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LOGICAL CONSEQUENCE IN MODAL LOGIC II: SOME SEMANTIC SYSTEMS FOR S4

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1 This paper is a continuation of the investigations reported in Corcoran and Weaver [1] where two logics $\mathcal{L}\Box$ and $\mathcal{L}\Box\Box$, having natural deduction systems based on Lewis's S5, are shown to have the usually desired properties (strong soundness, strong completeness, compactness). As in [1], we desire to treat modal logic as a "clean" natural deduction system with a conceptually meaningful semantics. Here, our investigations are carried out for several S4 based logics. These logics, when regarded as logistic systems (*cf.* Corcoran [2], p. 154), are seen to be equivalent; but, when regarded as consequence systems (ibid., p. 157), one diverges from the others in a fashion which suggests that two standard measures of semantic complexity may not be linked. Some of the results of [1] are presupposed here and the more obvious definitions will not be repeated in detail.

We consider the logics $\mathcal{L}1$, $\mathcal{L}2$, and $\mathcal{L}3$. These logics share the same language ($\mathcal{L}\Box\Box$) and deductive system ($\Delta'\Box\Box$) but each has its own semantics system ($\Sigma1$, $\Sigma2$, $\Sigma3$). $\Sigma1$ is an extension of the Kripke [3] semantics for S5 as modified in [1]. $\Sigma2$ is largely due to Makinson [5], and $\Sigma3$ is due to Kripke [4]. $\mathcal{L}\Box\Box$ is the usual modal sentential language with $\Box$, $\sim$ and $\supset$ as logical constants (see **2** below). $\Delta'\Box\Box$ (see **3** below), a modification of the natural deduction system given in [1], permits proofs from arbitrary sets of premises. For S a set of sentences and A a sentence, $S \vdash A$ means that A *is provable from* S, i.e., there is a proof (in $\Delta'\Box\Box$) of A whose premises are among the members of S. ($\vdash A$ means $S \vdash A$ where S is empty.) If $S \vdash A$, we sometimes say that the argument $\langle S, A\rangle$ is *demonstrable* and when, in addition, S is empty we say that A is *provable*.

Each semantic system includes a set of interpretations for $\mathcal{L}\Box\Box$ together with truth-valuations for each interpretation. As usual, if M is an interpretation (in Σi) and A is a sentence, M *is a model of* A iff A is true on M; if S is a set of sentences, M *is a model of* S iff M is a model of every member of S. $S \vDash A$ (in Σi) indicates that A *is a logical consequence of* S

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(in Σi), i.e., that every model of S is a model of A (alternatively, we say that the argument $\langle S, A\rangle$ is valid in Σi). $\vDash A$ indicates that A is *logically true* (in Σi), i.e., that A is true on every interpretation.

Given two semantic systems for a given language, we can naturally ask whether they are "equivalent" either in the sense (a) that the sets of logical truths are identical, or in the sense (b) that the sets of valid arguments are identical. More precisely, we say that two semantic systems are *strongly equivalent* if they have the same valid arguments; and *weakly equivalent* if they have the same logical truths. Strong equivalence implies weak equivalence but not conversely: $\Sigma 1$ is weakly but not strongly equivalent to $\Sigma 3$ (see **6.1** below).

The notions of completeness (weak and strong), soundness (weak and strong), and compactness are defined as usual. Recall that the semantic system of any strongly sound and strongly complete logic has a compact semantics. In the usual hierarchy of non-modal logics (sentential, first order with identity, second order, etc.) weak completeness, strong completeness, and compactness all fail at the same point. This leads to the question of whether there is some "inherent" property of logics which links weak completeness (essentially enumerability of logical truths) to compactness. Although $\mathcal{L}1$ is seen to be a counterexample to the positive conjecture on the issue (below), further investigation may reveal an intimate relation between compactness and the enumerability of the set of logical truths. In view of the fact that there are uncountably many valid arguments $\langle S, A\rangle$, compactness is an index of the complexity of a semantic system in much the same way that enumerability of logical truths is. One feels that these two indices are linked.

