Logarithmically Proportional Objective Function for Planar Surfaces Recognition in 3D Point Cloud

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***Abstract*—3D laser scanning is becoming a standard technology to generate building models of a facility's as-is condition. Since most constructions are constructed upon planar surfaces, recognition of them paves the way for automation of generating building models. This paper introduces a new logarithmically proportional objective function that can be used in both heuristic and metaheuristic (MH) algorithms to discover planar surfaces in a point cloud without exploiting any prior knowledge about those surfaces. It can also adopt itself to the structural density of a scanned construction. In this paper, a metaheuristic method, genetic algorithm (GA), is used to test this introduced objective function on a synthetic point cloud. The results obtained show the proposed method is capable to find all plane configurations of planar surfaces (with a wide variety of sizes) in the point cloud with a minor distance to the actual configurations.**

***Keywords*-logarithmic objective function; genetic algorithm; planar surface recognition; point cloud**

I. Introduction

Building models of the as-is conditions of a facility is a substantial need in the Architecture, Engineering, and Construction (AEC) domains for diverse applications, such as planning renovations, space usage planning, and managing building maintenance [1]; moreover, public security, navigation systems, and virtual tourism can also benefit from that [2]. Because of as-built/-is conditions of a construction generally can be different from the design drawings, constructing building models from available documents, e.g., CAD models, is not sufficient. Furthermore, that is neither an option in non-engineered buildings, nor an option in heritage constructions. 3D laser scanning is becoming a standard technology to generate building models of a facility's as-is condition, because of its rapid, dense, and accurate measurement [3]. The current techniques to create building models from laser-scanners data (point cloud) are manual, subjective, labor-intensive, time-consuming, and error-prone process. Since most frequent surface shape of man-made constructions is planar surface, recognition of that is often the first step to extract building information from 3D point clouds.

Planar surfaces recognition methods are often using bottom-up approach, where they first group points into locally similar patches (segments), and thereafter, patches that are sufficiently planar can be grouped together based on their similarity [3]. The objective of these techniques is to find a plane containing the maximum number of points, while the accuracy of the plane parameters, regarding to their relative points, is their second interest. Since these methods attempt to find a plane with the maximum number of points, if a segment contains more than one plane, these methods are highly affected by points that do not belong to the same plane. Increasing the number of segments to avoid this problem, at its turn, will expand the complexity of the post processing for joining small patches. Top-down methods formulate the problem without chopping up planar surfaces to small patches; it benefits them from avoiding post processing over discovered small patches.

This paper describes the design of a logarithmically proportional objective function that can adapt itself to scale of the scanned construction without using segmentation. The proposed function is sufficient to be used in top-down approaches; so it does not need to break down a surface to small pieces (segmentation is not needed). It also gives the same interest to the number of points fitted on a plane and the precision of the plane parameters. On the other hand, it provides a technique to prevent objective function from getting affected by points that are not in the region of a plane. It can also distinguish small changes among plane parameters. Furthermore, this new objective function can adapt itself to the structural density of a scanned construction.

The paper is organized as follows. Section II presents a brief review of related work that has been proposed to address the planar surfaces recognition and their objective functions. In Section III, we describe our logarithmically proportional objective function, as well as the technique to seize the effect of those points that are not in the region of a plane. With employing a genetic algorithm, we tested this objective function and results are presented and discussed in

Section IV. Finally, Section V concludes the paper.

II. Related Work

There are variety of techniques for planar surface recognition in 3D point cloud. Most of these methods are made robust by means of segmentation. In addition, if a point cloud contains more than one planar surface, segmentation is not an option, but a necessity. Their process is as the following. A point cloud at first is split into smaller parts (segments) by using a segmentation method, and thereafter one of these surface recognition methods is applied to each part separately. Finally, the similar planar surfaces, that were found in each segment, are merged by using plane growing [4], [5] or triangulated irregular network (TIN) meshes [6]. Three state-of-the-art approaches for planar surface recognition are region growing, Hough-transform, and RANSAC algorithms.

