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Logarithmic Level Comparison for Small Deviation Probabilities

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Log-level comparisons of the small deviation probabilities are studied in three different but related settings: Gaussian processes under the L^2 norm, multiple sums motivated by tensor product of Gaussian processes, and various integrated fractional Brownian motions under the sup-norm.

KEY WORDS: Small deviation probability; Gaussian process; tensor.
Classifications: 60G15; 60G10

1. INTRODUCTION

For a given continuous random process $X(t), t \in [0, 1]$, the small deviation probability concerns the asymptotic behavior of $\mathbb{P}(\|X\| < \varepsilon)$ as $\varepsilon \to 0^+$, where $\|\cdot\|$ is a norm on the space $C([0, 1])$. In the literature, small deviation probabilities of various types are studied and applied to many problems of interest under different names such as small ball probability, lower tail behaviors, two sided boundary crossing probability and the first exit time, etc. The survey paper of Ref. 18 for Gaussian processes, together with its extended references, covers much of the recent progress in this area. In particular, various applications and connections with other areas of probability and analysis are discussed.

In this paper, we study the log-level comparison of the type

$$\log \mathbb{P}(\|X\| < \varepsilon) \sim C \log \mathbb{P}(\|Y\| < \varepsilon)$$

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or

$$\log \mathbb{P}(\|X\| < C\varepsilon) \sim \log \mathbb{P}(\|Y\| < \varepsilon),$$

under easy to verify conditions on centered Gaussian processes X and Y, where the constant $C = C(\|\cdot\|, X, Y) \in (0, \infty)$. It is important to note that the main results in many works in this area determine only the log-level asymptotic behavior up to some constant factor in front of the rate. So it is very interesting and useful to find log-level comparison results with explicit constants. In many applications, one needs the small deviation rate and constant at logarithmic level. Our main results are in three different but related settings. Our methods of proofs are all different and can be applied to various related problems.

In the first setting, we consider the L_2 norm $\|\cdot\|_2$, arguably the simplest and well-studied case. By Karhunen–Loéve expansion, we have $\|X\|_2^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2$ where λ_n are the eigenvalues of the associated covariance operator. and ξ_n are i.i.d. standard normal random variables. Once the eigenvalues are known, the small deviation probability can be estimated (at least in principle) using a result of Ref. 21 see (2.1). However, eigenvalues are rarely found exactly. Often, one only knows the asymptotic approximation. Thus, a natural question is to study the relation between the small deviation of the original process and the one with approximated eigenvalues $\tilde{\lambda}_n$. This line of research started in Ref. 15 and continues in Refs. 9 and 11. (See also Refs. 13 and 7). Roughly speaking, the small deviation probabilities under L_2 norms are comparable if the infinite product $\prod \lambda_n/\tilde{\lambda}_n$ converges. Although under certain assumptions, there are complex analytic methods that enable one to find the aforementioned infinite product directly, without computing the eigenvalues (cf. Refs. 8 and 9), the typical case is that one has some rough estimate of the eigenvalues, which is not good enough to ensure the convergence of the infinite product. This is particularly true for multi-parameter processes. In Section 2, we show that the log-level comparison with constant holds under comparable finite rank approximation.

In the second setting, we consider multiple sums motivated by tensor product of Gaussian processes. The methods presented are general enough to handle nonnegative random variables other than squared L_2-norms of tensored Gaussian processes. Similar question has been studied by Karol *et al.*[13] and Fill and Torcaso[7] for tensored Gaussian random fields under the L_2-norm. However, our probabilistic argument allow us to handle the case that has been left open by using their methods.

In the third setting, we consider the comparison of small deviation under the sup-norm which is usually harder and more interesting. There seems to be no known method that handles the general case. So we only deal with the comparison among various integrated fractional Brownian motions which were studied recently under the L_2-norm in Refs 4, 7, 9, 10, 13, 19, and 20. Our method here works for general norms such as the sup-norm and the L_p-norm. It is a combination of techniques developed in Refs. 4, 16, and 17. More details are given in Section 4. In general, our method provides a systematic approach to log-level comparisons under general norms.

2. L_2-NORM

Given a continuous Gaussian process $X(t), t \in [0, 1]$, we have by Karhunen–Loéve expansion

$$\|X\|_2^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2$$

where λ_n are the eigenvalues of the associated covariance operator.

$$\mathcal{K} f(t) = \int_0^1 \sigma(t, s) f(s) ds, \quad \sigma(t, s) = \mathbb{E} X(t) X(s)$$

and ξ_n are i.i.d. standard normal random variables. Once the eigenvalues are known, the small deviation probability can be estimated (at least in principle) by using the following result of Ref. 21. Namely,

$$\mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leqslant \varepsilon^2\right) \sim (-2\pi t^2 h''(t))^{-1/2} \exp\{t h'(t) - h(t)\}, \tag{2.1}$$

where $h(t) = \frac{1}{2} \sum_{n=1}^{\infty} \log(1 + 2\lambda_n t)$ and $\varepsilon^2 = h'(t)$. This is the starting point of our result in this section.

Theorem 2.1. Let X and Y be two Gaussian processes with eigenvalues $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \cdots$ and $b_1 \geqslant b_2 \geqslant \cdots b_n \geqslant \cdots$, respectively. Suppose $\sum_{n>N} a_n \sim C^2 \sum_{n>N} b_n \sim r(N)$ where r is a decreasing function satisfying

$$\lim_{(a,x) \to (1,\infty)} \frac{r'(\alpha x)}{r'(x)} = 1 \quad \text{and} \quad r(x) = O(x r'(x)) \ as \ x \to \infty.$$

Then

$$\log \mathbb{P}(\|X\|_2 < C\varepsilon) \sim \log \mathbb{P}(\|Y\|_2 < \varepsilon).$$

Proof. We first need some analytic facts based on our assumptions. Given $\alpha > 0, \alpha \neq 1$, let N be large enough, so that $[\alpha N] \neq N$. If $\alpha > 1$, then

$$\sum_{n=N+1}^{[\alpha N]} a_n = r(N) - r([\alpha N]) + o(1) \cdot r(N) = ([\alpha N] - N) r'(\beta N) + o(1) \cdot r(N),$$

where $1 < \beta < \alpha$. Because $\{a_n\}$ is non-increasing, we have

$$a_N \geqslant -r'(\beta N) + o(1) \cdot \frac{r(N)}{[\alpha N] - N}.$$

Letting $N \to \infty$, and then $\alpha \to 1^+$, we have

$$\liminf_{N\to\infty} \frac{a_N}{-r'(N)} \geqslant \lim_{\alpha\to 1^+} \liminf_{N\to\infty} \frac{r'(\beta N)}{r'(N)} = 1.$$

If $\alpha < 1$, then

$$\sum_{n=[\alpha N]+1}^{N} a_n = (N - [\alpha N]) r'(\theta N) + o(1) \cdot r([\alpha N])).$$

Using the monotonicity of a_n, we have

$$a_N \leqslant -r'(\theta N) + o(1) \frac{r([\alpha N])}{N - [\alpha N]}.$$

Letting $N \to \infty$ and then $\alpha \to 1^-$, we obtain

$$\limsup_{N\to\infty} \frac{a_N}{-r'(N)} \leqslant \lim_{\alpha\to 1^-} \limsup_{N\to\infty} \frac{r'(\theta N)}{r'(N)} = 1.$$

Hence, $a_N \sim -r'(N)$. Similarly, we have $b_N \sim -C^{-2} r'(N)$. Therefore $a_n \sim C^2 b_n$.

