Logic of Determination of Objects : The Meaning of Variable in Quantification

Jean-Pierre Desclés*, Anca Pascu**
*Université de Paris-Sorbonne, LALICC CNRS, France
96, bd. Raspail, 75006, Paris, France
Jean-pierre.Descles@paris4.sorbonne.fr
**Université de Bretagne Occidentale, LALICC CNRS, France
20, rue Duquesne, 29200, Brest, France
Anca.Pascu@univ-brest.fr

Abstract

This article constitutes a contribution to an analysis of the notion of variable. Whithin the framework of Combinatory Logic as a formalism without bound variables, the Logic of Determination of Objects (LDO) provides an explanation for the necessary distinction between "whatever, any" and "indeterminate, indefinite" used by the introduction and elimination rules of quantifiers in Natural Deduction. The intension of a concept and typical and atypical occurrences of a concept are also introduced yielding new quantifiers which are more adequate to natural language processing (NLP) and to the study of natural inferences in common reasoning.

Keywords : Variable, Natural Deduction, Reasoning, Quantifiers, Arbitrary Object, Indefinite, Typical, Atypical, Logic of Determination of Objects.

Difficulties with the notion of "variable".

> The variable is perhaps the most distinctively mathematical of all notions ; it is certainly also one of the most difficult to understand. (Bertrand Russell, The Principles of Mathematics, 1903)

Natural languages can be seen as universal representation systems in the following sense : each artificial symbolic system, each formal system can be directly or indirectly interpreted in a natural language. Curry (Curry and Feys 1958) says that the opposition between natural languages and artificial languages is created by their "semiotic property": U (universal) for natural languages and A (artificial) for artificial languages : "The construction of a formal system has to be explained in a communicative language understood by both the speaker and the hearer. We call this language the U-language (the language being used); it is language in the usual sense of the word. It is well determined but not rigidly fixed; new locutions may be introduced in it by way of definition, old locutions may be made more precise, etc. Everything depends on the U-language; it can never be transcended; whatever we study by means of it. Of course, there is always vaguennes inherent in the U-language; but we can, by skilful use, to obtain any degree of precision by a process of successive approximation."

and :

"(...) given a certain presentation of a formal system, the A-language is that language which is constituted by the symbols and expressions used for the primitive ideas and their combinations. The symbols of the A-language are adjoined to the U-language to be used there; they perform grammatical functions therein." (Curry and Feys 1958)

But even if logic and algebraic formalisms have some autonomy using some formal calculus *on* and *with* a set of symbols, they are not completely independent of natural languages, because they can often be viewed as extensions, by a well-ordered addition, let us say "adjonction" of external symbols. These systems sometimes acquire such an autonomy that they become true "artificial languages" with their own morphologies, syntaxes and semantic interpretations.

The notion of “variable” is used in almost all fields, but with slightly different senses. This notion comes from logic and mathematics. It was borrowed by other fields, for example : computer sciences. In computer sciences there are difficulties in using variables (Hudak 1989). So, functional programming languages without bound variables were defined to avoid “side effects”.

In Curry’s combinatory logic (Curry and Feys 1958) the bound variable is not necessary. In this formalism it is possible to define complex concepts from more elementary concepts and also to develop the illative logic with quantifiers without using bound variables.

Ambiguity of “variable” in Natural Deduction

It will be shown that, in the presentation of classical logic in Natural Deduction given by Gentzen (Gentzen 1955), there is a crucial distinction between the two meanings of the variable.

