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Logical extensions of Aristotle's square

Dominique Luzeaux, Jean Sallantin and Christopher Dartnell

Abstract. We start from the geometrical-logical extension of Aristotle's square in [Bla66], [Pel06] and [Mor04], and study them from both syntactic and semantic points of view. Recall that Aristotle's square under its modal form has the following four vertices: A is $\Box\alpha$, E is $\Box\neg\alpha$, I is $\neg\Box\neg\alpha$ and O is $\neg\Box\alpha$, where α is a logical formula and $\Box$ is a modality which can be defined axiomatically within a particular logic known as $S5$ (classical or intuitionistic, depending on whether $\neg$ is involutive or not) modal logic.

[Béz03] has proposed extensions which can be interpreted respectively within paraconsistent and paracomplete logical frameworks. [Pel06] has shown that these extensions are subfigures of a tetraicosahedron whose vertices are actually obtained by closure of $\{\alpha, \Box\alpha\}$ by the logical operations $\{\neg, \wedge, \vee\}$, under the assumption of classical $S5$ modal logic. We pursue these researches on the geometrical-logical extensions of Aristotle's square: first we list all modal squares of opposition. We show that if the vertices of that geometrical figure are logical formulae and if the sub-alternation edges are interpreted as logical implication relations, then the underlying logic is none other than classical logic. Then we consider a higher-order extension introduced by [Mor04], and we show that the same tetraicosahedron plays a key role when additional modal operators are introduced. Finally we discuss the relation between the logic underlying these extensions and the resulting geometrical-logical figures.

Mathematics Subject Classification (2000). Primary 03B45; Secondary 06F99.

Keywords. Aristotle's square, modal logic.

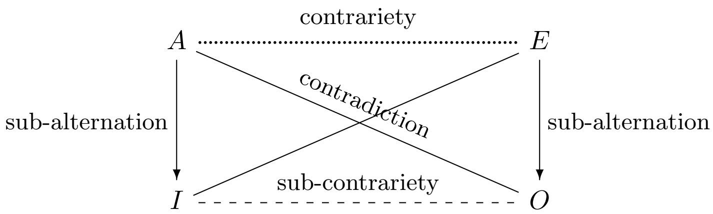
1. The traditional square of opposition and its lower-order generalizations

1.1. Aristotle's square

The doctrine of the square of opposition originated with Aristotle in the fourth century B.C. and has occurred in logic texts ever since. It relates various quantified propositions and their negation by introducing various notions of oppositions: *contradiction*, *contrariety* and *sub-contrariety*. *Contradiction* for two terms is defined as the impossibility for them to be both true and both false. *Contrariety* for two terms is the impossibility for them to be both true, but the possibility to be both false. *Sub-contrariety* is the impossibility to be both false, but the possibility to be both true. A last useful notion is *sub-alternation* between two terms, also better known as logical implication: it is defined as the impossibility of having the first term true without having also the second true. It will be denoted by an arrow, which is consistent with the logical interpretation of implication.

We should notice that although contrariety and sub-contrariety are discussed extensively in syllogistic, they are derived logical concepts when contradiction and sub-alternation are logically defined.

The square of oppositions synthesizes these notions within the following geometrical figure:



The column with A and I corresponds to affirmative propositions, while the column with E and O corresponds to negative propositions. The line with A and E corresponds to universal propositions, while the line with I and O corresponds to existential (also called particular) propositions. Medieval syllogistic has developed a bunch of rules based on the extensive use of this square, that encode true inferences between premisses and conclusions. However we will not need them, as a standard interpretation of truth with the aforementioned notions will suffice.

This square leads however to logical ambiguities, and in the last century it has been heavily criticized with the development of formal logic. Indeed Aristotle's theory of syllogism and all the Medieval syllogistic corpus developed until the 19th century should not be seen as a formal theory of logic, but as a theory of reasoning (based on the modus ponens and modus tollens rules, which form the basis of propositional calculus), that allows to infer the truth of certain propositions. Furthermore the process which supplies evidence for the validity, or for the invalidity, of certain inferences and conversions (of a proposition into its negative) is based on simple graphics. Therefore the square of opposition appears as a geometrization of the inference process.

Several extensions have been proposed in order to palliate the logical drawbacks and develop the inference capabilities of the traditional Aristotelian square. [Bla66] decorates the vertices with modalities [Gol05], and develops a theory of reasoning based on the square of modalities: A is $\Box\alpha$, E is $\Box\neg\alpha$, I is $\neg\Box\neg\alpha$ and O is $\neg\Box\alpha$, where α is a (classical logical) proposition. The modality $\Box$ is a universal modality, usually interpreted for instance as "necessarily" or "it is always known" or "it is proved" or "it is compulsory", depending on the modal logic used: necessity, epistemic, provability, deontic. The modality $\neg\Box\neg$ is usually denoted by $\Diamond$

and is interpreted in the former cases by "possibly", "it is sometimes known", "it is not refuted", "it is allowed"; $\neg$ stands for usual classical negation. In addition, [Bla66] introduces two additional vertices $I \wedge O$ and $A \vee E$, yielding a contrariety triangle A, E, $I \wedge O$ and a sub-contrariety triangle $A \vee E$, O, I. Furthermore there are obvious sub-alternation relations between A and $A \vee E$, E and $A \vee E$, $I \wedge O$ and I, $I \wedge O$ and O. Thus the square turns into a hexagon, as illustrated on the left part of figure 1.

Working on the geometrical aspects of this hexagon and its various modal decorations, [Béz03] introduces other hexagons: a paraconsistent one (in a paraconsistent logic there exists a proposition which is true and the negation of which is true, without implying triviality of the theory, i.e. the truth of any proposition), and a paracomplete one (in a paracomplete logic there exists a proposition which is false and the negation of which is false, without implying triviality of the theory). Both these hexagons are illustrated in figure 1.

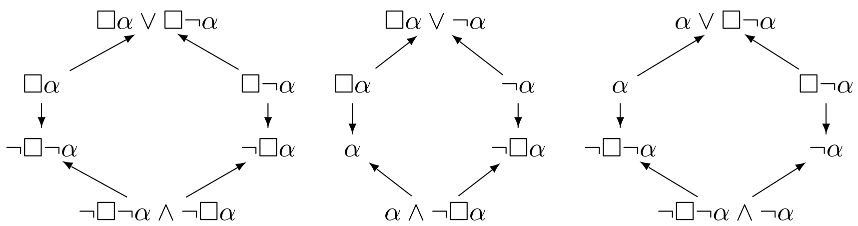


FIGURE 1. The three hexagons from left to right: classical, paraconsistent, paracomplete.

The interesting observation in these hexagons is the underlying square in each, and more specifically the column of negative terms EO. Looking at the three hexagons, we have thus the more general sub-alternation relation between these various negative terms: $\Box\neg\alpha \to \neg\alpha \to \neg\Box\alpha$. This is no surprise, since these terms are known as expressing various kinds of negation in classical and modal proposition logics with the corresponding weakening relations: [Béz05, Doš84a, Doš84b] show that $\Box\neg$ is an intuitionistic paracomplete negation, and [Béz02] shows that $\neg\Box$ is a paraconsistent negation.

It is worth noticing that the paraconsistent hexagon is obtained from the paracomplete hexagon, and conversely, whenever α is changed into $\neg\alpha$.

Questions that arise at that point are: are there other modal decorations on hexagons? are there higher-order geometrical figures beyond squares and hexagons, that can model other possibly richer opposition theories?