Next, we show that $a_n \sim C^2 b_n$ and $r(x) = O(x r'(x))$ imply

$$\log \mathbb{P}(\|X\|_2 < C\varepsilon) \sim \log \mathbb{P}(\|Y\|_2 < \varepsilon)$$

as $\varepsilon \to 0^+$. To this end, we note that by the result of Sytaya mentioned earlier, we have

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leqslant \varepsilon^2\right) \sim \gamma h_a'(\gamma) - h_a(\gamma) - \frac{1}{2}\log(2\pi\gamma^2 h''(\gamma)). \tag{2.2}$$

Because $h'' < 0$, and $h''' > 0$, we have

$$|\gamma h'(\gamma) - h(\gamma)| = -\int_0^\gamma t h''(t) dt \geqslant -\int_0^\gamma t h''(\gamma) dt \geqslant -\gamma^2 h''(\gamma)/2.$$

Thus, the third term on the right-hand side of (2.2) is of smaller order, and we have

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leqslant \varepsilon^2\right) \sim \gamma h_a'(\gamma) - h_a(\gamma). \tag{2.3}$$

By otherwise considering a_n/C^2, we can assume $a_n \sim b_n$. For any small $\varepsilon > 0$, let t and s be chosen such that $h_a'(t) = h_b'(s) = \varepsilon^2$. Note that $h_b''(s)ds/dt = h_a''(t)$. By L'Hospital's rule, we have

$$\frac{\log \mathbb{P}\left(\sum_{n=1}^{\infty} b_n \xi_n^2 \leqslant \varepsilon^2\right)}{\log \mathbb{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \leqslant \varepsilon^2\right)} \sim \frac{-sh_b'(s) + h_a(s)}{-th_a'(t) + h_a(t)} \sim \frac{-sh_b''(s)ds/dt}{-th_a''(t)} = \frac{s}{t} \sim 1,$$

provided that we show $t \sim s$.

To show $t \sim s$, we study the equation $h_a'(t) = h_b'(s)$, that is,

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{-1} + 2t} = \sum_{n=1}^{\infty} \frac{1}{b_n^{-1} + 2s}. \tag{2.4}$$

For $0 < \delta < 1$. Because $a_n \sim b_n$, there exists N_0 such that for $n > N_0$, $|a_n^{-1} - b_n^{-1}| < \delta b_n^{-1}$. For t fixed, choose N_1, N_2, so that $t \leqslant a_{N_1}^{-1} < 2t$, and $s \leqslant b_{N_2}^{-1} < 2s$. Without loss of generality, we assume $N_1 < N_2$. Thus, $s < 2(1+\delta)t$. By choosing t large enough we can assume $N_1 > 2N_0$ and $a_n > r'(n)$ for $n \geqslant N_1$. From (2.4) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|t-s|}{\left(a_n^{-1} + 2t\right)\left(b_n^{-1} + 2s\right)} &\leqslant \sum_{n=1}^{\infty} \frac{|a_n^{-1} - b_n^{-1}|}{\left(a_n^{-1} + 2t\right)\left(b_n^{-1} + 2s\right)} \\
&\leqslant \frac{N_0\left(a_{N_0}^{-1} + b_{N_0}^{-1}\right)}{4ts} + \sum_{n > N_0} \frac{\delta}{a_n^{-1} + 2t} \\
&\leqslant \frac{N_0\left(a_{N_0}^{-1} + b_{N_0}^{-1}\right)}{4ts} + \sum_{n \leqslant N_1} \frac{\delta}{2t} + \sum_{n > N_1} \frac{\delta}{a_n^{-1}} \\
&\leqslant \frac{N_0\left(a_{N_0}^{-1} + b_{N_0}^{-1}\right)}{4ts} + \delta N_1 a N_1 + \delta r\left(N_1\right).
\end{aligned}$$

Because $a_n \sim -r'(n)$ and $r(n) = O(nr'(n))$, there exists $M > 0$ such that

$$\left|1 - \frac{s}{t}\right| \sum_{n=1}^{\infty} \frac{t}{\left(a_n^{-1} + 2t\right)\left(b_n^{-1} + 2s\right)} \leqslant \frac{N_0\left(a_{N_0}^{-1} + b_{N_0}^{-1}\right)}{4ts} + M\delta N_1 a_{N_1}. \tag{2.5}$$

Note that

$$\sum_{n=1}^{\infty}\frac{t}{\left(a_n^{-1}+2t\right)\left(b_n^{-1}+2s\right)} \geqslant \sum_{n=N_0+1}^{N_1}\frac{t}{\left(a_n^{-1}+2t\right)\left(b_n^{-1}+2s\right)}$$
$$\geqslant \frac{N_1-N_0}{16\,(1+\delta)\,t}$$
$$\geqslant \frac{1}{128}N_1 a_{N_1}.$$

Also, it is easy to check that

$$\frac{N_0(a_{N_0}^{-1}+b_{N_0}^{-1})}{4ts}=o(1)\cdot\sum_{n=1}^{\infty}\frac{t}{(a_n^{-1}+2t)(b_n^{-1}+2s)}.$$

Thus, from (2.5) we obtain

$$\limsup_{t\to\infty}\left|1-\frac{s}{t}\right|\leqslant 128M\delta.$$

Because δ is arbitrary, we have $t\sim s$. This proves the theorem. □

We would like to remark that in Theorem 2.1 the two conditions on $r(x)$ are weak, and can be easily satisfied in most of the applications. Indeed, the first condition essentially says that $r(x)$ does not go to 0 too slow (at logarithmic level); while the second condition requires that it does not go to 0 too fast (exponentially). Readers interested in operator theory may have noticed that $r(N)$ is closely related to the so-called s-number, and is a measurement of the compactness of the covariance operator. When $r(N)$ decreases slowly, the operator is less compact, the corresponding Gaussian process is "less continuous", and has smaller small deviation probability; when $r(N)$ decreases fast, the covariance operator is closer to a finite rank operator, the corresponding Gaussian process is "smoother" and has larger small deviation probability.

