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Logic Based Merging

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Abstract Belief merging aims at combining several pieces of information coming from different sources. In this paper we review the works on belief merging of propositional bases. We discuss the relationship between merging, revision, update and confluence, and some links between belief merging and social choice theory. Finally we mention the main generalizations of these works in other logical frameworks.

Keywords Belief merging · Arbitration · Belief change

1 Introduction

Belief change theory has produced a lot of different operators that models the different ways the beliefs of one (or some) agent(s) evolve over time. Among these operators, one can quote revision [1, 44, 45, 61], update [52, 60], extrapolation [38], etc.

Belief revision has to be used if one wants to combine two pieces of information while giving the precedence to one of them. Belief merging [6, 7, 69, 74, 76–78, 89, 91] aims at combining several pieces of information when there are no strict precedence between them. The agent faces several conflicting pieces of

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information coming from several sources of equal reliability,[1] and he has to build a coherent description of the world from them. In that respect belief merging has some links with non-prioritized belief revision (see [51]), where the new input is not automatically accepted. But works on non-prioritized belief revision still focus on binary operators, and can not easily been generalized to the aggregation of n sources of information (see for instance the trivialization result in the case of commutative revision [74]).

The aim of this work is to give an account from the main tools developed in last years in the area of belief merging. We will focus on the case where the pieces of information have logical representations. Merging is also at work on numerical datas, but it is not the aim of this paper.[2] See [17] for an interesting global overview on (logical and numerical) merging.

Like in belief revision, rationality postulates have been proposed to characterize belief merging operators. These postulates are closely related to the revision ones. Nevertheless there is an important difference, namely the social aspect of merging: one needs some postulates to say how to solve the conflicts between the sources of information. So it is possible to distinguish different families of merging operators, depending of their behavior with respect to the sources, like a majority behavior for instance.

Similarly to belief revision, it is possible to state representation theorems that provide a constructive way to define merging operators satisfying all the desired logical properties.

They are numerous ways to define merging operators: model-based operators, that select the interpretations that are the closest from the set of sources; formula-based operators, that use a selection function on sets of formulae; DA^2 operators, that generalize model-based operators and allow to take into account inconsistent sources; disjunctive operators, that select the result of the merging inside the disjunction of the bases; conflict-based operators, that use a vector of conflict in order to represent the conflict instead of the numerical distance of model-based operators; default-based operators, that use renaming of the propositional variables of the language.

There are interesting relationships between belief revision, belief merging and other change operators. Actually, belief merging is an extension of belief revision. Furthemore the tight relationships between belief revision operators and update operators suggest that there is a class of operators that extend updates operators while having tights relationships with belief merging. These operators are the confluence operators. They can be seen as a sort of pointwise belief merging operators.

It is interesting to notice that merging operators show tight relationships with social choice theory, and in particular with voting methods. Studying these

[1] More generally the sources can have different reliability, but we will focus on the case where all the sources have the same reliability. There are already a lot to say in this case.

[2] See for instance [5, 16, 93] for some examples of numerical data fusion, or look at the Information Fusion Journal issues.

relationships can give rise to interesting problems and solutions. For instance from merging to social choice, it has been proposed to use merging operators in order to define judgment aggregation methods [84]. In the other way, it is interesting to study what are the consequences of concepts well known in social choice theory when applied to merging scenarios. We will give two such examples: strategy-proofness and truth-tracking.

Merging is a problem that occur in a lot of situations, some of them do not use propositional logic as representation language, but more structured languages. We will mention the extension of propositional merging operators to some of these frameworks, namely weighted logics, first order logic, logic programs, constraint networks and argumentation frameworks.

2 Notation

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the standard connectives, including $\top$ and $\bot$.

An interpretation ω is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is noted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $mod(\varphi)$ denotes the set of models of the formula φ, i.e., $mod(\varphi) = \{\omega \in \mathcal{W} \mid \omega \models \varphi\}$. When M is a set of models we denote by φ_M a formula such that $\mathrm{mod}\ (\varphi_M) = M$.

A *base* K is a finite set of propositional formulae $\{\varphi_1, \ldots, \varphi_n\}$. We denote by $\bigwedge K$ the conjunction of formulae of K, i.e., $\bigwedge K = \varphi_1 \wedge \ldots \wedge \varphi_n$. Often, in order to simplify the notations, we will identify[3] the base K with the formula $\varphi = \bigwedge K$ which is the conjunction of the formulae of K. We denote by $\mathcal{K}$ the set of bases.

A *profile* E is a non-empty multi-set (bag) of bases $E = \{K_1, \ldots, K_n\}$ (hence different agents are allowed to exhibit identical bases), and represents a group of n agents. We denote by $\mathcal{E}$ the set of profiles.

We denote by $\bigwedge E$ the conjunction of bases of $E = \{K_1, \ldots, K_n\}$, i.e., $\bigwedge E = \bigwedge K_1 \wedge \ldots \wedge \bigwedge K_n$. We denote by $\bigvee E$ the disjunction of bases of E, i.e., $\bigvee E = \bigwedge K_1 \vee \ldots \vee \bigwedge K_n$.

A profile E is said to be consistent if and only if $\bigwedge E$ is consistent. The multi-set union is noted $\sqcup$. By abuse of notation we will write $K \sqcup E$ instead of $\{K\} \sqcup E$. We denote by E^n the profile in which E appears n times, more precisely $E^n = \underbrace{E \sqcup \ldots \sqcup E}_{n}$. Two profiles are equivalent, denoted $E_1 \equiv E_2$, if there is a bijective function f from E_1 onto E_2 such that for any $K \in E_1$, $f(K) \equiv K$.

A base (formula) K is complete if it has only one model. A profile E is complete if all the bases of E are complete.

[3] This identification will be done when the approach is not sensitive to syntactical representation. When the approach is sensitive to syntactical representation, it will be important to distinguish between K and the conjunction of its formulae (see e.g. [66]).

If $\leq$ denotes a pre-order on $\mathcal{W}$ (i.e. a reflexive and transitive relation), then $<$ denotes the associated strict order defined by $\omega < \omega'$ if and only if $\omega \leq \omega'$ and $\omega' \not\leq \omega$. A pre-order is *total* if $\forall \omega, \omega' \in \mathcal{W}$, $\omega \leq \omega'$ or $\omega' \leq \omega$. A pre-order that is not total is called *partial*. Let $\leq$ be a pre-order on A, and $B \subseteq A$, then $min(B, \leq) = \{b \in B \mid \nexists a \in B\ a < b\}$.

If A is a set, we denote $|A|$ the cardinal of A. The symbol $\subseteq$ will denote set containment and $\subset$ strict set containment, i.e., $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

Merging operators we will consider are functions from the set of profiles and the set of propositional formulae (that will represent integrity constraints) to the set of bases, i.e. $\triangle : \mathcal{E} \times \mathcal{L} \mapsto \mathcal{K}$. We will use the notation $\triangle_\mu(E)$ instead of $\triangle(E, \mu)$.

3 The Logical Framework for Merging

We first study the logical properties of propositional merging operators and state a representation theorem for these operators in terms of pre-orders on interpretations.

3.1 Logical Properties

Just as for belief revision, it is possible to state some logical properties one could expect from any reasonable merging operator.

Definition 1 An operator $\triangle : \mathcal{E} \times \mathcal{L} \mapsto \mathcal{K}$ is said to be an Integrity Constraints merging operator (IC merging operator for short) iff the following properties hold:

(IC0) $\triangle_\mu(E) \vdash \mu$

(IC1) If μ is consistent, then $\triangle_\mu(E)$ is consistent

(IC2) If $\bigwedge E$ is consistent with μ, then $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$

(IC3) If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$

(IC4) If $K_1 \vdash \mu$ and $K_2 \vdash \mu$, then $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent

(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \vdash \triangle_\mu(E_1 \sqcup E_2)$

(IC6) If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \vdash \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$

(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(E)$

(IC8) If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \vdash \triangle_{\mu_1}(E)$

Some of these properties had been proposed by Revesz [91] in order to define *model fitting* operators. They have been extended in [67, 69].