1.2. n-opposition theory

Answers to the previous questions have been provided by [Mor04] and [Pel06]. The theory of n-opposition has been developed by A. Moretti [Mor04] in order to generalize the completions of Aristotle's square into the hexagons discovered by Blanché and Béziau. Each of these hexagons is a 2-bisimplex built from a 2-simplex (a triangle) of contrariety (two by two conjunctions of its vertices are false) and its symmetric by central symmetry, the sub-contrariety triangle (two by two disjunctions of its vertices are true).

In order to generalize the ternary contrariety relation to an n-ary opposition relation, Moretti has developed the theory of n-opposition relying on n-oppositional figures ($(n-1)$-bisimplices), and has created the notion of modal $n(m)$-graphs. These graphs are a convenient tool to generate n-oppositional figures, as shown in [Mor04]. Furthermore all n-oppositional figures generated by a given modal graph can be listed combinatorially, and the geometrical solid which includes all these n-oppositional figures is called a β-structure in Moretti's terminology. The formal definitions follow.

Definition 1.1. An n-opposition consists of an $(n-1)$-bisimplex formed by an $(n-1)$-simplex, called contrariety simplex, and by its symmetric by central symmetry, called sub-contrariety simplex, each vertex of the first simplex being connected by sub-alternation to each non-symmetric vertex of the second simplex. The symmetric vertices are contradictory. The contrariety simplex has the property that two by two conjunctions of its vertices are false. Dually, the sub-contrariety simplex has the property that two by two disjunctions of its vertices are true.

In order to decorate (i.e. put the right formulas on each vertex of the bi-simplex) modally such simplices, modal $n(m)$-graphs are introduced.

Definition 1.2. A modal $n(m)$-graph is the cartesian product in the category of graphs of an $(n-2)$ simplex by the oriented graph $1 \to 2 \to \ldots \to m$, i.e. m copies of the simplex, such that for all $j \leq n-1$ and $i \leq m$:

- the j-th vertex of the i-th copy is contradictory to the disjunction of all vertices of the $m-i+1$-th copy but the j-th;
- there is a sub-alternation relation between the j-th vertex of the i-th copy and the j-th vertex of the $m-i+1$-th copy for $i \leq m-i+1$.

The resulting graph is called modal since all vertices are labelled by modal operators, and the oriented edges correspond to logical implication.

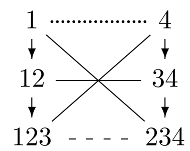
Definition 1.3. A β-structure associated to an $n(m)$-graph is the geometrical solid consisting of all p-oppositions exhibited by the $n(m)$-graph.

The second reference [Pel06] continues the previous work, by introducing a particular encoding. The modal $n(m)$-graphs – seen actually as "directed" $n(m)$-graphs, since the actual modal essence of the labels is not exploited – are translated into set theory, by identifying the vertices of the graph with subsets of a given set, consistently with the underlying lattice structure: an implication (resp. upper bound, resp. lower bound) between two vertices turns into an inclusion (resp. union, resp. intersection) of the corresponding subsets. With help of this translation, [Pel06] shows that an $n(m)$-graph is translated into the Boolean lattice corresponding to a set of cardinality $c_{n,m}$ equal to: $(n-1)\frac{m}{2}+1$ if m is even, and $(n-1)(\frac{m+1}{2})$ if m is odd.

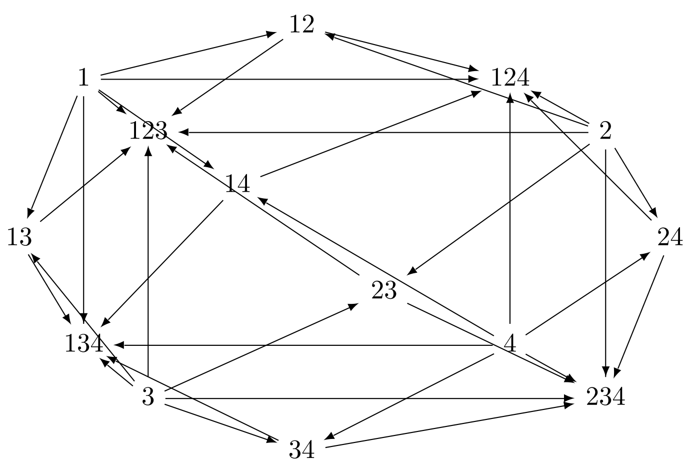
An immediate corollary [Pel06] is that an $n(m)$-graph and a $3(c_{n,m}-1)$ graph are translated into the same Boolean lattice corresponding to a set with $c_{n,m}$ elements. Thus the general problem can be reduced, up to this translation, to the study of all modal $3(m)$-graphs.

In particular, as illustrated in [Pel06], the study of the modal 3(3)-graph exhibits 3-bisimplices, which can be gathered into a geometrical figure, a β-structure in Moretti's terminology, known as a tetraicosahedron. Let us recall this, using Pellissier's encoding for sake of simplicity.

Recalling [Pel06], a 3(3)-graph can be translated into a Boolean lattice constructed on a set of cardinality $c_{3,3}=4$. Therefore one starts from the following modal 3(3) graph, with similar graphical representations as before (dots for contrariety, dashes for subcontrariety, lines for contradiction, arrows for sub-alternation):



and obtains a geometrical figure, known as a tetraicosahedron, composed of a cube with a square-based pyramid on each face of the cube. This directed tetraicosahedron, illustrated in figure 2, represents all sub-alternation relations that exist within the Boolean lattice generated by a 4-element set under the assumption of the previous 3(3)-graph.



(1.1)

FIGURE 2. The "Boolean" tetraicosahedron.

Actually, for sake of completeness, this figure should have two more vertices, labelled $\emptyset$ and 1234, with arrows from $\emptyset$ to all vertices, and from all vertices to 1234; but since their interpretation, as developed in the next sections, is trivial most of the time, we will not consider them systematically. When we do consider them, we use the denomination “completed” tetraicosahedron.

This tetraicosahedron with 14 vertices and 24 faces contains four “weak” and six “strong” hexagons as well as a “strong” cube, that form all possible geometrical figures within 3-opposition theory. The denomination “strong”, introduced in [Pel06], characterizes those figures where the contrariety simplex is built from terms that are not only contrary two by two (their conjunction two by two is false), but their global disjunction is true. The denomination “weak” corresponds to situations where the disjunction of all terms within the contrariety simplex is not necessarily true. Actually one of the originalities of [Pel06] is this distinction, as all previous references had always assumed “strong” constructions. Back to our tetraicosahedron, all geometrical figures are thus (hexagons are described by starting from the left uppermost vertex and turning clockwise):

- weak hexagons {1, 124, 2, 234, 3, 134}, {1, 123, 2, 234, 4, 134}, {1, 124, 4, 234, 3, 123}, {124, 2, 123, 3, 134, 4};
- strong hexagons {12, 2, 234, 34, 134, 1}, {12, 124, 4, 34, 3, 123}, {1, 124, 24, 234, 3, 13}, {123, 2, 24, 4, 134, 13}, {1, 123, 23, 234, 4, 14}, {124, 2, 23, 3, 134, 14};
- a strong cube whose faces are {1, 124, 2, 123} and {134, 4, 234, 3}.

2. Logical interpretations within classical $S5$ modal logic

2.1. The logical tetraicosahedron and the logical cube

Let us turn now to the logical interpretations of the previous notions. From now on, we will assume some kind of a logical framework is given (for instance classical propositional logic). The modal $n(m)$-graphs provide additional logical constraints on this given logic. The vertices of the n-oppositional figure are then obtained by the closure, under conjunction, disjunction and contradiction (negation), of the algebra generated by the $n(m)$-graph, and the oriented edges are the implications induced by the $n(m)$-graph between the corresponding vertices.