Two cases that are not covered by the theorem above are: (a) $a_n\sim Cn^{-1}[\log(n+1)]^{\beta}$ with $\beta<-1$; and (b) $\log a_n\sim -Cn^{\alpha}[\log(n+1)]^{\beta}$. The former case, the small deviation is super exponentially small, thus does not have sufficient interest in application. For the latter case, we have the following

Theorem 2.2. Let X and Y be two Gaussian processes with eigenvalues $a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant\cdots$ and $b_1\geqslant b_2\geqslant\cdots\geqslant b_n\geqslant\cdots$, respectively. Suppose $\log a_n\sim -n^{\alpha}J(n)$, where $J(x)$ is a slow varying function, then

$$\log\mathbb{P}(\|X\|_2<\varepsilon)\sim\frac{-\alpha 2^{1/\alpha}}{\alpha+1}\,\frac{\log^{1/\alpha+1}1/\varepsilon}{[J(\log^{1/\alpha}1/\varepsilon)]1/\alpha}.$$

Thus, if $\log b_n \sim C \log a_n$, then

$$\log \mathbb{P}(\|Y\|_2 < \varepsilon) \sim C^{-1/\alpha} \cdot \log \mathbb{P}(\|X\|_2 < \varepsilon).$$

Proof. For $0 < \delta < 1$ and large t, let N be smallest integer such that $a_N^{-1} < \delta t$. Then

$$\frac{N}{(2+\delta)t} \leqslant \sum_{n=1}^{\infty} \frac{1}{a_n^{-1} + 2t} \leqslant \frac{N}{2t}.$$

Because $\log a_n \sim -n^\alpha J(n)$, we have $N^\alpha J(N) \sim \log(\delta t)$, which implies

$$N \sim \left(\frac{\log(\delta t)}{J(\log^{1/\alpha}(\delta t))} \right)^{1/\alpha}.$$

Also, note that

$$\sum_{n>N} \frac{1}{a_n^{-1} + 2t} \leqslant \sum_{n>N} a_n = O\left(a_N\right) = O\left(\frac{1}{\delta t}\right) = o(1) \cdot \sum_{n=1}^{N} \frac{1}{a_n^{-1} + 2t}.$$

Thus,

$$\frac{1+o(1)}{(2+\delta)t} \left(\frac{\log(\delta t)}{J(\log^{1/\alpha}(\delta t))} \right)^{1/\alpha} \leqslant h_a'(t) \leqslant \frac{1+o(1)}{2t} \left(\frac{\log(\delta t)}{J(\log^{1/\alpha}(\delta t))} \right)^{1/\alpha}.$$

Because δ is arbitrary, we conclude that

$$h_a'(t) \sim \frac{1}{2t} \left(\frac{\log t}{J(\log^{1/\alpha} t)} \right)^{1/\alpha},$$

which implies that

$$h(t) \sim \frac{\alpha}{2(\alpha+1)} \frac{\log^{1/\alpha+1} t}{[J(\log^{1/\alpha}(t))]^{1/\alpha}}.$$

Clearly, $th'(t) = o(h(t))$. Thus, by (2.3) we have

$$\log \mathbb{P}(\|X\|_2 < \varepsilon) \sim -h(t),$$

where t satisfies $h_a'(t) = \varepsilon^2$. By the asymptotic estimate of $h_a'(t)$ obtained above, we have

$$t \sim 2^{1/\alpha - 1} \frac{1}{\varepsilon^2} \left(\frac{\log 1/\varepsilon}{J(\log^{1/\alpha} 1/\varepsilon)} \right)^{1/\alpha}.$$

Hence,

$$\log \mathbb{P}(\|X\|_2 < \varepsilon) \sim \frac{\alpha 2^{1/\alpha}}{\alpha+1} \frac{\log^{1/\alpha+1} 1/\varepsilon}{[J(\log^{1/\alpha} 1/\varepsilon)]^{1/\alpha}}.$$

□

3. MULTIPLE SUMS

In this section, we present a probabilistic comparison arguments for multiple sums of independent random variables. This is motivated by the study of the small deviation probabilities for tensored Gaussian random fields under the L_2-norm. Suppose we have two centered Gaussian processes $X(t)$ and $Y(t)$ on [0, 1] with continuous covariance function $\sigma_X(s,t)$ and $\sigma_Y(s,t)$, respectively. Then the tensored Gaussian process $X \otimes Y(t_1, t_2)$ on $[0, 1]^2$ has mean zero and continuous covariance function

$$\sigma_{X\otimes Y}((s_1, s_2), (t_1, t_2)) = \sigma_X(s_1, t_1) \cdot \sigma_Y(s_2, t_2), \quad 0 \leqslant s_1, s_2, t_1, t_2 \leqslant 1.$$

It is well known that $X \otimes Y(t_1, t_2)$ on $[0, 1]^2$ is continuous if $X(t)$ and $Y(t)$ are continuous on [0, 1], based on work initiated in Refs. 5 and 6 (see also Ref. 3). Detailed information can also be found in Ref. 14.

In particular, we have the following series representation. Assume the well-known Karhunen-Loève expansion

$$X(t) = \sum_{n\geqslant 1} a_n^{1/2} \xi_n e_n(t)$$
$$Y(t) = \sum_{m\geqslant 1} b_m^{1/2} \xi_m h_m(t)$$

where ξ_i denotes as usual i.i.d $N(0,1)$ sequences, $\{e_n(t), n \geqslant 1\}$ and $\{h_m(t), m \geqslant 1\}$ are complete orthonormal bases in $L_2[0, 1]$. Then we have Karhunen-Loève expansion

$$X \otimes Y(t_1, t_2) = \sum_{n\geqslant 1} \sum_{m\geqslant 1} a_n^{1/2} b_m^{1/2} \xi_{nm} e_n(t_1) h_m(t_2)$$

with

$$\|X \otimes Y(t_1, t_2)\|_2^2 = \sum_{n\geqslant 1} \sum_{m\geqslant 1} a_n b_m \xi_{mn}^2,$$

where ξ_{ij} denotes as usual a doubly indexed i.i.d $N(0,1)$ sequences.

There are various studies recently on L_2-norm small deviation for the above tensored Gaussian random fields via different analytic methods (see, e.g. Refs. 7, 13, and 15). The main goal of this section is to present a simple probabilistic argument for the small deviation probability

$$\log \mathbb{P}\left(\sum_{n\geqslant 1}\sum_{m\geqslant 1} a_n b_m X_{mn} \leqslant \varepsilon\right),$$

where $X_{mn} > 0$ are i.i.d random variables. Of course, our probabilistic method works also for multiple sums. To really make the basic ideas clear, we also restrict ourself to $X_{mn} = \xi_{mn}^2$ since similar arguments works for more general situation. Even in this tensored Gaussian random fields setting, our result covers a variety of interesting parameter ranges for sequences a_n, b_n, and thus fills a gap left open from the spectral methods used in Ref. 13 where many interesting examples can be found.

As discussed above, we assume for the remaining of this section that we are in the Gaussian setting. And it is easy to see that all our arguments work in general setting.

There are several ways to obtain the exact asymptotics at the logarithmic level. One is given in Ref. 15 based directly on Sytaja's Tauberian theorem and analytic computations. Another is given in a recent work by Karol *et al.*[(13)] based on spectral asymptotics for tensor products of compact self-adjoint operators. One of the most powerful technique is the Mellin transform developed by Fill and Torcaso.[(7)] Our probabilistic arguments below are different but depends on some canonical known analytic results.

We start with a well known Exponential Tauberian theorem that connects the asymptotic Laplace transform of a positive random variable V with the small deviation behavior of the positive random variable V near zero. Namely, for $\alpha > 0$ and $\beta \in \mathbb{R}$

$$\log \mathbb{P}(V \leqslant \varepsilon) \sim -C_V \varepsilon^{-\alpha} |\log \varepsilon|^{\beta} \quad \text{as } \varepsilon \to 0^+$$

if and only if

$$\log E \exp(-\lambda V) \sim -(1+\alpha)^{1-\beta/(\alpha+1)} \alpha^{-\alpha/(1+\alpha)} C_V^{1/(1+\alpha)} \lambda^{\alpha/(1+\alpha)} (\log \lambda)^{\beta/(1+\alpha)} \quad \text{as } \lambda \to \infty.$$

A slightly more general formulation is given in Theorem 4.12.9 of Ref. 2 and is called de Bruijn's exponential Tauberian theorem. Note that one direction between the two quantities is easy and follows from

$$\mathbb{P}(V \leqslant \varepsilon) = \mathbb{P}(-\lambda V \geqslant -\lambda\varepsilon) \leqslant \exp(\lambda\varepsilon)\mathbb{E}\exp(-\lambda V),$$

which is just Chebyshev's inequality.