We show (1) that $\mathcal{L}1$ is strongly sound; (2) that $\mathcal{L}1$ is not compact, hence not strongly complete; (3) that for every interpretation in $\Sigma 1$ there is an interpretation in $\Sigma 2$ making exactly the same sentences true; (4) that $\mathcal{L}2$ is strongly sound and strongly complete, and hence that $\Sigma 1$ and $\Sigma 2$ are not strongly equivalent; (5) that for every interpretation in $\Sigma 2$, there is an interpretation in $\Sigma 3$ making exactly the same set of sentences true; therefore $\mathcal{L}3$ is strongly complete; (6) that given any interpretation in $\Sigma 3$ and any infinite proper subset C' of the set of sentence constants, there is an interpretation in $\Sigma 1$ agreeing with it on all sentences whose constants are not in C'; hence (a) that $\mathcal{L}3$ is strongly sound and compact; (b) that $\Sigma 3$ is strongly equivalent to $\Sigma 2$ but not (strongly equivalent) to $\Sigma 1$; (c) that $\Sigma 2$ and $\Sigma 3$ are both weakly equivalent but not strongly equivalent to $\Sigma 1$; and (d) that $\mathcal{L}1$ is weakly complete. Finally, we note that both the Feys-von Wright system [5], M, and the Brouwerische system [5] (with semantics like $\Sigma 1$) are not strongly complete.

2 $\mathcal{L}\Box\Box$ is the result of closing C, a countably infinite set of sentential constants, under forming $\Box A$, $\sim A$, and $(A \supset B)$ for A, B already in the set. $\mathcal{L}$ indicates those sentences of $\mathcal{L}\Box\Box$ devoid of occurrences of $\Box$.

3 $\Delta'\Box\Box$: The deductive system $\Delta'\Box\Box$ closely resembles the system $\Delta\Box\Box$ for $\mathcal{L}\Box\Box$ given in [1]. As in [1] proofs are written left-to-right rather than up-to-down, and brackets,] [, are used as punctuation (see [1] for examples). The terms *proof expression, open in a proof expression, closed in a proof expression,* and *assumption of a proof expression* are defined as in [1] (p. 343). A *proof expression* is any string over $\mathcal{L}\Box\Box \cup \{], [\}$. Loosely, an occurrence of the sentence A is *closed in a proof expression* provided it occurs between matching brackets; an occurrence of A is *open in a proof expression* provided this occurrence is not closed; an occurrence of A is an *assumption in a proof expression* iff it is immediately preceded by a left bracket, [; a sentence is an *assumption in a proof expression* iff some occurrence of that sentence is an assumption. A is an *open assumption in a proof expression* provided that an occurrence of A is an assumption and open in the proof expression. In the following, A, B, and D are sentences of $\mathcal{L}\Box\Box$, and Π and σ are proof expressions. These symbols occur with or without subscripts and accents.

Definition For every proof expression Π, Π is a *proof* (in $\Delta'\Box\Box$) iff Π is constructed by a finite number of applications of the following rules:

(i) The six rules A, R, $\supset$E, $\supset$I, $\sim$E, $\Box$E (*cf.*, [1], p. 373).
(ii) ($\Box$I$'$) If Π is a proof, A occurs open in Π, and all open assumptions in Π are of the form $\Box B$, then $\Pi\Box A$ is a proof.

As in [1], C(Π) denotes the last line of Π and O(Π) denotes the *open assumptions* (or *premises*) of Π. For every set of sentences S and every sentence A, we say that *A is provable from S* in $\Delta'\Box\Box$ iff there exists a proof Π in $\Delta'\Box\Box$ such that $O(\Pi) \subseteq S$ and $C(\Pi) = A$. (As noted above, we indicate this by $S \vdash A$.) $\Delta'\Box\Box$ differs from the deductive system for $\mathcal{L}\Box\Box$ ([1], p. 373) only in the rule for the introduction of $\Box$, $\Box$I$'$ permits the inference of $\Box A$ in a proof Π provided that A has been inferred in Π and every premise in Π is of the form $\Box B$. The rule,[1] $\Box$I, of [1] allows the inference of $\Box A$ in Π provided that every premise in Π is a truth functional combination of sentences of the form $\Box B$ (and, of course, that A is inferred in Π). Obviously, any proof in $\Delta'\Box\Box$ is a proof in $\Delta\Box\Box$. The difference between $\Box$I and $\Box$I$'$ is the difference between S5 and S4. $\Box$I$'$ is essentially the rule for the introduction of $\Box$ given in [1], p. 380, [5], p. 74 and [3], p. 114.