Such a logical interpretation of the n-oppositional figures motivated Béziau’s, Moretti’s and Pellisier’s pioneering work, following Blanché’s initial research, and in the rest of this paper we will delve deeper into this connection between logical and geometrical aspects of n-opposition theory.

The first logic we will consider is classical propositional modal logic. Recall that the latter is not uniquely defined. However among all the diversity, there are a few commonly used systems, following Lewis’s seminal work in the 20th century.

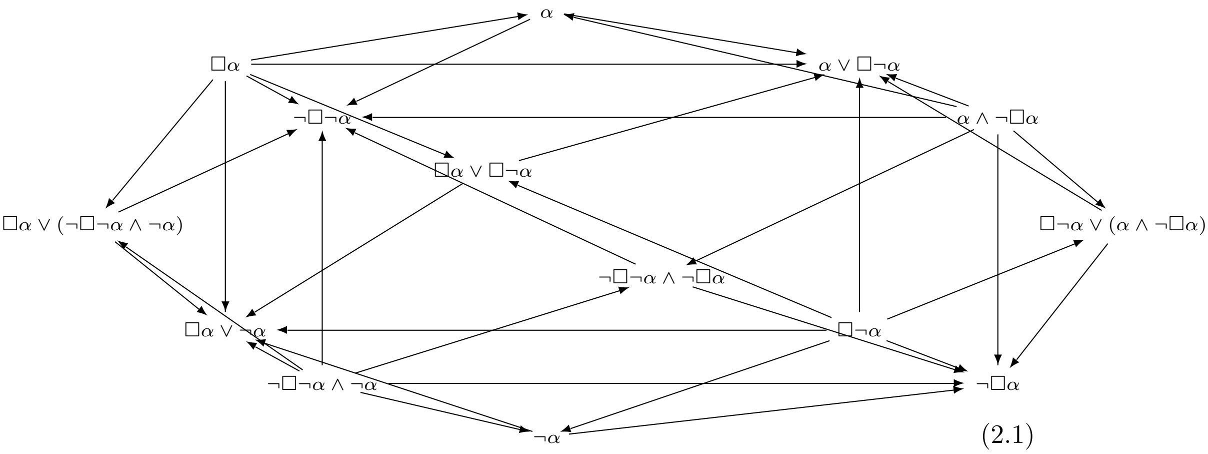
One starts from classical propositional logic (with the usual connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, and its usual theorems, such as de Morgan’s laws, contraposition and involutivity of negation). In this system there are only two modalities ($\cdot$ the identity and $\neg$ its negation) which yields in the classical interpretation: something is or is not. Then one adds a new modal operator $\Box$ (usually interpreted as necessity); two axiom schemata (α is any formula of classical propositional logic): $\Box\alpha \to \alpha$, known as axiom T, and $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$, known as axiom K; and the inference rule of necessitation: from any theorem α infer $\Box\alpha$. As a shortcut, the string of symbols $\neg\Box\neg$ is written $\Diamond$. This yields a system known as KT, that has in fact an infinity of non equivalent affirmative and negative modalities (affirmative modalities are finite iterates of $\Box$ and $\neg$ with an even number of $\neg$, whereas negative modalities have an odd number of $\neg$). If we add to KT the axiom $\Box\alpha \to \Box\Box\alpha$ (also known as 4, or reflexivity of necessity, or positive introspection: if I know something then I know that I know it), the system obtained is called classical $S4$ modal logic. If one adds to $S4$ (or equivalently to KT as is shown in any textbook on modal logic) the axiom $\Diamond\alpha \to \Box\Diamond\alpha$, i.e. $\neg\Box\neg\alpha \to \Box\neg\Box\neg\alpha$ (also known as 5, or negative introspection: if I do not know something then I know that I do not know it), the system obtained is called classical $S5$ modal logic. It is easy to see that the system $S4$ has actually 14 distinct irreducible modalities ($\cdot$, $\Box$, $\Diamond$, $\Diamond\Box$, $\Box\Diamond$, $\Box\Diamond\Box$, $\Diamond\Box\Diamond$, and their negations) while the system $S5$ has even fewer distinct irreducible modalities ($\cdot$, $\Box$, $\Diamond$, and their negations).

As the previous lines show, $S5$ is a rather strong system, due to its numerous additional axioms. It is largely used, as it is the "simplest" modal extension of classical logic, in the sense that any finite combination of $\neg$, $\Box$, $\Diamond$ in front of a formula can be reduced to a combination of at most one $\neg$ and one $\Box$. Furthermore, returning to our initial motivation of dealing with Aristotle's square, if one looks carefully at the modal interpretations of that square that led to medieval syllogistic, it appears that $S5$ was actually the logic used: although the left edge of the basic Aristotelian square ($\Box\alpha \to \Diamond\alpha$) needs only a very weak modal logic (known as the D system) which is of interest to those exploring the deontic interpretations of the modal operators, there is a need for stronger logics to address the usual modal perplexities by making all modal truths necessary (axioms 4 and 5), and $S5$ is the commonly employed system for this. Indeed, it is known as the system to capture the notion of Leibnizian possible worlds.

Back to our main thread, let us take the following modal 3(3)-graph and interpret it within classical $S5$ modal logic:

$$\begin{tikzcd}
\Box\alpha \arrow[r, dotted, no head] \arrow[d] \arrow[ddr, no head] & \Box\neg\alpha \arrow[d] \arrow[ddl, no head] \\
\alpha \arrow[r, no head] \arrow[d] & \neg\alpha \arrow[d] \\
\neg\Box\neg\alpha \arrow[r, dashed, no head] & \neg\Box\alpha
\end{tikzcd}$$

This yields the tetraicosahedron illustrated in figure 3 (in (1.1) replace 1 by $\Box\alpha$, 2 by $\alpha \wedge \neg\Box\alpha$, 3 by $\neg\Box\neg\alpha \wedge \neg\alpha$ and 4 by $\Box\neg\alpha$, and any combination of digits by the disjunction of the corresponding replacements).

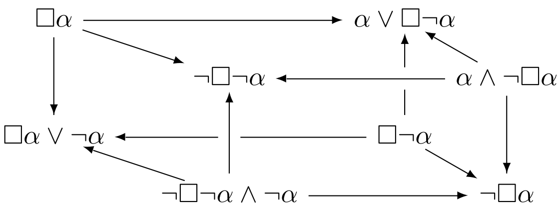


(2.1)

FIGURE 3. The $S5$ modal logical tetraicosahedron.

Since a sub-alternation relationship is the same as logical implication, it is easy to see that the vertices of the tetraicosahedron (2.1) are none else than all possible formulas (except $\top$ and $\perp$ which are the two additional vertices of the "completed" tetraicosahedron) one can build from arbitrary recursive application of $\wedge$, $\vee$, $\neg$, $\Box$ on α, using classical $S5$ modal logic axioms.

The various "strong" and "weak" subfigures exhibited before can be deduced easily. Among them, the strong logical cube, illustrated in figure 4, is a logical bi-simplex of dimension 3, built from two distinct tetrahedra: one of contrariety (any two of its vertices are contrary), one of sub-contrariety (any two of its vertices are sub-contrary). The vertices of the tetrahedron of contrariety are those from which the sub-alternation arrows start. Furthermore any vertex of the cube is contradictory to the furthest lying opposite vertex (i.e. obtained by central symmetry when the cube is considered as a geometrical figure).



(2.2)

FIGURE 4. The $S5$ modal logical cube.

Thus the logical cube is a three-dimensional generalization of the square of oppositions (where the contrariety simplex is a segment) and of the paracomplete and paraconsistent hexagons (where the contrariety simplex is a triangle). As such it can be used as a model of formal reasoning [ADL+07].

