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LOGICAL AND PROBABILITY ANALYSIS OF SYSTEMS

MITCHELL O. LOCKS

1 *Introduction* This paper is concerned with a method for symbolically describing an outcome, event, proposition or system of propositions, premises, assumptions, etc., which is a binary-valued function of n binary elements, and for calculating the probability of an event, based upon Boole's classic, *The Laws of Thought* [1]. We employ concepts and terminology borrowed from fields such as symbolic logic and Boolean algebra, lattice theory and partially ordered sets, which trace their origins to Boole's book. Some new terminology is added, however, for convenience in describing sets. Boole recognized that for a two-valued function of binary elements, the possible outcomes can be algebraically partitioned into two subsets: the "true" or "one" states which are consistent with all of the assumptions or premises of the system, and which we shall call in the sequel the "identity set $\mathfrak{I}$"; and the "false" or "zero" states which are not consistent and which we shall call the "zero set $\mathfrak{O}$." The method described in *Laws of Thought* was to first find the zero set directly from the premises, then subtract it from the universal set to obtain $\mathfrak{I}$. Both $\mathfrak{O}$ and $\mathfrak{I}$ are represented as Boolean polynomials. This is the same thing as building a truth table entirely algebraically, by first finding the false statements and then the true ones, without actually listing out the entire table. Probability assignments and calculations are meaningful only to subsets of $\mathfrak{I}$, because only outcomes included in $\mathfrak{I}$ are consistent with every single characteristic the system is assumed to have.

We follow Boole's method of representing $\mathfrak{O}$ in just about the same way it is done in his book. Each premise or proposition implies that a specified value (either "zero" or "one") for one or more binary elements cannot coexist, under that premise, with a specified value for some other element or elements. Thus, each proposition defines a zero-valued complete subset[1]

1. A complete subset of a partially ordered set contains both its greatest lower bound and its least upper bound, and all elements in the interval between these bounds. An interesting combinatorial result obtained in this paper is that the total number of complete subsets of the universal set is 3^n; this is believed to be new.

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of states or binary n-vectors, from the universal set. The union or join of the sets formed from all such propositions is the $\mathfrak{O}$ polynomial, each of whose terms represents a complete subset of $\mathfrak{O}$. By contrast to the relative ease with which the zero set is found, Boole's derivation of the identify set $\mathfrak{I}$ is not a convenient one to use. What we do instead of Boole's method is to complement the $\mathfrak{O}$ polynomial to obtain the $\mathfrak{I}$ polynomial, using de Morgan's theorems, followed by a simplification to get $\mathfrak{I}$ into its smallest (minimal) form.

A useful by-product of these results is that since the universal set is partitioned in such a way that states can easily be identified as being either consistent or inconsistent, subsets which are defined by the presence or absence of certain binary characteristics can also be described entirely algebraically, by Boolean polynomials. In particular, probability assignments can be made only to subsets of $\mathfrak{I}$ which are called "events". To derive the Boolean polynomial for an event E, it is only necessary to logically multiply the binary indicator or set of indicators which define that event, by the $\mathfrak{I}$ polynomial, and simplify. A probability calculation of an event E which is a subset of $\mathfrak{I}$ is straightforward, but it can be very tedious. A probability polynomial, a numerical-valued function of the probabilities of the components, is derived by Poincare's method (inclusion-exclusion) from the Boolean polynomial. The number of terms of the probability polynomial is potentially very large, but it can be substantially reduced by the fact that in an inclusion-exclusion expansion, certain terms are usually repeated many times, and terms can also be eliminated if there are contradictions. Machine derivation of the polynomial is essential for problems of any substantial size (say, $n \geq 10$). The probability calculation is made by substituting the probabilities of the elements into the polynomial.

CHAPTER I: LOGICAL ANALYSIS

2 *Concepts and definitions*

2.1 *Components, states, and the universal set* A system consists of n binary-valued atomic components or generators, 2^n distinct states (because each component has two possibilities) collectively called the *universal set*, $\mathfrak{U}$, and a number of specifications, propositions or statements which either specify the values of one of more components or else define the interrelationships between two or more components. The components may be either properties, animate or inanimate objects, or concepts, for example: "Socrates", "is a man", "functioning black box", "closed circuit". For each i, $i = 1, \ldots, n$, $x_i = 1$ denotes that the i-th component is "true" or "exists", and $x_i = 0$ or $\bar{x}_i = 1$ the opposite: "person other than Socrates", "not a man", "failed black box", "open circuit". Examples of specifications are: "Socrates (x_1) is a man (x_2) '$x_1\bar{x}_2 = 0$'"; "the circuit is open $(\bar{x}_3)$ if the black box fails $(\bar{x}_4)$ '$\bar{x}_3\bar{x}_4 = 1$'".