If logic is thout of as a codification of reasoning, then it should stay close to the practice of inference making, instead of being based on the notion of truth. Natural Deduction explores the non-semantic approach, by setting up a system for deriving conclusions from premises. Although this approach is of a formal nature, it is advisable to keep some interpretation in mind. It introduces rules, separated into introduction rules and elimination rules, which are used to derive new steps from hypotheses and already proved steps in a deduction. According to Gentzen (Gentzen 1955), the rules are associated with every propositional connective, negation and quantifier, to express an intuitive meaning of them. We follow the presentation given by Fitch (Fitch 1974) for introduction and elimination rules quantifiers below :

Universal quantifier :

$$\left|\begin{array}{l} \left.\begin{array}{ll} y & \\ \vdots & \\ & P(y) \end{array}\right| \\ (\forall x)\ P(x) \end{array}\right. \qquad [\text{ i - }\forall\ ]\ (1)$$

$$\left|\begin{array}{l} (\forall x)P(x) \\ \vdots \\ P(y) \end{array}\right| \qquad [\text{ e -}\forall\ ]\ (2)$$

It is necessary that y to be free for x.

The notion of free variable is explained as : In a general, we will say that y is free for x in P, if after writing y instead of x, no occurence which should be free becomes bound.

Existential quantifier :

$$\left|\begin{array}{l} P(y) \\ \vdots \\ \vdots \\ (\exists x)\ P(x) \end{array}\right. \qquad [\text{ i - }\exists\ ]\ (3)$$

$$\left|\begin{array}{ll} (\exists x)\ P(x) & \\ y & \left|\begin{array}{l} P(y) \\ \vdots \\ B \end{array}\right. \\ B & \end{array}\right. \qquad [\text{ e - }\exists\ ]\ (4)$$

The following condition is required: x is not free in B and in any hypothesis of a subderivation of B, other than $P(x)$.

It should be noted that the interpretation of the variable in these rules is not the same. The crucial difference is between the meaning of “any, whatever ” and the meaning of “indefinite, indeterminate”.

Following Van Dalen (Van Dalen 1991), the sense of the introduction rule of the universal quantifier is : If an arbitrary object x has the property P, then every object has the property P.

Van Dalen explains the notion of “arbitrary object” as follows: The problem is that none of the objects we know in mathematics can be considered “arbitrary”. So instead of looking for the “arbitrary object” in the real word (as far as mathematics is concerned), let us try to find a syntactic criterion. Consider a variable x (or a constant) in a derivation, are there reasonable grounds for calling x “arbitrary”? Here is a plausible suggestion: in the context of the derivation we shall call x arbitrary if nothing has been assumed concerning x. In more technical terms, x is arbitrary at its particular occurence in a derivation if the part of the derivation above it contains no hypotheses containing x free.

Desclés (Desclés and Cheong 2004) gives the introduction rule of the universal quantifier the following interpretation : .... It means that we carry out our reasoning with “any element” / “whatever” denoted by y (rule (1)). One has no hypothesis on y, one can not reiterate an expression containing free occurences of y in the sub-deduction following y introduction as hypothesis. If this is the case, then the element denoted by y will be not “any” / “whatever” anymore because of the fact that it has a property (for exemple $B(y)$). Let us suppose now that carrying out our reasoning with this “any object”, one obtains that it has the property P. Then we can state $(\forall x)\ P(x)$ that is we can release the choice of y by introducing the universal quantifier.

In the case of the introduction of the universal quantifier the reasoning is founded on the notion of “whatever”. In the elimination of the existential quantifier the reasoning runs with the variable interpreted as an “indeterminate (undefined , non-specified)” object. Indeed, in this reasoning we suppose the existence of an object with the property P. The exact denotation of this object is not known. An “indeterminate” object called y is introduced, and inferences about this object are obtained. If the sequence of inferences concludes to a proposition B, then it can be concluded that B follows from the hypothesis (∃ x) P(x).

In the two rules (universal quantifier introduction and existential quantifier elimination) the formalism use the same symbol (a variable y in the rules (1), (4)) but its meaning and its behavior is not the same.

“Whatever” and “indeterminate” in LDO

How can the two notions of “whatever” or “arbitrary object ” (see Fine 1985) and “indeterminate” or “non-specified” be distinguished?