2.2. Aristotelian-like modal squares

Due to its construction, the logical cube contains six squares of oppositions, and features several operators of contingency ($\alpha \wedge \neg\Box\alpha$ and $\neg\Box\neg\alpha \wedge \neg\alpha$) which refine the case of pure contingency $\neg\Box\alpha \wedge \neg\Box\neg\alpha$ of Aristotle's modal square, introduced by [Bla66]. These operators are those introduced in the paracomplete and paraconsistent hexagons discussed previously. Figure 5 lists these 6 squares of opposition.

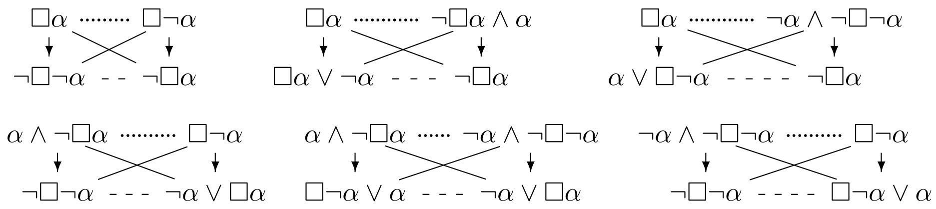


FIGURE 5. The 6 squares of opposition of the $S5$ modal logical cube.

In addition to these six squares which exhaust the $S5$ modal logical cube, there are twelve additional squares that are included in the $S5$ modal logical tetraicosahedron. They are listed in figure 6.

Together these eighteen squares exhaust all possible squares of opposition that can be written within classical $S5$ modal logic.

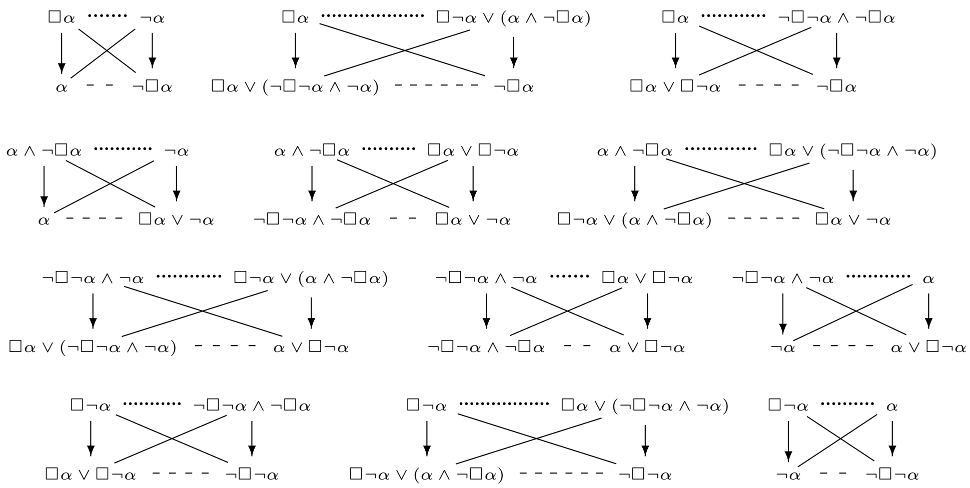


FIGURE 6. The 12 squares of opposition included in the $S5$ modal logical tetraicosahedron that are not in the $S5$ modal logical cube.

2.3. Towards semantic interpretations of the $S5$ modal logical cube

Following Aristotelian or Medieval tradition, we could use the $S5$ modal logical cube (2.2) as a model of reasoning within classical $S5$ modal logic. For instance, in [ADL+07], we have used it to supervise the entire process of theory formation while studying a phenomenon, as done by chemists studying properties of a new molecule such as absorption or toxicity.

Let us propose therefore interpretations in natural language of the various vertices. These interpretations are inspired from those usual within temporal, epistemic or doxastic logics:

- $\Box\alpha$: the fact described by formula α is proven;
- $\alpha \vee \Box\neg\alpha$: the fact is either observed or refuted; if the logic were classical, this formula would be rewritten as $\neg\alpha$ implies $\Box\neg\alpha$, which is interpreted as an excessive non-assertion, or hyperbolic doubt as exposed by Descartes;
- $\neg\Box\neg\alpha$: the fact is not refuted ; it is possible; given as an advice;
- $\alpha \wedge \neg\Box\alpha$: the fact is observed and not proven; it is conjectured;
- $\Box\alpha \vee \neg\alpha$: the fact is either proven or its negation has been observed; if the logic were classical, this formula would be rewritten as α implies $\Box\alpha$, which is interpreted as an excessive assertion;
- $\Box\neg\alpha$: the fact is refuted;
- $\neg\Box\neg\alpha\wedge\neg\alpha$: the fact is not refuted and not observed; it is taken as a postulate;
- $\neg\Box\alpha$: the fact is not proven; its negation is possible.

3. How modal is the logical tetraicosahedron?

3.1. Syntactic translation

The logical tetraicosahedron (1.1) is decorated by modalities from classical $S5$ modal logic. However we have the following result.

Proposition 3.1. *The $S5$ modal logical tetraicosahedron (1.1) can be translated into the logical tetraicosahedron built on all formulas generated by two propositional variables and admissible within classical propositional logic.*

Proof. Let us take $p = \alpha$ and $q = \Box\alpha \vee \Box\neg\alpha$. The tetraicosahedron (2.1) can be rewritten as the tetraicosahedron illustrated in figure 7.

One recognizes here the 14 non-trivial (i.e. all except the always true tautology and the always false contradiction, which correspond to both additional vertices mentioned in section 2.1) truth-value combinations one can build within classical logic with two propositions, which are respectively known as: assertion (respectively of p and of q), negation (respectively of p and of q), disjunction ($p \vee q$), direct implication ($p \supset q$, equal to $\neg p \vee q$), converse implication ($q \supset p$), direct non-implication ($\neg(p \supset q)$), converse non-implication ($\neg(q \supset p)$), equivalence ($p \equiv q$, equal to $(p \wedge q) \vee \neg(p \vee q)$), alternative ($p \vee\!\vee q$, equal to $(p \vee q) \wedge \neg(p \wedge q)$), incompatibility ($\neg p \vee \neg q$), rejection ($\neg p \wedge \neg q$). $\Box$

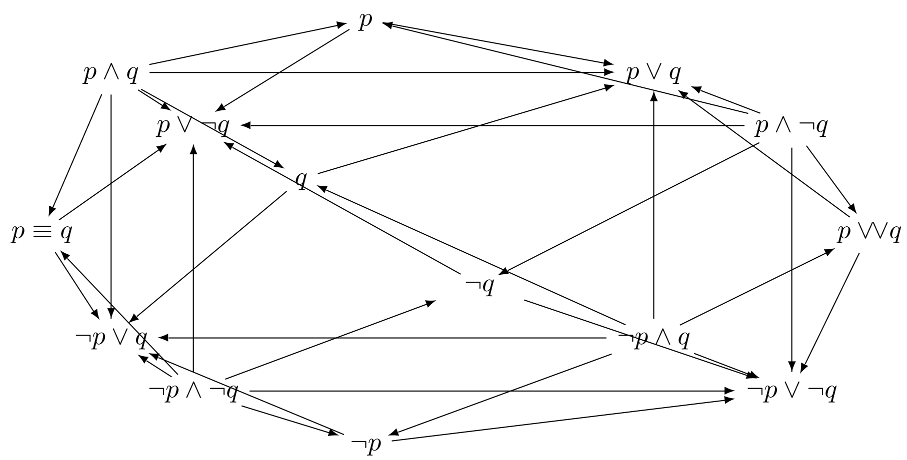


FIGURE 7. The classical logical tetraicosahedron.