2.2 *The identity and zero sets* A state of the system is an element of $\mathfrak{U}$; it is an n-tuple which specifies a value for each of the n binary components, and it also has a "truth value" which may be either one or zero, depending upon the specifications. Those states which are consistent with all specifications are assigned a value of one and collectively are the *identity set*, $\mathfrak{J}$; those with a zero value are in the *zero set*, $\mathfrak{O}$. The *null set* is denoted by the usual symbol, $\emptyset$. The difference between the zero and null sets is that each member of $\mathfrak{O}$ is inconsistent with at least one specification, whereas $\emptyset$ includes no states at all. If all states are consistent with all specifications, $\mathfrak{J} = \mathfrak{A}$, so that $\mathfrak{O} = \emptyset$, and the system is a *tautology* (Wittgenstein [2]). In the opposite case, no states are consistent with any specification, so that $\mathfrak{J} = \emptyset$ and $\mathfrak{O} = \mathfrak{A}$, and the system is a *contradiction*.

2.3 *Logical addition and multiplication: upper and lower bounds* The universal set $\mathfrak{A}$ is a *free Boolean algebra* (Halmos [3], p. 5 and p. 40). It contains the unique zero element $(0, \ldots, 0)$ and unique one element $(1, \ldots, 1)$. Addition, multiplication, and complementation of elements is performed component by component under the usual rules ($0 + 0 = 0$, $1 + 0 = 1 + 1 = 1$; $0 \cdot 0 = 0 \cdot 1 = 0$, $1 \cdot 1 = 1$; $\bar{0} = 1$, $\bar{1} = 0$). Addition or multiplication of n-tuples is performed either pairwise, denoted by "+" and "·", or in sequences, denoted by "$\sum$" and "$\prod$".

An element which is the Boolean sum of two or more elements of $\mathfrak{A}$ is also their least upper bound, l.u.b., and the Boolean product the greatest lower bound, g.l.b. The zero and the unit elements are respectively lower and upper bounds for all subsets of $\mathfrak{A}$, but not necessarily the "greatest" lower bound or "least" upper bound.

2.4 *Logical comparison* Comparison of elements is denoted by the symbol $\leqslant$. For any pair of them, $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, $X \leqslant Y$ (Y is said to include X) means that $X \cdot Y = X (x_i = 1 \Rightarrow y_i = 1)$. *Equality* ($X = Y$) means both $X \leqslant Y$ and $Y \leqslant X$. *Strict inclusion* ($X < Y$) means both $X \leqslant Y$ and $X \neq Y$; *strict noninclusion* ($X \not< Y$) means $X \cdot Y < X$.

2.5 *Lattices* A lattice $\mathfrak{L}$ (Birkhoff [4]) is a nonempty set for which every pair of elements has both a g.l.b. and a l.u.b. Since the sum of every pair or sequence of elements in $\mathfrak{A}$ is the l.u.b. and the product the g.l.b., the universal set $\mathfrak{A}$ and every subset of it are also lattices. $\mathfrak{L}$ is *closed* if it contains both its g.l.b. and l.u.b. A closed lattice is *complete* if it contains all of the elements in the interval between the bounds.[2]

When a lattice, $\mathfrak{L}$, is expressed as either the union or the intersection of lattices $\mathfrak{L}_1, \ldots, \mathfrak{L}_m$, both the g.l.b. of $\mathfrak{L}$ and the l.u.b. of $\mathfrak{L}$ are related to the corresponding bounds of $\mathfrak{L}_1, \ldots \mathfrak{L}_m$. We have:

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2. An example of a free Boolean algebra is the set $\mathfrak{A} = \{00, 01, 10, 11\}$ of 2-tuples; the g.l.b. is "00" and the l.u.b. is "11." The subset $0 \cdot = \{00, 01\}$ is a complete sublattice with a fixed zero value for the first component; the g.l.b. is "00" and the l.u.b. is "01." Similarly, the subset $\cdot 0 = \{00, 10\}$ is a complete sublattice with a g.l.b. of "00" and l.u.b. of "10." The set $0 \cdot \vee \cdot 0 = \{00, 01, 10\}$ is neither closed nor complete; it has a g.l.b. of "00," which is in the set and a l.u.b. of "11" which is not. The lattice $0 \cdot \vee\ 11 = \{00, 01, 11\}$ contains both bounds, but is not complete.

$$\text{if } \mathfrak{L} = \bigcup_{j=1}^{m} \mathfrak{L}_j, \text{ g.l.b. } \mathfrak{L} = \prod_{j=1}^{m} \text{g.l.b. } \mathfrak{L}_j. \qquad (1)$$

A proof is by contradiction: assume that g.l.b. $\mathfrak{L} < \prod_{j=1}^{m}$ g.l.b. $\mathfrak{L}_j$; then g.l.b. $\mathfrak{L}$ is not a "greatest" lower bound for $\mathfrak{L}$ because a greater one can be found. If the reverse inequality were true, then g.l.b. $\mathfrak{L}$ would not be a lower bound for $\mathfrak{L}$. There are three statements which are dual to (1) and to each other:

$$\text{if } \mathfrak{L} = \bigcup_{j=1}^{m} \mathfrak{L}_j, \text{ l.u.b. } \mathfrak{L} = \sum_{j=1}^{m} \text{l.u.b. } \mathfrak{L}_j,$$

$$\text{if } \mathfrak{L} = \bigcap_{j=1}^{m} \mathfrak{L}_j, \text{ g.l.b. } \mathfrak{L} = \sum_{j=1}^{m} \text{g.l.b. } \mathfrak{L}_j,$$

$$\text{if } \mathfrak{L} = \bigcap_{j=1}^{m} \mathfrak{L}_j, \text{ l.u.b. } \mathfrak{L} = \prod_{j=1}^{m} \text{l.u.b. } \mathfrak{L}_j.$$

This duality is explained by the fact that the symbols in any two of the three pairs $(\bigcup, \bigcap)$, (l.u.b., g.l.b.) and (Σ, Π) may be reversed in any one of these four statements to obtain one of the other three.