In our opinion, LDO (Logic of Determination of Object) provides a solution to this question. LDO is defined inside the framework of combinatory logic with functional types (Desclés 2002, Pascu 2001, Freund and alii 2004). In this approach a concept (see Frege 1893) is a function from a domain of objects into truth values. With every concept f the following are canonically associated :

- an object called “typical object”, $\tau(f)$ which represents the concept f as an object . This object is completely undetermined;
- a function, $\delta(f)$ defined on objects : the image-object is more determined than the argument-object for this function;
- the intesion of the concept, Int(f) conceived as the class of all concepts that the concept f “includes”, that is a semantic network of concepts structured by the relation “IS-A”;
- the expanse of the concept, Exp(f) which contains all “more or less determined objects” such that the concept f applies to;
- a part of the expanse is the extension of the concept, Ext (f) which contains all completely determined objects such that the concept f applies to.

In LDO objects are :“more or less determined objects” and “completely determined objects”. LDO captures two kinds of objects : typical objects and atypical objects. Typical objects in Exp(f) inherit all concepts of Int f; atypical objects in Exp(f) inherit only some concepts of Int(f).

In LDO star quantifiers are defined (Desclés and Guentcheva 2001, Pascu 2001). They are considered as determiners of objects in Exp(f). These determiners are $\Pi^\star$ et $\Sigma^\star$. They are different to the usual quantifiers Π and Σ from the illative version (Curry 1958) of Frege’s quantifiers (Frege 1879) in combinatory logic. Indeed, in LDO, $\Pi^\star$ is the universal quantifier retricted to typical objects and Σ (Frege’s existential quantifier) is the existential quantifier, but also restricted to typical objects. Π and $\Sigma^\star$ are respectively, the universal quantifier and the existential quantifier not restricted to typical objects, that is, they work on Ext(f).

In LDO the object $\Pi^\star(\tau(f))$ starting from $\tau(f)$ is constructed by applying the $\Pi^\star$ operator (figure 1).

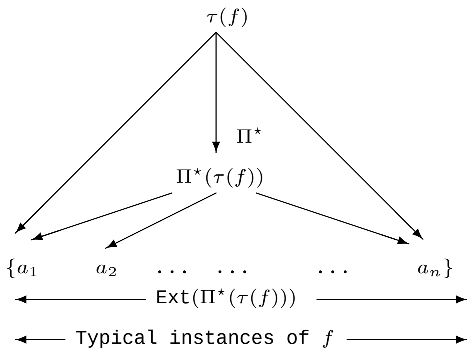

Figure 1: The construction of the object "whatever, any"

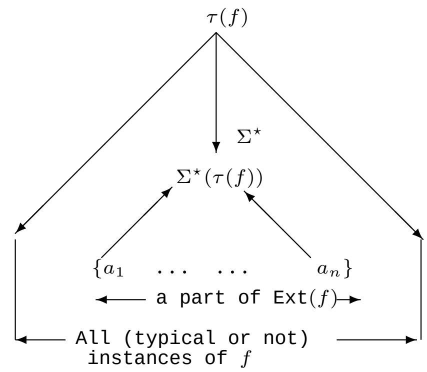

Figure 2: The construction of the "indeterminate object"

In figure 1, a_1, a_2,...., a_n are completely determined objects, which can be substituted for the undetermined object $\Pi^\star(\tau(f))$.

$\Pi^\star(\tau(f))$ is an object of Exp(f) – Ext(f). It represents a "whatever typical object" to which f applies. This object is not in Ext(f) but it can be identified with any typical object of Ext(f). $\Pi^\star(\tau(f))$ is introduced as a "whatever object". In the rule of universal quantifier introduction (1) it is the object corresponding to the variable "y". In fact, this rule, says that : if we can state, by the inference process, the proposition P($\Pi^\star(\tau(f))$), then we can state the universal quantifier introduction.

In LDO the object $\Sigma^\star(\tau(f))$ is constructed starting from $\tau(f)$ by applying the $\Sigma^\star$ operator (figure 2).