Since this translation maps the $S5$ modal logical tetraicosahedron into the full classical tetraicosahedron that can be obtained from classical logic, it means that the $S5$ modal logical tetraicosahedron takes its modal flavor only because it is modally decorated, but not from its inherent logic.

This was already noticed in [Pel06], but we have exhibited here another translation which is interesting from a historical point of view, since Blanché has introduced part of the classical logical tetraicosahedron in [Bla57] and parts of the $S5$ modal logical tetraicosahedron in [Bla66] but did not connect them as we have just done.

3.2. Semantic translation

Whereas the previous translation was a mere rewriting of the vertices, we introduce now another translation based on a possible-world semantic interpretation of classical $S5$ modal logic, which interprets that logic within a multivalued logic [Res69, DDT78, Tha88].

This will yield another translation of the $S5$ modal logical tetraicosahedron into the classical logical tetraicosahedron built on the Boolean lattice obtained from the set $\{1, 2, 3, 4\}$, i.e. the "Boolean" tetraicosahedron (1.1).

Proposition 3.2. *The embedding of classical $S5$ modal logic within a four-valued logic yields a translation of the $S5$ modal logical tetraicosahedron (2.1) into the "Boolean" tetraicosahedron (1.1).*

Proof. The proof is based on the interpretation of classical $S5$ modal logic within a multivalued logic, obtained by considering a possible-world semantic with 2 worlds: the "current" world X and a "possible" world Y.

Let V be a 4-valued valuation, i.e. a mapping from the set of formulas into $\{v_1, v_2, v_3, v_4\}$. To any proposition, a logical value among these four values can be attributed depending on its truth or falsity in X and Y: v_1 if it is true in X and Y (meaning necessarily true), v_2 if it is true in X and false in Y (meaning currently true but not necessarily true), v_3 if it is false in X and true in Y (meaning currently false but not necessarily false), v_4 if it is false in X and Y (meaning necessarily false).

Let $\mathcal{D}$ be the set of designated values $\{v_1, v_2\}$ and $\mathcal{U}$ be the set of undesignated values $\{v_3, v_4\}$. Remember that in a multivalued logic designated values are the ones which "count as true" when one is not interested in the fine structure of the truth set, and the undesignated values "count as false".

With this interpretation, v_1 and v_2 count as true, v_3 and v_4 as false, v_1 and v_4 as non contingent, v_2 and v_3 as contingent, v_1 as necessarily true, v_4 as necessarily false, v_1, v_2 and v_3 as possibly true (not necessarily false), v_2, v_3 and v_4 as possibly false (not necessarily true).

This yields the following truth table for a formula α:

α	$\neg\alpha$	$\Box\alpha$	$\Box\neg\alpha$	$\neg\Box\alpha$	$\neg\Box\neg\alpha$
v_1	v_4	v_1	v_4	v_4	v_1
v_2	v_3	v_4	v_4	v_1	v_1
v_3	v_2	v_4	v_4	v_1	v_1
v_4	v_1	v_4	v_1	v_1	v_4

Conjunction and disjunction of formulas are interpreted by introducing a total order on $\mathcal{D} \cup \mathcal{U}$ defined as $v_4 < v_3 < v_2 < v_1$, and interpreting conjunction as the lower bound and disjunction as the upper bound.

A canonical notion of entailment $\models$ can be then defined by saying that a formula ψ follows from a formula ϕ, denoted by $\phi \models \psi$, whenever all models of ϕ are also models of ψ, that is $V(\psi) \in \mathcal{D}$ whenever $V(\phi) \in \mathcal{D}$. A look at the previous truth table shows that: $\Box\alpha \models \alpha \models \neg\Box\neg\alpha$ and $\Box\neg\alpha \models \neg\alpha \models \neg\Box\alpha$. One recognizes here the semantic interpretation of the modal 3(3)-graph introduced previously.

Let us now map any formula ϕ into the set of indices of the designated values taken by $V(\alpha)$ for which $V(\phi) \in \mathcal{D}$. This map is well-defined since all formulas ϕ considered are recursively built from conjunctions, disjunctions, negations, and modal decorations of α. This yields for instance: $\Box\alpha \mapsto \{1\}$, $\alpha \mapsto \{1, 2\}$, $\neg\Box\neg\alpha \mapsto \{1, 2, 3\}$, $\Box\neg\alpha \mapsto \{4\}$, $\neg\alpha \mapsto \{3, 4\}$, $\neg\Box\alpha \mapsto \{2, 3, 4\}$. More generally if a formula α_1 maps into some subset of $\{1, 2, 3, 4\}$, and a formula α_2 maps into some subset of $\{1, 2, 3, 4\}$, then $\alpha_1 \vee \alpha_2$ maps into the set of indices of the designated values taken by $V(\alpha)$ for which $V(\alpha_1 \vee \alpha_2) \in \mathcal{D}$ or equivalently for which $V(\alpha_1) \cup V(\alpha_2) \in \mathcal{D}$ (by definition of the interpretation of the disjunction), which is none else that the union of the subsets into which α_1 and α_2 map. Mutatis mutandis, conjunction corresponds to intersection of the subsets.

Thus this mapping is none else than Pellissier's encoding, which led to the "Boolean" tetraicosahedron. $\square$

4. A higher-order generalization

The previous constructions have shown all the extensions of Aristotle's square that can be obtained by adding the modality $\Box$. However one could also add further modal operators, as suggested in the construction of higher-order geometrical figures of opposition in [Mor04].

Let us thus define some classical propositional multimodal logic, and following [Mor04], let us start from an arbitrary modal 4(3)-graph:

$$\begin{tikzcd}
A^1 \arrow[rr, dotted, no head] \arrow[dr, dotted, no head] \arrow[dd] & & A^3 \arrow[dd] \\
& A^2 \arrow[ur, dotted, no head] \arrow[dd] & \\
B^1 \arrow[rr, no head] \arrow[dr, no head] \arrow[dd] & & B^3 \arrow[dd] \\
& B^2 \arrow[ur, no head] \arrow[dd] & \\
C^1 \arrow[rr, dashed, no head] \arrow[dr, dashed, no head] & & C^3 \\
& C^2 \arrow[ur, dashed, no head] &
\end{tikzcd}$$

where arrows stand for sub-alternation, A's, B's and C's are formulas, and opposition is defined geometrically as: the disjunction of two vertices of any of the three triangles is contradictory to the vertex that can be obtained by central symmetry relatively to the center of gravity of the middle triangle. In other words, if (i, j, k) is any permutation of $\{1, 2, 3\}$, $A^i \vee A^j$ is contradictory to C^k, $C^i \vee C^j$ is contradictory to A^k, $B^i \vee B^j$ is contradictory to B^k.

As before, we will try now to construct the geometrical figure composed of all the combinations of the modalities obtained by applying $\wedge$, $\vee$ and $\neg$. We have the following proposition.