Intuitively $\triangle_\mu(E)$ is the closest belief base to the profile E satisfying the integrity constraint μ. This idea is what the postulates try to capture. The meaning of the postulates is the following: (IC0) assures that the result of the merging satisfies the integrity constraints. (IC1) states that if the integrity constraints are consistent, then the result of the merging will be consistent. (IC2) states that if possible, the result of the merging is simply the conjunction of the belief bases with the integrity constraints. (IC3) is the principle of irrelevance of syntax, i.e. if two profiles are equivalent and two integrity constraints bases are logically equivalent then the belief bases result of the two merging will be logically equivalent. (IC4) is the fairness postulate, the point is that when we merge two belief bases, merging operators must not give preference to one of them. (IC5) expresses the following idea: if two groups E_1 and E_2 agree on some alternatives then these alternatives will be chosen if we join the two groups. (IC5) and (IC6) together state that if one could find two subgroups which agree on at least one alternative, then the result of the global merging will be exactly those alternatives the two groups agree on. (IC7) and (IC8) are a direct generalization of the (R5–R6) postulates for revision [61]. They state some conditions about integrity constraints conjunctions. Actually, they ensure that the notion of *closeness* is well-behaved. For instance, if an alternative A is chosen among a set of alternatives, then if the set of alternatives is narrowed but the alternative A remains in this set, the alternative A will be still chosen. This quite natural property appears in different theories of choice (social choice, decision, etc).

These properties are the basic ones one could expect from merging operators. But it is still possible to ask additional constraints on the behavior of the operators. For instance two important subclasses of IC merging operators are majority operators and arbitration operators.

A majority merging operator is an IC merging operator that satisfies the following *majority* postulate:

(Maj) $\exists n \ \triangle_\mu (E_1 \sqcup E_2^n) \vdash \triangle_\mu(E_2)$

This postulate expresses the fact that if a subgroup is repeated sufficiently many times then it is the opinion of this subgroup that will prevail. Notice that this property is quite general. It doesn't say the exact number of times a subprofile has to appear to prevail.

The majority merging operators aims to satisfy the group as a whole. Unlike these operators, arbitration operators, aim to satisfy each individual of the group as far as possible.

An arbitration operator is an IC merging operator that satisfies the following postulate:

(Arb) If $\triangle_{\mu_1}(K_1) \equiv \triangle_{\mu_2}(K_2)$, $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\{K_1, K_2\}) \equiv (\mu_1 \leftrightarrow \neg\mu_2)$, $\mu_1 \nvdash \mu_2$, and $\mu_2 \nvdash \mu_1$, then $\triangle_{\mu_1 \vee \mu_2}(\{K_1, K_2\}) \equiv \triangle_{\mu_1}(K_1)$

This postulate says that if a set of alternatives preferred among one set of integrity constraints μ_1 for a base K_1 corresponds to the set of alternatives preferred among another set of integrity constraints μ_2 for a base K_2, and if the alternatives that belong to a set of integrity constraints but not to the other are equally preferred for the whole group ($K_1 \sqcup K_2$), then the subset of preferred alternatives among the disjunction of integrity constraints will coincide with the preferred alternatives of each base among their respective integrity constraints. This property is much more intuitive when it is expressed in a model-theoretical way (cf. condition 8 of a fair syncretic assignment in Definition 2). It states that the median possible choices are preferred.

3.2 Representation Theorem

Now that we have a logical definition of IC merging operators, we will state a representation theorem that gives a more constructive way to define IC merging operators. More precisely we will show that to each IC merging operator corresponds a family of pre-orders on interpretations.

First we have to introduce the notion of *syncretic assignment*.

Definition 2 A *syncretic assignment* is a function mapping each profile E to a total pre-order $\leq_E$ over interpretations such that for any profiles E, E_1, E_2 and for any belief bases K, K' the following conditions hold:

1. If $\omega \models E$ and $\omega' \models E$, then $\omega \simeq_E \omega'$
2. If $\omega \models E$ and $\omega' \not\models E$, then $\omega <_E \omega'$
3. If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$
4. $\forall \omega \models K \; \exists \omega' \models K' \; \omega' \leq_{K \sqcup K'} \omega$
5. If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega \leq_{E_1 \sqcup E_2} \omega'$
6. If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega <_{E_1 \sqcup E_2} \omega'$

A **majority syncretic assignment** is a syncretic assignment which satisfies the following condition:

7. If $\omega <_{E_2} \omega'$, then $\exists n \; \omega <_{E_1 \sqcup E_2^n} \omega'$

A **fair syncretic assignment** is a syncretic assignment which satisfies the following condition:

8. If $\omega <_{K_1} \omega'$, $\omega <_{K_2} \omega''$, and $\omega' \simeq_{K_1 \sqcup K_2} \omega''$, then $\omega <_{K_1 \sqcup K_2} \omega'$

The two first conditions ensure that the models of the profile (if any) are the most plausible interpretations for the pre-order associated to the profile. The third condition states that two equivalent profiles have the same associated pre-orders. These three conditions are very close to the ones existing in belief revision for faithful assignments [61]. The fourth condition states that, when merging two belief bases, for each model of the first one, there is a model of the second one that is at least as good than the first one. It ensures that the two bases receive the same treatment in the merging process. The fifth condition says that if an interpretation ω is at least as plausible as an interpretation ω' for a profile E_1 and if ω is at least as plausible as ω' for a profile E_2, then if one joins the two profiles, then ω will still be at least as plausible as ω'. The sixth condition strengthen the previous condition by saying that an interpretation ω

is at least as plausible as an interpretation ω' for a profile E_1 and if ω is strictly more plausible than ω' for a profile E_2, then if one joins the two profiles, then ω will be strictly more plausible than ω'. These two previous conditions are very close to Pareto conditions in Social Choice Theory [4, 62]. Condition 7 says that if an interpretation ω is strictly more plausible than an interpretation ω' for a profile E_2, then there is a quorum n of repetitions of the profile from which ω will be more plausible than ω' for the larger profile $E_1 \sqcup E_2^n$. This condition seems to be the weakest form of "majority" condition one could state. Condition 8 states that the median choices are preferred by the group. More precisely, if an interpretation ω is more plausible than an interpretation ω' for a belief base K_1, if ω is more plausible than ω'' for another base K_2, and if ω' and ω'' are equally plausible for the profile $K_1 \sqcup K_2$, then ω has to be more plausible than ω' and ω'' for $K_1 \sqcup K_2$.

Now we can state the following representation theorem for IC merging operators:

Theorem 3 ([69]) *An operator $\triangle$ is an IC merging operator (or majority merging operator or arbitration operator respectively) if and only if there exists a syncretic assignment (or majority syncretic assignment or fair syncretic assignment respectively) that maps each profile E to a total pre-order $\leq_E$ such that $mod(\triangle_\mu(E)) = min(mod(\mu), \leq_E)$.*

This theorem has been generalized to the framework of infinite propositional logic in [23].

4 Main Families of Merging Operators

We will now give in this Section a short overview of the main families of belief merging operators.

4.1 Model-Based Operators

Model-based operators are defined by selecting in the model of the constraints the interpretations that are the closest from the profile, i.e.

Definition 4 A model based merging operator $\triangle$ is defined by :

$$mod(\triangle_\mu(E)) = min(mod(\mu), \leq_E)$$

Note that this definition is directly inspired by the representation theorem.

Usually this closeness is represented using a distance between interpretations and an aggregation function, in the following way.

Definition 5 A **(pseudo-)distance**[4] between interpretations is a function $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{R}^+$ such that for any $\omega, \omega' \in \mathcal{W}$:

- $d(\omega, \omega') = d(\omega', \omega)$, and
- $d(\omega, \omega') = 0$ iff $\omega = \omega'$.

Definition 6 An **aggregation function** f is a function mapping for any positive integer n, each n-tuple of positive reals into a positive real such that for for any $x_1, \ldots, x_n, x, y \in \mathbb{R}^+$:

- if $x \leq y$, then $f(x_1, \ldots, x, \ldots, x_n) \leq f(x_1, \ldots, y, \ldots, x_n)$ *(monotony)*
- $f(x_1, \ldots, x_n) = 0$ iff $x_1 = \ldots = x_n = 0$ *(minimality)*
- $f(x) = x$ *(identity)*

Definition 7 Let d and f be a distance between interpretations and an aggregation function respectively. The model based merging operator $\triangle^{d,f}$ is defined by $mod(\triangle_{\mu}^{d,f}(E)) = min(mod(\mu), \leq_E)$, where the total pre-order $\leq_E$ on $\mathcal{W}$ is defined in the following way:

- $\omega \leq_E \omega'$ iff $d(\omega, E) \leq d(\omega', E)$,
- $d(\omega, E) = f(d(\omega, K_1), \ldots, d(\omega, K_n))$, where $E = \{K_1, \ldots, K_n\}$,
- $d(\omega, K) = min_{\omega' \models K} d(\omega, \omega')$

The operators studied in [78, 91] are particular cases when the distance used is the Hamming distance (the Hamming distance d_H is the number of propositional letters on which the two interpretations differ) and the aggregation function is the sum or the maximum. The results in [69] show that any distance produce merging operators with good logical properties. Another particular distance is the drastic distance d_D, defined as $d_D(\omega_1, \omega_2) = 0$ if $\omega_1 = \omega_2$, or 1 otherwise.