Let $S \subseteq \mathcal{L}\Box\Box$. *$S$ is inconsistent* (in $\Delta'\Box\Box$) iff there is a sentence A such that $S \vdash A$ and $S \vdash \sim A$; *S is consistent* (in $\Delta'\Box\Box$) otherwise, *S is maximally consistent* (in $\Delta'\Box\Box$) iff S is consistent and for all A, either $A \in S$ or $\sim A \in S$. Let Δ be the subsystem of $\Delta\Box\Box$ consisting of just those proofs over the language $\mathcal{L}$; for $S \subseteq \mathcal{L}$, $A \in \mathcal{L}$, we define *$S \vdash A$ in Δ, $\vdash A$ in Δ, S is consistent in Δ,* and *S is maximally consistent in Δ as above* (*mutatis mutandis*). The following facts about maximally consistent sets can easily be verified.

Lemma 1 *For all $S \subseteq \mathcal{L}\Box\Box$, if S is maximally consistent in $\Delta'\Box\Box$ then for all A, $B \in \mathcal{L}$*

1. *$A \in S$ iff $S \vdash A$;*
2. *$A \in S$ or $\sim A \in S$;*

3. $(A \supset B) \in S$ *iff* $A \notin S$ *or* $B \in S$;
4. $S \cap \mathcal{L}$ *is maximally consistent in* Δ.

Lemma 2 [Lindenbaum]. *Every consistent set (in* $\Delta'\Box\Box$*) can be extended to a maximally consistent set (in* $\Delta'\Box\Box$*).*

If one has a "standard" enumeration of the sentences of $\mathcal{L}\Box\Box$, each of the usual proofs of Lemma 2 provides a functional process for associating with each consistent S a unique maximally consistent S^+. We assume such a standard enumeration and a functional process so that for each consistent S, S^+ indicates a maximally consistent extension of S. For every S, we define $\Box(S) = \{A : \Box A \in S\}$ and $\overline{\Box}(S) = \{A : \sim\Box A \in S\}$. Let $S + A = S \cup \{A\}$. The following are easily verified.

Lemma 3 *For every proof* Π *in* $\Delta'\Box\Box$ *there exists a proof* Π' *such that* $\Pi' = [B_1[B_2[\ldots[B_n \sigma A$ *where* $O(\Pi) = \{B_1, \ldots, B_n\} = O(\Pi')$ *and* $C(\Pi) = A = C(\Pi')$.

Fact 1 *For all* S, A, $S + \sim A$ *is inconsistent iff* $S \vdash A$.

Lemma 4 *Let* S *be maximally consistent in* $\Delta'\Box\Box$. *Then*

(a) $\Box(S)$ *and* $\overline{\Box}(S)$ *are disjoint*;
(b) *For all* A, $A \in \Box(S)$ *iff* $\Box(S) \vdash A$;
(c) *For all* A, *if* $A \in \overline{\Box}(S)$, *then* $\Box(S) + \sim A$ *is consistent.*