2.6 *Maximal and minimal elements* A *maximal* element of $\mathfrak{L}$ is a binary n-vector $V \epsilon \mathfrak{L}$ such that $V < X$ for no $X \epsilon \mathfrak{L}$, that is, a maximal element is the l.u.b. of a complete sublattice in $\mathfrak{L}$ which is not a proper subset of any other sublattice in $\mathfrak{L}$. A *minimal* element of $\mathfrak{L}$ is a $V \epsilon \mathfrak{L}$ such that $X < V$ for no $X \epsilon \mathfrak{L}$, that is, the g.l.b. of a complete sublattice in $\mathfrak{L}$ which is not a proper subset of any other sublattice in $\mathfrak{L}$. If $\mathfrak{L}$ is complete, there are just one minimal element and one maximal element, respectively the g.l.b. and the l.u.b. If $\mathfrak{L}$ is not complete, however, there are either more than one minimal element, more than one maximal element, or both; furthermore, there can be several (many) different configurations of complete subsets of $\mathfrak{L}$. Note that $\mathfrak{L}$ has exactly one l.u.b. and exactly one g.l.b., neither of which necessarily belongs to $\mathfrak{L}$, and these are both unique. The maximal and minimal elements of $\mathfrak{L}$, however, can be different for each different configuration of complete subsets.

2.7 *Complete subsets: ideals and filters* Every lattice has a representation as a Boolean polynomial, frequently known as the *disjunctive normal form*. For a complete lattice, this is a single term (monomial). If the system has n binary-valued components or generators, a complete sublattice $\mathfrak{L}$, which is a proper subset of $\mathfrak{A}$, is represented by m binary indicators, $m < n$, each of which is either zero (0) or one (1) in every one of the 2^{n-m} states in the lattice. The monomial $x_1 \ldots x_m$, $x_i = 0$ or 1, is frequently called a "minterm". For the g.l.b. (l.u.b.) the $n - m$ free elements are all zero-valued (unit-valued).

Nonempty complete sublattices whose indicators all have the same value are either ideals or filters (dual-ideals). For an ideal $L_|$, all indicators are zero-valued; from the definition of an ideal: if $X \epsilon L_|$, $Y \epsilon \mathfrak{A}$,

$X \cdot Y_{\mathrm{I}} \in L_{\mathrm{I}}$ (*cf.* Bell and Slomson [5], p. 12 and Halmos [3], pp. 45-50). For a filter L_{F}, all indicators are unit-valued; if $X \in L_{\mathrm{F}}$, $Y \in \mathfrak{U}$, $X + Y \in L_{\mathrm{F}}$.

2.8 *Counting complete subsets* There are exactly

$$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + 1 = 2^n - 1$$

different ideals, because there are n different ways of selecting zero indicators one at a time, $\binom{n}{2}$ two at a time, etc. Similarly there are exactly $2^n - 1$ different filters.[3] The total number of complete subsets of $\mathfrak{U}$, including ideals, filters, and subsets with mixed indicators, is 3^n. This can be shown by counting first, according to the number of fixed components and second, according to the number of different ways binary-valued indicators can be obtained, then by accumulation. The only set with no fixed components is the universal set. With one fixed component there are n ways of choosing the components; for each component there are two choices, hence $2n$ complete subsets. For two fixed components there are $\binom{n}{2}$ ways of selecting the components and 2^2 different ways of obtaining binary indicators. Finally, with n fixed components there are 2^n subsets. It can be seen therefore that the total number of complete subsets of $\mathfrak{U}$ is

$$\binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \ldots + \binom{n}{n}2^n = (1 + 2)^n = 3^n.$$

2.9 *Boole's method* In his classic *Laws of Thought Boole* described a calculus employing lattice polynomials for analyzing a system of logical propositions, to determine which states of the universal set $\mathfrak{U}$ are not consistent with the system. In the sequel we call this set of "inconsistent (false)" states the zero set $\mathfrak{O}$. However, Boole did not have a convenient way of symbolically describing the set of states which are consistent (true) with the system, which we call the "identity set $\mathfrak{I}$". Starting from what he called the "Principle of Contradiction (that an event and its complement cannot both occur)" p. 49, he proceeded on pp. 71-80 through some rather complex steps, including a Taylor-series expansion of a logical function, to derive a truth function. In the sequel we follow Boole's procedure for obtaining $\mathfrak{O}$; to obtain $\mathfrak{I}$, however, we complement the $\mathfrak{O}$ polynomial using de Morgan's theorems, and then employ a Quine [8] simplification, to get the $\mathfrak{I}$ polynomial into its smallest (minimal) form.