Region growing is a simple segmentation method which intends to group points into locally similar patches [7]. It examines neighboring points of an initial point, *seed point*, and determines whether the point neighbors should be added to the region. It uses local estimate of surface shape, e.g., flatness [8] or surface curvature [9], to group points that locally represent similar surface. Thereafter, those local planar patches are grouped based on their normal vectors similarity to create a bigger surface. Identifying suitable seed points is crucial to the success of this approach.

The 3D Hough-transform is an extension of the 2D Hough-transform [10] which is used for the line recognition problem in 2D images. In the 3D Hough-transform, each point in a point cloud defines a plane, $z = s_x x + s_y y + d$ in the 3D parameter space; where s_x and s_y are the slopes with x and y axes, respectively, and d is the vertical distance of the plane to the origin $(0, 0, 0)$. If a point cloud contains points of a planar surface, planes of those points in the parameter space will intersect at the position that corresponds to the slopes and distance of their relative planar surface. The best found plane by this method is not the most probable plane calculated according to the least squares theory [11], [12], but instead it means the plane containing the maximum number of points [13], [14], [5].

The principle of RANdom SAmple Consensus (RANSAC) algorithm [15] is as the following. It starts with picking up 3 points randomly from a point cloud, and calculating the parameters of the corresponding plane. Thereafter, with respect to a given threshold, it detects those points that belong to the calculated plane. It repeats this procedures N times and each time, it compares the obtained results with the previous one. In case of improvement, it replaces the previous plane configuration with the new one. This method is a problem-dependent technique; it can not find the number of trial N by itself, and it is often calculated based on a pure probability law, which yields different N from one point cloud to another. Nonetheless, it provides satisfying results in compare to Hough-transform methods [14].

III. Objective Function

This section introduces a new logarithmically proportional objective function which can be used in both heuristic and metaheuristic algorithms. This function has a dynamic layout that enables it to adapt itself to the scale of the scanned construction without using segmentation. It also describes a technique to seize the influence of those points that are not in the region of a sampled plane on the objective function.

The proposed objective function is used in the process to find a closest planar surface to a hypothesized planar surface in the 3D point cloud. The most accurate way to do so is to calculate the volume between two planar surfaces with closed boundaries. The sampled plane with the smallest volume is the most closest to the hypothesized planar surface. The volumetric technique can be applied if and only if sampled and hypothesized planar surfaces are defined by continuous functions. Unlike sampled planar surfaces that are represented by 4 variables of the geometric-plane formula, $ax + by + cz + d = 0$, planar surfaces in the 3D point cloud (hypothesized planar surfaces) have discrete topology, because points are isolated from each others; hence, the volumetric technique cannot be applied to the problem at hand.

The total least squares (TLS) criterion are often used for this purpose [12]. It represents the sum of squares of Euclidean distances between all points in the point cloud (or a segment of it) and the sampled plane. Point clouds with different structural densities are treated in the same way by using TLS. Since it uses square form of Euclidean distances, points that are far from the sampled plane have more effect than those are close to it. In other words, noises (those points that are far from a sampled planar surface) have a very high impact on the objective function evaluation, while small changes around a solution do not have much effect on TLS. We propose a logarithmic function that behaves the opposite way TLS does. It is very sensitive about small changes around a sampled plane, while noises do not have much impact on the function evaluation.

In the next subsections, we first describe the logarithmic objective function, and thereafter we give a formula to calculate the base of the logarithmic function according to the structural density of the scanned construction to guarantee the dynamic layout of the function.

A. Logarithmic Objective Function

Since points in point clouds are isolated from each other and are defined in a discrete space, their relative Euclidean distances to a sampled plane are also defined in the discrete space. We want to transform those discrete distances to one value in the continuous space, which makes different sampled planar surfaces possible to be compared with each other. We sum the logarithmic form of Euclidean distances for this purpose.