The first clause is obvious and the last clause is implied by the first two as follows. Suppose that $A \in \overline{\Box}(S)$ but that $\Box(S) + \sim A$ is inconsistent; then $\Box(S) \vdash A$ and, by the second, $A \in \Box(S)$. This contradicts the first. To see the second note that the "only if" part is trivial, then suppose $\Box(S) \vdash A$; i.e., that there is a proof Π of A from some finite subset, $B_1, \ldots, B_n$, of $\Box(S)$. By Lemma 3 there exists a proof $\Pi' = [B_1[B_2[\ldots[B_n \sigma A$. Let σ' be the result of deleting the left brackets ([) which occur immediately before the occurrences of B_i which are *assumptions* in Π'. Now form $[\Box B_1[\Box B_2[\ldots[\Box B_n \sigma'$. The latter is a proof of A from $\Box B_1, \ldots, \Box B_n$ and by applying $(\Box I')$ $[\Box B_1[\Box B_2[\ldots[\Box B_n \sigma \Box A$ is a proof. Since S is maximally consistent and $\Box B_i \in S$, $\Box A \in S$.

4 $\Sigma 1$: The interpretations of $\Sigma 1$ are all triples $\langle a, P, R\rangle$ where a is an ordinary truth-value assignment (a function from C into $\{\mathbf{t}, \mathbf{f}\}$), P is a set of such functions containing a, and R is a reflexive and transitive relation on P. Below a, b, a', b', etc. are assignments, P, P', etc. are sets of assignments and R, R', etc. are relations. We use aPR to indicate interpretations $\langle a, P, R\rangle$. For each interpretation aPR we define the function V^{aPR} from $\mathcal{L}\Box\Box$ into $\{\mathbf{t}, \mathbf{f}\}$ as follows:

Definition For every sentential constant A, $\mathrm{V}^{aPR}(A) = a(A)$; if $A = \sim B$, $\mathrm{V}^{aPR}(A) = N(\mathrm{V}^{aPR}(B))$; if $A = (B \supset D)$, then $\mathrm{V}^{aPR}(A) = \mathrm{C}(\mathrm{V}^{aPR}(B), \mathrm{V}^{aPR}(D))$ (where N is the truth function associated with negation and C that associated with material conditional); if $A = \Box B$, then $\mathrm{V}^{aPR}(A) = \mathbf{t}$ iff for all $b \in P$ such that aRb, $\mathrm{V}^{aPR}(B) = \mathbf{t}$.

The following is easily seen,

Theorem 1 *For all* S, S', A, B:

S0.0 *if* $S \vDash A$ *for all* $A \in S'$ *and* $S' \vDash B$, *then* $S \vDash B$;
S0.1 *if* $S \subseteq S'$ *and* $S \vDash A$, *then* $S' \vDash A$;
S.1 $S + A \vDash A$;
S.2 *if* $S \vDash A$ *and* $S \vDash (A \supset B)$, *then* $S \vDash B$;
S.3 *if* $S + A \vDash B$, *then* $S \vDash (A \supset B)$;
S.4 *if* $S + {\sim}A \vDash B$ *and* $S + {\sim}A \vDash {\sim}B$, *then* $S \vDash A$;
S.5 *if* $S \vDash \Box A$, *then* $S \vDash A$;
S.6 *if all members of* S *are of the form* $\Box A$ *and* $S \vDash B$, *then* $S \vDash \Box B$.

4.1 Strong Soundness of $\mathcal{L}1$: The strong soundness of $\mathcal{L}1$ is immediate from the following lemma whose proof using Theorem 1 parallels that of of Lemma 1.4 of [1] (p. 376).

Lemma 1 *For every proof* Π *in* $\Delta\Box\Box$, *if* A *occurs open in* Π, *then* $O(\Pi) \vDash A$.

Theorem 1 (*Strong Soundness of* $\mathcal{L}1$): $S \vdash A$ *in* $\Delta'\Box\Box$ *implies* $S \vDash A$ *in* $\Sigma 1$.

