Logically Possible Worlds and Counterpart Semantics for Modal Logic

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Abstract. The paper reviews the technical results from modal logic as well as their philosophical significance. It focuses on possible worlds semantics in general and on the notion of a possible world, of accessibility, and object.

Contents

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> Since to belong and to belong of necessity and to be possible to belong are different (for many things belong, but nevertheless not of necessity, while others neither belong of necessity nor belong at all, but it is possible for them to belong), it is clear that there will also be different deductions of each and that their terms will not be alike: rather, one deduction will be from necessary terms, one from terms which belong, and one from possible terms.
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> Aristotle: *Prior Analytics, Book I, 8*[1]

Introduction. Modal logic has a remarkable history indeed. Originally conceived as the logic of necessity and possibility, its philosophical roots go back at least as far as Aristotle and the Stoic Diodorus Cronus.[2] While Diodorus's 'Master Argument' could be considered an early example of modal propositional, more precisely temporal, reasoning,[3] the Stagirite's investigations into modal syllogisms probably constitute the earliest serious essay on combining modality with quantification.[4] Interest into modal distinctions kept very much alive during Mediaeval Scholastics, notably in the form of ontological arguments such as St. Anselm's,[5] and only seems to have slowed down with the dawn of the Renaissance. While Leibniz, being an eminent contributor to the development of modern logic, is usually credited with the introduction of the notion of a 'possible world', the birth of modal logic as a technical discipline of mathematical logic is

[1] English translation by Robin Smith [4].

[2] Classic (and encyclopaedic) texts on the early history of logic are Bocheński's [19] and William and Martha Kneale's [91].

[3] For the 'Master Argument' compare [197], as well as, for prominent modern reconstructions, Arthur Prior's [151], Oskar Becker's [13], and Jaakko Hintikka's [79].

[4] Modern attempts at interpreting and understanding Aristotle's still puzzling elaborations in Prior Analytics include Łukasiewicz's [126] in the 1950ies, [188, 189], [146] and [162].

[5] Of which we find variants in the work of, for example, Descartes and Leibniz, and a modern counterpart in Kurt Gödel's ontological argument, a descendent of the argument given by Leibniz, being sketched in higher-order modal logic (second- or third-order depending on the interpretation) and which was only posthumously published in [70]. For details compare Melvin Fitting's [53].

often considered to be Clarence Irving Lewis's investigations into the paradoxes of material implication in 1918[6], and, in the early thirties, Kurt Gödel's interpretation [68, 69] of Heyting's version [77] of Brouwer's intuitionistic propositional logic [22, 23] in terms of the modality 'it is provable'[7]. Despite Quine's early attempts at discrediting the intelligibility of modal discourse [154, 155], its technical development went hand in hand with the rise of analytic philosophy, and both its technical sophistication as well as its philosophical significance virtually exploded in the 1970s with the unlikely success of Kripke's possible worlds semantics. In the early 21st century, modal logic is not only an important tool in philosophy and linguistics, it also provides, in its many disguises, key formalisms in Artificial Intelligence and Knowledge Representation.

In this contribution, however, we shall not review the history of modal logic, nor do we delve deeply into the discussion of its metaphysics.[8]

Rather, we shall concentrate on the interplay between formal results and analytical thought. As argued already in [75], formal semantic theories such as Lewisian counterpart theory or Kripkean possible worlds semantics are compatible with a wide spectrum of metaphysical views concerning possible worlds, the nature of objects and possibility, and so on. It is therefore pointless to ask whether or not a given formal semantics is ontologically adequate. Besides, many of these questions quickly take a different turn: they become questions of physical nature whose answers often elude the specialists themselves. That theme will be touched upon when we discuss the nature of time. Our main interest here is, however, different and more modest. We shall simply ask in what ways the formal structures reflect the intuitions that they are claimed to model, and in what ways technical results can be interpreted from a philosophical point of view. The most common semantics of modal logics introduce the notions of a **world**, a **possibility** or a **situation**, and impose further structure by means of more or less complex relations, e.g., by the notion of an **accessibility**.

The nature and existence of 'worlds' has been an intensely debated topic in analytic philosophy and metaphysics. Also, the nature of the accessibility relation has largely been unscrutinised. Although nowadays the idea of a binary relation between worlds is the most popular one,

[6] C. I. Lewis created his first correct modal calculus **S3** of strict implication in 1920 [116] (an emendation of a system proposed earlier in his [115]), compare also [142].

[7] In fact, a similar interpretation can be found even earlier in Orlov's [141].