As a simple application of the Tauberian theorem, we have the following lemma for sums of independent random variables.

Lemma 3.1. If $V_i, 1 \leqslant i \leqslant m+l$, are independent nonnegative random variables such that

$$-\log \mathbb{P}(V_i \leqslant \varepsilon) \sim d_i \varepsilon^{-\alpha} |\log \varepsilon|^{\beta}, \quad 1 \leqslant i \leqslant m,$$

and

$$-\log \mathbb{P}(V_i \leqslant \varepsilon) = o\left(\varepsilon^{-\alpha} |\log \varepsilon|^{\beta}\right), \quad m+1 \leqslant i \leqslant m+l$$

for $0 < \alpha < \infty, \beta \in \mathbb{R}$ and $0 \leqslant d_i < \infty$, then

$$-\log \mathbb{P}\left(\sum_{i=1}^{m+l} V_i \leqslant \varepsilon\right) \sim \left(\sum_{i=1}^{m} d_i^{1/(1+\alpha)}\right)^{1+\alpha} \varepsilon^{-\alpha} |\log \varepsilon|^{\beta}.$$

Proof. We can first write down equivalent statements for both assumptions and conclusions in terms of the asymptotic behaviors of Laplace transform by using the above exponential Tauberian theorem. The desired result then follows from

$$\log \mathbb{E} \exp\left(-\lambda \sum_{i=1}^{m+l} V_i\right) = \sum_{i=1}^{m+l} \log \mathbb{E} \exp(-\lambda V_i)$$

for independent random variables $V_i, 1 \leqslant i \leqslant m+l$. □

Our second lemma is a well-known fact and a detailed proof can be found in Ref. 13 in the case $\theta \geqslant 0$. In general, it follows simply from Theorem 2.1. Below we give a simple and direct argument which also serves as a warm up for proof of Lemma 3.3.

Lemma 3.2. Assume as $n \to \infty$,

$$\lambda_n \sim C n^{-\gamma} (\log n)^{\theta}$$

for $\gamma > 1$ and $\theta \in \mathbb{R}$. Then we have as $\varepsilon \to 0$,

$$\log \mathbb{P}(\|X\| \leqslant \varepsilon) = \log \mathbb{P}\left(\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leqslant \varepsilon^2\right) \sim -D \cdot C^{1/(\gamma-1)} \varepsilon^{-2/(\gamma-1)} |\log \varepsilon|^{\theta/(\gamma-1)}$$

where

$$D = ((\gamma-1)/2)^{(\gamma-\theta-1)/(\gamma-1)} (\pi \gamma^{-1} \csc(\pi/\gamma))^{\gamma/(\gamma-1)}. \tag{3.1}$$

Proof. By Theorem 2.1, it suffices to estimate the probability $\log \mathbb{P}(V \leqslant \varepsilon^2)$, where

$$V = \sum_{n=1}^{\infty} n^{-\gamma} [\log(n+1)]^{\theta} \xi_n^2.$$

Note that as $\lambda \to \infty$,

$$\begin{aligned}
\log \mathbb{E} \exp(-\lambda V) &= -\frac{1}{2} \sum_{n=1}^{\infty} \log[1 + 2n^{-\gamma} (\log(n+1))^{\theta} \lambda] \\
&\sim -\frac{1}{2} \int_0^{\infty} \log[1 + 2x^{-\gamma} (\log(x+1))^{\theta} \lambda] dx \\
&\sim -\frac{1}{2} \gamma^{-1-\theta/\gamma} \lambda^{1/\gamma} (\log \lambda)^{\theta/\gamma} \cdot \int_0^{\infty} t^{1/\gamma - 1} \log(1 + 2/t) dt \\
&= -2^{1/\gamma - 1} \gamma^{-\theta/\gamma} \pi \csc(\pi/\gamma) \cdot \lambda^{1/\gamma} (\log \lambda)^{\theta/\gamma}.
\end{aligned}$$

Hence Lemma 3.2 follows from the exponential Tauberian theorem. □

Our next lemma follows from similar arguments outlined in Ref. 15 and/or the much more powerful Mellin transform techniques developed by Fill and Torcaso[7] in the case when θ_a and θ_b are nonnegative integers. Here we give a direct argument based on estimates of Laplace transform and the exponential Tauberian theorem.

Lemma 3.3. For any fixed positive integer K,

$$\log \mathbb{P}\left(\sum_{n=K+1}^{\infty} \sum_{k=K+1}^{\infty} n^{-\gamma_a} (\log n)^{\theta_a} \cdot k^{-\gamma_b} (\log k)^{\theta_b} \xi_{nk}^2 \leqslant \varepsilon^2 \right)$$
$$\sim \begin{cases} -D_1 \left(\sum_{k=K+1}^{\infty} k^{-\gamma_b/\gamma_a} (\log k)^{\theta_b/\gamma_a}\right)^{\gamma_a/(\gamma_a-1)} \varepsilon^{-2/(\alpha-1)} |\log \varepsilon|^{\theta_a/(\gamma_a-1)} & \text{if } \gamma_b > \gamma_a > 1 \\ -D_2 \varepsilon^{-2/(\gamma-1)} |\log \varepsilon|^{(\gamma+\theta_a+\theta_b)/(\gamma-1)} & \text{if} \gamma_b = \gamma_a = \gamma > 1 \quad \text{and } \theta_a, \theta_b \geqslant 0 \end{cases}$$

where

$$D_1 = ((\gamma_a - 1)/2)^{(\gamma_a - \theta_a - 1)/(\gamma_a - 1)} (\pi \gamma_a^{-1} \csc(\pi/\gamma_a))^{(\gamma_a/(\gamma_a-1))}, \tag{3.2}$$
$$D_2 = (2/(\gamma - 1))^{(1+\theta_a+\theta_b)/(\gamma-1)} (B(1+\theta_a/\gamma, 1+\theta_b/\gamma)\pi \gamma^{-1} \csc(\pi/\gamma))^{\gamma/(\gamma-1)} \tag{3.3}$$

and $B(x, y)$ is the Beta function.