3. Examples of both ideals and filters arise in the reliability analysis of coherent systems [6, 7]. If $x_j = 1$, the j-th component operates successfully; otherwise $x_j = 0$. The universal set has two types of outcomes, system failure states or "cuts" and success states or "paths." A "minimal cut" is defined as a complete set of cuts which is associated with the failure of a specified set of components, and which is not wholly contained within any other complete set of cuts. The minimal cuts are ideals and each one of them includes the zero state $(0, \ldots, 0)$, all components failed. Similarly a "minimal path" is a complete set of paths not wholly contained within any other such complete set of paths. Every minimal path includes $(1, \ldots, 1)$ and is a filter.

3 *Examples*

3.1 *A syllogism*[4] The well known syllogism "all men are mortal; Socrates is a man; therefore Socrates is mortal" is used to illustrate how to develop the $\mathfrak{O}$ and $\mathfrak{I}$ polynomials. Denote Socrates as x_1, a man as x_2 and a mortal as x_3. For every i let x_i denote the unit value and $\bar{x}_i$ the zero value. The major and minor premises are, $x_2 \supset x_3$ and $x_1 \supset x_2$. These are restated respectively as the zero equations $x_2\bar{x}_3 = 0$ and $x_1\bar{x}_2 = 0$. The term $x_2\bar{x}_3$ represents a complete sublattice with two states, $\bar{x}_1x_2\bar{x}_3$ and $x_1x_2\bar{x}_3$. Similarly the term $x_1\bar{x}_2$ represents a complete sublattice with two states: $x_1\bar{x}_2x_3$, $x_1\bar{x}_2\bar{x}_3$. The zero set $\mathfrak{O}$ is the join of these two sets

$$\mathfrak{O} = x_1\bar{x}_2 \vee x_2\bar{x}_3. \tag{2}$$

Invert (2) by de Morgan's theorems; after simplification we have the complement of $\mathfrak{O}$, the identity polynomial

$$\mathfrak{I} = \bar{x}_1\bar{x}_2 \vee x_2x_3. \tag{3}$$

The identity set $\mathfrak{I}$ has four elements, $x_1x_2x_3$, $\bar{x}_1x_2x_3$, $\bar{x}_1\bar{x}_2x_3$, and $\bar{x}_1\bar{x}_2\bar{x}_3$. Each of these represents a proposition which is consistent with both the major and the minor premises. The conclusion "Socrates, a man, is mortal" is implied by $x_1x_2x_3$. Other conclusions are:

$\bar{x}_1x_2x_3$: a non-Socrates man is mortal.
$\bar{x}_1\bar{x}_2x_3$: a non-Socrates non-man could be mortal.
$\bar{x}_1\bar{x}_2\bar{x}_3$: a non-Socrates non-man might not be mortal.

Note that of the four propositions in $\mathfrak{I}$, two have probability 1, and two have probability less than one. $x_1x_2x_3$ ($x_3 = 1$ if both $x_1 = 1$ and $x_2 = 1$) and $\bar{x}_1x_2x_3$ ($x_3 = 1$ if $x_1 = 0$ and $x_2 = 1$) are both implied by the premises with probability 1. Since the premises imply nothing with certainty about "non-men", the mortality of $\bar{x}_1\bar{x}_2$ (non-Socrates non-men) has a probability less than 1.

It is instructive to summarize the results of this analysis in the form of a truth table, with additional remarks. (See page 129, Table 1.)

3.2 *Epimenides paradox* "Epimenides the Cretan speaks the truth; he says that all Cretans are liars". The system has three components: x_1, Epimenides; x_2, Cretans; and x_3, liars. The three premises inferred are interpreted as zero equations: (1) $x_1\bar{x}_2 = 0$, (2) $x_1x_3 = 0$, and (3) $x_2\bar{x}_3 = 0$. The zero set is

$$\mathfrak{O} = x_1\bar{x}_2 \vee x_2\bar{x}_3 \vee x_1x_3.$$

4. Since the Aristotelean syllogism is the gest known example of a system of propositions, it has been studied extensively. Gardner [9] has an interesting discussion of the history of attempts to solve the syllogism by various geometrical, electromechanical and algebraic means. Lewis Carroll (better known as the author of "Alice in Wonderland") used both Boolean polynomials and "logic boards" [10], pp. 72-80. For a modern symbolic treatment of the syllogism without Boolean polynomials, refer to Kleene [11], example 19, p. 135.