In figure 2 : a_1, a_2,...., a_n are completely determined objects, such that :

$$f(a_1) = f(a_2) = \ldots f(a_n) = \top$$

Moreover the following is an axiom :

$$f(\Sigma^\star(\tau(f)) = \top \iff \text{Ext}(f) \neq \emptyset$$

$\Sigma^\star(\tau(f))$ is also an object of Exp(f) – Ext(f). It represents as an "undetermined object" a non-empty part of Ext(f). $\Sigma^\star(\tau(f))$ is such that new propositions (like B) from the proposition P ($\Sigma^\star(\tau(f))$) that is, by existential quantifier elimination. This object is the object corresponding to "y" in rule (4). Both objects $\Pi^\star(\tau(f))$ and $\Sigma^\star(\tau(f))$ are "more or less determined objects" in the sense of the LDO : they belong to Exp(f), they are not completely determined objects. But $\Pi^\star(\tau(f))$ is the "any object / whatever" object, as for $\Sigma^\star(\tau(f))$ it is the "indeterminate / non-specified" object. (figure 3)

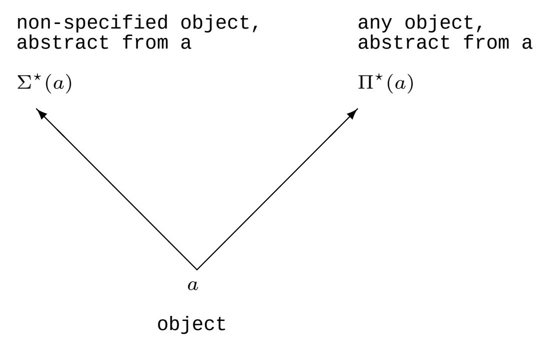

Figure 3: "whatever / any" object versus "indeterminate / non-specified object"

Examples

Some examples from natural language are given to prove that :

- Using classical logic in formalization of the language the typicality /atypicality cannot be captured;
- The indefinite article from a language with the variable from classical logic cannot be identified.

Let us consider the following examples :

1. An Alsatian drinks beer. (a typical Alsatian) /*Un alsacien boit de la bière.*

In LDO, the statement :

$$(to-drink-beer)(\Pi^*(\tau(to-be-an-Alsatian)))$$

is true, but the folowing statements :

$$(\forall x)(Alsatian\ x) \supset ((to-drink-beer)\ x)^1$$

or equivalently,

$\Pi(to-be-Alsatian)(to-drink-beer)^2$ are false.

1*. Anca saw an Alsatian in the street. (an indeterminate Alsatian, one does not specify him) / *Anca a vu un alsacien dans la rue.*

The applicative expression associated is :

$$(((SS)(\Sigma^*(\tau(to-be-an-Alsatian))))\ Anca)$$

where SS stands for $to-see-in-the-street$.

2. A mathematician know how to solve this problem. (a typical mathematician) / *Un mathématicien sait résoudre ce problème.*

$$(KSP)(\Pi^*(\tau(to-be-a-mathematician)))$$

where KSP stands for $to-know-solving-this-problem$.

2*. Luc is a strange mathematician. (not necessarily typical) / *Luc est un mathématicien bizarre.*

$$(((to-be)((\delta\ (to-be-strange))(\Sigma^*(\tau M))))\ Luc)$$

where M stands for $to-be-a-mathematician$

2**. Luc has a mathematician in his family. (not necessarily typical) / *Luc a un mathématicien dans sa famille.*

$$(((to-have-in-his-family)(\Sigma^*(\tau M)))Luc)$$

3. A triangle has the sum of its angles equal to $180°$. (whatever typical triangle) / *Un triangle a la somme de angles égale à 180°.*

$$(S-180)(\Pi^*(\tau T))$$

where T stands for $to-be-a-triangle$ and S-180 stands for $to-have-the-sum-of-angles-equal-to-180$.