In [69] the use of the aggregation function Gmax (leximax) was proposed, leading to arbitration operators. Using as aggregation functions the sum of nth powers allows to modulate the consensual degree of the operator [70]. Finally the use of the aggregation function Gmin (leximin) has been recently proposed [41], and give rise to disjunctive operators (see Section 4.4).

If the aggregation function f has good properties, like most common aggregation functions (the sum, the leximax, the sum of *nth* powers, the leximin), the model based merging operator associated (for any distance) are IC merging operators. More precisely we have the following results [66]:

Theorem 8 *Let d and f be a distance between interpretations and an aggregation function respectively. The operator $\triangle^{d,f}$ satisfies the postulates (IC0), (IC1), (IC2), (IC7) and (IC8).*

[4]The triangle inequality is not required.

Theorem 9 *Let d and f be a distance between interpretations and an aggregation function respectively. The operator $\triangle^{d,f}$ satisfies the postulates (IC0–IC8) iff the aggregation function f satisfies the following properties:*

– *For any permutation σ, $f(x_1, \ldots, x_n) = f(\sigma(x_1, \ldots, x_n))$* (symmetry)
– *If $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$, then $f(x_1, \ldots, x_n, z) \leq f(y_1, \ldots, y_n, z)$* (composition)
– *If $f(x_1, \ldots, x_n, z) \leq f(y_1, \ldots, y_n, z)$, then $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$* (decomposition)

Let us now give as example two families of IC merging operators: Σ operators and GMAX operators.

Let d be any distance between interpretations, the Σ operator $\triangle^{d,\Sigma}$ (where Σ denotes the usual sum) is a majority merging operator:

Theorem 10 ([69]) *Every operator $\triangle^{d,\Sigma}$ is a majority merging operator.*

In order to define the other family. Let us introduce the function GMAX. It is a generalization of the operators defined with the aggregation function max, which are not IC merging operators [69] (operators defined with the max function are IC quasi-merging operators [69]).

Let $E = \{K_1, \ldots, K_n\}$ be a profile. For each interpretation ω we build the list $(d_1^\omega, \ldots, d_n^\omega)$ of distances between this interpretation and the n belief bases in E, i.e. $d_j^\omega = d(\omega, K_j)$. Let L_ω^E be the list obtained from $(d_1^\omega, \ldots, d_n^\omega)$ by sorting it in descending order. Then these vectors are compared with the lexicographical order. This method can be identified with an aggregation function[5] GMAX such that

$$\text{GMAX}(d_1^\omega, \ldots, d_n^\omega) \leq \text{GMAX}(d_1^{\omega'}, \ldots, d_n^{\omega'}) \quad \text{iff} \quad L_\omega^E \leq_{lex} L_{\omega'}^E$$

This aggregation function has also the properties of Theorem 9 (see [69]). So the model-based operator build from this function (whatever the distance d), called $\triangle^{d,\text{GMAX}}$ satisfies the postulates (IC0)–(IC8). Indeed, these operators satisfy also the arbitration postulate:

Theorem 11 ([69]) *Every operator $\triangle^{d,\text{GMAX}}$ is an arbitration operator.*

It is interesting to notice that the families of arbitration operators and majority operators are not disjoint. Actually if d_D is the drastic distance we have $\triangle^{d_D,\Sigma} = \triangle^{d_D,\text{GMAX}}$.

The following example, in which d_H will be the Hamming distance, illustrates the behavior of these classes of operators.

[5]See for instance [66] for a formal mathematical definition of GMAX, we prefer to give the intuition here.

Example 12 At a meeting of a block of flat co-owners, the chairman proposes for the coming year the construction of a swimming-pool, of a tennis-court and of a private-car-park. But if two of these three items are build, the rent will increase significantly. We will denote by S, T, P respectively the construction of the swimming-pool, the tennis-court and the private-car-park. We will denote I the rent increase.

The chairman outlines that building two items or more will have an important impact on the rent: $\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$.

There are four co-owners $E = \{K_1 \sqcup K_2 \sqcup K_3 \sqcup K_4\}$. Two of the co-owners want to build the three items and do not care about the rent increase: $K_1 = K_2 = S \wedge T \wedge P$. The third one thinks that building any item will caused at some time an increase of the rent and wants to pay the lowest rent so he is opposed to any construction: $K_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$. The last one thinks that the block really needs a tennis-court and a private-car-park but does not want a high rent increase : $K_4 = T \wedge P \wedge \neg I$.

The propositional letters S, T, P, I will be considered in that order for the interpretations:

$$mod(\mu) = \omega \setminus \{(0,1,1,0),(1,0,1,0),(1,1,0,0),(1,1,1,0)\}$$
$$mod(K_1) = \text{mod }(K_2) = \{(1,1,1,1),(1,1,1,0)\}$$
$$mod(K_3) = \{(0,0,0,0)\} \qquad mod(K_4) = \{(1,1,1,0),(0,1,1,0)\}$$

We sum up the calculations in Table 1, for each interpretation we give the distances between this interpretation and the four belief bases and the distance between this interpretation and the profile according to the $\triangle^{d_H,\Sigma}$ and the $\triangle^{d_H,\text{Gmax}}$ operators. The lines shadowed correspond to the interpretations rejected by the integrity constraints. Thus the result has to be found among the interpretations that are not shadowed.

If one takes the decision according to the majority wishes, using the $\triangle^{d_H,\Sigma}$ operator we have 5 as minimum distance, so $mod(\triangle_\mu^{d_H,\Sigma}(E)) = \{(1,1,1,1)\}$,

Table 1 $\triangle^{d_H,\Sigma}$ and $\triangle^{d_H,\text{Gmax}}$ operators

	K₁	**K₂**	**K₃**	**K₄**	Σ	Gmax
(0, 0, 0, 0)	3	3	0	2	8	(3,3,2,0)
(0, 0, 0, 1)	3	3	1	3	10	(3,3,3,1)
(0, 0, 1, 0)	2	2	1	1	6	**(2,2,1,1)**
(0, 0, 1, 1)	2	2	2	2	8	(2,2,2,2)
(0, 1, 0, 0)	2	2	1	1	6	**(2,2,1,1)**
(0, 1, 0, 1)	2	2	2	2	8	(2,2,2,2)
(0, 1, 1, 0)	1	1	2	0	4	(2,1,1,0)
(0, 1, 1, 1)	1	1	3	1	6	(3,1,1,1)
(1, 0, 0, 0)	2	2	1	2	7	(2,2,2,1)
(1, 0, 0, 1)	2	2	2	3	9	(3,2,2,2)
(1, 0, 1, 0)	1	1	2	1	5	(2,1,1,1)
(1, 0, 1, 1)	1	1	3	2	7	(3,2,1,1)
(1, 1, 0, 0)	1	1	2	1	5	(2,1,1,1)
(1, 1, 0, 1)	1	1	3	2	7	(3,2,1,1)
(1, 1, 1, 0)	0	0	3	0	3	(3,0,0,0)
(1, 1, 1, 1)	0	0	4	1	**5**	(4,1,0,0)

and the decision that satisfies the majority in the group is to build the three items and to increase the rent. But, with an arbitration operator, such as Gmax, we have $mod(\triangle_{\mu}^{d_H,\text{Gmax}}(E)) = \{(0, 0, 1, 0), (0, 1, 0, 0)\}$, so the decision that best fit the group and that is allowed by the integrity constraints is to build either the tennis-court or the private-car-park, without increasing the rent.

See [2] for a study of properties of the consequence relations that can be defined from model-based merging operators.

4.2 Formula-Based Operators

Formula-based merging operators are sensitive to the syntax of the formulae of the belief bases to be merged. When the belief bases are sets of formulae, usual formula-based merging operators select in the union of the bases some maximal consistent subsets of formulae [7, 9]. This method has as drawback that the sources of information are lost in the process, so these operators can not take into account the distribution of the information for computing the merging. This is a problem if one wants to take majority notions into account for instance.

In [63] it has been proposed to use selection functions, close to the *transitively relational partial meet contractions functions* [1] used for belief revision/contraction, and to use them in order to take the distribution of information into account. This allow to define formula-based merging operators with a better behaviour and thus better logical properties. Let us define these operators formally:

Definition 13 Let $\text{Maxcons}(K, \mu)$ be the set of the maximal (for set inclusion) consistent subsets (**maxcons**) of $K \cup \{\mu\}$ that contain μ. So $\text{Maxcons}(K, \mu)$ is the set of M such that :

- $M \subseteq K \cup \{\mu\}$,
- $\mu \in M$,
- $M \nvdash \bot$,
- if $M \subset M' \subseteq K \cup \{\mu\}$, then $M' \vdash \bot$.