Element	Member of	Remarks
$x_1x_2x_3$	$\mathfrak{I}$	x_2 if x_1 by minor premise; x_3 if x_2 by major premise
$x_1x_2\bar{x}_3$	$\mathfrak{O}$	excluded by major premise
$x_1\bar{x}_2x_3$	$\mathfrak{O}$	excluded by minor premise
$\bar{x}_1x_2x_3$	$\mathfrak{I}$	x_3 if x_2 by major premise
$x_1\bar{x}_2\bar{x}_3$	$\mathfrak{O}$	excluded by minor premise
$\bar{x}_1x_2\bar{x}_3$	$\mathfrak{O}$	excluded by major premise
$\bar{x}_1\bar{x}_2x_3$	$\mathfrak{I}$	a possible state of "non-Socrates non-human mortals"
$\bar{x}_1\bar{x}_2\bar{x}_3$	$\mathfrak{I}$	a possible state of "non-Socrates non-human non-mortal"

Table 1

By inverting $\mathfrak{O}$ and simplifying, we obtain the identity set

$$\mathfrak{I} = \bar{x}_1\bar{x}_2 \vee \bar{x}_1 x_3.$$

Thus no statement whatever can be made about Epimenides, because there are no states in $\mathfrak{I}$ with $x_1 = 1$. Other elements, however, are consistent with this system such as "non-Cretans who are liars", "non-Cretans who are not liars", etc. In other words, the paradox is not a logical contradiction, because some elements are consistent with it. The truth table is as follows:

Element	Member of	Remarks
$x_1x_2x_3$	$\mathfrak{O}$	excluded by (2)
$x_1x_2\bar{x}_3$	$\mathfrak{O}$	excluded by (3)
$x_1\bar{x}_2x_3$	$\mathfrak{O}$	excluded by both (1) and (2)
$\bar{x}_1x_2x_3$	$\mathfrak{I}$	non-Epimenides Cretan can be (is) a liar
$x_1\bar{x}_2\bar{x}_3$	$\mathfrak{O}$	excluded by (1)
$\bar{x}_1x_2\bar{x}_3$	$\mathfrak{O}$	excluded by (3)
$\bar{x}_1\bar{x}_2x_3$	$\mathfrak{I}$	non-Epimenides non-Cretan can be a liar
$\bar{x}_1\bar{x}_2\bar{x}_3$	$\mathfrak{I}$	non-Epimenides non-Cretan can tell the truth

Table 2

4 *Subsets of the identity set* Every subset of $\mathfrak{A}$ can be represented by a Boolean polynomial. A subset is identified by a specified binary value for a characteristic or else by a specified set of binary values for specified characteristics. Only subsets of $\mathfrak{I}$, however, are meaningful for further analysis, such as, for example, probability calculations, because only elements in $\mathfrak{I}$ are consistent with every proposition of the system. In order to derive the polynomial for a subset of $\mathfrak{I}$, the binary-valued characteristic which defines the subset is logically multiplied by the polynomial. For the foregoing syllogism, an example of a defining characteristic is "man", "Socrates" or "mortal", or some combination of these. Let "$x_i = 1$" be the specified characteristic. Then x_i defines a complete sublattice of $\mathfrak{A}$ with 2^{n-1} elements. In order to find the subset of $\mathfrak{I}$ which is defined by x_i, we logically multiply the $\mathfrak{I}$ polynomial by x_i, to obtain the subset of $\mathfrak{I}$ for which $x_i = 1$. Similarly, if the defined characteristic is "$x_i = 0$", the $\mathfrak{I}$ polynomial is multiplied by $\bar{x}_i$ to obtain the desired result. For example, with the syllogism (3), there are six such possibilities:

Socrates (x_1):	$x_1x_2x_3$
non-Socrates ($\bar{x}_1$):	$\bar{x}_1\bar{x}_2 \vee \bar{x}_1x_2x_3$
men (x_2):	x_2x_3
non-men ($\bar{x}_2$):	$\bar{x}_1\bar{x}_2$
mortals (x_3):	$\bar{x}_1\bar{x}_2x_3 \vee x_2x_3$
non-mortals ($\bar{x}_3$):	$\bar{x}_1\bar{x}_2\bar{x}_3$.

To extend the example, an event E can be defined by the intersection of fixed values for two or more components. Let x_i and x_j be two fixed-valued components which define E; the polynomial is the logical product of $x_i x_j$ by $\mathfrak{I}$. We have, for example:

Socrates non-mortals $(x_1\bar{x}_3)$: zero (*not* null)
men mortals (x_2x_3): x_2x_3.

An event E can be defined as the "inclusive or" for two or more fixed-valued components. Let x_i and x_j be two such components; the polynomial is the logical product of $x_i \vee x_j$ by $\mathfrak{I}$. For the example, we have:

man or mortal $(x_2 \vee x_3)$: $\bar{x}_1\bar{x}_2x_3 \vee x_2x_3$
non-man or mortal $(\bar{x}_2 \vee x_3)$: $\mathfrak{I}$
Socrates or non-mortal $(x_1 \vee \bar{x}_3)$: $x_1x_2x_3 \vee \bar{x}_1\bar{x}_2\bar{x}_3$.

Clearly, the event E can be defined in terms of even more complex combinations of "ors" and "ands" than are shown in this section.

CHAPTER II: PROBABILITY ANALYSIS

The objective of this section is to develop the methodology for determining the probability of an event, E, as a function of the probabilities of the components of the system, structuring this function, a polynomial, after the lattice polynomial for the event. We first describe the probability space for the components and the states. The probability of a state in a complete lattice is a product of the conditional probabilities of the indicators. Then Poincare's method is used to obtain the probability of the event.

