $l = 2$; the resulting equation implies $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(A_j(v; w_1, \ldots, w_n)) = 0$ for all $v \in V$, $w_1 \in W_1, \ldots, w_n \in W_n$. ∎

**Proposition 4.1.4.** *For any $w_1 \in W_1, \ldots w_n \in W_n$, we have the $L(-1)$-derivative property*

$$\begin{aligned} &\frac{\partial}{\partial z_i} F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\ &\qquad = F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_{j-1}, L(-1)w_j, w_{j+1}, \ldots, w_n; z_1, \ldots, z_n; q) \end{aligned} \tag{4.1.1}$$

*Proof.* This is an immediate consequence of Lemma 3.4.8. ∎

**Proposition 4.1.5.** *Consider grading restricted generalized $V$-modules $W_i$, $\tilde{W}_i$ for $i = 1, \ldots, n$, with $\tilde{W}_0 = \tilde{W}_n$, and intertwining operators $\mathcal{Y}_i$ of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ for $i = 1, \ldots, n$. Moreover, suppose $\tilde{W}_0$ is a $V$-$P$-bimodule projective as right $P$-module for some algebra*

*$P$ equipped with a symmetric linear function $\phi$. Then for any homogeneous elements $w_1 \in W_1, \dots, w_n \in W_n$, and any $j = 1, \dots, n$, we have*

$$\left((2\pi i)^2 q\frac{\partial}{\partial q} + \tilde{G}_2(q)\sum_{i=1}^{n} \mathrm{wt}\ w_i + \tilde{G}_2(q)\sum_{i=1}^{n} z_i\frac{\partial}{\partial z_i} - \sum_{i\neq j}\tilde{\wp}_1(z_i - z_j; q)\frac{\partial}{\partial z_i}\right)\cdot$$

$$\cdot F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q)$$

$$+ \tilde{G}_2(q)\sum_{i=1}^{n} F_{\mathcal{Y}_2,\dots,\mathcal{Y}_n}(w_1, \dots, w_{i-1}, L(0)_n w_i, w_{i+1}, \dots, w_n; z_1, \dots, z_n; q)$$

$$= F_{\mathcal{Y}_2,\dots,\mathcal{Y}_n}(w_1, \dots, w_{j-1}, L(-2)w_j, w_{j+1}, \dots, w_n; z_1, \dots, z_n; q)$$

$$- \sum_{k=1}^{\infty}\tilde{G}_{2k+2}(q)\cdot$$

$$\cdot F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_{j-1}, L(2k)w_j, w_{j+1}, \dots, w_n; z_1, \dots, z_n; q)$$

$$+ \sum_{i\neq j}\sum_{m=1}^{\infty}(-1)^m\tilde{\wp}_{m+1}(z_i - z_j); q)\cdot$$

$$\cdot F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_{i-1}, L(m-1)w_i, w_{i+1}, \dots, w_n; z_1, \dots, z_n; q) \qquad (4.1.2)$$

*Proof.* This follows from (3.4.16) and the definition of $\psi_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}$, using Lemma 3.1.11 and (4.1.1). ∎

**Remark 4.1.6.** This formula differs from the one obtained in [H2], by the presence of the additional term

$$G_2(q)\sum_{i=1}^{n} F_{\mathcal{Y}_2,\dots,\mathcal{Y}_n}(w_1, \dots, w_{i-1}, L(0)_n w_i, w_{i+1}, \dots, w_n; z_1, \dots, z_n; q)$$

due to the non-semisimplicity of the operator $L(0)$.

We will now consider a second grading on the space $T$ by "modular weights": define the modular weight on the ring $R$ by assigning weight $2k$ to the function $G_{2k}(q)$ and weight $m$ to $\tilde{\wp}_m(z; q)$, and denote by $R_p$ the homogeneous subspace of degree $p$.
Now if $t = f \otimes w_1 \otimes \dots \otimes w_n \in T$, with homogeneous $f \in R_p$ and $w_i \in W_i$, $i = 1, \dots, n$, we assign $t$ modular degree $p + \sum_{i=1}^{n} \mathrm{wt}\ w_i$. Clearly from this definition, if $v \in V$ and $w_i \in W_i$, $i = 1, \dots, n$ are homogeneous, then the element $A(v; w_1, \dots, w_n) \in J$ has modular weight $\mathrm{wt}\ v + \sum_{i=1}^{n} \mathrm{wt}\ w_i + 1$. As a consequence, we have

**Proposition 4.1.7.** *The ideal $J$ is graded by modular weights; in particular, this grading induces a grading on the quotient module $T/J$.*

**Proposition 4.1.8.** *Let $W_1, \ldots W_n$ be generalized modules for the vertex operator algebra $V$, and consider homogeneous $w_i \in W_i$ for $i = 1, \ldots n$. Then there exist elements $a_{p,i} \in R_p$ for $p = 1, \ldots, m$ such that for any $V$-$P$-bimodules $\tilde{W}_j$ and $P$-intertwining operators $\mathcal{Y}_j$ of type $\binom{\tilde{W}_{j-1}}{W_j \tilde{W}_j}$, $j = 1, \ldots, n$, with $\tilde{W}_0 = \tilde{W}_n$ projective as a right $P$-module, the series*

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdot \ldots \cdot \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q^{L(0) - \frac{c}{24}}$$

*satisfies the differential equations*

$$\frac{\partial^m}{\partial z_i^m}\varphi + \sum_{p=1}^{m} a_{p,i}(z_1, \ldots, z_n; q) \frac{\partial^{m-p}}{\partial z_i^{m-p}}\varphi = 0. \tag{4.1.3}$$

*for $i = 1, \ldots, n$, in the region $1 > |q_{z_1}| > \ldots > |q_{z_1}| > |q| > 0$.*

*Proof.* Consider the submodule $M_i$ of $T/J$ generated by the elements

$$1 \otimes w_1 \otimes \ldots \otimes w_{i-1} \otimes L(-1)^k w_i \otimes w_{i+1} \otimes \ldots w_n + J$$

for $k \in \mathbb{N}$. Since $R$ is Noetherian and $T/J$ is finitely generated, $M_i$ is also finitely generated. Therefore there exists an integer $m$ and elements $a_{p,i} \in R$ such that

$$\begin{aligned}
&1 \otimes w_1 \otimes \ldots \otimes w_{i-1} \otimes L(-1)^m w_i \otimes w_{i+1} \otimes \ldots w_n + J \\
&\quad = \sum_{i=1}^{m} a_{p,i}(z_1, \ldots, z_n; q) \cdot \\
&\qquad \cdot 1 \otimes w_1 \otimes \ldots \otimes w_{i-1} \otimes L(-1)^{m-p} w_i \otimes w_{i+1} \otimes \ldots w_n + J.
\end{aligned} \tag{4.1.4}$$

Note that since the modular weight of $1 \otimes w_1 \otimes \ldots \otimes w_{i-1} \otimes L(-1)^k w_i \otimes w_{i+1} \otimes \ldots w_n + J$ is $\sum_{i=1}^{n} \mathrm{wt}\ w_i + k$, we can choose the element $a_{p,i}$ to have degree $p$. The conclusion follows by applying the map $\psi_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}$ to both sides and using the $L(-1)$ derivative property. ∎