Let $\text{Maxcons}(E, \mu) = \text{Maxcons}(\bigcup_{K_i \in E} K_i, \mu)$. When maximality is defined with respect to cardinality, we will use the subscript "card", i.e. we will note $\text{Maxcons}_{\text{card}}(E, \mu)$.

Then one can define the following **formula-based merging** operators:

Definition 14 Let E be a profile and μ be a formula:

$$\triangle_{\mu}^{C1}(E) = \bigvee \text{Maxcons}(E, \mu)$$
$$\triangle_{\mu}^{C3}(E) = \bigvee \{M \mid M \in \text{Maxcons}(E, \top) \text{ and } M \cup \{\mu\} \text{ consistent}\}$$

Table 2 Properties for operators with selection functions based on $\triangle^{C1}$

	IC0	IC1	IC2	IC3	IC4	IC5	IC6	IC7	IC8	MI	Maj
$\triangle^{C1}$	√	√	√	–	√	√	–	√	–	√	–
$\triangle^{d}$	√	√	√	–	√	√	–	√	–	–	√
$\triangle^{S,\Sigma}$	√	√	√	–	√	–	–	√	√	–	√
$\triangle^{\cap,\Sigma}$	√	√	√	–	–	√	√	√	√	–	√

$$\triangle_\mu^{C4}(E) = \bigvee \textsc{Maxcons}_{\text{card}}(E,\mu)$$
$$\triangle_\mu^{C5}(E) = \bigvee\{M\cup\{\mu\} \mid M\in\textsc{Maxcons}(E,\top) \text{ and } M\cup\{\mu\} \text{ consistent}\}$$
$$\text{if this set is non-empty and } \mu \text{ otherwise}$$

Operators $\triangle_\mu^{C1}(E)$, $\triangle_\mu^{C3}(E)$ and $\triangle_\mu^{C4}(E)$ correspond respectively to operators $Comb1(E,\mu)$, $Comb3(E,\mu)$ and $Comb4(E,\mu)$ defined in [7]. $\triangle^{C5}$ is a slight modification of $\triangle^{C3}$ in order to have better logical properties [63].

These operators make the union of the bases, and then they try to obtain a consistent result from this inconsistent union. This is very close to inference relations based on maximal consistent subsets [10, 11, 88]. But this is not satisfactory from a merging point of view since we do not take into account the localization of the pieces of information among the different bases of the profile. That is exactly this point that makes a distinction between merging and inference under inconsistency. We call these operators **combination operators** (this name was used in the original papers of [6, 7]), to make a distinction with **merging operators**.

Then it has been proposed in [63] to select only the maxcons that best satisfy a merging criterion. These selection functions are inspired from the ones used in belief revision to define *partial meet contraction functions* [1]. In both cases, the aim of the selection fonctions is to select only the "best" maxcons. The idea in the merging case is that these fonctions bring some social evaluation (i.e. they take into account the distribution of the information among the bases).

Three particular criteria has been studied in [63]. The first one ($\triangle^d$) selects the maxcons that are consistent with the highest number of bases. The second one ($\triangle^{S,\Sigma}$) selects the maxcons that have the smallest differences (for cardinality) with the bases. The third one ($\triangle^{\cap,\Sigma}$) selects the maxcons that have the biggest intersection (for cardinality) with the bases.

Table 2 shows, using the combination operator[6] $\triangle^{C1}$, that the use of selection functions allows to obtain better properties.

One can check that none formula-based merging operator satisfy all the properties of IC merging (see Section 4.4 for a possible alternative).

The study of other operators and selection functions seems interesting. In particular it seems sensible to expect a representation theorem for IC merging using selection functions, that would be a generalisation of the ones for *partial meet contraction functions* in the belief revision framework [1].

[6]See [63] for the logical properties of the other combination operators.

4.3 DA^2 Operators

DA^2 operators [66] are parametrized by a distance and two aggregation functions. These operators are a generalization of usual model-based operators, but they also capture some formula-based operators. The drawback of usual model-based merging operators is that they can not take into account inconsistent belief bases. But in some cases the information from these bases can be useful.

Definition 15 Let d be a distance between interpretations and f and g be two aggregation functions. The **DA^2 merging** operator $\triangle^{d,f,g}$ is defined as $mod(\triangle_{\mu}^{d,f,g}(E)) = min(mod(\mu), \leq_E)$, where the pre-order $\leq_E$ on $\mathcal{W}$ is defined as:

- $\omega \leq_E \omega'$ if and only if $d(\omega, E) \leq d(\omega', E)$, where
- $d(\omega, E) = f(d(\omega, K_1) \ldots, d(\omega, K_n))$, where $E = \{K_1, \ldots, K_n\}$.
- $d(\omega, K_i) = g(d(\omega, \varphi_1) \ldots, d(\omega, \varphi_{m_i}))$, where $K_i = \{\varphi_1, \ldots, \varphi_{n_i}\}$.

The first aggregation function g allows to extract a coherent piece of information from any base K_i even if the base is inconsistent. Then the second function f is used for the (usual) inter-source aggregation.

From a complexity point of view these operators are at the same level of the polynomial hierarchy than model-based merging operators (so this generalisation has no cost from a complexity point of view).

For more details on computational complexity of merging operators, and exact complexity of the main merging operators see [66].

4.4 Disjonctive Operators and Unaninity

Quota merging operators [41] define the models of the merging as the interpretations that satisfy sufficiently many bases. These operators are a good compromise between several important criteria for merging operators: logical properties, computational complexity, strategy-proofness (see Section 6.1) and inferential power.

Gmin operators are model-based operators (Section 4.1) with a leximin aggregation function [41] that are related to quota merging operators since they have a better inferential power and they satisfy more logical properties.[7]

All these operators are disjunctive operators, that means that the result of the merging is selected among the disjunction (union) of the bases.

This property is not mandatory for merging operators, since it prevents from finding compromise solutions, that have not been proposed by any base. But in some cases it can be justified to require this disjunction property, especially for belief merging (cf. Section on belief merging versus goal merging

[7] On the other hand this has to be paid by a higher computational complexity.

(Section 6.2)). For instance suppose that several doctors propose different medical treatments for a given patient. It seems clear that it is preferable to choose among the possible treatments rather than to melt the treatments. Note that the commutative revision operators of Liberatore and Schaerf [74] are also disjunctive operators.

Another justification of this disjunction property is that it can be explained as the translation of a unanimity property. Unanimity is a classical property when one aggregates pieces of information, and it is considered as a major requirement for voting methods for instance. This property intuitively means that if all the agents judge a candidate as the best one, then he has to be the best one for the group. If we consider unanimity for belief merging, this property can be expressed in two different ways [41]. The most direct one is the **unanimity on Models**, that can be expressed this way:

(UnaM) If $\omega \models \mu$ and $\forall K \in E, \omega \models K$, then $\omega \models \triangle_\mu(E)$

This condition is a consequence of property (IC2), so it is satisfied by any IC merging operator. But, if one consider a base as the set of its logical consequences, then we can express an **unanimity on Formulae**:

(UnaF) If $\exists K \in E$ s.t. $\mu \wedge K$ is consistent, then if $\forall K \in E$, $K \models \alpha$, then $\triangle_\mu(E) \models \alpha$

The additional condition of (UnaF) just ensures that it is possible to select a result in the disjunction of the bases that is compatible with the integrity constraints.

And (UnaF) is equivalent to the following disjunction property [41]:

(Disj) If $\bigvee E$ is consistent with μ, then $\triangle_\mu(E) \models \bigvee E$

This (Disj) property is also the main reason for motivating the use of formula-based merging operators (see Section 4.2). Since, compared to model-based operators, these operators satisfy less logical properties, and often have a higher computational complexity.

This work suggests that Gmin operators are a good substitute to formula-based operators: they also satisfy the disjunction property, but they have better logical properties (they are IC merging operators), and a lower computational complexity.

4.5 Conflict-Based Operators

Model-based merging operators (Section 4.1) are based on a notion of proximity between models. This proximity notion is captured by a (numerical) distance, such as the Hamming distance for instance. An other possibility is to consider as "distance" the set of all propositional variables that differ between the two interpretation (this distinction exists in the framework of belief revision, if one looks at the link between the Dalal operator [29] and the Borgida operator [20] for instance). This allows to be more precise than with

Table 3 Differences between ω_1 and ω_2?

	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, K_3)$	$\mathsf{diff}(\omega, K_4)$
ω_1	$\{\{a\}\}$	$\{\{a\}\}$	$\{\{a\}\}$	$\{\{a\}\}$
ω_2	$\{\{a\}\}$	$\{\{b\}\}$	$\{\{c\}\}$	$\{\{d\}\}$

the Hamming distance. So it is possible to define **conflict-based operators**, that generalize model-based operators.