5 *Probability spaces for components and states* Let $X = (x_1, \ldots, x_n)$, $x_i = 0$ or 1, $i = 1, \ldots, n$, be any state. Corresponding to X there is an n-tuple of conditional probabilities whose components are respectively

$$\text{pr}(x_1),\ \text{pr}(x_2 \mid x_1),\ \text{pr}(x_3 \mid x_1, x_2),\ \ldots,\ \text{pr}(x_n \mid x_1, \ldots, x_{n-1}), \tag{4}$$

all of which are defined as real numbers between zero and one, inclusive. It is assumed that there is a complementary relationship such that

$$\text{pr}(x_i = 1 \mid x_1, \ldots, x_{i-1}) = 1 - \text{pr}(x_i = 0 \mid x_1, \ldots, x_{i-1}),\ i = 1, \ldots, n.$$

The probability for the state, X, is the joint probability for all the values of its components, or the numerical product of the conditional probabilities in (4)

$$\text{pr}(X) = \text{pr}(x_1)\ \text{pr}(x_2 \mid x_1) \ldots \text{pr}(x_n \mid x_1, \ldots, x_{n-1}). \tag{5}$$

When the components are all independent (5) becomes simply

$$\text{pr}(x_1)\ \text{pr}(x_2) \ldots \text{pr}(x_n).$$

The description above holds for any chosen permutation or ordering of the generators $x_1, \ldots, x_n$. Depending upon the ordering selected, the values of the factors of (5) change because of the altered selection of components upon which to condition the probabilities. It is assumed, however, that $\text{pr}(X)$ does not depend upon how the atoms are ordered. It can be shown that the sum of the probabilities for all 2^n states in $\mathfrak{A}$ is one.

Since all probability measure is assigned to the identity set, $\mathfrak{I}$ constitutes a "sample space".

6 *Probability of a complete subset* For a system having n atomic elements a complete subset is identified by m fixed-valued indicators, $m < n$, and has 2^{n-m} states. If the n components are renumbered, for convenience, so that these m indicators are first, thus: $x_1, \ldots, x_m$, the probability that the actual outcome of a random experiment is one of these 2^{n-m} states is the product of the conditional probabilities

$$\mathrm{pr}(x_1)\ \mathrm{pr}(x_2 \mid x_1) \ldots \mathrm{pr}(x_m \mid x_1, \ldots, x_{m-1}), \tag{6}$$

if these probabilities are all defined. If the components are totally independent (6) becomes

$$\mathrm{pr}(x_1)\ \mathrm{pr}(x_2) \ldots \mathrm{pr}(x_m).$$

Independence of components is possible only if the identity set $\mathfrak{I}$ is identical with the universal set $\mathfrak{A}$, so that all states are possible; this condition "$\mathfrak{I} = \mathfrak{A}$" is known in logic as a "tautology". The dual condition that the identity set is empty "$\mathfrak{I} = \emptyset$" is known as a "contradiction"; since no states at all can occur by assumption, no probability assignment can be made at all.

7 *Probability of a set (lattice)* Every lattice $\mathfrak{L}$ can be characterized as a join of the m complete sublattices $\mathfrak{L}_1, \ldots, \mathfrak{L}_m$, which are all proper subsets of $\mathfrak{L}$. Therefore, the probability of $\mathfrak{L}$ is the probability that at least one of $\mathfrak{L}_1, \ldots, \mathfrak{L}_m$ occurs. The principle of inclusion-exclusion (Feller [12], Chapter 4), also called Poincare's Theorem (Riordan) ([13], p. 52) is used to derive the probability equation, a numerical-valued polynomial in the probabilities of the components. Let S_1 denote the sum of the probabilities of the m subsets taken one at a time, S_2 the sum of the $\binom{m}{2}$ joint probabilities of the subsets taken two at a time, S_3 the sum of the $\binom{m}{3}$ joint probabilities of the subsets three at a time, etc. Then the probability desired is

$$\mathrm{pr}(\mathfrak{L}) = S_1 - S_2 + S_3 - \ldots + (-1)^{m-1} S_m. \tag{7}$$

Equation (7) has a maximum of $2^m - 1$ terms, which can often be a very large number. The problem of data processing and bookkeeping in order to generate Equation (7) is usually simplified considerably by the fact that most terms cancel, either because of contradictions or because in the expansion the same lattice is obtained in different ways. Electronic computation is necessary for this purpose if the problem is of any substantial size. If m is very large and if there is not enough cancellation of terms in the process of generating Equation (7) to operate within a reasonably bounded computer memory allocation, it may be uneconomical to generate the complete polynomial. The Bonferroni inequality (Feller [12]) can be used to set up a criterion for termination with a reasonably good approximation. If only the first $r-1$ terms of the right-hand side of (7) are used, viz:

$$\mathrm{pr}(\mathfrak{L}) \simeq S_1 - S_2 + S_3 + \ldots + (-1)^{r-2} S_{r-1},$$

the error (true value minus approximation) is smaller in absolute value than S_r.