4.2 Non-compactness of $\Sigma 1$: In order to see that $\Sigma 1$ is not compact it is sufficient to notice that the set T (below) has no models despite the fact that all of its finite subsets have models. T is the union of the following four sets: T_1 is the set of sentential constants, T_2 is all sentences ${\sim}\Box B$ for B in T_1, T_3 is all sentences ${\sim}\Box B$ for B in T_2, and T_4 is all sentences $\Box(P_i \equiv P_{i+1})$ where the indices are given by some standard enumeration of the sentential constants. Thus, T is the union of all sets $\{P_i, {\sim}\Box P_i, {\sim}\Box{\sim}\Box P_i, \Box(P_i \equiv P_{i+1})\}$. It follows that $\mathcal{L}1$ is not strongly complete. In a sense one might say that $\mathcal{L}1$ is not compact because two assignments make the same sentences true iff they are identical.[2] If assignments are considered representations of "possible worlds," the above can be rephrased as follows: two worlds are different iff they can be internally distinguished.[3]

5 $\Sigma 2$: The interpretations of $\Sigma 2$ are just those triples $\langle S, K, R\rangle$ where K is a set of maximally consistent sets of sentences (in $\Delta'\Box\Box$), $S \in K$, and R is a relation on K such that for all T, T' in K, TRT' iff for all A, if $\Box A \in T$, $A \in T'$; and for all $T \in K$, if ${\sim}\Box B \in T$ there exists T' with TRT' and ${\sim}B \in T'$. Note that R is symmetric and transitive since $\Box A \vdash A$ and $\Box A \vdash \Box\Box A$ for every A.[4] As above, SKR indicates $\langle S, K, R\rangle$; and for each SKR we define $V^{SKR}(A)$ as follows: if A is a sentential constant $V^{SKR}(A) = \mathbf{t}$ iff $A \in S$; $V^{SKR}({\sim}B) = N(V^{SKR}(B))$; $V^{SKR}(A \supset B) = C(V^{SKR}(A), V^{SKR}(B))$; $V^{SK\overline{R}}(\Box B) = \mathbf{t}$ iff for all T with SRT, $V^{TKR}(B) = \mathbf{t}$.

Since $\mathcal{L}1$ is sound the set of sentences true on aPR is maximally consistent in $\Delta'\Box\Box$. Moreover, for each interpretation in $\Sigma 1$ there is an interpretation in $\Sigma 2$ making exactly the same set of sentences true. For aPR in $\Sigma 1$, let $\overline{aPR}$ be those sentences true on aPR. Let $S = \overline{aPR}$ and $K = \{\overline{bPR}: b \in P\}$. Define the relation $\overline{R}$ on K as follows where $\overline{bPR}\ \overline{R}\ \overline{b'PR}$ iff bRb'. It is trivial to verify that $SK\overline{R}$ is an interpretation in $\Sigma 2$ and for all A, $V^{aPR}(A) = V^{SK\overline{R}}(A)$.

5.1 The Strong Soundness of $\mathcal{L}2$:

Lemma 1 *For all SKR and A*, $V^{SKR}(A) = \mathbf{t}$ *iff* $A \in S$.

Proof: Let Z be the set of sentences which satisfy the lemma. Every sentential constant is in Z, as are all truth-functional combinations of sentences in Z. Let $A \in Z$ and let SKR be arbitrary. Suppose $\Box A \in S$; then by definition for all T such that SRT, $A \in T$. Therefore, by hypothesis, $V^{TKR}(A) = \mathbf{t}$, and $V^{SKR}(\Box A) = \mathbf{t}$. Suppose $\Box A \notin S$. Then $\sim\Box A \in S$ and there exists T such that SRT and $\sim A \in T$. Therefore, by hypothesis, $V^{SKR}(\Box A) = \mathbf{f}$.

Theorem 1 $\mathcal{L}2$ *is strongly sound.*

5.2 The Strong Completeness of $\mathcal{L}2$: Consider the following relation W on sets of sentences. SWS' iff $S' = (\Box(S) + \sim A)^+$ for some $A \in \overline{\Box}(S)$. Let S be a maximally consistent set of sentences. The *Makinson set for S*, $\mathsf{M}(S)$, is the smallest class of sets which contains S (as a member) and which is closed under W. Given S and $\mathsf{M}(S)$, we can easily form the interpretation $S\mathsf{M}(S)R$ where TRT' iff for all $\Box A$, if $\Box A \in T$, $A \in T'$. Note that for each S, $\mathsf{M}(S)$ is countable. We can easily establish the following:

Theorem 1 *Every consistent set of sentences in* $\Delta'\Box\Box$ *has a model in* $\Sigma 2$.