Let us see an example in order to illustrate how the definition of a **vector of conflict** allows a better discrimination of interpretations than when the conflict is summed up by a numerical distance.

Example 16 Consider a language that contains the variables a, b, c, d, a profile $E = \{K_1, \ldots, K_4\}$ and an integrity constraint μ such that $mod(\mu) = \{\omega_1, \omega_2\}$. Define $\mathsf{diff}(\omega, \omega') = \{a \in \mathcal{P} \mid \omega(a) \neq \omega'(a)\}$, and $\mathsf{diff}(\omega, K) = min_{\omega' \models K}(\mathsf{diff}(\omega, \omega'), \subseteq)$.

Table 3 shows the minimal conflict between each model of the constraints and each base of the profile. For instance the conflict between ω_1 and K_1 is about the variable a, this means that the model of K_1 that is the closest from ω_1 only disagree about the truth value of variable a.

Clearly the Hamming distance does not allow to discriminate these two interpretations, since they both are at a distance 1 of each base. We obtain two vectors $\langle 1, 1, 1, 1\rangle$ that are indistinguishable for any aggregation function. If we use vectors of conflict we obtain in a case $\langle a, a, a, a\rangle$, and in the other $\langle a, b, c, d\rangle$, so there is a clear difference between the two situations. In ω_1 all the agents agree that the problem is about the variable a, whereas this is not the case in ω_2. So these two interpretations can be treated differently by conflict-based merging operators.

Let us define formally the vectors of conflict and the corresponding merging operators [40]:

Definition 17 The conflict between two interpretations is defined as $\mathsf{diff}(\omega, \omega') = \{p \in \mathcal{P} \mid \omega(p) \neq \omega'(p)\}$. The conflict between an interpretation and a base is $\mathsf{diff}(\omega, K) = min(\{\mathsf{diff}(\omega, \omega') \mid \omega' \models K\}, \subseteq)$. And the **vector of conflict** between an interpretation and a profile $E = \langle K_1, \ldots, K_n\rangle$ is $\mathsf{diff}(\omega, E) = \{\langle c_1^\omega, c_2^\omega, \ldots, c_n^\omega\rangle \mid c_i^\omega \in \mathsf{diff}(\omega, K_i)\}$.

Then one can define a comparison relations $\preceq_R$ between vector of conflicts, and use it to define a conflict-based merging operator:

Definition 18 Let $E = \langle K_1, \ldots, K_n\rangle$ be a profile, μ some integrity constraints and let $\preceq_R$ be a relation on conflict vectors of dimension n. We define[8] $mod(\Delta_\mu^{\mathsf{diff}, R}(E)) = min([\mu], \leq_R^E)$.

[8]In [40] the relation $\leq_R^E$ over $\mathcal{W}$ is defined by $\omega \leq_R^E \omega'$ iff $\exists c \in \mathsf{diff}(\omega, E)$ s.t. $\forall c' \in \mathsf{diff}(\omega', E)$, we have $c \preceq_R c'$. But one can consider other lifting policies.

This is easy to show that usual model-based merging operators can be obtained as special cases. It is also possible to define logical refinements of these operators [40].

4.6 Default-Based Operators

In [34] **default-based merging operators** have been proposed. The idea is to use a specific language for each base, in order to ensure that the union of these bases is consistent. Then to try to add as many default rules as possible in order to identify the corresponding variables of each language (this idea is close to the one used to define a paraconsistent inference relation proposed by Besnard and Schaub [15]).

These operators are some links with conflict-based operators, since they look at the conflict variable by variable. But they treat it differently. So they give rise to distinct operators.

The formal definition is the following one [34]:

Definition 19 A ***i*-renaming** of a language $\mathcal{L}$ is the language $\mathcal{L}^i$, build from a set of propositional variables $\mathcal{P}^i = \{p^i \mid p \in \mathcal{P}\}$, where for each $\alpha \in \mathcal{L}$, α^i is the result of renaming in α every propositional variable $p \in \mathcal{P}$ by the corresponding variable $p^i \in \mathcal{P}^i$. Let K be a base, the i-renaming of (formulae of) K is denoted K^i.

Definition 20 Consider a profile $E = \{K_1, \dots, K_n\}$.

Let EQ be a subset of $\{p^k \Leftrightarrow p^l \mid p \in \mathcal{L} \text{ and } k, l \in \{1 \dots n\}\}$ maximal (for set inclusion) such that $(\bigwedge_{K_i \in E} K_i^i) \wedge EQ$ is consistent. Then $\{\alpha \mid \forall j \in \{1 \dots n\}\ (\bigwedge_{K_i \in E} K_i^i) \wedge EQ \models \alpha^j\}$ is a **symmetric consistent extension** of E. The skeptical merging $\Delta_s(E)$ of E is the intersection of every symmetric consistent extension of E.

Let EQ be a subset of $\{p^j \Leftrightarrow p \mid p \in \mathcal{L} \text{ and } j \in \{1 \dots n\}\}$ maximal (for set inclusion) such that $(\bigwedge_{K_i \in E} K_i^i) \wedge EQ$ is consistent. Then $(\bigwedge_{K_i \in E} K_i^i) \wedge EQ$ is a **projected consistent extension** of E. The skeptical merging $\nabla_s(E)$ of E is the intersection of every projected consistent extension of E.

Let us see how it works on a small example:

Example 21 Consider the profile $E = \{K_1, K_2, K_3\}$, with $K_1 = (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$, $K_2 = (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r)$ and $K_3 = \neg q \wedge \neg r$.

There are four maximal consistent sets of equivalences for $\Delta_s(E)$:

$$EQ_1 = \{p^1 \Leftrightarrow p^2, p^1 \Leftrightarrow p^3, p^2 \Leftrightarrow p^3, r^1 \Leftrightarrow r^2, r^1 \Leftrightarrow r^3, r^2 \Leftrightarrow r^3, q^2 \Leftrightarrow q^3\}$$
$$EQ_2 = \{p^1 \Leftrightarrow p^3, r^1 \Leftrightarrow r^2, r^1 \Leftrightarrow r^3, r^2 \Leftrightarrow r^3, q^1 \Leftrightarrow q^2\}$$
$$EQ_3 = \{p^2 \Leftrightarrow p^3, r^1 \Leftrightarrow r^2, r^1 \Leftrightarrow r^3, r^2 \Leftrightarrow r^3, q^1 \Leftrightarrow q^2\}$$
$$EQ_4 = \{p^1 \Leftrightarrow p^2, p^1 \Leftrightarrow p^3, p^2 \Leftrightarrow p^3, r^2 \Leftrightarrow r^3, q^1 \Leftrightarrow q^2\}$$

So, $\Delta_s(E) \equiv \neg r \vee (\neg p \wedge q)$.

For ∇_s, the maximal consistent sets of equivalences are the following ones ($p \Leftrightarrow p^1 \Leftrightarrow p^2 \Leftrightarrow p^3$ will be used as an abbreviation for $p \Leftrightarrow p^1, p \Leftrightarrow p^2, p \Leftrightarrow p^3$) :

$EQ_1 = \{p \Leftrightarrow p^1 \Leftrightarrow p^2 \Leftrightarrow p^3, r \Leftrightarrow r^1 \Leftrightarrow r^2 \Leftrightarrow r^3, q \Leftrightarrow q^2 \Leftrightarrow q^3\}$
$EQ'_1 = \{p \Leftrightarrow p^1 \Leftrightarrow p^2 \Leftrightarrow p^3, r \Leftrightarrow r^1 \Leftrightarrow r^2 \Leftrightarrow r^3, q \Leftrightarrow q^1\}$,
$EQ_2 = \{p \Leftrightarrow p^1 \Leftrightarrow p^3, r \Leftrightarrow r^1 \Leftrightarrow r^2 \Leftrightarrow r^3, q \Leftrightarrow q^1 \Leftrightarrow q^2\}$
$EQ_3 = \{p \Leftrightarrow p^2 \Leftrightarrow p^3, r \Leftrightarrow r^1 \Leftrightarrow r^2 \Leftrightarrow r^3, q \Leftrightarrow q^1 \Leftrightarrow q^2\}$
$EQ_4 = \{p \Leftrightarrow p^1 \Leftrightarrow p^2 \Leftrightarrow p^3, r \Leftrightarrow r^2 \Leftrightarrow r^3, q \Leftrightarrow q^1 \Leftrightarrow q^2\}$
$EQ'_4 = \{p \Leftrightarrow p^1 \Leftrightarrow p^2 \Leftrightarrow p^3, r \Leftrightarrow r^1, q \Leftrightarrow q^1 \Leftrightarrow q^2\}$

So, $\nabla_s(E) \equiv (p \wedge \neg r) \vee (\neg p \wedge q)$.