Proof. First, assume $\gamma_b > \gamma_a > 1$. Let

$$V_k = \sum_{n=K+1}^{\infty} n^{-\gamma_a} (\log n)^{\theta_a \xi_{nk}^2} \quad \text{and} \quad V = \sum_{k=K+1}^{\infty} k^{-\gamma_b} (\log k)^{\theta_b} V_k.$$

Pick integer $\Lambda \sim \lambda^{\eta}$, where $(\gamma_a - 1)/(\gamma_b - 1) \cdot \gamma_a^{-1} < \eta < \gamma_b^{-1}$. Then,

$$1 \ll \lambda \Lambda^{-\gamma_b} (\log \Lambda)^{\theta_b} \quad \text{and} \quad \lambda \Lambda^{1-\gamma_b} (\log \Lambda)^{\theta_b} \ll \lambda^{1/\gamma_a}.$$

By proof of Lemma 3.2, we have

$$\begin{aligned}
\log \mathbb{E} \exp(-\lambda V) &= \sum_{k=K+1}^{\infty} \log \mathbb{E} \exp(-\lambda V_k) \\
&\sim -2^{1/\gamma_a - 1} \gamma^{-\theta_a/\gamma_a} \pi \csc(\pi/\gamma_a) \\
&\quad \sum_{k=K+1}^{\Lambda} (\lambda k^{-\gamma_b} (\log k)^{\theta_b})^{1/\gamma_a} [\log(\lambda k^{-\gamma_b} (\log k)^{\theta_b})]^{\theta_a/\gamma_a} \\
&\quad - \sum_{n=K+1}^{\infty} \sum_{k=\Lambda+1}^{\infty} n^{-\gamma_a} (\log n)^{\theta_a} \cdot k^{-\gamma_b} (\log k)^{\theta_b} \lambda.
\end{aligned}$$

Because the second term on the right-hand side is of order $O(\lambda \Lambda^{1-\gamma_b} (\log \Lambda)^{\theta_b})$, which is lower than the first term, by letting $\Lambda \to \infty$ we obtain

$$\begin{aligned}
\log \mathbb{E} \exp(-\lambda V) \sim -2^{1/\gamma_a - 1} \gamma_a^{-\theta_a/\gamma_a} \pi \csc(\pi/a) \sum_{k=K+1}^{\infty} k^{-\gamma_b/\gamma_a} (\log k)^{\theta_b/\gamma_a} \\
\times \lambda^{1/\gamma_a} (\log \lambda)^{\theta_a/\gamma_a}.
\end{aligned}$$

When $\gamma_a = \gamma_b => 1$, the argument is slightly different.

$$\begin{aligned}
\log \mathbb{E} \exp(-\lambda V) &= -\frac{1}{2} \sum_{n=K+1}^{\infty} \sum_{k=K+1}^{\infty} \log[1 + 2n^{-\gamma} k^{-\gamma} (\log n)^{\theta_a} (\log k)^{\theta_b} \lambda] \\
&\sim -\frac{1}{2} \int_K^{\infty} \int_K^{\infty} \log[1 + 2x^{-\gamma} y^{-\gamma} (\log x)^{\theta_a} (\log y)^{\theta_b} \lambda] dx\, dy.
\end{aligned}$$

Let $\log y = \frac{w}{r}\log(\lambda z)$ and $\log x = \frac{1-w}{r}\log(\lambda z)$, then

$$\begin{aligned}
&\log \mathbb{E}\exp(-\lambda V)\\
&\sim -\frac{1}{2}\int_{K_2\gamma/\lambda}^{\infty}\int_0^1 \log\left(1+\frac{2w^{\theta_b}(1-w)^{\theta_a}(\log\lambda z)^{\theta_a+\theta_b}}{z\gamma^{\theta_a+\theta_b}}\right)\gamma^{-2}\lambda^{1/\gamma}z^{1/\gamma-1}\log\lambda z dw dz\\
&\sim -\frac{1}{2}\gamma^{-2}\lambda^{1/\gamma}\log\lambda\int_0^1\int_0^{\infty}\log\left(1+\frac{2w^{\theta_b}(1-w)^{\theta_a}(\log\lambda)^{\theta_a+\theta_b}}{z\gamma^{\theta_a+\theta_b}}\right)z^{1/\gamma-1}dz\,dw\\
&= -2^{1/\gamma-1}\pi\csc(\pi/\gamma)\lambda^{1/\gamma}\left(\frac{\log\lambda}{\gamma}\right)^{1+(\theta_a+\theta_b)/\gamma}\int_0^1 w^{\theta_b/\gamma}(1-w)^{\theta_a/\gamma}dw\\
&= -2^{1/\gamma-1}\pi\ \csc\ (\pi/\gamma)\gamma^{-1-(\theta_a+\theta_b)/\gamma}B(1+\theta_a/\gamma, 1+\theta_b/\gamma)\cdot\lambda^{1/\lambda}(\log\lambda)^{1+(\theta_a+\theta_b)/\gamma}.
\end{aligned}$$

The lemma now follows from the exponential Tauberian theorem. □

Theorem 3.4. Assume as $n\to\infty$,

$$a_n \sim C_a n^{-\gamma_a}(\log n)^{\theta_a}, \qquad b_n \sim C_b n^{-\gamma_b}(\log n)^{\theta_b}$$

for $\gamma_b \geqslant \gamma_a > 1$. Then we have as $\varepsilon\to 0$,

(i) for $\gamma_b > \gamma_a > 1$,

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \sim -D_1\cdot C_a^{1/(\gamma_a-1)}\left(\sum_{k=1}^{\infty} b_k^{1/\gamma_a}\right)^{\gamma_a/(\gamma_a-1)} \varepsilon^{-2/(\gamma_a-1)}|\log\varepsilon|^{\theta_a/(\gamma_a-1)}$$

where the constant D_1 is given in (3.2).

(ii) for $\gamma_b = \gamma_a = \gamma > 1$ and $\theta_a, \theta_b\lambda \geqslant 0$,

$$\log \mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \sim -D_2(C_a C_b)^{1/(\gamma-1)}\varepsilon^{-2/(\gamma-1)} |\log\varepsilon|^{(\gamma+\theta_a+\theta_b)/(\gamma-1)}$$

where the constant D_2 is given in (3.3).

Proof. We first treat the case (i). For the upper bound, we have for any positive integer $K \geqslant 1$

$$\begin{aligned}
&\log \mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right)\\
&\le \log \mathbb{P}\left(\sum_{k=1}^{K} b_k \sum_{n=1}^{\infty} a_n \xi_{nk}^2 \leqslant \varepsilon^2\right)\\
&\sim -D_1 C_a^{1/(\gamma_a-1)}\left(\sum_{k=1}^{K} b_k^{1/\gamma_a}\right)^{\gamma/(\gamma_a-1)} \varepsilon^{-2/(\gamma_a-1)}|\log\varepsilon|^{\theta_a/(\gamma_a-1)}, \quad (3.4)
\end{aligned}$$

where the last line follows from Lemma 3.1 and the fact that for each $1 \leqslant k \leqslant K$,

$$\log \mathbb{P}\left(b_k \sum_{n=1}^{\infty} a_n \xi_{nk}^2 \leqslant \varepsilon^2\right) \sim -D_1 (C_a b_k)^{1/(\gamma_a-1)} \varepsilon^{-2/(\gamma_a-1)} |\log\varepsilon|^{\theta_a/(\gamma_a-1)}$$

based on Lemma 3.2. Note that we had to be careful here since we have ε^2 rather than just ε in the Lemma 3.2. Taking $K \to \infty$, we obtain the desired upper bound in the case $\gamma_b > \gamma_a > 1$.