Logarithmic functions are often used in science and engineering to scale very large/small numbers into numbers that are easier to comprehend and compare [16]. They are continuous functions, and according to intermediate value theorem, a continuous function that produces two values m and n also produces any value that lies between m and n. We want that value to be obviously distinguishable from m and n while they are very small and very close to each other; a logarithmic function which is strictly decreasing as $x \to 0^+$ can fulfill this aim. The other reason of using logarithm is to decrease the impact of noises. In $h(x) = \log_b x$, though $h(x) \to \pm\infty$ on the interval $]0, +\infty[$ and when $b > 1$, the decay rate for $x \leq b$ as $x \to 0^+$ is dramatically higher than the growth rate for $x > b$ as $x \to +\infty$. Thus, in order to decrease the impact of noise, we need to formulate the problem in which very close points to a sampled plane yield values on the interval $]-\infty, 0]$, and far points (noises) yield positive values.

This proposed objective function has two critical constructional parameters that need to be determined by an architect or whomever is familiar with the scanned construction as well as its structural density. Those parameters are as follows:

- Plane tolerance threshold: it is the maximum acceptable distance between a sampled plane and those points in the point cloud that are identified as fitted on the sampled planes. In real data, this parameter is related to the altimetric accuracy of the point cloud [14], used building materials in constructing planar surfaces, and precision in constructing planar surfaces.
- Plane region: since point clouds often contain several planar surfaces, not all the points in the point cloud should play a part in the objective function evaluation of a sampled plane. Only those that are in the region of a sampled plane must be considered for this evaluation. Note that the radius of plane region must be bigger than the plane tolerance threshold. The radius of sampled-plane region needs also to be determined by an expert familiar with the structural density of the scanned construction.

With respect to the above-mentioned factors, the objective function is formulated in two steps as follows:

$$\beta = \frac{1}{|R|} \left(\sum_{x_i \in R,\ x_i \neq 0} (\log_b x_i - \log_b t) \right), \quad |R| \neq 0\,, \quad (1)$$

$$obj.\ fun. = \begin{cases} \frac{1}{|F|} \times \beta & \text{if } \beta > 0\,, \\ |F| \times \beta & \text{if } \beta < 0\,, \\ f(x) & \text{if } \beta = 0, \text{ or } |F| = 0\,, \end{cases} \quad (2)$$

where x_i is the Euclidean distance between a point and a sampled plane, t is the *plane tolerance threshold*. The base of the logarithm, b, is a constant value, an its calculation based on the radius of the *plane region* and the *plane tolerance threshold* will be studied in the next subsection. F and R are two subsets of a point cloud in which set F represents fitted points on the sampled plane according to the plane tolerance threshold and set R represents points inside of the plane region; hence, the relationship between them is as $F \subseteq R$. $|F|$ and $|R|$ are the cardinalities of those subsets, i.e. the numbers of points in subsets F and R, respectively.

Equation (1) calculates the average of logarithmic form of Euclidean distances of those points that are placed in the region of a plane. In addition, in order to decrease the impact of noises, the logarithmic function is translated by using $-\log_b t$; so that, fitted points on the sampled plane yield negative values, while those none-fitted points inside of the plane region $(R - F)$ yield positive values. Since (1) calculates the average, the planar surface's size in a point cloud (number of the points on the planar surface) does not stress on β. Rather, it gets tremendously affected by the closeness of hypothesized and sampled planes. The closer the two planes are, the better β is; and β is better as it tends toward $-\infty$.

Once the average is obtained by Equation (1), if $|F| \neq 0$, β will be weighted with using the number of fitted points on a sampled plane, as shown in Equation (2), in favor of sampled planes with the higher number of fitted points. Since the objective function gives smaller values to better sampled planes, if $\beta > 0$, which can only happen when there are a few number of fitted points on the sampled plane, β should proportionally decrease by dividing it by $|F|$; if $\beta < 0$, which can happen when there are more fitted points on the sampled plane, β should proportionally decrease by multiplying it by $|F|$. Hence, unlike Equation (1), the planar surface's number of points in a point cloud has a direct impact on Equation (2) (objective function); in simpler terms, if two sampled planes have the same distance to their relative hypothesized planes, the plane with more points would be evaluated better. In consequence, those heuristic and metaheuristic algorithms that use this objective function generally would find planar surfaces with bigger number of points sooner than planar surfaces with smaller number of points. This results in searching through smaller point cloud as long as the search process carries on.