Given a logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, for generalized modules

$W_1$, $W_2$, $W_3$, we will consider the map

$$\begin{aligned}\mathcal{Y}^{(k)} : & W_1 \otimes W_2 \to W_3\{x\}[\log x] \\ & w_1 \otimes w_2 \mapsto \mathcal{Y}(L(0)_n^k w_1, x) w_2\end{aligned}$$

for $k \in \mathbb{N}$. Since $L(0)_n$ acts on $W_1$ as a $V$-module endomorphism, $\mathcal{Y}^{(k)}$ is itself an intertwining operator of the same type. Also, for $j = 1, \ldots, n$ define

$$\begin{aligned}
&Q_j(1 \otimes w_1 \otimes \ldots \otimes w_n) = \\
&\quad w_1 \otimes \ldots \otimes w_{j-1} \otimes L(-2) w_j \otimes w_{j+1} \otimes \ldots \otimes w_n \\
&\quad - \sum_{k=1}^{\infty} \tilde{G}_{2k+2}(q) \cdot \\
&\qquad \cdot w_1 \otimes \ldots \otimes w_{j-1} \otimes L(2k) w_j \otimes w_{j+1} \otimes \ldots \otimes w_n \\
&\quad + \sum_{i \neq j} \sum_{m=1}^{\infty} (-1)^m \tilde{\wp}_{m+1}(z_i - z_j; q) \cdot \\
&\qquad \cdot w_1 \otimes \ldots \otimes w_{i-1} \otimes L(m-1) w_i \otimes w_{i+1} \otimes \ldots \otimes w_n.
\end{aligned}$$

Then, according to (4.1.2), we have

$$\begin{aligned}
&\psi_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}(Q_j(1 \otimes w_1 \otimes \ldots \otimes w_n)) = \\
&\quad \left( (2\pi i)^2 q \frac{\partial}{\partial q} + \tilde{G}_2(q) \sum_{i=1}^{n} \text{wt } w_i + \tilde{G}_2(q) \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} - \sum_{i \neq j} \tilde{\wp}_1(z_i - z_j; q) \frac{\partial}{\partial z_i} \right) \cdot \\
&\qquad \cdot F_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots z_n; q) \\
&\quad + \tilde{G}_2(q) \sum_{i=1}^{n} f(q) F_{\mathcal{Y}_1, \ldots, \mathcal{Y}_{i-1}, \mathcal{Y}_i^{(1)}, \mathcal{Y}_{i+1}, \ldots \mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q). \qquad (4.1.5)
\end{aligned}$$

For $\alpha \in \mathbb{C}$ and $j = 1, \ldots, n$, we define the differential operator

$$\mathcal{O}_j(\alpha) = \left( (2\pi i)^2 q \frac{\partial}{\partial q} + \tilde{G}_2(q) \alpha + \tilde{G}_2(q) \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} - \sum_{i \neq j} \tilde{\wp}_1(z_i - z_j; q) \frac{\partial}{\partial z_i} \right) \qquad (4.1.6)$$

and introduce the notation

$$\begin{aligned}
&\mathcal{D}_j(\alpha) F_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\quad = \mathcal{O}_j(\alpha) F_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q)
\end{aligned}$$

$$+ \tilde{G}_2(q) \sum_{i=1}^{n} F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_{i-1}, L(0)_n w_i, w_{i+1}, \ldots, w_n; z_1, \ldots, z_n; q)$$

and inductively

$$\begin{aligned}
&\left(\prod_{l=1}^{k} \mathcal{D}_j(\alpha_l)\right) F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\quad = \mathcal{O}_j(\alpha_1) \left( \left(\prod_{l=2}^{k} \mathcal{D}_j(\alpha_l)\right) F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \right) \\
&\qquad + \tilde{G}_2(q) \sum_{i=1}^{n} \left(\prod_{l=2}^{k} \mathcal{D}_j(\alpha_l)\right) \cdot \\
&\qquad\qquad \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_{i-1}, L(0)_n w_i, w_{i+1}, \ldots, w_n; z_1, \ldots, z_n; q).
\end{aligned}$$

**Remark 4.1.9.** Using this notation, (4.1.5) can be written as

$$\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(\mathcal{Q}_j(1 \otimes w_1 \otimes \ldots \otimes w_n)) = \mathcal{D}_j\left(\sum_{i=1}^{n} \mathrm{wt}\ w_i\right) \psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(1 \otimes w_1 \otimes \ldots \otimes w_n).$$

We now want to extend $Q_j$ to a map $\mathcal{Q}_j : T \to T$ such that, for any $t \in T$ of modular weight $\alpha$,

$$\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(\mathcal{Q}_j(t)) = \mathcal{D}_j(\alpha) \psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(t).$$

This is necessary since we will need to apply $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$ to repeated iterations of the map $\mathcal{Q}_j$ on the element $1 \otimes w_1 \otimes \ldots \otimes w_n$ for homogeneous elements $w_1, \ldots, w_n$; and while the elements of $T$ obtained this way are not homogeneous in the conformal grading, they have a well defined modular weight.

**Definition 4.1.10.** We define functions $\vartheta_j : R \to R$ for $j = 1, \ldots, n$ in the following way: let

$$\vartheta_j(\tilde{G}_k(q)) = (2\pi i)^2 q \frac{\partial}{\partial q} \tilde{G}_k(q) + k \tilde{G}_2(q) \tilde{G}_k(q),$$

and for the formal series $\tilde{\wp}_m(z_r - z_s; q_\tau)$ with $1 \le r \ne s \le n$ and any $m \ge 2$, and $j = 1, \ldots, n$, we define $\vartheta_j(\tilde{\wp}_m)$ by

$$\begin{aligned}
&\vartheta_j(\tilde{\wp}_m(z_r - z_s; q)) \\
&\quad = (2\pi i)^2 q \frac{\partial}{\partial q} \tilde{\wp}_m(z_r - z_s; q) + m \tilde{G}_2(q) \tilde{\wp}_m(z_r - z_s; q) \\
&\qquad - m \tilde{G}_2(q)(z_r - z_s) \tilde{\wp}_{m+1}(z_r - z_s; q)
\end{aligned}$$

$$+ m\tilde{\wp}_{m+1}(z_r - z_s; q)\left(\tilde{\wp}_1(z_r - z_j; q) - \tilde{\wp}_1(z_s - z_j; q)\right)$$

if $j \notin \{r, s\}$; and by

$$\begin{aligned}
&\vartheta_j(\tilde{\wp}_m(z_j - z_s; q)) \\
&\quad = (2\pi i)^2 q\frac{\partial}{\partial q}\tilde{\wp}_m(z_j - z_s; q) + m\tilde{G}_2(q)\tilde{\wp}_m(z_j - z_s; q) \\
&\quad\quad - m\tilde{G}_2(q)(z_j - z_s)\tilde{\wp}_{m+1}(z_j - z_s; q) \\
&\quad\quad - m\tilde{\wp}_{m+1}(z_j - z_s; q)\tilde{\wp}_1(z_s - z_j; q)
\end{aligned}$$

if $j = r$, and we extend $\vartheta_j$ as a derivation on the ring $R$.