For the lower bound in case (i), we split the summation region into three disjoint parts so that we have three independent sums. For any $\delta > 0$ small, there exists positive integer K such that for any $n, k \geqslant K+1$,

$$(1-\delta) C_a n^{-\gamma_a} (\log n)^{\theta_a} \leqslant a_n \leqslant (1+\delta) C_a n^{-\gamma_a} (\log n)^{\theta_a}, \quad (3.5)$$

$$(1-\delta) C_b k^{-\gamma_b} (\log k)^{\theta_b} \leqslant b_k \leqslant (1+\delta) C_b k^{-\gamma_b} (\log k)^{\theta_b}. \quad (3.6)$$

With δ and K defined above, we have by the independence of three disjoint sums

$$\begin{aligned}
&\mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \geqslant \mathbb{P}\left(\sum_{k=1}^{K} b_k \sum_{n=1}^{\infty} a_n \xi_{nk}^2 \leqslant (1-\delta^2)\varepsilon^2\right)\cdot\\
&\cdot \mathbb{P}\sum_{n=1}^{K}\left(\sum_{n=1}^{K} a_n \sum_{k=K+1}^{\infty} b_k \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right)\cdot\left(\sum_{n=K+1}^{\infty}\sum_{k=K+1}^{\infty} a - n b_k \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right)\cdot
\end{aligned}$$

Thus we have again by Lemmas 3.1–3.3, with $\phi(\varepsilon) = \varepsilon^{2/(\gamma_a-1)} |\log \varepsilon|^{-\theta_a/(\gamma_a-1)}$,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \phi(\varepsilon) \log \mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \\
&\geqslant \liminf_{\varepsilon\to 0} \phi(\varepsilon) \log \mathbb{P}\left(\sum_{k=1}^{K} b_k \sum_{n=1}^{\infty} a_n \xi_{nk}^2 \leqslant (1-\delta^2)\varepsilon^2\right) \\
&+ \liminf_{\varepsilon\to 0} \phi(\varepsilon) \mathbb{P}\left(\sum_{n=1}^{K} a_n \sum_{k=K+1}^{\infty} b_k \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right) \\
&+ \liminf_{\varepsilon\to 0} \phi(\varepsilon) \mathbb{P}\left(\sum_{n=K+1}^{\infty}\sum_{k=K+1}^{\infty} (1+\delta)^2 C_a n^{-\gamma_a} (\log n)^{\theta_a} C_b k^{-\gamma_b} (\log k)^{\theta_b} \xi_{nk}^2 \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. \leqslant 2^{-1}\delta^2\varepsilon^2\right) \\
&= -D_1 C^{1/(\gamma_a-1)} \left(\sum_{k=1}^{K} b_k^{1/\gamma_a}\right)^{\gamma_a/(\gamma_a-1)} (1-\delta^2)^{-1/(\gamma_a-1)} + 0 \\
&- D_1 (2 C_a C_b)^{1/(\gamma_a-1)} (1+\delta^{-1})^{2/(\gamma_a-1)} \left(\sum_{k=K+1}^{\infty} k^{-\gamma_b/\gamma_a} (\log k)^{\theta_b/\gamma_a}\right)^{\gamma_a/(\gamma_a-1)}.
\end{aligned}$$

Taking $K \to \infty$ first and then $\delta \to 0$, we obtain the lower bound in (i).

Next we turn into the more interesting and harder case (ii). For the upper bound, we have for any positive integer $K \geqslant 1$ determined in (3.5) and (3.6),

$$\begin{aligned}
&\log \mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \\
&\leqslant \log \mathbb{P}\left(\sum_{n=K+1}^{\infty}\sum_{k=K+1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \\
&\leqslant \log \mathbb{P}\left(\sum_{n=K+1}^{\infty}\sum_{k=K+1}^{\infty} (1-\delta)^2 C_a n^{-\gamma_a} (\log n)^{\theta_a} C_b k^{-\gamma_b} (\log k)^{\theta_b} \xi_{nk}^2 \leqslant \varepsilon^2\right) \\
&\sim -D_2 (C_a C_b)^{1/(\gamma-1)} (1-\delta)^{2/(\gamma-1)} \varepsilon^{-2/(\gamma-1)} |\log \varepsilon|^{(\gamma+\theta_a+\theta_b)/(\gamma-1)}
\end{aligned}$$

where the last line follows from Lemma 3.3. Taking $\delta \to 0$, we obtain the desired upper bound in the case $\gamma_a = \gamma_b = \gamma > 1$.

For the lower bound in case (ii), we again split the summation region into three disjoint parts like we did in the case (i) but with different weights on their contributions. For any $\delta > 0$ small and K large such that the relation (3.4) holds, we have by the independence of three disjoint sums,

$$\mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right) \geqslant \mathbb{P}\left(\sum_{k=1}^{K} b_k \sum_{n=1}^{\infty} a_n \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right)$$
$$\times \mathbb{P}\left(\sum_{n=1}^{K} a_n \sum_{k=K+1}^{\infty} b_k \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right) \cdot \mathbb{P}\left(\sum_{n=K+1}^{\infty}\sum_{k=K+1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant (1-\delta^2)\varepsilon^2\right).$$

Thus, we have again by Lemmas 3.1–3.3 with $\psi(\varepsilon) = \varepsilon^2/(\gamma_a - 1) |\log \varepsilon|^{-(\gamma+\theta_a+\theta_b)/(\gamma_a-1)}$,

$$\liminf_{\varepsilon\to 0} \psi(\varepsilon) \log \mathbb{P}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k \xi_{nk}^2 \leqslant \varepsilon^2\right)$$
$$\geqslant \liminf_{\varepsilon\to 0} \psi(\varepsilon) \log \mathbb{P}\left(\sum_{k=1}^{K} b_k \sum_{n=1}^{\infty} a_n \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right)$$
$$+ \liminf_{\varepsilon\to 0} \psi(\varepsilon) \mathbb{P}\left(\sum_{n=1}^{\infty} a_n \sum_{k=K+1}^{\infty} b_k \xi_{nk}^2 \leqslant 2^{-1}\delta^2\varepsilon^2\right)$$
$$+ \liminf_{\varepsilon\to 0} \psi(\varepsilon)$$
$$\times \mathbb{P}\left(\sum_{n=K+1}^{\infty}\sum_{k=K+1}^{\infty} (1+\delta)^2 C_a n^{-\gamma_a} (\log n)^{\theta_a} C_b k^{-\gamma_b} (logk)^{\theta_b} \xi_{nk}^2 \leqslant (1-\delta^2)\varepsilon^2\right)$$
$$= 0 + 0 - D_1 \cdot \left((-\delta^2)^{-1}(1+\delta)^2 C_a C_b\right)^{1/(\gamma-1)}.$$

Taking $\delta \to 0$, we obtain the lower bound in (ii) and hence finish the proof. □

4. SUP-NORM

Consider integrated fractional Brownian motion processes

$$W_{H,m}(t) = W_{H,m}^{[\beta_1,\dots,\beta_m]}(t) = (-1)\beta_1 + \cdots + \beta_m \int_{\beta_m}^{t} \int_{\beta_m-1}^{t_{m-1}} \cdots \int_{\beta_1}^{t_1} W_H(t_0)dt_0 \cdots dt_{m-1}$$

on the interval [0, 1] where any β_k equals either zero or one. There has been a lot of study recently for the Brownian motion case $H = 1/2$ under the L_2-norm (see Refs. 10, 19, and 20). It is known that

$$\lim_{\varepsilon \to \infty} \varepsilon^{1/(m+H)} \log \mathbb{P}\left(\int_0^1 |W_{H,m}(t)|^2 dt \leqslant \varepsilon^2\right) = -K_{H,m},$$

where

$$K_{H,m} = \frac{(m+H)[\Gamma(2H+1)\sin(\pi H)]^{\frac{1}{2m+2H}}}{\left[(2m+2H+1)\sin\left(\frac{\pi}{2m+2H+1}\right)\right]^{\frac{2m+2H+1}{2m+2H}}}$$

is a positive constant independent of the choices of β_k. Our goal of this section is to deal with the sup-norm case, which we only know the existence of the constant.