In this objective function there are three cases that might very rarely happen, but worth to be considered and give solutions to them. First, when a sampled plane does not contain any point in its region, i.e. sampled plane does not pass through the point cloud ($|R| = 0$); thus, it will be evaluated by the worst value, i.e. $+\infty$. Second, $\beta = 0$ which happens either when all the points in the point cloud have the same distance as the *tolerance threshold* to a sampled plane, or when the sum of the logarithmic form of distances becomes zero. Since the sampled planes' configurations and the points' coordinates are represented by real values, the occurrence of $\beta = 0$ is scarce; and if it happens the result of Equation (1) would be considered as the objective function value without weighting. The final special case is when $x_i = 0$, which means a point is exactly located on the sampled plane (having an Euclidean distance of zero). It is important to note that it does not mean the sampled plane is the best match to any hypothesized plane. Speaking informally, with using real values that might occur once in a blue moon. Nevertheless, if it happens, that point won't be involved in the objective function evaluation.

After elaborating the objective function, here we give an example to compare the behavior of this proposed objective function with total least squares. For the sake of simplicity, assume there are only two parallel planar surfaces in the point cloud with the same shape, size, number of points, right in front of each other, and of course inside of plane region with a radius much smaller than the size of plane. In spite of the fact that the best planar surfaces' configurations should have the same configurations as those planar surfaces in the point cloud, the optimal planar surface according to TLS criterion is a planar surface, parallel to those two planar surfaces in the point could, and exactly in the middle of them. In the same problem, if the logarithmically proportional objective function is applied, the location of any of those planar surfaces in the point cloud is the best solution without any privilege; and after locating one and removing its corresponding points from the point cloud, we can go for the second one. The logarithmically proportional objective function makes any of those planar surfaces to see the other one as noise. Note that in general, from the perspective of any planar surface in a point cloud, other surfaces are noises. This example shows that using the logarithm makes the objective function able to effectively deal with this fact.

B. Base of the Logarithmic Objective Function

In the previous subsection we have explained the proposed objective function and its features. Now we are ready to present you a formula to calculate the base of the logarithm of the Equation (1) according to the structural density of a scanned construction.

The logarithmic part of Equation (1) is as the following:

$$f(x) = \log_b x - \log_b t \,, \tag{3}$$

where x is the Euclidean distance. Since $\log_b t$ is a constant value, the derivative of (3) is expressed as:

$$\frac{d}{dx} f(x) = \frac{1}{x \ln b} \,. \tag{4}$$

Equation (4) indicates that the decay/growth rates of (3) on the interval $]0, r]$, where r is the given radius of a plane region, depends on b. In other words, the log function has a very high growth rate under horizontal axis, where x is on the interval $]0, t]$, and above of the axis as $x \to +\infty$ that rate extremely decreases; and we want to cut the log function from where it still has an effective growth rate. The inverse of log function, f^{-1}, reverse its behavior. That means it has very low growth rate and then tremendously increase the rate as $x \to +\infty$; hence, r should be located on the intersection of f and f^{-1}. The inverse function of Equation (3) is:

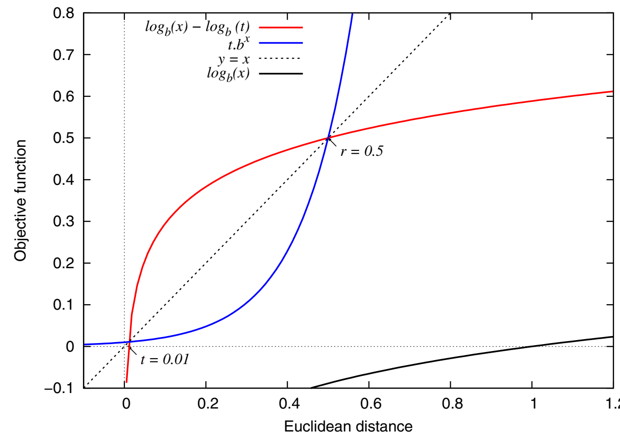

Figure 1. Intersection of Equations (3), and (5), where $t = 0.01m$ and $r = 0.5m$. Applying the given plane tolerance threshold (t) and radius of plane region (r) to Equation (6) yields a base of 2500. The effect of using $-\log_b t$ in Equation (3) is also shown in this figure.