Proof: Let S be consistent and let S^+ be the maximally consistent extension of S. By Lemma 1 of **5.1** $S^+\mathsf{M}(S^+)R$ *is a model of* S.

Since results of this section imply that $\Sigma 2$ is compact it follows that $\Sigma 1$ and $\Sigma 2$ are not strongly equivalent.

6 $\Sigma 3$: For a truth value assignment, a, and a natural number n, let a_n denote the pair $\langle a, n\rangle$. Interpretations in $\Sigma 3$ are triples a_iPR where P is a subset of pairs, R is a reflexive and transitive relation on P and $a_i \in P$. For each a_iPR, we define V^{a_iPR} as follows: $V^{a_iPR}(A) = a(A)$ for A a sentential constant; $V^{a_iPR}(A \supset B) = C(V^{a_iPR}(A), V^{a_iPR}(B))$; $V^{a_iPR}(\sim A) = N(V^{a_iPR}(A))$; $V^{a_iPR}(\Box A) = \mathbf{t}$ iff for all b_j such that a_iRb_j, $V^{b_jPR}(A) = \mathbf{t}$. $\Sigma 3$ is essentially the semantics given by Kripke for **S4** in [3]. $\Sigma 3$ differs from $\Sigma 1$ only in distinguishing "assignments" which make exactly the same sentences true.

Let SKR be any interpretation in $\Sigma 2$, where K is countable. For each $T \in K$, $T \cap L$ is maximally consistent in Δ; therefore $T \cap L$ has a model a. Let $T_0 = S$ and $\{a_i\}$ be an enumeration of the models of $\{T_i \cap L\}$. Also let $P(K) = \{a_0, a_1, \ldots\}$ and take R^* to be the relation on $P(K)$ such that $a_iR^*a_j$ iff T_iRT_j. $a_0P(K)R^*$ is obviously an interpretation in $\Sigma 3$. More interestingly, we have the following:

Lemma 1 *For all SKR in* $\Sigma 2\,\Box$ *and all A*, $V^{SKR}(A) = V^{a_iP(K)R^*}(A)$.

Theorem 2 $\mathcal{L}3$ *is strongly complete.*

Proof: This follows from the fact that for all S, A, if $S \models A$ in $\Sigma 3$, $S \models A$ in $\Sigma 2$ and $\Sigma 2$ is strongly complete.

Thus $\Sigma 3$ is strongly equivalent to $\Sigma 2$ but not strongly equivalent to $\Sigma 1$.

6.1 The Weak Equivalence of $\Sigma 1$ and $\Sigma 3$: Let C' be any subset of C, where all but a countable number of members of C are outside C'; $\mathcal{L}C'\Box\Box$ indicates those sentences of $\mathcal{L}\Box\Box$ whose constants are in C'.

Lemma 1 *For all a_iPR in $\Sigma 3$, there exists $a_iP'R'$ in $\Sigma 3$ such that P' is a countable subset of P, $R' = R|P'$, and for all A in $\mathcal{L}\Box\Box$, $V^{a_iP'R'}(A) = V^{a_iPR}(A)$.*

Proof: Let a_iPR be in $\Sigma 3$. For each $a_j \in P$, we define the following:

(i) $\bar{\Box}(a_jPR) = \{A : V^{a_jPR}(\Box A) = \mathbf{f}\}$;
(ii) for all $A \in \bar{\Box}(a_jPR)$, $A(a_j) = \{a_k : a_jRa_k \text{ and } V^{a_kPR}(A) = \mathbf{f}\}$;
(iii) $T(a_j) = \{A(a_j): A \in \bar{\Box}(a_jPR)\}$;
(iv) Ca_j the choice function for $T(a_j)$ where $Ca_j(A(a_j)) = \overline{a_j}$;
(v) $\Delta_0(a_j) = \{Ca_j(A(a_j)): A(a_j) \in T(a_j)\}$
$\Delta_m(a_j) = \Delta_{m-1}(a_j) \cup \{\Delta_0(a_k): a_k \in \Delta_{m-1}(a_i)\}$;
(vi) $P' = \bigcup_m \Delta_m(a_i)$;
(vii) $R' = R|P'$.