**Proposition 4.1.11.** *Let $\varphi(z_1, \dots, z_n; q) = \vartheta_j(\tilde{\wp}_m(z_r - z_s; q))$; then $\varphi(z_1, \dots, z_n; q_\tau)$ converges uniformly to an elliptic function in the variables $z_1, \dots, z_n$ with possible poles at $z_r = z_s + m\tau + n$, $n, m \in \mathbb{Z}$, $r, s = 1, \dots, n$. Moreover, for any $g \in SL_2(\mathbb{Z})$, if*

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

*we have*

$$\varphi\left(\frac{z_1}{\gamma\tau + \delta}, \dots, \frac{z_n}{\gamma\tau + \delta}; \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = (\gamma\tau + \delta)^{m+2}\,\varphi(z_1, \dots, z_n; \tau).$$

*Proof.* Easy computation using transformation properties of $\wp_m$ and $G_k$ under the action of $SL_2(\mathbb{Z})$. ∎

In particular, $\vartheta_j$ is indeed a map from $R$ to $R$ and if $f \in R$ has modular weight $p$, then $\vartheta_j(f)$ has modular weight $p + 2$. We then consider the $\mathbb{C}$-linear maps $\mathcal{Q}_j : T \to T$ for $j = 1, \dots, n$ defined by

$$\begin{aligned}
&\mathcal{Q}_j(f(q) \otimes w_1 \otimes \ldots \otimes w_n) \\
&\quad = f(q) \cdot Q_j(1 \otimes w_1 \otimes \ldots \otimes w_n) + \vartheta_j(f(q)) \otimes w_1 \otimes \ldots \otimes w_n;
\end{aligned}$$

note that if the modular weight of $t \in T$ is $\alpha$, then the modular weight of $\mathcal{Q}_j(t)$ is $\alpha + 2$.

**Proposition 4.1.12.** *Let $t \in T$ be an element of modular weight $\alpha$: then*

$$\psi_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(\mathcal{Q}_j(t)) = \mathcal{D}_j(\alpha) \cdot \psi_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(t).$$

*Proof.* It is enough to prove this for elements $t$ of the form $\tilde{G}_k(q) \otimes w_1 \otimes \ldots \otimes w_n$ or $\tilde{\wp}_m(z_i - z_j; q) \otimes w_1 \otimes \ldots \otimes w_n$. Suppose $t$ is of the first form with $\sum_{i=1}^n \mathrm{wt}\ w_i = s$ with $k + s = \alpha$. Then

$$\begin{aligned}
&\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(\mathcal{Q}_j(t)) \\
&\quad = \psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(\tilde{G}_k(q)Q_j(1 \otimes w_1 \otimes \ldots \otimes w_n) + \theta(\tilde{G}_k(q)) \otimes w_1 \otimes \ldots \otimes w_n) \\
&\quad = \tilde{G}_k(q)\mathcal{D}_j(s)F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\qquad + \left((2\pi i)^2 q\frac{\partial}{\partial q}\tilde{G}_k(q) + k\tilde{G}_2(q)\tilde{G}_k(q)\right) F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\quad = \mathcal{D}_j(s + k)\tilde{G}_k(q)F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\quad = \mathcal{D}_j(\alpha)\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(t).
\end{aligned}$$

The proof of the other case is a similar computation. ∎

**Proposition 4.1.13.** *Let* $\alpha = \sum_{i=1}^n \mathrm{wt}\ w_i$*; then for any* $s \in \mathbb{N}$*, we have*

$$\begin{aligned}
&\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(Q_j^s(1 \otimes w_1 \otimes \ldots \otimes w_n)) \\
&\quad = \prod_{i=1}^{s} \mathcal{D}_j(\alpha + 2(s - i)) \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q). \qquad (4.1.7)
\end{aligned}$$

*Proof.* We proceed by induction on $s$; the base case $s = 0$ follows by definition of $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$. Now suppose the claim holds for $s-1$; then by (4.1.2), and since $Q^k(1 \otimes w_1 \otimes \ldots \otimes w_n)$ has modular weight equal to $\alpha + 2(k - 1)$,

$$\begin{aligned}
&\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(Q_j^s(1 \otimes w_1 \otimes \ldots \otimes w_n)) \\
&\quad = \mathcal{O}_j(\alpha + 2(s - 1))\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(Q_j^{s-1}(1 \otimes w_1 \otimes \ldots \otimes w_n)) \\
&\qquad + \tilde{G}_2(q)\sum_{i=1}^{n} \psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_{i-1},\mathcal{Y}_i^{(1)},\mathcal{Y}_{i+1},\ldots,\mathcal{Y}_n}(Q_j^{s-1}(1 \otimes w_1 \otimes \ldots \otimes w_n)) \\
&\quad = \mathcal{O}_j(\alpha + 2(s - 1))\left(\prod_{l=1}^{s-1} \mathcal{D}_j(\alpha + 2(s - 1 - l))\right) \cdot \\
&\qquad\qquad \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\qquad + \tilde{G}_2(q)\sum_{i=1}^{n}\left(\prod_{l=1}^{s-1} \mathcal{D}_j(\alpha + 2(s - 1 - l))\right) \cdot \\
&\qquad\qquad \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_{i-1}, L(0)_n w_i, w_{i+1}, \ldots, w_n; z_1, \ldots, z_n; q)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}_j(\alpha + 2(s-1)) \left( \prod_{l=2}^{s} \mathcal{D}_j(\alpha + 2(s-l)) \right) \cdot \\
&\qquad\qquad \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q) \\
&\qquad + \tilde{G}_2(q) \sum_{i=1}^{n} \left( \prod_{l=2}^{s} \mathcal{D}_j(\alpha + 2(s-l)) \right) \cdot \\
&\qquad\qquad \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_{i-1}, L(0)_n w_i, w_{i+1}, \ldots, w_n; z_1, \ldots, z_n; q) \\
&= \prod_{l=1}^{s} \mathcal{D}_j(\alpha + 2(s-l)) \cdot F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q),
\end{aligned}
$$

which concludes the proof. ∎

**Proposition 4.1.14.** *Let $W_1, \ldots W_n$ be generalized modules for the vertex operator algebra $V$, and consider homogeneous $w_i \in W_i$ for $i = 1, \ldots n$; let $\alpha = \sum_{i=1}^{n} \mathrm{wt}\ w_i$. Then there exist elements $b_{p,i} \in R_{2p}$ for $p = 1, \ldots, m$ such that for any $V$-$P$-bimodules $\tilde{W}_j$ and $P$-intertwining operators $\mathcal{Y}_j$ of type $\binom{\tilde{W}_{j-1}}{W_j \tilde{W}_j}$, $j = 1, \ldots, n$, with $\tilde{W}_0 = \tilde{W}_n$ projective as right $P$-module, the series*

$$
\varphi = F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots z_n; q)
$$