Proposition 4.1. *The* 4*-oppositional figure generated from an arbitrary modal* 4(3)*-graph is a tetraicosahedron. More precisely, all formulas generated by* 4*-opposition lie on the vertices or the edges of a "completed" tetraicosahedron.*

Proof. We will first consider only the A's and the C's. In the next paragraphs, we adopt the following convention: when we write an expression such as $A^i \wedge A^j$, we really mean different exponents which can take any value within $\{1, 2, 3\}$; whereas $A^i \wedge A^i$ refers to the same A^i. Using the opposition properties applied to C's and B's, the previous graph can be rewritten as:

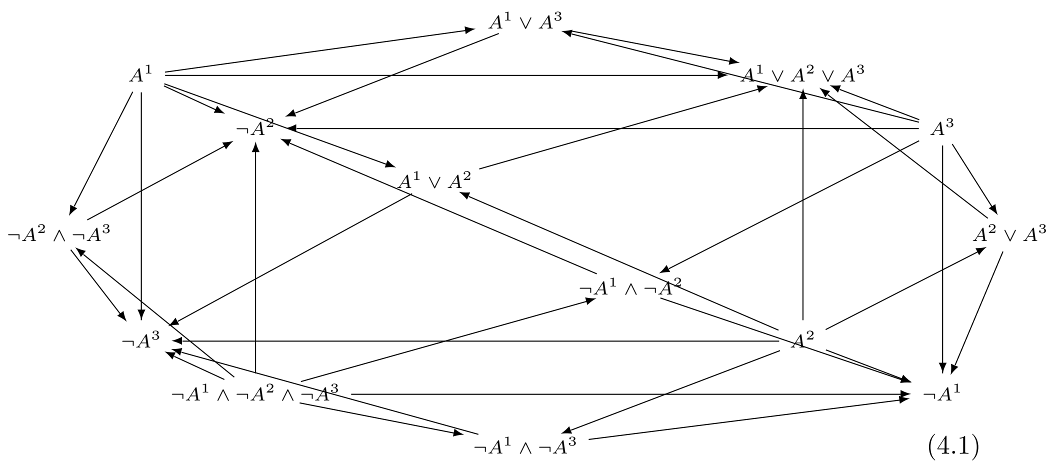
$$\begin{tikzcd}
A^1 \arrow[rr, dotted, no head] \arrow[dr, dotted, no head] \arrow[dd] & & A^3 \arrow[dd] \\
& A^2 \arrow[ur, dotted, no head] \arrow[dd] & \\
B^1 \arrow[rr, no head] \arrow[dr, no head] \arrow[dd] & & \neg B^1 \wedge \neg B^2 \arrow[dd] \\
& B^2 \arrow[ur, no head] \arrow[dd] & \\
\neg A^2 \wedge \neg A^3 \arrow[rr, dashed, no head] \arrow[dr, dashed, no head] & & \neg A^1 \wedge \neg A^2 \\
& \neg A^1 \wedge \neg A^3 \arrow[ur, dashed, no head] &
\end{tikzcd}$$

Sub-alternation reads as $A^i \to \neg A^j \wedge \neg A^k$, which implies $A^i \to \neg A^j$. Thus $A^i \wedge \neg A^j = A^i$ and $A^i \wedge A^j \to \neg A^j \wedge A^j = \perp$, whence $A^i \wedge A^j = \perp$, while obviously $A^i \wedge A^i = A^i$.

From that follows $A^1 \wedge A^2 \wedge A^3 {=} \perp$. It is also easy to verify that if we express each vertex (indexed by i) of the upper triangle as the negation of the disjunction of both opposite vertices $C^j = \neg A^i \wedge \neg A^k$ and $C^k = \neg A^i \wedge \neg A^j$ (opposition should define the triangle of C's from the triangle of A's and conversely), then we obtain A^i as expected. Hence all opposition requirements do not introduce any new constraints on the A's or the B's.

Applying negation to the previous results, we have $\neg A^i \vee \neg A^j = \top$, $\neg A^i \vee A^j = \neg A^i$ and $\neg A^i \vee \neg A^j \vee \neg A^k = \top$.

To sum up, the closure of the A's by $\{\wedge, \vee, \neg\}$ yields the terms $\{\perp, \top, A^i, \neg A^i, A^i \vee A^j, \neg A^i \wedge \neg A^j, A^i \vee A^j \vee A^k, \neg A^i \wedge \neg A^j \wedge \neg A^k\}$ with the sub-alternation relations between terms illustrated in figure 8.



(4.1)

FIGURE 8. The tetraicosahedron corresponding to the A's and C's of a modal 4(3)-graph.

Actually, this tetraicosahedron has two additional vertices corresponding to $\top$ and $\perp$, as mentioned in section 2.1, with the obvious sub-alternation relations: $\perp \to \neg A^1 \wedge \neg A^2 \wedge \neg A^3$ and $A^1 \vee A^2 \vee A^3 \to \top$.

This is no surprise, since up to now, we have restricted our study to A's and C's, i.e. we have considered actually a modal 4(2)-graph: but if we remember the corollary from [Pel06] recalled in a previous section, a modal 4(2)-graph and a modal 3(3)-graph are translated into the same Boolean lattice corresponding to a set of 4 elements, which is the tetraicosahedron (1.1).

Let us see now how the B's fit in this framework (in the sequel we do not use the fact that $B^3 = \neg B^1 \wedge \neg B^2$ and use only the notation B^3, in order to have as much symmetry as possible in the notations).

Let us introduce, with the name E, the collection of the following edges: e_1 stands for any of the three edges $A^i \to \neg A^j \wedge \neg A^k$, e_2 stands for any of the three edges $A^i \vee A^j \to \neg A^k$, e_3 is the edge $\bot \to \neg A^i \wedge \neg A^j \wedge \neg A^k$, e_4 is the edge $A^i \vee A^j \vee A^k \to \top$. Notice that e_3 and e_4 involve the "completed" tetraicosahedron.

Since $A^i \to B^i \to \neg A^j \wedge \neg A^k$, we have $A^j \vee A^k \to \neg B^i \to \neg A^i$ and:

$A^i \wedge A^j = \bot \to B^i \wedge B^j \to \neg A^i \wedge \neg A^j \wedge \neg A^k$,

$A^i \vee A^j \to B^i \vee B^j \to (\neg A^j \wedge \neg A^k) \vee (\neg A^i \wedge \neg A^k) = \neg A^k$,

$A^i \wedge (A^i \vee A^k) = A^i \to B^i \wedge \neg B^j \to \neg A^j \wedge \neg A^k$.

Therefore the terms B^i, $B^i \wedge B^j$, $B^i \vee B^j$, $B^i \wedge \neg B^j$ and their negations lie on the edges of E. Notice that E is closed under "negation", i.e. whenever a term lies on an edge defined by two vertices, the negation of the term lies on the edge defined by the negations of both vertices, and this edge is still in E.

For sake of simplicity, we adopt now the following convention: $e_1 \wedge A^l$ (resp. $e_1 \vee A^l$) means the set of edges obtained from any edge e_1 by composing its source and target by conjunction (resp. disjunction) with A^l. Remember that i, j, k are mute indices within $\{1, 2, 3\}$ and that when we write different (resp. the same) indices, we really mean they are different (resp. the same).