These operators have been implemented in the COBA system [33].

A possible criticism about this approach is that, just like formula-based merging operators, these operators do not take into account the distribution of the information among the sources. In particular they are not majority operators, and a piece of information that is believed by all the bases except one will not be in the result. But, as for formula-based operators (see Section 4.2), it seems possible to improve the behaviour of these operators, by adding a selection function for choosing the maximal subsets of equivalence EQ.

5 Merging, Revision and other Change Operators

We focus in this section on the study of relationships between belief merging operators, belief revision operators and update operators. There are close links between them. This is particularly clear when looking at the technical definitions.

There are close relationship between revision [1, 44, 61] and KM (for Katsuno and Mendelzon) update operators [60]. The first ones looking at the beliefs of the agents globally, the second ones looking at them locally.

Theorem 22 *If $\circ$ is a revision operator (i.e. it satisfies (R1)–(R6)), then the operator $\diamond$ defined by:*

$$K \diamond \mu = \bigvee_{\omega \models K} \varphi_{\{\omega\}} \circ \mu$$

is an update operator that satisfies (U1)–(U9).

Moreover, for each update operator $\diamond$, there exists a revision operator $\circ$ such that the previous equation holds.

See [38, 52, 72] for more discussions on update and its links with revision. There is also a close connection between revision and merging operators. In fact revision operators can be seen as particular cases of merging operators (see [69] for more details).

Fig. 1 Revision - Update - Merging - Confluence

$$\begin{tikzcd}
\text{Revision} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{Update} \arrow[d, leftrightarrow] \\
\text{Merging} \arrow[r, leftrightarrow] & \text{Confluence}
\end{tikzcd}$$

Theorem 23 *If $\triangle$ is an IC merging operator (it satisfies (IC0–IC8)), then the operator $\circ$, defined as $K \circ \mu = \triangle_\mu(K)$, is an AGM revision operator (it satisfies (R1–R6)).*

From these two facts a very natural question arises: What is the family of operators that are a generalization of update operators in the same way merging operators generalize revision operators? Or, equivalently, what are the operators that can be considered as pointwise merging, just as KM update operators can be considered as pointwise belief revision. This can be outlined by Fig. 1.

Confluence operators have been proposed in [71]. In order to illustrate the need of these new operators and also the difference of behaviour between merging and confluence let us consider a small example.

Example 24 Mary and Peter are planning to buy a car. Mary does not like a German car nor an expensive car. She likes small cars. Peter hesitates between a German, expensive but small car or a car which is not German, nor expensive and is a big car. Taking three propositional variables `German_car`, `Expensive_car` and `Small_car` in this order, Mary's desires are represented by $\mathrm{mod}\,(A) = \{001\}$ and Peter's desires by $mod(B) = \{111, 000\}$. Most of the merging operators[9] give as solution (in semantical terms) the set $\{001, 000\}$. That is the same solution obtained when we suppose that Peter's desires are only a car which is not German nor expensive but a small car ($mod(B') = \{000\}$). The confluence operators will take into account the disjunctive nature of Peter's desires in a better manner and they will incorporate the interpretations that are a trade-off between 001 and 111. For instance, the worlds 011 and 101 will be also in the solution if one uses the confluence operator $\diamondsuit^{d_H,\textsc{Gmax}}$.

This kind of operators is particularly adapted when the base describes a situation that is not perfectly known, or that can evolve in the future. For instance Peter's desires can either be imperfectly known (he wants one of the two situations but we do not know which one), or can evolve in the future (he will choose later between the two situations). In those situations the solutions proposed by confluence operators will be more adequate than the

[9]Such as $\triangle^{d_H,\Sigma}$ and $\triangle^{d_H,\textsc{Gmax}}$ [69].

one proposed by merging operators. The solutions proposed by the confluence operators can also be seen as all possible agreements in a negotiation process.

Definition 25 An operator $\Diamond$ is a confluence operator if it satisfies the following properties:

(UC0) $\diamond_\mu(E) \vdash \mu$
(UC1) If μ is consistent and E is p-consistent,[10] then $\diamond_\mu(E)$ is consistent
(UC2) If E is complete and E is consistent and $\bigwedge E \vdash \mu$, then $\diamond_\mu(E) \equiv \bigwedge E$
(UC3) If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\diamond_{\mu_1}(E_1) \equiv \diamond_{\mu_2}(E_2)$
(UC4) If K_1 and K_2 are complete formulae , and $K_1 \vdash \mu$, $K_2 \vdash \mu$, then $\diamond_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only $\diamond_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent
(UC5) $\diamond_\mu(E_1) \wedge \diamond_\mu(E_2) \vdash \diamond_\mu(E_1 \sqcup E_2)$
(UC6) If E_1 and E_2 are complete profiles, and $\diamond_\mu(E_1) \wedge \diamond_\mu(E_2)$ is consistent, then $\diamond_\mu(E_1 \sqcup E_2) \vdash \diamond_\mu(E_1) \wedge \diamond_\mu(E_2)$
(UC7) $\diamond_{\mu_1}(E) \wedge \mu_2 \vdash \diamond_{\mu_1 \wedge \mu_2}(E)$
(UC8) If E is a complete profile and if $\diamond_{\mu_1}(E) \wedge \mu_2$ is consistent then $\diamond_{\mu_1 \wedge \mu_2}(E) \vdash \diamond_{\mu_1}(E) \wedge \mu_2$
(UC9) $\diamond_\mu(E \sqcup \{K \vee K'\}) \equiv \diamond_\mu(E \sqcup \{K\}) \vee \diamond_\mu(E \sqcup \{K'\})$

Some of the (UC) postulates are exactly the same than (IC) ones, just like some (U) postulates for update are exactly the same than (R) ones for revision. In fact, (UC0), (UC3), (UC5) and (UC7) are exactly the same than the corresponding (IC) postulates. So the specificity of confluence operators lay in postulates (UC1), (UC2), (UC6), (UC8) and (UC9). (UC1) is close to (U3) for update (conversely to (IC1) for merging, consistent integrity constraint is not enough to ensure consistency of the result). (UC4), (UC6) and (UC8) are close from the corresponding (IC) postulates, but hold for complete profile only. The present formulation of (UC2) is quite similar to formulation of (U2) for update. Remark that in the case of complete profile the hypothesis of (UC2) is equivalent to ask coherence with the constraints, *i.e.* the hypothesis of (IC2). Postulates (UC8) and (UC9) are the main difference with merging postulates, and correspond also to the main difference between revision and KM update operators. (UC9) is the most important postulate, that defines confluence operators as pointwise aggregation, just like (U8) defines update operators as pointwise revision. This will be expressed more formally in the next Section (Lemma 1).

5.1 Representation Theorem for Confluence Operators

In order to state the representation theorem for confluence operators, we first have to be able to "localize" the problem. For update this is done just by

[10] A profile $E = \{K_1, \ldots K_n\}$ is *p-consistent* if all its bases are consistent, i.e $\forall K_i \in E$, K_i is consistent.

looking to each model of the base, instead of looking at the base (set of models) as a whole. So for "localize" the aggregation process, we have to find what is the local view of a profile. That is what we call a state.

Definition 26 A multi-set of interpretations will be called a *state*. We use the letter e, eventually with subscripts, for denoting states. If $E = \{K_1, \ldots, K_n\}$ is a profile and $e = \{\omega_1, \ldots, \omega_n\}$ is a state such that $\omega_i \models K_i$ for each i, we say that e is a state of the profile E and, for short, this will be denoted by $e \models E$. If $e = \{\omega_1, \ldots, \omega_n\}$ is a state, we define the profile E_e by putting $E_e = \{K_{\{\omega_1\}}, \ldots, K_{\{\omega_n\}}\}$.

State is an interesting notion. If we consider each base as the current point of view (goals) of the corresponding agent (that can be eventually strengthened in the future) then states are all possible negotiation starting points.

States are the points of interest for confluence operators (like interpretations are for update), as stated in the following Lemma:

Lemma 1 *If $\diamond$ satisfies (UC3) and (UC9) then $\diamond$ satisfies the following* $\diamond_\mu(E) \equiv \bigvee_{e \models E} \diamond_\mu(E_e)$

Like revision's faithful assignments that have to be "localized" to interpretations for update, merging's syncretic assignments have to be localized to states for confluence.