*satisfies the differential equations*

$$
\begin{aligned}
&\prod_{l=1}^{m} \mathcal{D}_j(\alpha + 2(m-l))\varphi \\
&\qquad + \sum_{p=1}^{m} b_{p,j}(z_1, \ldots, z_n; q) \prod_{l=1}^{m-p} \mathcal{D}_j(\alpha + 2(m-p-l))\varphi = 0
\end{aligned}
\tag{4.1.8}
$$

*for $j = 1, \ldots, n$, in the region $1 > |q_{z_1}| > \ldots > |q_{z_1}| > |q| > 0$.*
*Moreover, for any $k = 1, \ldots, n$, the series*

$$
\begin{aligned}
&\varphi_k(z_1, \ldots, z_n; q) \\
&\qquad = F^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_{k-1}, L(0)_n w_k, w_{k+1}, \ldots, w_n; z_1, \ldots z_n; q)
\end{aligned}
$$

*also satisfies (4.1.8), for the same choice of elements $b_{p,i}$.*

*Proof.* Consider the $R$-submodule $M$ of $T/J$ generated by the elements

$$
\mathcal{Q}_j^k(1 \otimes w_1 \otimes \ldots \otimes w_n) + J
$$

for $k \in \mathbb{N}$. Since $T/J$ is finitely generated and $R$ is noetherian, $M$ is also finitely generated; therefore, there exists $m$ and elements $b_{p,j}(z_1, \ldots, z_n; q) \in R$, $p = 1, \ldots, m$ such that

$$\begin{aligned} &\mathcal{Q}_j^m(1 \otimes w_1 \otimes \ldots \otimes w_n) \\ &\qquad + \sum_{p=1}^{m} b_{p,j}(z_1, \ldots, z_n; q) \mathcal{Q}_j^{m-p}(1 \otimes w_1 \otimes \ldots \otimes w_n) \in J. \end{aligned}$$

Since the modular weight of $\mathcal{Q}_j^i(1 \otimes w_1 \otimes \ldots \otimes w_n)$ is $\alpha + 2i$, we can choose $b_{p,j}$ in $R_{2p}$; then applying $\psi_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}$ to both sides of the equation and applying (4.1.7), we obtain (4.1.8).

The last part of the proposition follows by applying the first part to the intertwining operators $\mathcal{Y}_1, \ldots, \mathcal{Y}_{k-1}, \mathcal{Y}_k^{(1)}, \mathcal{Y}_{k+1}, \ldots, Y_n$. ∎

**Remark 4.1.15.** Note that the coefficients $b_{p,j}$ only depend on the elements $w_1, \ldots, w_n$; in particular, they do not depend on the choice of algebra $P$ and symmetric function $\phi$ in the definition of the pseudotrace.

**Remark 4.1.16.** For $i_j \in \mathbb{N}$, and $i = 1, \ldots, n$, let

$$\begin{aligned} &\varphi_{i_1,\ldots,i_n}(z_1, \ldots, z_n; q) \\ &\qquad = F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(L(0)_n^{i_1} w_1, \ldots, L(0)_n^{i_n} w_n; z_1, \ldots, z_n; q), \end{aligned}$$

The differential equation (4.1.8) depends on all the functions $\varphi_{i_1,\ldots,i_n}$; therefore, we obtain a system of differential equations of which $\{\varphi_{i_1,\ldots,i_n} | i_j \in \mathbb{N}, j = 1, \ldots, n\}$ is a solution. Since $L(0)_n$ is locally nilpotent, this system of equations is finite; if the modules $W_1, \ldots, W_n$ all have length smaller than $l$, then we obtain a system for the $l(n+1)$ functions $\{\varphi_{i_1,\ldots,i_n} | i_j = 0, \ldots, l, j = 1, \ldots, n\}$. If the modules are ordinary modules the system decouples and we obtain the equations in [H2].

**Proposition 4.1.17.** *The series*

$$F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q_\tau) \tag{4.1.9}$$

*is absolutely convergent in the region* $1 > |q_{z_1}| > \ldots > |q_{z_n}| > |q_\tau| > 0|$ *and can be extended to a multivalued analytic function in the region* $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ *for* $i \neq j$, $l, m \in \mathbb{Z}$.

*Proof.* For fixed $z_1, \ldots, z_n$ such that $|q_{z_1}| > \ldots > |q_{z_n}| > 0$, the coefficients in the variable $q$, $\log q$ of the series

$$F_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q)$$

are absolutely convergent, and the series satisfies a system of differential equations with a regular singular point at $q = 0$ and analytic coefficients. Since the coefficients of the differential equations are analytic functions in $z_1, \ldots, z_n, q_\tau$, with possible singularities in the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j$ and $l, m \in \mathbb{Z}$, the solutions of the system (4.1.8) can be extended to an analytic (multivalued) function in the same region. ∎

We will call *genus-one correlation functions* the analytic extensions of (4.1.9) to the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j$, $l, m \in \mathbb{Z}$, and we will denote them by

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; q_\tau).$$

**Proposition 4.1.18** (Genus-one commutativity)**.** *Let $W_i$ be $V$-modules, $\tilde{W}_i$ be $V$-$P$ bimodules and $\mathcal{Y}_i$ intertwining operators of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_{i-1}}$ for $i = 1, \ldots, n$, with $\tilde{W}_0 = \tilde{W}_n$ projective as right $P$-module. Then for any $k \leq n-1$, there exists a $V$-$P$-bimodules $\hat{W}_k$, and intertwining operators $\hat{\mathcal{Y}}_k$, $\hat{\mathcal{Y}}_{k+1}$ of type $\binom{\hat{W}_k}{W_k \tilde{W}_{k+1}}$, $\binom{\tilde{W}_{k-1}}{W_{k+1} \hat{W}_k}$ such that*

$$\begin{aligned}
&\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; \tau)\\
&\quad = \overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\hat{Y}_k,\mathcal{Y}_{k+2},\ldots\mathcal{Y}_n}(w_1, \ldots, w_{k-1}, w_{k+1}, w_k, w_{k+2} \ldots, w_n;\\
&\qquad\qquad\qquad z_1, \ldots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \ldots, z_n; \tau)
\end{aligned}$$

*as multivalued analytic functions.*

*Proof.* Follows from commutativity for $P$ intertwining operators. ∎

**Proposition 4.1.19** (Genus-one associativity)**.** *Let $W_i$ be $V$-modules, $\tilde{W}_i$ be $V$-$P$-bimodules and $\mathcal{Y}_i$ intertwining operators of type $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_{i-1}}$ for $i = 1, \ldots, n$, with $\tilde{W}_0 = \tilde{W}_n$ projective as right $P$-module. Then for any $k \leq n-1$, there exists a $V$-$P$-bimodules*