3*. I found in the mathematical literature a triangle such that the sum of its angles is different from $180°$. (it is not whatever, it is indeterminate, but it is atypical versus the property "sum of angles equal to $180°$") / *J'ai trouvé dans la littérature mathématique un triangle dont la somme des angles est différente de 180°.*

$$(((to-find\ \ldots)((\delta\ (S-180))(\Sigma^*(\tau T))))\ I)$$

4. A student pay his university fees. (an whatever student and typical).[3] /*Un étudiant paie ses droits universitaires.*

$$(Pay(\Pi^*(\tau(to-be-a-student))))$$

where Pay stands for $to-pay-his-university-fees$. One can see that from $(Pay(\Pi^*(\tau(to-be-a-student))))$ we cannot deduce $\Sigma(to-be-student)(Pay)$

The indefinite article in languages such as English and French cannot be identified (as a notion) with the variable of classical logic. We can see that the value of the indefinite article "a, an" ("un, une" in French) is either "whatever/any" object as in examples 1, 2, 3, 4, or "indeterminate/non-specified" object as in examples 1*, 2*, 2**, 3*.

Conclusion

LDO extends classical logic since classical logic does not take into account Intension or typical and atypical occurrences of a concept. In this latter case Expanse is reduced to Extension, even identical. Moreover, the distinction between "whatever" and "indeterminate, non-specified" is confused. This distinction is essential in the study of natural deduction as a cognitive process, but also for Artificial Intelligence where

[1] This is the logical form of this sentence in the classical logic.

[2] This is the form of this sentence in the Curry's logic.

[3] In France, in public universities all students have to pay their university fees except the children of university teachers.

reasoning about typical or atypical occurrences leads to solving local contradictions. LDO tries to explain this distinction. More generally, LDO gives a logical foundation to categorization and inference processes with semantic networks. This distinction is very useful in capturing the meaning of indefinite articles and linguistic quantifiers in natural languages.

References

Curry, H. B. and Feys, R. 1958. *Combinatory Logic*. North Holland.

Desclés, J.-P. and Guentceva Z. 2001. Quantification Without Bound Variables. In Bőttner, Thümmel (eds), *Variable-free Semantic*, 210–233. Osnabrück : Secolo Verlag, Rolandsmauer 13-14.

Desclés, J.-P., 2002. Categorization : A Logical Approach of a Cognitive Problem. In *Journal of Cognitive Science* 3(2): 85–137.

Desclés, J.-P. and Cheong, K.-S., 2004. Analyse critique de la notion de variable, Technical Report, LALICC, Sorbonne University. Forthcoming. In Mathématiques et Sciences Humaines 74 pages.

Freund, M., Desclés, J.-P., Pascu, A. and Cardot, J. Typicality, Contextual Inferences and Object Determination Logic. In Proceedings of the International Florida Artificial Intelligence Research Society Conference, 491-495. Menlo Park, Calif.: AAAI Press.

Fine, K. 1985. *Reasoning with Arbitrary Objects* Aristotelian Society Series, vol 3, Oxford :Blackwell.

Fitch, F. 1974. *Elements of Combinatory Logic*. Yale University Press.

Frege, G. 1879. *Begriffsschrift. Eine Formelsprache des reinen Denkens* . Halle.

Frege, G. 1893. *The Basic Laws of Arithmetic* (translated and edited with an introduction by Montgomery Furth in 1967. University of California Press.

Gentzen, G.1955. *Untersuchungen über das logische Schließen - Recherches sur la déduction logique (translated in French by Feys R. and Ladriere J*. Paris : Presses Universitaires de France.

Hudak, P. 1989. Conception, Evolution, and Application of Functional Programming Languages. In *ACM Computing Surveys* 21:360–411.

Pascu, A. 2001. Logique de Détermination d'Objets: concepts de base et mathématisation en vue d'une modélisation objet. PhD diss., Département de Logique, Informatique et Cognition (LALICC), Université de Paris-Sorbonne.

Russell, B. 1903. *Principles of Mathematics*. London, 1956. George Allen.

Van Dalen, D. 1994. *Logic and Structure*. Springer-Verlag.