$e_1 \wedge A^i = A^i \to A^i$	$e_2 \wedge A^i = A^i \to A^i$
$e_1 \wedge A^j = \bot \to \bot$	$e_2 \wedge A^k = \bot \to \bot$
$e_1 \wedge (A^i \vee A^j) = A^i \to A^i$	$e_2 \wedge (A^i \vee A^j) = A^i \vee A^j \to A^i \vee A^j$
$e_1 \wedge (A^j \vee A^k) = \bot \to \bot$	$e_2 \wedge (A^i \vee A^k) = A^i \to A^i$
$e_1 \wedge (A^i \vee A^j \vee A^k) = A^i \to A^i$	$e_2 \wedge (A^i \vee A^j \vee A^k) = A^i \vee A^j \to A^i \vee A^j$
$e_1 \vee A^i = A^i \to \neg A^j \wedge \neg A^k$	$e_2 \vee A^i = A^i \vee A^j \to \neg A^k$
$e_1 \vee A^j = A^i \vee A^j \to \neg A^k$	$e_2 \vee A^k = A^i \vee A^j \vee A^k \to \top$
$e_1 \vee (A^i \vee A^j) = A^i \vee A^j \to \neg A^k$	$e_2 \vee (A^i \vee A^j) = A^i \vee A^j \to \neg A^k$
$e_1 \vee (A^j \vee A^k) = A^i \vee A^j \vee A^k \to \top$	$e_2 \vee (A^i \vee A^k) = A^i \vee A^j \vee A^k \to \top$
$e_1 \vee (A^i \vee A^j \vee A^k) = A^i \vee A^j \vee A^k \to \top$	$e_2 \vee (A^i \vee A^j \vee A^k) = A^i \vee A^j \vee A^k \to \top$
$e_3 \wedge A^i = \bot \to \bot$	$e_4 \wedge A^i = A^i \to A^i$
$e_3 \wedge (A^i \vee A^j) = \bot \to \bot$	$e_4 \wedge (A^i \vee A^j) = A^i \vee A^j \to A^i \vee A^j$
$e_3 \wedge (A^i \vee A^j \vee A^k) = \bot \to \bot$	$e_4 \wedge (A^i \vee A^j \vee A^k) = A^i \vee A^j \vee A^k \to A^i \vee A^j \vee A^k$
$e_3 \vee A^i = A^i \to \neg A^j \wedge \neg A^k$	$e_4 \vee A^i = A^i \vee A^j \vee A^k \to \top$
$e_3 \vee (A^i \vee A^j) = A^i \vee A^j \to \neg A^k$	$e_4 \vee (A^i \vee A^j) = A^i \vee A^j \vee A^k \to \top$
$e_3 \vee (A^i \vee A^j \vee A^k) = A^i \vee A^j \vee A^k \to \top$	$e_4 \vee (A^i \vee A^j \vee A^k) = A^i \vee A^j \vee A^k \to \top$

TABLE 1. Proof of the closure of E under composition with admissible A terms: note that an edge like $A^i \to A^i$ is actually reduced to the vertex A^i.

Table 1 shows how the edges of E are transported when composed by conjunction and disjunction with A^i, $A^i \vee A^j$, $A^i \vee A^j \vee A^k$. For instance, $e_1 \wedge A^i$

stands for any of the three edges $A^i \wedge A^i \to \neg A^j \wedge \neg A^k \wedge A^i$, i.e. $A^i \to A^i$, since $\neg A^j \wedge (\neg A^k \wedge A^i) = \neg A^j \wedge A^i = A^i$.

Looking at all cases, we see that E is closed under the compositions with the various terms that can be built with the A's. Since E is closed under negation, we conclude that all the terms that can be built from A's and B's with any of $\{\wedge, \vee, \neg\}$ lie on some edge of E. □

From the previous proof it is straightforward to determine the edges and vertices of the tetraicosahedron (4.1) on which the composite terms with A's and B's lie. This is helpful when additional relations are given between the various modalities, as in the next section.

5. Special cases of the higher-order generalizations

We show now how the tetraicosahedron (4.1) encompasses all previous logical figures. This will be achieved by taking special values for A^1, A^2, A^3, B^1, B^2, using the usual modal operator $\Box$ as well as new modal operators.

Let us take $B^1 = \Box\alpha$ and $B^2 = \Box\neg\alpha$, where $\Box$ satisfies the axioms of classical $S5$ modal logic. If we assume the triangle of A's to be also "strong", then $A^3 = \neg A^1 \wedge \neg A^2$, and the sub-alternation relation $A^1 \to \Box\alpha \to \neg A^2 \wedge \neg(\neg A^1 \wedge \neg A^2) = \neg A^2 \wedge A^1 = A^1$ implies $A^1 = \Box\alpha$. In a similar way, $A^2 = \Box\neg\alpha$. It is easy to see that the tetraicosahedron (4.1) collapses to the classical hexagon from figure 1.

Let us take $A^1 = \Box\alpha$, $A^2 = \Box\neg\alpha$, $A^3 = \alpha \wedge \neg\Box\alpha$. Take also $B^1 = \Box\alpha$ and $B^2 = \Box\neg\alpha$ which is consistent with all sub-alternation relations. Then the cube underlying the tetraicosahedron (4.1) yields the strong logical cube (2.2).

Let us now take $A^1 = \Box\alpha$, $A^2 = \Box\neg\alpha$, and $A^3 = \sim\alpha$ where $\sim$ is a new modal operator. The choice of $B^1 = \Box\alpha$ and $B^2 = \Box\neg\alpha$ is consistent with all sub-alternation relations. This assignment of modalities corresponds to the modal 4(3)-graph mentioned in [Mor04], where: $\beta = \Box$, $| = \sim$ and $\beth = \Box\neg$, and the cube underlying the tetraicosahedron (4.1) yields:

$$\begin{tikzcd}
\Box\alpha \arrow[rr] \arrow[dr] \arrow[dd] & & \Box\alpha \vee \sim\alpha \vee \Box\neg\alpha & \\
& \neg\Box\neg\alpha & & \sim\alpha \arrow[ul] \arrow[ll] \arrow[dd] \\
\neg\sim\alpha & & \Box\neg\alpha \arrow[uu] \arrow[ll] \arrow[dr] & \\
& \neg\Box\alpha \wedge \neg\sim\alpha \wedge \neg\Box\neg\alpha \arrow[ul] \arrow[uu] \arrow[rr] & & \neg\Box\alpha
\end{tikzcd} \tag{5.1}$$

which is constructed from the six squares of opposition illustrated in figure 9.

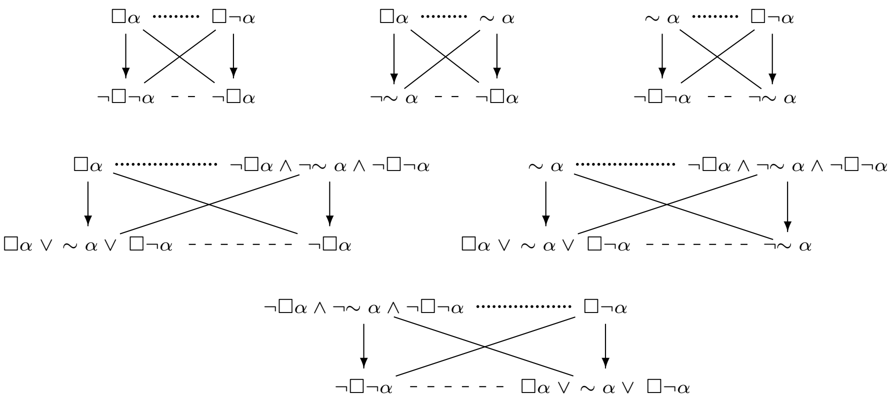


FIGURE 9. The 6 squares included in the multimodal logical cube (5.1).

By analogy with the Aristotelian square, we can have the intuition from the second and third top squares of figure 9 that $\sim$ should have the flavor of a negation (negative particular assertion) and of a necessity modality (affirmative universal assertion). Obviously classical negation does not work in this case, however an interpretation within society semantics [CLM05] is adequate, as we elaborate below.

6. Semantics associated to the modal operator $\sim$

In the next paragraphs, we give a Kripke-based semantics for the modal operator $\sim$ introduced previously. A Kripke model K is a triple $(W, \prec, V)$, where W is a non-empty set, $\prec$ a binary relation on W, and V is a mapping (the valuation) that assigns a subset $V(p)$ of W to each propositional variable p. W is meant to be the set of possible worlds, $w \prec v$ is meant to say that v is reachable from w, and $V(p)$ is intended to be the set of worlds at which p is true under the valuation V.