*$\hat{W}_k$, and intertwining operators $\hat{\mathcal{Y}}_k$, $\hat{\mathcal{Y}}_{k+1}$ of type $\binom{\hat{W}_k}{W_k W_{k+1}}$, $\binom{\tilde{W}_{k-1}}{\hat{W}_k \tilde{W}_{k+1}}$ such that the series*

$$\begin{aligned}
&\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_n}(w_1,\dots,w_{k-1},\hat{\mathcal{Y}}(w_k,z_k-z_{k+1})w_{k+1},\\
&\qquad\qquad\qquad\qquad w_{k+2},\dots,w_n;z_1,\dots,z_n;\tau)\\
&\quad=\sum_{r\in\mathbb{R}}\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_n}(w_1,\dots,w_{k-1},P_r(\hat{\mathcal{Y}}(w_k,z_k-z_{k+1})w_{k+1}),\\
&\qquad\qquad\qquad\qquad w_{k+2},\dots,w_n;z_1,\dots,z_n;\tau)
\end{aligned}$$

*is absolutely convergent in the region*

$$1>|q_{z_1}|>\dots|q_{z_{k-1}}|>|q_{z_{k+1}}|\dots>|q_{z_n}|>|q_\tau|>0$$

*and $1>|q_{(z_k-z_{k+1})}|>0$ and converges to $\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n;z_1,\dots,z_n;\tau)$ when $1>|q_{z_1}|>\dots>|q_{z_n}|>|q_\tau|>0$ and $|q_{(z_k-z_{k+1})}|>1>|q_{(z_k-z_{k+1})}-1|>0$.*

*Proof.* By associativity for $P$ intertwining operators, there exist a $V$-$P$-bimodule $\tilde{W}_k$ and intertwining operators $\hat{\mathcal{Y}}_k$, $\hat{\mathcal{Y}}_{k+1}$ of type $\binom{\hat{W}_k}{W_k W_{k+1}}$, $\binom{\tilde{W}_{k-1}}{\hat{W}_k \tilde{W}_{k+1}}$ such that for any $z_1,\dots,z_n\in\mathbb{C}$ satisfying $1>|q_{z_1}|>\dots>|q_{z_n}|>0$ and $|q_{z_{k+1}}|>|q_{z_k}-q_{z_{k+1}}|>0$, and for any element $\tilde{w}'_n\in\tilde{W}'_n$, $\tilde{w}_n\in\tilde{W}_n$

$$\begin{aligned}
&\langle w'_n,\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\dots\mathcal{Y}_{k-1}(\mathcal{U}(q_{z_{k-1}})w_{k-1},q_{z_{k-1}})\cdot\\
&\qquad\cdot\hat{\mathcal{Y}}_{k+1}(\mathcal{U}(q_{z_{k+1}})\hat{\mathcal{Y}}_k(w_k,z_k-z_{k+1})w_{k+1},q_{z_{k+1}})\cdot\\
&\qquad\cdot\mathcal{Y}_{k+2}(\mathcal{U}(q_{z_{k+2}})w_{k+2},q_{z_{k+2}})\dots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})\tilde{w}_n\\
&\quad=\langle\tilde{w}'_n,\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\dots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})\tilde{w}_n\rangle
\end{aligned}$$

and therefore as series in $q$ and $\log q$,

$$\begin{aligned}
&\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\dots\mathcal{Y}_{k-1}(\mathcal{U}(q_{z_{k-1}})w_{k-1},q_{z_{k-1}})\cdot\\
&\qquad\cdot\hat{\mathcal{Y}}_{k+1}(\mathcal{U}(q_{z_{k+1}})\hat{\mathcal{Y}}_k(w_k,z_k-z_{k+1})w_{k+1},q_{z_{k+1}})\cdot\\
&\qquad\cdot\mathcal{Y}_{k+2}(\mathcal{U}(q_{z_{k+2}})w_{k+2},q_{z_{k+2}})\dots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)-\frac{c}{24}}\\
&\quad=\mathrm{tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1,q_{z_1})\dots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n,q_{z_n})q^{L(0)-\frac{c}{24}}.
\end{aligned}$$

The right hand side is absolutely convergent when $q=q_\tau$ and $1>|q_{z_1}|>\dots>|q_{z_1}|>|q_\tau|>0$, the right hand side is also absolutely convergent if $|q_{z_k}|>|q_{z_k}-q_{z_{k+1}}|>0$, and satisfies the same system of differential equations as

$$\mathrm{tr}^{\phi}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \dots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q^{L(0)-\frac{c}{24}}.$$

So in this region the left hand side converges absolutely to a function that can be extended to the multivalued analytic function

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n, \tau),$$

which concludes the proof. ∎

## 4.2 Modular invariance of the space of solutions

In this section we consider a space of functions which contains the solutions of the system of differential equations (4.1.8), and we define an action of the group $SL_2(\mathbb{Z})$ on the elements of this space. We prove that the space of solutions of (4.1.8) is invariant under this action.

We will introduce the following notations: let $\chi$ be the space of sequences (indexed by $n$ indices) of analytic multivalued functions in the variables $z_1, \dots, z_n, \tau$, on the region $\Im(\tau) > 0$, $z_i \neq z_j + n\tau + m$ for $n, m \in \mathbb{N}$, $i \neq j$

$$\Phi = \Phi(z_1, \dots, z_n; \tau) = (\phi_{i_1,\dots,i_n}(z_1, \dots, z_n; \tau))_{i_1,\dots,i_n \in \mathbb{N}}$$

with preferred branches on the region $1 > |q_{z_1}| > \dots > |q_{z_n}| > |q_\tau| > 0$, such that $\phi_{i_1,\dots,i_n} \equiv 0$ whenever $\max\{i_1, \dots, i_n\}$ is sufficiently large. We will also denote an element $\Phi$ of $\chi$ by $(\phi_{i_1,\dots,i_n})$ or using a multi-index notation $(\phi_\mu)$ for $\mu$ ranging over $\mathbb{N}^n$. The sum of sequences of this kind is defined component by component, so that if $\Phi^1$, $\Phi^2$ are two elements of $\chi$, the $\mu$-th component of $\Phi^1 + \Phi^2$ is $(\phi^1_\mu + \phi^2_\mu)$ and similarly we can define the product by another analytic function. Moreover, we extend differential operators to $\chi$ component-wise:

$$\left(\frac{\partial}{\partial z_i}\Phi\right)_\mu = \frac{\partial \phi_\mu}{\partial z_i}, \; i = 1, \dots, n.$$

For $j = 1, \dots, n$, let $d_j : \chi \to \chi$ be the shift operator on the $j$-th coordinate defined by

$$(d_j\Phi)_{i_1,\dots,i_n} = \phi_{i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots i_n}.$$

Note that for any $\Phi \in \chi$, $d_j^k \Phi = 0$ if $k$ is large enough; therefore for any function $f(z_1, \dots, z_n; \tau)$, the operator

$$e^{f(z_1,\dots,z_n;\tau)d_j} = \sum_{k=0}^{\infty} \frac{f^k(z_1, \dots, z_n; \tau)}{k!} d_j^k$$

is well defined.