Theorem 4.1. There exists a constant $C_{H,m} \in (0, \infty)$ independent of the choices of $\beta_k \in \{0, 1\}$, $1 \leqslant k \leqslant m$, such that

$$\lim_{\varepsilon \to 0} \varepsilon^{1/(m+H)} \log \mathbb{P}\left(\sup_{0 \leqslant t \leqslant 1} |W_{H,m}(t)| \leqslant \varepsilon\right) = -C_{H,m}. \tag{4.1}$$

Furthermore, we have

$$\begin{aligned} &\frac{(m+H)[\Gamma(2H+1)\sin(\pi H)^{\frac{1}{2m+2H}}}{\left[(2m+2H+1)\sin\left(\frac{\pi}{2m+2H+1}\right)\right]^{\frac{2m+2H+1}{2m+2H}}} \leqslant C_{H,m} \\ &\leqslant \left(\frac{\pi}{2}\right)^{1/(m+H)} \frac{(m+H)[\Gamma(2H+1)\sin \pi H]^{\frac{1}{2m+2H-1}}}{\left[(2m+2H-1)\sin\left(\frac{\pi}{2m+2H+1}\right)\right]^{\frac{2m+2H-1}{2m+2H}}} \end{aligned} \tag{4.2}$$

Proof. Our proof consists three steps. We first show the limit in (4.1) exists for the special choice of β_k, $1 \leqslant k \leqslant m$, i.e. the so-called standard mth integrated fractional Brownian motion. To be more precise, define

$$\begin{aligned} W_{H,m}^s(t) &= \int_0^t \int_0^{t_{m-1}} \cdots \int_0^{t_1} W_H(t_0)dt_0 \cdots dt_{m-1} \\ &= \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} W_H(s)ds. \end{aligned}$$

The key fact is the scaling property

$$W_{H,m}^s(ct) = c^{m+H} W_{H,m}^s(t), \quad t \geqslant 0$$

in distribution as processes, for any fixed constant $c>0$. This allows us to show the existence of a constant $C_{H,m}\in(0,\infty)$ such that

$$\lim_{\varepsilon\to 0}\varepsilon^{1/(m+H)}\log\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}|W^{s}_{H,m}(t)|\leqslant\varepsilon\right)=-C_{H,m}. \tag{4.3}$$

The arguments are similar to those given for the first time in Ref.17 for the existence of small deviation constant for fractional Brownian motion and the related Riemann-Liouville type processes $\int_0^t(t-s)^{\alpha}dB(s)$. To be more precise, we use the very useful representation

$$W_H(t)=a_H(X_H(t)+Z_H(t))\quad t\geqslant 0, \tag{4.4}$$

where

$$X_H(t)=\frac{1}{\Gamma(H+1/2)}\int_0^t(t-s)^{H-1/2}dB(s),$$
$$Z_H(t)=\frac{1}{\Gamma(H+1/2)}\int_{-\infty}^0\{(t-s)^{H-1/2}-(-s)^{H-1/2}\}dB(s)$$

and the constant

$$a_H=\Gamma(H+1/2)\left((2H)^{-1}+\int_{-\infty}^0((1-s)^{H-1/2}-(-s)^{H-1/2})^2ds\right)^{-1/2}. \tag{4.5}$$

Furthermore, $X_H(t)$ is independent of $Z_H(t)$. Observe that the centered Gaussian process $X_\beta(t)$ is defined for all $\beta>0$ as a fractional Wiener integral. Hence we have the independent sum representation

$$W^{s}_{H,m}(t)=a_H X_{m+H}(t)+\frac{a_H}{(m-1)!}\int_0^t(t-s)^{m-1}Z_H(s)ds. \tag{4.6}$$

From Ref. 17, the small deviation constants exists for the process $X_{m+H}(t)$ under the sup-norm. The estimates for the part $\int_0^t(t-s)^{m-1}Z_H(s)ds$ can be found in Ref. 1. We omit details since these are well-known arguments now.

Our second step is to show the limit in (4.1) is the same for all choices of $\beta_k\in\{0,1\}$, $1\leqslant k\leqslant m$. We compare $W_{H,m(t)}$ with $W^{s}_{H,m(t)}$. Let us assume that not all $\beta_k=0$ and define

$$j=\inf\{k:\beta_k=1,1\leqslant k\leqslant m\},\quad 1\leqslant j\leqslant m.$$

Then we have

$$\int_{\beta_m}^{t}\int_{\beta_{m-1}}^{t_{m-1}}\cdots\int_{\beta_1}^{t_1} W_H(t_0)dt_0\cdots dt_{m-1}$$
$$=\int_{\beta_m}^{t}\cdots\int_{\beta_j=1}^{t_j}\int_0^{t_{j-1}}\cdots\int_0^{t_1} W_H(t_0)dt_0\cdots dt_{m-1}$$
$$=-\int_{\beta_m}^{t}\cdots\int_{\beta_{j+1}}^{t_j+1}\int_0^{t_j}\cdots\int_0^{t_1} W_H(t_0)dt_0\cdots dt_{m-1}$$
$$+\int_{\beta_m}^{t}\cdots\int_{\beta_{j+1}}^{t_{j+1}}\left(\int_0^1\int_0^{t_{j-1}}\cdots\int_0^{t_1} W_H(t_0)dt_0\cdots dt_{j-1}\right)dt_j\cdots dt_{m-1}$$
$$=-\int_{\beta_m}^{t}\cdots\int_{\beta_{j+1}}^{t_j+1}\int_0^{t_j}\cdots\int_0^{t_1} W_H(t_0)dt_0\cdots dt_{m-1}+g_{m-j}(t)\cdot Y_j$$

where

$$g_{m-j}(t)=\int_{\beta_m}^{t}\cdots\int_{\beta_{j+1}}^{t_j+1} dt_j\cdots dt_{m-1},$$
$$Y_j=\int_0^1\int_0^{t_{j-1}}\cdots\int_0^{t_1} W_H(t_0)dt_0\cdots dt_{j-1}. \tag{4.7}$$

Note that the function $g_{m-j}(t)$ is a polynomial of degree $m-k$ and Y_j is a Gaussian random variable.