The following notation is usual: $w \models_K p$ iff $w \in V(p)$. The forcing relation $\models_K$ is defined recursively on all formulas by: $w \models_K \neg A$ iff $w \not\models_K A$; $w \models_K A \vee B$ iff $w \models_K A$ or $w \models_K B$; $w \models_K A \wedge B$ iff $w \models_K A$ and $w \models_K B$; $w \models_K A \rightarrow B$ iff $w \not\models_K A$ or $w \models_K B$; $w \models_K \Box A$ iff for every v such that $w \prec v$, $v \models_K A$.

The semantics presented below is inspired form [Car00] and [CLM05]. It is given by a local forcing relation $\models_{1,2}$ based on two different Kripke models defined on the same relation $\prec$, $\models_1$ and $\models_2$, which can be interpreted as two different observation or measurement processes that yield potentially different results, due to their intrinsic difference. The formal definition is:

- $w \models_{1,2} p$ iff $w \models_1 p$ or $w \models_2 p$
- $w \models_{1,2} \neg A$ iff $(w \not\models_{1,2} A)$
- $w \models_{1,2} A \wedge B$ iff $w \models_{1,2} A$ and $w \models_{1,2} B$
- $w \models_{1,2} A \vee B$ iff $w \models_{1,2} A$ or $w \models_{1,2} B$

- $w \models_{1,2} A \to B$ iff $w \not\models_{1,2} A$ or $w \models_{1,2} B$
- $w \models_{1,2} \Box A$ iff $w \models_1 \Box A$ and $w \models_2 \Box A$
- $w \models_{1,2} {\sim} A$ iff $(\exists v_1, w \prec v_1)\ (v_1 \models_1 A)$ and $(\exists v_2, w \prec v_2)\ (v_2 \models_2 \neg A)$

In plain words, ${\sim}A$ means that we might observe A with measurement process 1 and $\neg A$ with measurement process 2 (not necessarily at the same time though). Therefore, as stated previously intuitively, the modality $\sim$ has both a positive and a negative touch, and it behaves in a paraconsistent way, since we can find some A such that $x \models_{1,2} {\sim}A$ and $x \models_{1,2} {\sim}\neg A$, without trivializing the whole logical framework.

We deduce from the former definitions: $w \models_{1,2} \neg\Box A$ iff $w \not\models_{1,2} \Box A$ iff $w \not\models_1 \Box A$ or $v \not\models_2 \Box A$ iff $(\exists u_1, w \prec u_1)\ (u_1 \models_1 \neg A)$ or $(\exists u_2, w \prec u_2)\ (u_2 \models_2 \neg A)$.

We also deduce: $w \models_{1,2} \neg\Box\neg A$ iff $(\exists u_1, w \prec u_1)\ (u_1 \models_1 A)$ or $(\exists u_2, w \prec u_2)\ (u_2 \models_2 A)$.

From this, it is easy to see that $w \models_{1,2} {\sim}A$ implies $w \models_{1,2} \neg\Box A$, as well as $w \models_{1,2} \neg\Box\neg A$. This shows that the consistency relations imposed by the subalternation relations of the modal 4(3)-graph are satisfied.

This semantics is actually an extension of what was discussed in section 2.3 with two potentially disagreeing observers instead of a unique observer.

7. Changing the logical framework

7.1. From classical modal $S5$ logic to classical modal $S4$ modal logic

In the previous paragraphs, the logic considered in order to interpret the various formulas was the classical $S5$ modal logic, which relies on the chain of affirmative modalities $\Box \to \cdot \to \neg\Box\neg$ and the chain of negative modalities $\Box\neg \to \neg \to \neg\Box$.

Instead of $S5$, one can consider classical $S4$ modal logic, which relies on the following chain of affirmative and negative modalities:

$$\begin{tikzcd}
 & & & \Box \arrow[dll] \arrow[ddr] & & & \Box\neg \arrow[ddl] \arrow[drr] & & & \\
 & \Box\Diamond\Box \arrow[dl] \arrow[dr] & & & & & & & \Box\Diamond\Box\neg \arrow[dl] \arrow[dr] & \\
\Diamond\Box \arrow[dr] & & \Box\Diamond \arrow[dl] & & \cdot \arrow[ddl] & \neg \arrow[ddr] & & \Diamond\Box\neg \arrow[dr] & & \Box\Diamond\neg \arrow[dl] \\
 & \Diamond\Box\Diamond \arrow[drr] & & & & & & & \Diamond\Box\Diamond\neg \arrow[dll] & \\
 & & & \Diamond & & & \Diamond\neg & & &
\end{tikzcd}$$

What has been done previously within classical $S5$ modal logic can then be done within classical $S4$ modal logic: the n-oppositional figures are obtained by considering all the formulas generated from arbitrary (admissible) recursive application of the modal operators, conjunction and negation on some α, and using the lattice properties (conserved through conjunction and disjunction with any formula) between the modalities in order to deduce the logical implication between all these formulas. It is easy to see that the set of generated formulas is finite as there is a trivial injective morphism from it into the algebra of all subsets of a set whose cardinality is equal to the number of all possible positive and negative modalities.

Of course, if one switches then to weaker logics, the n-oppositional figures might not be finite since there might be an infinite number of admissible modalities, as was recalled earlier for some modal logics.

7.2. From classical modal logic to intuitionistic modal logic

Instead of considering classical modal logics, i.e. logics built on classical propositional logic, it is possible to change that too, and switch for instance to intuitionistic $S5$ or $S4$ modal logics, where the negation operator is no longer involutive. This breaks the duality between affirmative and negative modalities.

[Fon86] shows that intuitionistic $S5$ modal logic has 5 affirmative modalities

$$(\begin{tikzcd}[row sep=tiny] & \cdot \arrow[dr] & \\ \Box \arrow[ur] \arrow[dr] & & \neg\neg \arrow[r] & \neg\Box\neg) \\ & \Box\neg\neg \arrow[ur] & \end{tikzcd}$$

and 4 negative modalities $(\Box\neg \to \neg \to \neg\Box\neg\neg \to \neg\Box)$. The same reference shows that intuitionistic $S4$ modal logic has a system of 17 affirmative modalities and 14 negative modalities, with a lack of symmetry (or duality) between these two groups.

Therefore the lattice built from the closure of both these sets of modalities through conjunction and disjunction will yield more complicated geometrical figures than previously, since some of the inferences made in the previous proofs, which reduced the number of non-equivalent terms, are not valid any longer (e.g. double negation does not imply affirmation, and among de Morgan's laws $\neg(\phi \wedge \psi) \to \neg\phi \vee \neg\psi$ does not hold).

However since these logics yield interesting epistemological views on diverse positive and negative introspection capabilities, the obtained n-oppositional figures could provide a model of formal reasoning on propositions. The challenge is more to find an application, than to exhibit the logical geometrical figures, since the latter are only a matter of combinatorics.

8. Future work

In this paper we have seen how Aristotle's square could be generalized, and we have exhibited various translations between some generalizations. We have only hinted at how the latter could be used as models for formal reasoning by proposing particular semantics for the generalizations proposed. Obviously many other logical frameworks can be defined. Furthermore the methods introduced in this paper can be applied to the higher-order extensions of [Mor04] in a straightforward way: the only difficulty associated with the introduction of additional modal operators (a modal $n(m)$-graph with larger n and m implies only bigger simplices and more copies of them) is more cumbersome notations and fastidious proofs. More complex geometrical figures arise, and more complex semantics based on society semantics – with the right juggling of the Kripke forcing relations – are needed.

This paves the way for a more general theory of reasoning where Aristotle's square and its generalizations do not model only the reasoning abilities of a unique agent, but of several interacting agents. The modalities between various such agents can be related together, or can arise as the interaction between the agents, which leads then to a model of learner-teacher interactions.

These various ideas are currently formalized using category theory, representing an agent as a topos and the interaction between agents as an adjunction between the corresponding topoi [SLS07].

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