Now, for $\alpha \in \mathbb{C}$ and $j = 1, \dots n$, we define $\mathcal{D}_j(\alpha) : \chi \to \chi$ by

$$\mathcal{D}_j(\alpha) = \mathcal{O}_j(\alpha) + G_2(\tau) \sum_{i=1}^{n} d_i$$

where $\mathcal{O}_j(\alpha)$ is defined as in (4.1.6) (with $(2\pi i)^2 q \frac{\partial}{\partial q} = 2\pi i \frac{\partial}{\partial \tau}$) and extended component-wise to $\chi$.

**Remark 4.2.1.** Using the notation as in remark (4.1.16), for fixed elements $w_i \in W_i$, $i = 1, \dots, n$, and intertwining operators $\mathcal{Y}_1, \dots \mathcal{Y}_n$, one can consider the element of $\chi$

$$\Phi = (\varphi_{i_1,\dots,i_n})_{i_1,\dots,i_n \in \mathbb{N}}.$$

Then if $b_{p,j}(z_1, \dots, z_n; \tau)$ $j = 1, \dots, n$ are defined as in Proposition 4.1.14, by the same proposition we have

$$\left( \prod_{l=1}^{m} \mathcal{D}_j(\alpha + 2(m-l)) \right.$$
$$\left. + \sum_{p=1}^{m} b_{p,j}(z_1, \dots, z_n; q) \prod_{l=1}^{m-p} \mathcal{D}_j(\alpha + 2(m-p-l)) \right) \Phi = 0. \tag{4.2.10}$$

**Definition 4.2.2** ($SL_2(\mathbb{Z})$ action)**.** Let $a \in \mathbb{C}$, and consider an element $g$ in $SL_2(\mathbb{Z})$,

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

For $\Phi \in \chi$, we define

$$\Phi|_{g,a}(z_1, \dots, z_n; \tau)$$
$$= \left( \frac{1}{\gamma\tau + \delta} \right)^a \prod_{i=1}^{n} e^{-\log(\gamma\tau+\delta)d_i} \Phi \left( \frac{z_1}{\gamma\tau + \delta}, \dots, \frac{z_n}{\gamma\tau + \delta}; \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) \tag{4.2.11}$$

**Proposition 4.2.3.** *For any $a \in \mathbb{C}$, (4.2.11) defines an action of the group $SL_2(\mathbb{Z})$ on the space $\chi$.*

We will also use the notation $z'$, $\tau'$ to denote $\frac{z}{\gamma\tau+\delta}$ and $\frac{\alpha\tau+\beta}{\gamma\tau+\delta}$ respectively. Note that for any function $f(z_1,\dots,z_n;q)$, $g\in SL_2(\mathbb{Z})$, and any $\Phi\in\chi$,

$$(f\Phi)|_{g,a}=f(z_1',\dots,z_n',\tau')\Phi|_{g,a};$$

in particular, if $f\in R_p$, then $f(z_1',\dots,z_n';\tau')=(\gamma\tau+\delta)^p f(z_1,\dots,z_n;\tau)$ and thus

$$(f\Phi)|_{g,a}=f(z_1,\dots,z_n,\tau)\Phi|_{g,a-p}.$$

**Proposition 4.2.4.** *Let $\Phi\in\chi$, $a\in\mathbb{C}$ and $g\in G$. Then*

$$\mathcal{D}_j(a)(\Phi|_{g,a})=(\mathcal{D}_j(a)\Phi)|_{g,a+2}. \tag{4.2.12}$$

*Proof.* This is just a straightforward computation, using the transformation properties of the functions $G_2(\tau)$ and $\wp_1(z;\tau)$: for simplicity we use the notation

$$e^d=\prod_{i=1}^n e^{-\log(\gamma\tau+\delta)d_i}$$

$$\begin{aligned}
\mathcal{D}_j(a)(\Phi|_{g,a}) &= \mathcal{D}_j(a)\left(\left(\frac{1}{\gamma\tau+\delta}\right)^a e^d\Phi(z_1',\dots,z_n';\tau')\right)\\
&= \left((2\pi i)\frac{\partial}{\partial\tau}+G_2(\tau)\left(a+\sum_{i=1}^n d_i\right)+G_2(\tau)\sum_{i=1}^n z_i\frac{\partial}{\partial z_i}-\sum_{i\neq j}\wp_1(z_i-z_j;\tau)\frac{\partial}{\partial z_i}\right)\cdot\\
&\qquad\cdot\left(\left(\frac{1}{\gamma\tau+\delta}\right)^a\prod_{i=1}^n e^{-\log(\gamma\tau+\delta)d_i}\Phi(z_1'\dots,z_n';\tau')\right)\\
&= -(2\pi i)\gamma a\left(\frac{1}{\gamma\tau+\delta}\right)^{a+1} e^d\Phi(z_1',\dots,z_n',\tau')\\
&\quad-(2\pi i)\gamma\left(\frac{1}{\gamma\tau+\delta}\right)^{a+1} e^d\sum_{i=1}^n d_i\Phi(z_1',\dots,z_n'\tau')\\
&\quad-(2\pi i)\gamma\left(\frac{1}{\gamma\tau+\delta}\right)^{a+1} e^d\sum_{i=1}^n z_i'\frac{\partial\Phi}{\partial z_i}(z_1',\dots,z_n',\tau')\\
&\quad+(2\pi i)\left(\frac{1}{\gamma\tau+\delta}\right)^{a+2} e^d\frac{\partial\Phi}{\partial\tau}(z_1'\dots,z_n',\tau')\\
&\quad+\left(G_2(\tau')\left(\frac{1}{\gamma\tau+\delta}\right)^2+2\pi i\gamma\left(\frac{1}{\gamma\tau+\delta}\right)\right)\left(a+\sum_{i=1}^n d_i\right)\cdot\\
&\qquad\cdot\left(\frac{1}{\gamma\tau+\delta}\right)^a e^d\Phi(z_1',\dots,z_n';\tau')\\
&\quad+\left(G_2(\tau')\left(\frac{1}{\gamma\tau+\delta}\right)^2+2\pi i\gamma\left(\frac{1}{\gamma\tau+\delta}\right)\right)\cdot
\end{aligned}$$

$$\cdot \left(\frac{1}{\gamma\tau+\delta}\right)^{a} e^{d} \sum_{i=1}^{n} z_i' \frac{\partial \Phi}{\partial z_i}(z_1', \ldots, z_n', \tau')$$
$$- \left(\frac{1}{\gamma\tau+\delta}\right)^{a+2} e^{d} \sum_{i \neq j} \wp_1(z_i' - z_j'; \tau') \frac{\partial \Phi}{\partial z_i}(z_1', \ldots, z_n', \tau')$$
$$= \left(\frac{1}{\gamma\tau+\delta}\right)^{a+2} e^{d} \cdot$$
$$\cdot \left( (2\pi i) \frac{\partial \Phi}{\partial \tau}(z_1', \ldots, z_n', \tau') + G_2(\tau') \left(a + \sum_{i=1}^{n} d_i\right) \Phi(z_1', \ldots, z_n'; \tau') \right.$$
$$\left. + G_2(\tau') \sum_{i=1}^{n} z_i' \frac{\partial \Phi}{\partial z_i}(z_1', \ldots, z_n', \tau') - \sum_{i \neq j} \wp_1(z_i' - z_j'; \tau') \frac{\partial \Phi}{\partial z_i}(z_1', \ldots, z_n', \tau') \right)$$

which is equal to $(\mathcal{D}_j(a)\Phi)|_{g,a+2}$, concluding the proof. ∎

We can then prove the following

**Proposition 4.2.5.** *Let $\Phi$ be a solution of the system of differential equations (4.2.10). Then for any $g \in SL_2(\mathbb{Z})$, $\Phi|_{g,\alpha}$ is also a solution of the same system.*