Note: For all $a_j \in P$ and all m that $\Delta_m(a_j) \subseteq \Delta_{m+1}(a_j)$ and $\Delta_n(a_j)$ is countable, hence P' is countable.

A simple induction argument will show that for every $a_j \in P'$ and $A \in \mathcal{L}\Box\Box$, $V^{a_jPR}(A) = V^{a_jP'R'}(A)$. We need only consider the case where $V^{a_jPR}(\Box A) = \mathbf{f}$. Suppose that for all $a_k \in P'$, $V^{a_kPR}(A) = V^{a_kP'R'}(A)$ and that $V^{a_jPR}(\Box A) = \mathbf{f}$. Therefore, $A \in \bar{\Box}(a_jPR)$, $A(a_j) \in T(a_j)$ and $Ca_j(A(a_j)) = \bar{a}_j \in A(a_j)$. Hence $a_jR\bar{a}_j$ and $V^{\bar{a}_jPR}(A) = \mathbf{f}$. We need only show that $\bar{a}_j$ is in P'. Since $a_j \in P'$, there exists m with $a_j \in \Delta_m(a_i)$ and $\bar{a}_j \in \Delta_0(a_j)$. Obviously, $\bar{a}_j \in \Delta_{m+1}(a_i) = \Delta_m(a_i) \cup \bigcup\{\Delta_0(a_k): a_k \in \Delta_m(a_i)\}$, hence $\bar{a}_j \in P'$.

Lemma 2 *For all a_iPR in $\Sigma 3$ and all C' as above, there exists $aP'R'$ in $\Sigma 1$ such that for all A in $\mathcal{L}C'\Box\Box V^{a_iPR}(A) = V^{aP'R'}(A)$.*

Proof: Let a_iPR be in $\Sigma 3$. By Lemma 1 we may assume that P is countable. Let $\{A_j\}$ be an enumeration of $C - C'$ and let $\{(a_n)_j\}$ be an enumeration of P. Let f be defined on P so that for all j, $f(a_n)j$ is a truth-value assignment agreeing with $(a_n)_j$ except on A_j. Form $aP'R'$ where a, P' and R' are images under f, respectively, of a_i, P, and R.

Theorem 1 *$\mathcal{L}3$ is strongly sound.*

Proof: Let $S \vdash A$. Then there exists $S' \subseteq S$, S' finite, such that $S' \vdash A$. Let a_iPR be any model of S' in $\Sigma 3$. Since $S' + A$ is finite there are countably many constants not occurring in sentences of $S' + A$. By Lemma 2 there is $a'P'R'$ in $\Sigma 1$ such that it agrees with a_iPR on $S' + A$; and, since $\mathcal{L}1$ is sound, $a'P'R'$ is a model of A and $S \vDash A$ in $\Sigma 3$.

As immediate consequences of Theorem 1 we obtain that $\Sigma 2$ is strongly equivalent to $\Sigma 3$; that $\Sigma 1$ is weakly equivalent to $\Sigma 2$ and hence that $\mathcal{L}1$ is weakly complete.

7 Remarks The strong completeness of the Kripke formulation of S4 depends essentially on treating ordinary truth-value assignments which are identical as being distinct. However, as Kripke points out [5] (p. 69), this restriction is inessential for weak completeness.

The system M can be given a semantics where the interpretations are triples aPR where P is a set of assignments, $a \in P$ and R is a relation which is reflexive. We can easily show that this semantics is not compact, by proving the analogues of Lemmas 1 and 2 of **4.2** for M.