$$f(x)^{-1} = tb^x \,. \tag{5}$$

The graph of f and f^{-1} are reflection of one another about the line $y = x$ (Figure 1). Thus, the x value of the second intersection of the Equations (5), and line $y = x$ should be r, and the base of logarithm with applying $x = y$ to (5) is obtained as the following:

$$b = \sqrt[r]{r/t} \,. \tag{6}$$

The base of this logarithmic function has the same unit as given Euclidean distances. That means t and r should have the same unit as Euclidean distances to apply to (6).

IV. Experimental Results

This section presents the experimental results from testing the proposed objective function on the synthetic point cloud by means of GA. Although the aim of this work is not to study the GA behavior on the problem at hand, which can be a topic of another study, it is interesting to see the logarithmically proportional objective function in practice.

The synthetic point cloud that is used in this experiment represents a one-floor house with two rooms, and a hip-roof on a rectangular plan; furthermore, it contains a total of 31657 points. Those points represent 22 planar surfaces with different sizes from $\sim$200 points to $\sim$10000 points (Figure 2). Points are uniformly distributed on the surfaces and there are almost 50 points per square meter. Since we deal with synthetic point cloud, the tolerance threshold

0.01m is the maximum distance that we accept to consider points as laid on the sampled plane. Half of the narrowest width of a construction space, i.e. 0.50m, is picked as the radius of plane region. With applying the given plane tolerance threshold (t), and radius of plane region (r) to the Equation (6), it gives 2500 as the base of logarithm (see Figure 1).

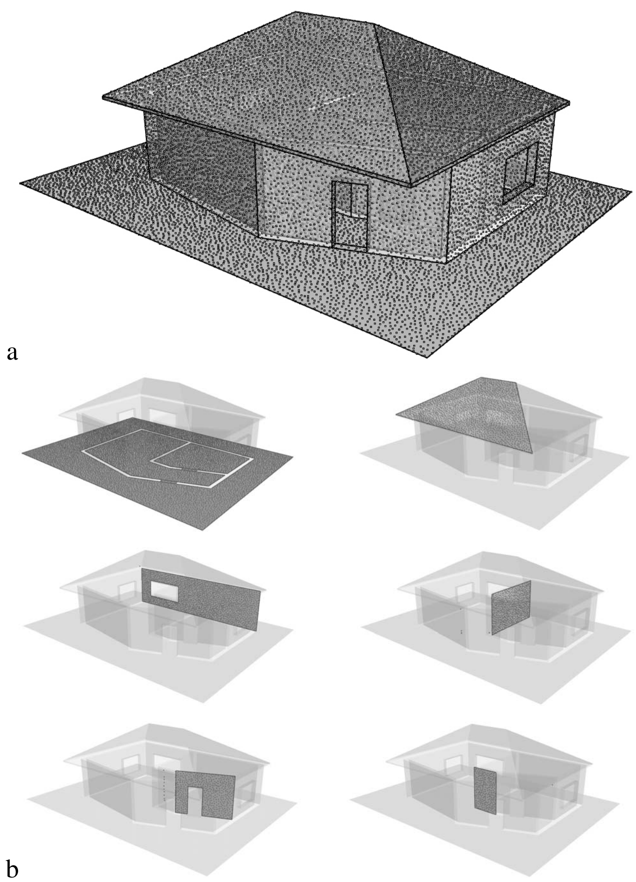

Figure 2. (a) An outline of the house which is built of eight walls, each of which is represented by two planar surfaces, one base rectangular plan that contraction is on it, and a four-sides hip-roof on one rectangular plan. (b) shows some of the discovered planar surfaces by using logarithmic objective function, going from bigger to smaller.