*Proof.* Just apply $|_{g,\alpha+2m}$ to both sides of (4.2.10); since $b_{p,j}$ belongs to $R_{2p}$, we find

$$\left( b_{p,j}(z_1, \ldots, z_n; q) \prod_{l=1}^{m-p} \mathcal{D}_j(\alpha + 2(m-p-l))\Phi \right)\Bigg|_{g,\alpha+2m}$$
$$= b_{p,j}(z_1, \ldots, z_n; q) \left( \prod_{l=1}^{m-p} \mathcal{D}_j(\alpha + 2(m-p-l))\Phi \right)\Bigg|_{g,\alpha+2(m-p)}.$$

Now applying (4.2.12) several consecutive times, we obtain

$$\left( \prod_{l=1}^{m} \mathcal{D}_j(\alpha + 2(m-l)) \right.$$
$$\left. + \sum_{p=1}^{m} b_{p,j}(z_1, \ldots, z_n; q) \prod_{l=1}^{m-p} \mathcal{D}_j(\alpha + 2(m-p-l)) \right) \Phi|_{g,\alpha} = 0$$

which concludes the proof. ∎

## 4.3 Towards modular invariance for intertwining operators

In this section we will sketch the roadmap to prove modular invariance for intertwining operators using the results from the previous sections. We recall properties of the $N$-th

Zhu's algebra $A_N(V)$ (see [DLM1], [HY]) and of the $A_N(V)$-bimodule $A(W)$ associated with a grading restricted generalized $V$-module $W$ ([HY]). Fix a positive integer $N$ and a vertex operator algebra $V$. We can define a product $*_N$ on $V$ by

$$u *_N v = \sum_{m=0}^{N} (-1)^m \binom{m+N}{N} \mathrm{Res}_x x^{-N-m-1} \mathcal{Y}((1+x)^{L(0)+N} u, x) v$$

for $u, v \in V$; let $O_N(V)$ be the subspace of $V$ spanned by the elements of the form $(L(-1) + L(0))u$ for $u \in V$ and

$$\mathrm{Res}_x x^{-2N-1-n} Y((1+x)^{L(0)+N} u, x) v$$

Then $O_N(V)$ is a two sided ideal of $V$ under $*_N$, and the product $*_N$ defines a structure of an associative algebra on $A_N(V) = V/O_N(V)$ with identity element $\mathbf{1} + O_N(V)$; moreover, the element $\omega + O_N(V)$ belongs to the center of $A_N(V)$.

For any left $A_N(V)$-module $U$, we can construct a lower bounded generalized $V$-module $S_N(U)$ such that the $N-th$ graded piece of $S_N(U)$ is equal to $U$ and for any $v \in V$, $o(v)$ acts on this piece by the action given by $A_N(V)$ on $U$ (for details, see [DLM1], [HY]).

We still need to prove that given an $n$-point genus-one function

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau),$$

and any element $g \in SL_2(\mathbb{Z})$,

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

the function

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}\left((\gamma\tau+\delta)^{-L(0)} w_1, \dots, (\gamma\tau+\delta)^{-L(0)} w_n; z_1', \dots, z_n'; \tau'\right)$$

can be expressed as linear combination of $n$-point functions

$$\overline{F}^{\phi_i}_{\mathcal{Y}_{i,1},\dots,\mathcal{Y}_{i,n}}(w_1, \dots, w_n; z_1, \dots, z_n; \tau),$$

for certain intertwining operator $\mathcal{Y}_{i,n}$ and symmetric functions $\phi_i$. Using genus-one associativity, we can reduce this problem to the case of 1-point functions

$$\overline{F}^{\phi}_{\mathcal{Y}}(w; z; \tau)$$

for intertwining operator $\mathcal{Y}$ of type $\binom{\tilde{W}}{W,\tilde{W}}$. For these 1-point functions we have

$$\frac{\partial}{\partial z}\overline{F}^{\phi}_{\mathcal{Y}}(w;z;\tau) = 0.$$

Then by Proposition 4.2.4, the function

$$\begin{aligned}
&\overline{F}^{\phi}_{\mathcal{Y}}((\gamma\tau+\delta)^{-L(0)}w;z;\tau)\\
&\quad = \left(\frac{1}{\gamma\tau+\delta}\right)^{\mathrm{wt}\ w}\sum_{i=0}^{\infty}\frac{(-\log(\gamma\tau+\delta))^i}{i!}\overline{F}^{\phi}_{\mathcal{Y}}(L(0)_n^i w;z;\tau)
\end{aligned}$$

satisfies the system (4.1.8) with $\alpha = \mathrm{wt}\ w$. Therefore,

$$\overline{F}^{\phi}_{\mathcal{Y}}((\gamma\tau+\delta)^{-L(0)}w;z;\tau) = \sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{m\in\mathbb{N}} C_{k,l,m}(w)\tau^k q_\tau^{r_l+m}$$

for some numbers $r_l \in \mathbb{R}$ such that $r_i - r_j \notin \mathbb{Z}$ if $i \neq j$. Let $S_{k,l,m}(w) = C_{k,l,m}(\mathcal{U}(1)^{-1}w)$.

**Lemma 4.3.1.** *We have $\mathcal{U}(1)O_N(W) \subseteq O_N(W)$; moreover, the map $S_{0,l,N}(w)$ defines a symmetric function on the $A_N(V)$-bimodule $A(W)$ such that*

$$S_{0,l,N}\left(\left(\omega - N - r_l - \frac{c}{24}\right)^s \star w\right) = 0.$$

Let $s \in \{0,\ldots,L\}$: by Theorem 2.3.5, there exist basic symmetric algebras $P_i$ with symmetric functions $\phi_i$, $A_N(V)$-$P_i$-bimodules $U_i$ and functions $f_i \in \mathrm{Hom}_{A_N(V),P_i}(A(W)\otimes_{A_N(V)} U_i, U_i)$, $i = 1,\ldots,n$ such that $\left(\omega - N - r_s - \frac{c}{24}\right)^s U_i = 0$ and

$$S_{0,s,N}(w) = \sum_{i=1}^{n}\phi^{f_i}_{U_i}(w)$$

for all $w \in W$.