Moreover, the Brouwerische system, given the semantics whose interpretations are triples aPR, P as above, $a \in P$ and R a reflexive and symmetric relation on P, can also be shown to be non-compact. Let T be the union of the following infinite sets:

$$\{P_1, \ldots, P_n, \ldots\}$$
$$\{\Box P_1, \ldots, \Box P_n, \ldots\}$$
$$\{\sim\Box\Box P_1, \ldots, \sim\Box\Box P_n, \ldots\}$$

T has no model although each of its finite subsets does.

NOTES

1. The rule $\Box$I′ is a "rigorous" rule of inference ([2], pp. 171-175) in that it (1) is effective; (2) is sound; (3) introduces or eliminates just one logical constant (and not both); and (4) involves in its application just one logical constant (type). But while ($\Box$I) satisfies the first three conditions, it fails to satisfy the fourth. Thus ($\Box$I) gives a counterexample to the conjecture suggested ([2], p. 172) that any rule satisfying condition (3) would also satisfy condition (4). In connection with the notion of "rigor" introduced in [2] we may note that John Myhill has suggested a rule which satisfies all four conditions but which is clearly not rigorous in any (correct) informal sense.

   Myhill's Rule: Let Π and $\Pi[(A \supset B)\sigma A$ be proofs where the indicated occurrence of $(A \supset B)$ is the right-most open assumption in the latter. Then $\Pi[(A \supset B)\sigma A]\, A$ is a proof.

   Myhill's Rule permits the following proof of "Peirce's Law"

   $$[((A \supset B) \supset A)\,[(A \supset B)A]\,A](((A \supset B) \supset A) \supset A)$$

2. Thus $\Sigma 1$ cannot be thought of as an abstraction gotten from a first order modal logic (without quantification into modal contexts) by treating quantified sentences as "unanalysed." The reason is that in such cases one will have (elementarily) equivalent interpretations (of the non-modal language) which are not identical.

3. It should be noted that $\Sigma 1$ violates the tentative guidelines recently suggested by Dana Scott in his speculative article on modal logic ([8], esp. p. 149). Scott suggests that one *should* work in a framework rich enough to "distinguish" identical worlds. In this situation Scott's advice could be realized by postulating a large (at least uncountable) index set I for the set of assignments. A "general interpretation," let us say, would again be a triple $\langle a_i, P, R\rangle$ but here P would be a subset of I and R would be a transitive, reflexive relation on P. The definition of truth here is obvious. $\Sigma 1$ would then be strongly equivalent, not to the general semantics including all general interpretations, but rather to the special semantics which included exactly the interpretations a_iPR where the indexing is one-one restricted to P. In light of construction of the obvious models for the finite subsets of T it is a triviality to produce a "general model" for T.

4. Our remark in section **1** (above) avowing a desire to deal with "conceptually meaningful" semantic systems should not be taken to imply that we regard $\Sigma 2$ as such a system. Indeed, the traditional principle which bans use of semantic notions in the definition of a deductive system may profitably be amended to include a ban on use of deductive notions in defining semantic systems. Moreover (see below), unless soundness (or at least consistency) of $\Delta'\Box\Box$ had already been established we would have little reason for believing that the set of interpretations from $\Sigma 2$ is *not null.*

REFERENCES

[1] Corcoran, J., and G. Weaver, "Logical consequence in modal logic: Natural deduction in S5," *Notre Dame Journal of Formal Logic*, vol. X (1969), pp. 370-384.

[2] Corcoran, J., "Three logical theories," *Philosophy of Science*, vol. 36 (1969), pp. 153-177.

[3] Curry, H. B., *A Theory of Formal Deducibility*, Notre Dame (1957).

[4] Kripke, S., "A completeness theorem in modal logic," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 1-14.

[5] Kripke, S., "Semantical analysis of modal logic I," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 9 (1963), pp. 67-96.

[6] Makinson, Donald, "On some completeness theorems in modal logic," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 12 (1966), pp. 379-384.

[7] Prawitz, Dag, *Natural Deduction: A Proof-Theoretical Study*, Stockholm (1965).

[8] Scott, Dana, "Advice on modal logic," *Philosophical Problems in Logic*, ed. by K. Lambert, Dordrecht (1970).

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