Repeating the above procedure, we obtain the representation

$$(-1)^{\beta_1+\cdots+\beta_m}\int_{\beta_m}^{t}\int_{\beta_{m-1}}^{t_{m-1}}\cdots\int_{\beta_1}^{t_1} W_H(t_0)dt_0\cdots dt_{m-1}$$
$$=W^s_{H,m}(t)+\sum_{j:\beta_j=1}\pm g_{m-j}(t)\cdot Y_j. \tag{4.8}$$

Note that

$$\sup_{0\leqslant t\leqslant 1}\left|\sum_{j:\beta_j=1}\pm g_{m-j}(t)\cdot Y_j\right| \leqslant \sum_{j:\beta_j=1}\sup_{0\leqslant t\leqslant 1}|g_{m-j}(t)|\cdot|Y_j|$$
$$\leqslant \max_{1\leqslant k\leqslant m}\sup_{0\leqslant t\leqslant 1}|g_k(t)|\cdot\sum_{j=1}^{m}|Y_j|$$

and hence there exists a constant $\delta_m>0$ small such that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\left|\sum_{j:\beta_j=1}\pm g_{m-j}(t)\cdot Y_j\right|\leqslant\varepsilon\right)\geqslant\mathbb{P}\left(\max_{1\leqslant k\leqslant m}\sup_{0\leqslant t\leqslant 1}|g_k(t)|\cdot\sum_{j=1}^{m}|Y_j|\leqslant\varepsilon\right)\geqslant\delta_m\varepsilon^m$$

for $\varepsilon > 0$ small. This implies

$$\lim_{\varepsilon\to 0} \varepsilon^{1/(m+H)} \log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\left|\sum_{j:\beta_j=1} \pm g_{m-j}(t)\cdot Y_j\right| \leqslant \varepsilon\right) = 0.$$

Thus (4.1) follows from a very general theorem below, which is given in Ref. 16 based on a weaker Gaussian correlation inequality. The key point is that two Gaussian random elements X and Y are *not* necessarily independent but with different small ball rates.

Lemma 4.2. For any joint Gaussian random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ in a Banach space satisfying

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma} \log \mathbb{P}(||X|| \leqslant \varepsilon) = -C_X, \qquad \lim_{\varepsilon\to 0} \varepsilon^{\gamma} \log \mathbb{P}(||Y|| \leqslant \varepsilon) = 0$$

with $0 < \gamma < \infty$ and $0 < C_X < \infty$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma} \log \mathbb{P}(||X+Y|| \leqslant \varepsilon) = -C_X.$$

Our third step is the estimates given in (4.2). The lower bound for $C_{H,m}$ or the upper bound for associated probability follows from the standard L_2-norm estimates. Namely,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W_{H,m}(t)| \leqslant \varepsilon\right) \leqslant \mathbb{P}\left(\int_0^1 |W_{H,m}(t)|^2 dt \leqslant \varepsilon^2\right).$$

Thus,

$$C_{H,m} \geqslant K_{H,m} = \frac{(m+H)[\Gamma(2H+1)\sin(\pi H)]^{\frac{1}{2m+2H}}}{[(2m+2H+1)\sin(\frac{\pi}{2m+2H+1})]^{\frac{2m+2H+1}{2m+2H}}}.$$

The upper bound for $C_{H,m}$ or the lower bound for associated probability follows from a nice technique developed in Ref. 4, based again on a slightly different L_2-norm estimates.

Let X and Y be any two centered Gaussian random vectors in a separable Banach space E with norm $||\cdot||$. We use $|\cdot|_\mu$ to denote the inner product norm of the reproducing kernel Hilbert space of $\mu = \mathcal{L}(X)$. Then the following general connection between small ball probabilities is discovered in Ref. 4. It provides a powerful tool to estimate small ball probabilities under any norm via a relative easier L_2-norm estimate.

Lemma 4.3. For any $\lambda > 0$ and $\varepsilon > 0$,

$$\mathbb{P}(||Y|| \leqslant \varepsilon) \geqslant \mathbb{P}(||X|| \leqslant \lambda\varepsilon) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2 |Y|_\mu^2\}. \tag{4.9}$$

Now back to proof of (4.2) in Theorem 4.1. Let $X = W(t)$, the Brownian motion. It is well known that

$$\log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W(t)| \leqslant \varepsilon\right) \sim -(\pi^2/8)\varepsilon^{-2}.$$

Take $Y = W^s_{H,m}(t)$ in Lemma 4.3. Because Wiener measure $\mu(W)$ satisfies $|f|^2_\mu = \int_0^1 (f'(s))^2 ds$, Lemma 4.3 gives

$$\mathbb{P}(||W^s_{H,m}|| \leqslant \varepsilon) \geqslant \mathbb{P}(||W(t)|| \leqslant \lambda\varepsilon) \cdot \mathbb{E}\ \exp\left\{-\frac{\lambda^2}{2}\int_0^1 [W^s_{H,m-1}(t)]^2 dt\right\} \tag{4.10}$$

Taking $||\cdot||$ to be the sup-norm on $C[0, 1]$ and $\lambda = \lambda_\varepsilon = \alpha\varepsilon^{1/(2m+2H)-1}$ in (4.10) with $\alpha > 0$ to be fixed later, it follows from the existence of the constants that

$$\begin{aligned}
-C_{H,m} &= \lim_{\varepsilon\to 0} \varepsilon^{1/(m+H)} \log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W^s_{H,m}(t)| \leqslant \varepsilon\right)\\
&\geqslant \lim_{\varepsilon\to 0} \varepsilon^{1/(m+H)} \log \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |W(t)| \leqslant \alpha\varepsilon^{1/(2m+2H)}\right)\\
&\quad + \lim_{\varepsilon\to 0} \varepsilon^{1/(m+H)} \log \mathbb{E}\ \exp\left\{-\frac{\alpha^2}{2}\varepsilon^{1/(m+H)-2}\int_0^1 [W^s_{H,m-1}(t)]^2 dt\right\}\\
&= -\frac{\pi^2}{8\alpha^2} - \frac{2m+2H-1}{2m+2H-2}((m+H-1)\alpha^2)^{1/(2m+2H-1)}(K_{H,m-1})^{1-1/(2m+2H-1)}\\
&= -\frac{\pi^2}{8\alpha^2} - \frac{\alpha^{2/(2m+2H-1)}(\Gamma(2H+1)\sin \pi H)^{1/(2M+2H-1)}}{2\sin(\frac{\pi}{2m+2H-1})}
\end{aligned}$$

Now pick the best $\alpha > 0$, we obtain

$$C_{H,m} \leqslant \min_{\alpha>0}\left(\frac{\pi^2}{8\alpha^2} - \frac{\alpha^{2/(2m+2H-1)}(\Gamma(2H+1)\sin \pi H)^{1/(2M+2H-1)}}{2\sin(\frac{\pi}{2m+2H-1})}\right)$$

$$= \left(\frac{\pi}{2}\right)^{1/(m+H)} \frac{(m+H)[\Gamma(2H+1)\sin(\pi H)]^{\frac{1}{2m+2H-1}}}{[(2m+2H+1)\sin(\frac{\pi}{2m+2H+1})]^{\frac{2m+2H-1}{2m+2H}}},$$

which is the upper bound for $C_{H,m}$ in (4.2). □

Remark. If in the third step of the proof we let $X = W^s_{H,m-1}(t)$, instead of $X = W(t)$, we will obtain an upper bound of $C_{H,m}$ in terms of $C_{H,m-1}$. Such an upper bound is slightly better than the one obtained in Theorem 4.1 However, either one is sharp. Finally, we point out that similar results like Theorem 4.1 also hold for L_p-norm, $1 \leqslant p \leqslant \infty$ and other related norms such as Holder norm. The proofs are also similar and we omit the details.

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