For any $i = 1,\ldots,n$, let $S_N(U_i)$ be the $V$-module generated by the $A_N(V)$-module $U_i$: from the construction of $S_N(U_i)$, it is easy to see that the action of $P_i$ on $U_i$ extends to a right action of $P_i$ on $S_N(U_i)$ which commutes with the action of $V$. Then by the results in [Miy2] and [Ar], we have

**Proposition 4.3.2.** *Suppose all the irreducible $V$-modules are infinite dimensional; then if $N$ is large enough, then the $V$-module $S_N(U_i)$ is projective as right $P_i$-module.*

Although the infinite dimensionality condition on the irreducible modules is not a natural one, it holds in particular for all vertex operator algebras of central charge zero.

Using the results in [HY], we can construct logarithmic intertwining operators $\mathcal{Y}^{f_i}$ of type $\binom{S_N(U_i)}{W, S_N(U_i)}$ for $i = 1, \ldots, n$ such that for all $w \in W$ and $u \in U_i$,

$$\mathrm{Res}_x x^{\mathrm{wt}\ w-1} \mathcal{Y}^{f_i}(w, x)u = f_i(w \otimes u)$$

(here Res denotes the coefficient of the monomial in $x^{-1} \log(x)^0$). Then

$$\begin{aligned} &\bar{F}^{\phi}_{\mathcal{Y}}((\gamma\tau + \delta)^{-L(0)} w; z; \tau) - \sum_{i=1}^{n} \bar{F}^{\phi_i}_{\mathcal{Y}_i}(w; z; \tau) \\ &= \sum_{k=0}^{K} \sum_{l=0}^{L} \sum_{m \in \mathbb{N}} \tilde{C}_{k,l,m}(w) \tau^k q_\tau^{r_l + m} \end{aligned}$$

with $\tilde{C}_{0,s,N}(w) = 0$ for all $w \in W$. Repeating this argument several times and using the fact that vertex algebras satisfying the $C_2$-cofiniteness condition have finitely many irreducible modules, we find that there exist symmetric basic algebras $P_i$, equipped with symmetric linear functions $\phi_i$, $V$-$P$-bimodules $W_i$, projective as right $P$-modules, and intertwining operators $\mathcal{Y}_i$ of type $\binom{W_i}{W, W_i}$, for $i = 1, \ldots, t$, such that

$$\bar{F}^{\phi}_{\mathcal{Y}}((\gamma\tau + \delta)^{-L(0)} w; z; \tau) = \sum_{i=1}^{t} \bar{F}^{\phi_i}_{\mathcal{Y}_i}(w; z; \tau).$$

# References


[AF] F. Anderson and K. Fuller, *R*ings and Categories of Modules, Graduate Texts in Mathematics 13, Springer-Verlag, New York, Heidelberg, Berlin (1991).

[AM1] D. Adamović and A. Milas, Logarithmic intertwining operators and $\mathcal{W}(2,2p-1)$-algebras, *Journal of Math. Physics* **48**, 073503 (2007).

[AM2] D. Adamović and A. Milas, On the triplet vertex algebra $\mathcal{W}(p)$, *Adv. Math.* **217**, 2664-2699 (2008).

[AM3] D. Adamović and A. Milas, On $\mathcal{W}$-algebras associated to $(2,p)$ minimal models and their representations, *IMRN* **20**, 3896-3934 (2010).

[AM4] D. Adamović and A. Milas, An analogue of modular BPZ equation in logarithmic (super)conformal field theory, *Contemp. Math.* **497**, (2009) 1-17.

[Ar] Y. Arike, Some remarks on symmetric linear functions and pseudotrace maps, *Proc. Japan Acad. Ser. A Math. Sci.* **86** (2010), 119-124.

[AN] Y. Arike and K. Nagatomo, Some remarks on pseudo-trace functions for orbifold models associated with symplectic fermions. *Int. J. Math.* **24**, (2013) 1350008

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci.* **83**, 3068-3071 (1986).

[Br] M. Broué, Higman criterion revisited, *Mich. J. Math.* **58**, (2009) 125-179.

[DLM1] Y. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206**, 67-96 (1998).

[DLM2] Y. Dong, H. Li and G. Mason, Modular invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214**, 1-56 (2000).

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs Amer. Math. Soc.*, **104,** 1993.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, New York, 1988.

[H1] Y.-Z. Huang, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, Vol. 148, Birkhäuser, Boston, 1997.

[H2] Y.-Z. Huang, Differential equations, duality and modular invariance, *Comm. Contemp. Math.* **7** (2005), 649-706.

[H3] Y.-Z. Huang, A theory of tensor product for module categories for a vertex operator algebra, IV, *J. Pure Appl. Algebra* **100** (1995), 173-216.

[H4] Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, *Comm. Contemp. Math.* **10** (2008), 103–154.

[H5] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.* **10** (2008), 871–911.

[HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor product for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699-756.

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor product for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757-786.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor product for module categories for a vertex operator algebra, III, *J. Pure Appl. Algebra* **100** (1995), 141-171.

[HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, in: *Conformal Field Theories and Tensor Categories, Proceedings of a Workshop Held at Beijing International Center for Mathematics Research*, ed. C. Bai, J. Fuchs, Y.-Z. Huang, L. Kong, I. Runkel and C. Schweigert, Mathematical Lectures from Beijing University, Vol. 2, Springer, New York, 2014, 169-248.

[HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators, arXiv:1012.4196.

[HLZ3] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, III: Intertwining maps and tensor product bifunctors, arXiv:1012.4197.

[HLZ4] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, IV: Constructions of tensor product bifunctors and the compatibility conditions, arXiv:1012.4198.

[HLZ5] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, V: Convergence condition for intertwining maps and the corresponding compatibility conditions, arXiv:1012.4199.

[HLZ6] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms, arXiv:1012.4202.

[HLZ7] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps, arXiv:1110.1929.

[HLZ8] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra, arXiv:1110.1931.

[HY] Y.-Z. Huang and J. Yang, Logarithmic intertwining operators and associative algebras, *J. Pure Appl. Alg.* **216** (2011), 1467-1492.

[L] S. Lang, *Elliptic functions*, Graduate Texts in Mathematics, Vol. 112, Springer-Verlag, New York, 1987.

[LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, VOl. 227, Birkhäuser, Boston, 2003.

[M1] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory*, ed. S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemp. Math., Vol. 297, American Mathematical Society, Providence, RI, 2002, 201225.

[M2] A. Milas, Logaritmic intertwining operators and vertex operators, *Comm. Math.Phys.* **277** (2008), 497-529.

[Miy1] M. Miyamoto, Intertwining operators and modular invariance, preprint, arXiv:math.QA/0010180

[Miy2] M. Myiamoto, Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness, *Duke Math. J.* **122** (2004), 51-91.

[MS1] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177-254.

[MS2] G. Moore and N. Seiberg, Polynomial equations for rational conformal field theories, *Phys. Lett.* **B212** (1988), 451-460.

[W] C. Weibel *An Introduction to Homological Algebra*, Cambridge Studies in Adv. Math., Vol 38, Cambridge University Press, Cambridge, 1994.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-307.