8 *Conditional probability of an event* The conditional probability of one event, given another event, can be determined in the usual way. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be any two lattices, both proper subsets of the identity set, $\mathfrak{I}$. The conditional probability of $\mathfrak{L}_2$ given $\mathfrak{L}_1$ is

$$\mathrm{pr}(\mathfrak{L}_2 \mid \mathfrak{L}_1) = \frac{\mathrm{pr}(\mathfrak{L}_1 \cap \mathfrak{L}_2)}{\mathrm{pr}(\mathfrak{L}_1)}, \ \mathrm{pr}(\mathfrak{L}_1) > 0.$$

9 *Examples*

9.1 *The syllogism* The major and minor premises can be interpreted, respectively

(i) a mortal (x_3) if a man (x_2), with probability one
(ii) a man (x_2) if Socrates (x_1), with probability one.

Therefore we have

$$\mathrm{pr}(x_3 \mid x_2) = \mathrm{pr}(x_2 \mid x_1) = 1.$$

Since the event "Socrates" (x_1), is identical with "Socrates-man-mortal" ($x_1x_2x_3$), the probability

$$\mathrm{pr}(x_1x_2x_3) = \mathrm{pr}(x_3 \mid x_2)\ \mathrm{pr}(x_2 \mid x_1)\ \mathrm{pr}(x_1) = \mathrm{pr}(x_1)$$

is identical with the probability of x_1. However, if we are given that the man selected is Socrates

$$\mathrm{pr}(x_2x_3 \mid x_1) = \mathrm{pr}(x_3 \mid x_2)\ \mathrm{pr}(x_2 \mid x_1) = 1.$$

9.2 *Boole's examples revisited* In Chapter XVIII, Boole gave a number of illustrations of the use of the general method for deriving probability functions. In this section, some of these examples are redone using the techniques described in this paper. In [1], p. 276:

> 2. Ex. 1.—The probability that it thunders upon a given day is p, the probability that it both thunders and hails is q, but of the connexion of the two phenomena of thunder and hail, nothing further is supposed to be known. Required the probability that it hails on the proposed day

Using the notation of this paper rather than Boole's, the example is continued: Let x_1 denote the event "it thunders" and let x_2 denote the event "it hails", and let $\bar{x}_1$ and $\bar{x}_2$ represent their respective complementary events. A zero space is not specified; therefore the identity lattice, $\mathfrak{I}$, is identical with the universal set, $\mathfrak{A}$, and the system is a tautology. Since $\mathfrak{I} = \mathfrak{A}$, the event "it hails" is the free lattice with indicator x_2. We write

$$x_2 = x_1x_2 \vee \bar{x}_1x_2.$$

It is given that $\mathrm{pr}(x_1x_2) = q$ and also that $\mathrm{pr}(x_1) = p$, so that $\mathrm{pr}(\bar{x}_1) = 1 - p$. What is needed is $\mathrm{pr}(x_2|\bar{x}_1)$; this Boole calls "c" and he gives the correct answer

$$\mathrm{pr}(x_2) = q + \mathrm{c}(1 - p).$$

9.2.1 *Analysis of Dr. S. Clarke's metaphysical system* Boole was very interested in metaphysical problems such as scientific proof of the existence of Deity, etc. He used Dr. Samuel Clarke's tract "Demonstration of the Being and Attributes of God" [14] as an example of how to construct a logical system. In the discussion below we reproduce the relevant portion of Boole's text setting up the problem and then, using essentially his interpretation and notation, construct probability functions both for the affirmation and the denial of the existence of a presiding intelligence in the universe. This example is interesting for two reasons: first, it is used to derive the probability function for an event in a relatively complex system; secondly, it is shown that there are enough contradictions in the premises that under reasonable assumptions neither affirmation nor denial is possible. The portion of the text reproduced below starts on p. 219 of [1]:

> 1. If matter is a necessary being, either the property of gravitation is necessarily present, or it is necessarily absent.
> 2. If gravitation is necessarily absent, and the world is not subject to any presiding intelligence, motion does not exist.
> 3. If gravitation is necessarily present, a vacuum is necessary.
> 4. If a vacuum is necessary, matter is not a necessary being.
> 5. If matter is a necessary being, the world is not subject to a presiding intelligence.
>
> If, as before, we represent the elementary propositions by the following notation, viz:
>
> x = Matter is a necessary being.
> y = Gravitation is necessarily present.
> w = Motion exists.
> t = Gravitation is necessarily absent.
> z = The world is merely material, and not subject to a presiding intelligence.
> v = A vacuum is necessary.
>
> We shall on expression of the premises and elimination of the indefinite class symbols (q), obtain the following system of equations:
>
> . . .
>
> $$xyt + x\bar{y}\bar{t} + y\bar{v} + vx + x\bar{z} + tzw = 0$$