Definition 27 A *distributed assignment* is a function mapping each state e to a total pre-order $\leq_e$ over states such that:

1. $\omega <_{\{\omega,\ldots,\omega\}} \omega'$ if $\omega' \neq \omega$
2. $\omega \simeq_{\{\omega,\omega'\}} \omega'$
3. If $\omega \leq_{e_1} \omega'$ and $\omega \leq_{e_2} \omega'$, then $\omega \leq_{e_1 \sqcup e_2} \omega'$
4. If $\omega <_{e_1} \omega'$ and $\omega \leq_{e_2} \omega'$, then $\omega <_{e_1 \sqcup e_2} \omega'$

Now we can state the representation theorem for confluence operators [71].

Theorem 28 *An operator $\diamond$ is a confluence operator if and only if there exists a distributed assignment that maps each state e to a total pre-order $\leq_e$ such that*
$$\mathrm{mod}\,(\diamond_\mu(E)) = \bigcup_{e \models E} min(\mathrm{mod}\,(\mu), \leq_e).$$

5.2 Confluence Versus Update and Merging

So now we are able to state the proposition that shows that update is a special case of confluence, just as revision is a special case of merging.

Theorem 29 *If $\diamond$ is a confluence operator (i.e. it satisfies (UC0–UC9)), then the operator $\diamond$, defined as $K \diamond \mu = \diamond_\mu(K)$, is an update operator (i.e. it satisfies (U1–U9)).*

Concerning merging operators, one can see easily that the restriction of a syncretic assignment to a complete profile is a distributed assignment. From that we obtain the following result (the one corresponding to Theorem 22 for revision and update):

Theorem 30 *If $\triangle$ is an IC merging operator (i.e. it satisfies (IC0–IC8)) then the operator $\diamond$ defined as $\diamond_\mu(E) = \bigvee_{e \models E} \triangle_\mu(E_e)$ is a confluence operator (i.e. it satisfies (UC0–UC9)).*

Moreover, for each confluence operator $\diamond$, there exists a merging operator $\triangle$ such that the previous equation holds.

It is interesting to note that this theorem shows that every merging operator can be used to define a confluence operator, and explain why we can consider confluence as a pointwise merging.

As a corollary of the representation theorem we obtain the following

Corollary 31 *If $\diamond$ is a confluence operator then the following property holds:*

(half IC2) *If $\bigwedge E \vdash \mu$ and E consistent, then $\bigwedge E \wedge \mu \vdash \diamond_\mu(E)$*

But it is not (generally) the case that $\diamond_\mu(E) \vdash \bigwedge E \wedge \mu$.

Note that this "(half IC2)" property is similar to the "(half R2)" satisfied by update operators.

This corollary is interesting since it underlines an important difference between merging and confluence operators. If all the bases agree (i.e. if their conjunction is consistent), then a merging operator gives as result exactly the conjunction, whereas a confluence operator will give this conjunction plus (eventually) additional results. This is useful if the bases do not represent interpretations that are considered equivalent by the agent, but uncertain information about the agent's current or future state of mind. For examples of confluence operators see [71].

6 Belief Merging and Social Choice Theory

Merging operators show tight relationships with social choice theory [3], and in particular with voting methods. It is interesting to study what are the consequences of well known social choice concepts when applied to merging scenarios. We will give two such examples in this Section: strategy-proofness and truth-tracking. Strategy-proofness is about the resistance of strategic manipulation from the sources/agents. The truth-tracking issue study if the merging/voting methods are capable to identify the true state of the world if the sources/agents are sufficiently reliable.

6.1 Strategy-Proofness

Merging operators allow to define the beliefs/goals of a group of agents. But if an agent is capable to make some reasoning about the result of the merging and on the impact that he can have on it, he can be tempted to try to modify the result of the merging in order to best fit his interests. This strategy-proofness issue for merging operators have been studied in [39].

Unsurprisingly, it is quite difficult to obtain strategy-proofness. Belief merging is quite close to preference aggregation, and a seminal result in social choice, the Gibbard-Sattherwaite theorem [48, 92], shows that it is not possible to define a preference aggregation method that is strategy-proof. So it is normal to obtain very few strategy-proof results in the belief merging framework.

This definition of strategy-proofness for merging is quite standard. The difference with usual preference aggregation where the comparison of situations is direct, is that for propositional merging we need to use a **satisfaction index** to evaluate the satisfaction[11] of an agent with respect to a merging result [39]:

Definition 32 Let i be a satisfaction index, i.e., a function from $\mathcal{L} \times \mathcal{L}$ to $\mathbb{R}^+$:

- A profile E is manipulable by a base K for the index i given the operator $\triangle$ and the integrity constraint μ if and only if there is a base K' such that $i(K, \Delta_\mu(E \sqcup \{K'\})) > i(K, \Delta_\mu(E \sqcup \{K\}))$.
- A merging operator Δ is **strategy-proof** for i if and only if there is no integrity constraint μ and no profile $E = \{K_1, \dots, K_n\}$ such that E is manipulable for i.

We focused on three satisfaction index, that are the most natural ones if we do not have any additional information:

Definition 33

- **Weak drastic index** : $i_{dw}(K, K_\triangle) = \begin{cases} 1 & \text{if } K \wedge K_\triangle \nvdash \bot \\ 0 & \text{otherwise} \end{cases}$
- **Strong drastic index** : $i_{ds}(K, K_\triangle) = \begin{cases} 1 & \text{if } K_\triangle \vdash K, \\ 0 & \text{otherwise} \end{cases}$
- **Probabilistic index**:[12] $i_p(K, K_\triangle) = \frac{|mod(K) \cap mod(K_\triangle)|}{|mod(K_\triangle)|}$

[11] The bigger the index, the more satisfied the agent.

[12] When $|mod(K_\triangle)| = 0$, then $i_p(K, K_\triangle) = 0$.

So with the weak drastic index an agent is satisfied if the result of the merging is consistent with his base. With the strong drastic index the result of the merging must imply the base. The probabilistic index allows a more progressive measure of satisfaction, that depends on the proportion of common models between the result of the merging and the base of the agent. See [39] for detailed strategy-proofness results.

Chopra, Ghose and Meyer have studied strategy-proofness for merging of weighted formulae [24].

6.2 Belief Merging Versus Goal Merging

Since the beginning of this paper we use the generic term "belief merging". But the works on logical characterizations of merging operators hold both for proper belief merging and for goal merging. In fact all the proposed postulates seem quite sensible in both cases. Even if it can seem strange than two concepts as different as beliefs and goals can be treated the same way for aggregation, the adequacy of IC merging operators (for belief or goal merging) has not been yet challenged by the proposition of new postulates that would allow to distinguish the two types of merging.[13]

The main nuance that we can find is that it is easier to justify the use of disjunctive merging operators for merging beliefs than for merging goals.

It is still possible to define an interpretation of belief merging, that allows to make a distinction. An interesting question for merging, coming from social choice theory, is to know if belief merging operators are good for *truth tracking*. In fact, this question leads to the definition of two interpretations for merging operators: the synthetic point of view versus the epistemic point of view [42].

Synthetic view: Under the synthetic point of view, the aim of merging is to define a base that best represents the information of the profile. This is the usual view for merging operators.

Epistemic view: Under the epistemic point of view, the aim of merging is to try to identify the true state of the world, so to remove as far as possible the uncertainty of the group.

Then it is interesting to note that the epistemic view, i.e. the truth tracking issue, is a way to distinguish belief merging from goal merging. Indeed, whereas the truth tracking issue is perfectly sensible for belief merging, it seems difficultly justifiable for goals, since it is clear that the notion of "true goal", that would be equivalent of the "true world" for beliefs, has little sense.

In social choice theory, the result that justifies decisions taken by a majority in a group is the Condorcet's jury theorem [80]. This theorem states that in order to answer to a binary (yes/no) question, if the member of the group are

[13] However see [31] for an interesting study of the interactions of merging beliefs, desires and goals.

reliable (they have more than 50% chances to be right), and independent, then listening to the majority is the good choice (in particular when the number of agents tends to infinity the probability of error tends to 0).

This result is what justifies the use of committees for taking decisions (in law court, etc.). But it is easy to see that the hypotheses of this theorem are very restrictive. In particular the question is only a binary one, and the agents can not be uncertain (they can not hesitate between yes or no).

In [42] a generalization of the Condorcet's jury theorem **under uncertainty** is shown. In this case the question is on any number of alternatives, and the agents can face uncertainty (they provide a set of alternatives, not a single one). It is shown that *approval voting* [22], that allows to vote for any number of candidates and that elects the candidate that receives the most votes, is the voting method to use in this case. The theorem and its consequences for belief merging operators are also studied in [42].

Truth-tracking has been also studied in the framework of judgment aggregation, that is also a form of logical aggregation, and that is related to belief merging. See [79] for an introduction to judgment aggregation. And see [21, 85] for papers on truth-tracking for judgment aggregation.