The geometric-plane formula, $ax+by+cz+d=0$, is defined by four parameters, a, b, c, and d. A candidate solution for the GA is therefore represented by a chromosome vector with four genes, each a real number. We used restricted tournament selection (RTS) proposed by Harick [17] with *windowsize* of 88, uniform crossover [18], and Polynomial mutation which was designed by Deb and Goyal with distribution index 100 [19]. The crossover probability was set to 0.5 with uniform exchange rate of 0.5, and the mutation rate was set to $1/l$, i.e. 0.25. For replacement we join current population and the newly generated solutions together, sort them, and keep the best half. This replacement strategy makes the GA elitist, never losing the best solution found so far. This GA employs a restarting mechanism which works as the following; when the population fully converges, that is when all the individuals have the same number of points in their tolerance threshold, it saves the best solution, removes its fitted points from the point cloud, and restarts the population. It keeps doing that until there is no more point left in the point cloud. The experiments were performed with populations of size 100, 200, 300, 400, and 500 individuals, and for each size, 100 independent runs were executed.

The planes' configurations collected by using GA were compared with the ground truth in order to find their corresponding planar surfaces in the point cloud. It was done by calculating the *dihedral angle* between any discovered plane and all the planar surfaces in the point cloud (the ground truth). The one with the lowest angle which also share at least one point with the discovered plane and having the same number of points in its plane tolerance threshold is the corresponding one. Thereafter, we calculated the success rate, the percentage of runs able to locate correctly all the 22 planar surfaces in the point cloud, as well as the average distance error, over the 100 runs, between points in the point cloud and their relative discovered planar surfaces. The results are summarized in Table I.

Table I
RESULTS FROM TESTING THE LOGARITHMICALLY PROPORTIONAL OBJECTIVE FUNCTION ON THE SYNTHETIC POINT CLOUD BY MEANS OF GA. THEY ARE AVERAGED OVER 100 INDEPENDENT RUNS.

Population size	Success rate (%)	Average of function evaluations	Average of distance errors (cm)
100	95.4	97618	0.0594
200	99.7	156794	0.0287
300	99.9	208773	0.0166
400	100.0	255496	0.0130
500	100.0	315135	0.0113

Table I shows that increasing the population size does help the GA to obtain higher success rates, and reach a 100% success rate in population sizes 400, and 500. In accordance to population sizing theory of GAs [20], larger populations sizes tend to produce a better solution quality, but also at the expense of more processing time. The improvement of solutions quality by increasing population size can be also observed in the average distance errors.

V. Summary and Conclusion

This paper presented a new objective function for planar surface recognition in 3D point cloud that can be used in any heuristic and metaheuristic algorithms without having any prior knowledge about those surfaces or applying segmentation. By paying close attention to the features of most part of built environments, one can realize that planar surfaces occur quite often. Most of the methods that are being used to deal with the problem of planar surface recognition are basically dependent on segmentation.

This paper approaches the problem from a different angle. The proposed logarithmic objective function makes any planar surface in the point cloud to see other surfaces as noise. Thus, segmentation won't be needed as well as this method can be used in any type of point cloud data. The use of the logarithmic form of Euclidean distances decreases the impact of noises on the objective function evaluation, while it increases the impact of points very close to the sampled plane. This function gives the same interest to the number of points fitted on a sampled plane and their precision. However, it tends to find bigger planar surfaces (surfaces with more points) at the early stage of run, which also benefits the algorithm from searching through smaller point cloud as long as it's going forward. The given formula for calculating the base of the logarithm according to the plane tolerance threshold and plane region gives a dynamic layout to the objective function that enables it to adapt itself to the scale of the scanned construction. The results presented in this paper do give evidence that the proposed objective function can perfectly cope with the problem as well as with choosing a large-enough population size it can find all the planar surfaces with a very minor error to the actual configurations. More important, it could find planar surfaces with different sizes and different dihedral angles between them.

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