The last expression (from p. 220) will be recognized as the zero polynomial, $\mathfrak{O}$, in the notation of this paper, if "v" is substituted for "+". It can also be recognized that $y = 1$ and $t = 1$ are essentially contradictory events, so that $yt = 0$, and the first term can therefore be simplified to just "yt". Thus, we may write the zero polynomial

$$\mathfrak{O} = yt \vee x\bar{y}\bar{t} \vee y\bar{v} \vee vx \vee x\bar{z} \vee tzw.$$

By complementation we obtain the identity polynomial

$$\mathfrak{J} = \bar{x}\bar{y}\bar{t} \vee \bar{x}\bar{y}\bar{z} \vee \bar{x}\bar{y}\bar{w} \vee \bar{x}\bar{t}v \vee \bar{y}t\bar{v}z\bar{w}.$$

If $z = 0$, that is, $\bar{z} = 1$, the world does have a presiding intelligence. The event $\bar{z}$ has the defining polynomial

$$\bar{z} = \bar{x}\bar{y}\bar{z} \vee \bar{x}\bar{t}v\bar{z}.$$

The corresponding probability function is

$$\mathrm{pr}(\bar{z}) = \mathrm{pr}(\bar{x}\bar{y}\bar{z}) + \mathrm{pr}(\bar{x}\bar{t}v\bar{z}) - \mathrm{pr}(\bar{x}\bar{y}\bar{z}\bar{t}v).$$

However, if matter exists ($x = 1$), $\bar{z}$ is in the zero set and this probability is zero. Similarly, $z = 1$ (which denies a presiding intelligence) has the polynomial

$$z = \bar{x}\bar{y}\bar{t}z \vee \bar{x}\bar{y}\bar{w}z \vee \bar{x}\bar{t}vz \vee \bar{y}t\bar{v}z\bar{w}.$$

The probability function for z is obtained in four stages:

$$\begin{aligned} p_1 &= \mathrm{pr}(\bar{x}\bar{y}\bar{t}z) \\ p_2 &= p_1 + \mathrm{pr}(\bar{x}\bar{y}\bar{w}z) - \mathrm{pr}(\bar{x}\bar{y}\bar{t}\bar{w}z) \\ p_3 &= p_2 + \mathrm{pr}(\bar{x}\bar{t}vz) - \mathrm{pr}(\bar{x}\bar{y}\bar{t}vz) \\ \mathrm{pr}(z) = p_4 &= p_3 + \mathrm{pr}(\bar{y}t\bar{v}z\bar{w}) - \mathrm{pr}(\bar{x}\bar{y}\bar{w}zt\bar{v}) \\ &= \mathrm{pr}(\bar{x}\bar{y}\bar{t}z) + \mathrm{pr}(\bar{x}\bar{y}\bar{w}z) + \mathrm{pr}(\bar{x}\bar{t}vz) + \mathrm{pr}(\bar{y}t\bar{v}z\bar{w}) \\ &\quad - \mathrm{pr}(\bar{x}\bar{y}\bar{t}\bar{w}z) - \mathrm{pr}(\bar{x}\bar{y}\bar{t}vz) - \mathrm{pr}(\bar{x}\bar{y}\bar{w}zt\bar{v}). \end{aligned}$$

If both matter ($x = 1$) and gravitation ($y = 1$) are axiomatically true, z is in the zero space and this probability is zero. Hence, if one accepts the existence both of matter and of gravitation, no inference can be made from the system as to the affirmation or denial of the existence of a presiding intelligence.

10 *Extensions* The procedure described herein, which closely follows Boole's, is to first define the complete subsets of the zero set formed by the propositions, "join" them to obtain the zero set, and then complement the zero set to find the identity set. As an alternative, the dual procedure can be used to produce the identity set directly. For example, the implication $x \supset y$ can be written either as "$x\bar{y} = 0$ (Boole's method)" or alternatively as its dual "$\bar{x} \vee y = 1$". A system such as a syllogism with two propositions, $x_1 \supset x_2$ and $x_2 \supset x_3$, can be written in the identity form

$$(\bar{x}_1 \vee x_2) \cdot (\bar{x}_2 \vee x_3) = \mathfrak{I}.$$

When this expression is expanded, you obtain

$$\bar{x}_1\bar{x}_2 \vee x_2x_3 = \mathfrak{I},$$

which is identical to Equation (3).

Either of the two dual forms of the propositional connectives commonly used in logic can be used in the same way as described above, with similar interpretations. For example "equivalence, $x \sim y$" becomes

$$\begin{aligned} x\bar{y} \vee \bar{x}y &= 0 \\ xy \vee \bar{x}\bar{y} &= 1; \end{aligned}$$

"conjunction, $x\&y$" is

$$\begin{aligned} xy &= 1 \\ \bar{x} \vee \bar{y} &= 0; \end{aligned}$$

and "disjunction $x \vee y$" is

$$\begin{aligned} x \vee y &= 1 \\ \bar{x} \cdot \bar{y} &= 0. \end{aligned}$$

Thus Boolean polynomials can be built up even for fairly complicated systems of propositions using the methods outlined in this paper, either for the zero set or the identity set, whichever is most convenient.

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Oklahoma State University
Stillwater, Oklahoma