7 Merging in Other Frameworks

Merging has also been studied in other representation frameworks. Operators discussed above are defined in the propositional logic framework, when all the bases have the same importance/priority/fiability. But it can be necessary to merge information with more structure than the one of propositional logic. In this case new problems and new possibilities arise. Let us review the closest works in this Section.

7.1 Prioritized Merging, Merging and Iterated Revision

In [32] Delgrande, Dubois and Lang propose an interesting discussion on **prioritized merging** operators. The idea is to merge a set of weighted formulae. The weights are used to stratify the formulae (a formula with a greater weight is more important, even if they are a large number of formula with smaller weights that contradict it).

Delgrande, Dubois and Lang motivate the generality of their approach by showing that classical merging operators (on unweighted formulae) and iterated belief revision operators (à la Darwiche and Pearl [30]) can be considered as two extreme cases of this weighted merging framework.

Their discussion on iterated belief revision operators is particularly interesting, and can be related to the warning of Friedman and Halpern on the problem of defining change operators without specifying their ontology [43]. The main argument is that if one makes the hypothesis that the new pieces of information that come successively in an iterated revision process are about a static world (the usual hypothesis), then there is no reason to give the preference to the last ones. If these information have different reliability, then this can be represented explicitly with the weights of the formulae, in order to take this difference of reliability in the iterated "revision" process if they do not come in the order corresponding to their relative reliability. And the correct way to do that is to make a prioritized merging.

This discussion is interesting since in several papers on iterated revisions, it seems that the authors do not make any distinction between the hypothesis to have more and more recent pieces of information, and the hypothesis to have more and more reliable pieces of information.

The framework of Delgrande, Dubois and Lang identifies the epistemic states as the sequences of formulae that the agent receives. This identification as already been proposed in [68, 73].

Delgrande, Dubois and Lang show that the postulates for iterated belief revision can be obtain as special case of their postulates for weighted merging. They show that they can also lead to some postulates of IC merging.

This work is interesting since it opens a way for logical characterization of prioritized merging. It could be interesting to try to find a representation theorem in this case, and to look at the generalization of IC merging operators in this prioritized merging framework.

7.2 Merging of Weighted Formulae

When all the pieces of information belonging to the bases do not have the same importance, one needs to use weighted approaches.

The most qualitative approach is to consider for each source/agent a set of bases (totally) ordered in different strata, from the least important to the most important one.

This is usually represented using **possibilistic logic** [36] or **ordinal conditional functions** [94]. In this case we associate an ordinal (usually finite, i.e. a natural number)[14] to each formula.

In the framework of possibilistic logic Benferhat, Dubois, Kaci and Prade studied several merging operators [8, 59]. They have also studied an extension of the logical properties of IC merging in this framework [12]. (See [87] for a generalization of IC merging operators in a weighted framework and see [57, 86, 87] for other developments of merging of weighted formulae).

In the framework of ordinal conditional functions, Meyer defined **combination operators** [82]. Some of them are translation of model-based operators in this framework, but some others are quite far from IC merging operators.

All these works that use weights face the same problem of **interpersonal comparison of utility**, i.e. this requires to suppose that the same weight used by different agents has the same meaning. This is also called the commensurability assumption.

[14] See [65] for a discussion.

This commensurability assumption can be sensible if the sources are similar, for instance if we consider several identical sensors. But in a lot of cases, when the sources are reasoning agents for instance, this hypothesis seems unrealistic. In particular, when working with weighted bases, one is quite close to the framework of voting methods in social choice theory. And in this domain the commensurability assumption is highly criticized [4]. For voting methods only the ordinal preferences of the agents are taken into account.

So if one wants to define merging operators in this weighted bases framework without the commensurability assumption, especially for majority operators, then the correct framework seems to use voting methods [3, 4].

Benferhat, Lagrue and Rossit studied some non-majority operators without commensurability assumption [13, 14]. Of course, this leads to operators much more cautious than in the commensurability case.

7.3 Merging of First Order Bases

Lang and Bloch propose to define model-based merging operators using the maximum as aggregation function ($\triangle^{d,\max}$) by using a **dilation**[15] process [18]. One can note that in the original Dalal paper [29], he defines his revision operator with such a dilation function rather than with a distance.

Gorogiannis and Hunter [50] extend this approach in order to define others model-based merging operators using dilations. So, in addition to $\triangle^{d,\max}$, they define $\triangle^{d,\Sigma}$, $\triangle^{d,\textsc{Gmax}}$ and $\triangle^{d,\textsc{Gmin}}$ operators.

The interest of this definition of these operators is that it can be easily extended to first order logic. The usual definition of model-based merging operators is based on the computation of distances between interpretations. So when using logics where the number of interpretations is infinite, this approach is not the more appropriate. The interest of defining these operators with dilations is that they can also be used in this case. This only needs to use the good dilation function. See [50] for a discussion and some examples of dilation functions in the first order logic case.

7.4 Merging of Logic Programs

There are some works on merging operators for logic programs, and more exactly for Answer Set Programming.

The approach of Hué et al. [53] relies on the deletion of a given set of formulae in the union of the bases, selected by a selection function (close to the idea used in [63]). These operators satisfy only some logical properties of IC merging.

Delgrande et al. [35] have proposed other merging operators for ASP. Their operators are based on the definition of a distance between stable models.

[15] Roughly speaking dilation allows to reach the points/worlds in the neighborhood of a point/world. See [18] to see how to define this formally.

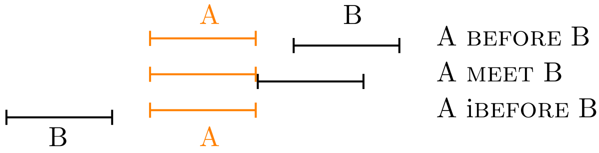

Fig. 2 Allen interval algebra

It is possible to compare these two approaches by making a parallel with propositional merging. Hué, Papini and Würbel operators correspond to formula-based operators, whereas operators proposed by Delgrande, Schaub, Tompits and Woltran correspond to model-based operators. So this is not surprinsing if the second approach allows to obtain more merging logical properties.

All these operators have been implemented. There are quite few implementations of belief merging operators. Apart from these two ones, one can mention the COBA system [33], the BReLS system [75] of Liberatore and Schaerf, and the BDD implementation of Gorogiannis and Hunter [49]. But a comparison of the computational performance of these different implementations is missing.

7.5 Merging of Constraint Networks

Condotta, Kaci, Marquis and Schwind studied the merging of qualitative constraint networks [25, 26]. These methods can be useful for merging constraint networks that represent spatial regions, for instance for Geographical Information Systems it can be necessary to merge spatial databases that come from different sources.

Conflicts that arise in this framework are more subtle that the binary ones in the propositional framework. In this case conflicts can be more or less important. For instance, if we use the Allen algebra, that allows to represent spatial information on segments on a line, namely relations as A BEFORE B, A AFTER B, A MEET B among other. A conflict between sentences A BEFORE B and A MEET B seems much less important than the one between A BEFORE B and A iBEFORE B (Fig. 2).

This "intensity" that we feel between conflicts allows to define more various merging policies than in the propositional framework.

One can also look at [27, 90] to see two examples of merging of spatial regions using logical representations.

7.6 Merging of Argumentation Frameworks

There are a lot of works on argumentation as a way to reason about contradictory pieces of information. The basic idea is to use a set of arguments and an attack relation between relations. This is the starting point of Dung argumentation framework [37]. But these works on argumentation consider only a single agent. In [28] the problem of merging of argumentation frameworks, where the arguments are distributed among several agents, have been studied.

This requires to define a new representation frameworks for argumentation: Partial Argumentation Frameworks, where there are three possible relations between two arguments A and B. Either the agent believes that A attacks B, or he believes that A does not attacks B, or he does not know if A attacks B or not. This last case is necessary to represent the fact that an agent ignores a given argument.

8 Final Remarks

We have presented an overview of logic based merging. Propositional merging operators have been extensively studied in recent years. Two currents and future paths of development of these works seem to be the modelization of negotiation processes and the development of merging operators in more expressive representation frameworks.

We quoted some of these works on merging in more complex representation framework in Section 7. There are also related works towards applications of these merging techniques for instance for text processing [54, 55, 58] and XML documents [56, 83], or requirement engineering [47]. We can expect more developments on this path in the future.

As for the modelization of negotiation using belief merging and belief revision tools, confluence operators, studied in Section 5, are a good example. But there are also other works in this direction, like the ones on iterated merging and conciliation [46], social contraction and belief negotiation/game models [19, 64], concession [81], or logical bargaining [95–97]. We think that there are still a lot of things to discover on modelization of negotiation.

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