[8] But compare [73] for the historical development of mathematical modal logic in the 20th century and [16] for the state-of-the-art as of 2005, [35] for the genesis of possible worlds semantics, and [122] for a discussion of various early systems of quantified modal logic. A good starting point for the reader interested in the metaphysics of modality would be the anthologies [123] and [124], or John Divers's [39].

many alternatives have been tried (and are being tried). There is, however, a problem with Kripke-semantics, which is both of technical and of philosophical nature: it is incomplete. There exist different logics with the same class of Kripke-frames. This means that the choice of Kripke-semantics encapsulates metaphysical commitments whose precise nature is unfortunately difficult to explicate. This is unproblematic when the semantics is given beforehand and the logic is derived from it. In all other cases, however, we have committed ourselves to the complete logics from the start without knowing whether that is justified. What is more, criteria for completeness are almost impossible to give (a notable exception is Kit Fine's [46]). The remedy is to restrict the domain of interpretation to admissible sets, but this approach is philosophically unsatisfactory as long as there is no clear account of where these sets originate.

Another notion that has been met with suspicion is that of an **object**. One problem with the notion of an object is the problem of **transworld identification**. If there are possible worlds, then which objects in a given world are the same as other objects in another world? The answer depends in part on what we think an object is. Saul Kripke and many with him have assumed that objects are transcendental, to use a rather old-fashioned terminology here. This choice was influenced by a particular school of thought which regards possible worlds as man made, constructed out of the things that we know, namely the objects. The worlds contain more or less the same objects, only the facts about these objects may change. This view faces philosophical problems of its own, and has been challenged on these grounds. Technically, its main disadvantage is that it is highly incomplete. Most propositional logics become incomplete when we move to predicate logic.

A different conception of objects is to view them as entirely world-bound; this is embodied in the semantics of counterpart theory as originally conceived by David Lewis (see [117]). It can be shown to be more general than Kripke-semantics, but it too carries ontological commitments. The third way is two view objects (in type theoretic terms) as **individual concepts**. This makes them both transcendental and world bound. It can be shown that this semantics (in its generalised form) is actually complete in the most general sense. A fourth way is to abandon the notion of object altogether. This is the semantics advocated for by Shehtman and Skvortsov [177]. In this semantics, a structure is an infinite sequence of frames. These frames combine both the notion of object and world in one; worlds can be identified with the elements at level 0; objects are only approximated to a finite degree at each level. It is possible to construct a frame with worlds and objects from it ([11]).

Part 1. Worlds without Objects

§1. Basic Concepts of Modal Propositional Logic. The language of **modal propositional logic** consists of a countable set $V = \{p_i : i \in \mathbb{N}\}$ of sentence letters, a set C of propositional constants, the connectives $\top$, $\neg$, and $\wedge$, and a set of so-called **modalities** or **modal operators**. A modality can have an arbitrary finite arity. We consider here only the case of a single operator $\Box$ of arity 1. Many of the philosophical considerations do not depend on that choice. For the technical side see [99]. A **modal logic** is a set L which contains all Boolean tautologies, and is closed under 'modus ponens' (MP): from $\varphi \in L$ and $\varphi \to \psi \in L$ infer $\psi \in L$, and substitution (sub): from $\varphi \in L$ infer $\sigma(\varphi) \in L$. Here, σ is defined from a function from variables to formulae; $\sigma(\varphi)$ is obtained by replacing each occurrence of a propositional variable p_i by some formula $\sigma(p_i)$. (This means that substitution is an admissible rule; a rule is **admissible** in L if the tautologies of L are closed under this rule. (MP) is actually a derived rule, that is, a rule applicable in reasoning from premisses.) L is **classical** if the following rule is admissible.

$$(1) \qquad \text{(RE)} \quad \frac{\varphi \leftrightarrow \chi}{\Box\varphi \leftrightarrow \Box\chi}$$

L is **normal** if it is classical, and additionally

1. $\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1) \in L$,
2. and (MN) is admissible in L:

$$(2) \qquad \text{(MN)} \quad \frac{\varphi}{\Box\varphi}$$

A **Kripke-frame** is a triple $\langle F, R, \mathfrak{I}\rangle$, where F is a set, the set of **worlds**, $R \subseteq F^2$ the **accessibility relation**, and $\mathfrak{I} : C \to \wp(F)$ a function mapping propositional constants to subsets of F. In what is to follow, we shall usually assume $C = \varnothing$.[9] A **valuation** is a function $\beta : V \to \wp(F)$ which assigns a set of worlds to each sentence letter. A **pointed Kripke-model** is a triple $\langle \mathfrak{F}, \beta, w\rangle$, where $\mathfrak{F}$ is a (Kripke-) frame, β a valuation, and w a world. **Truth** of formulae in a model is then defined as follows: $\langle \mathfrak{F}, \beta, w\rangle \vDash \top$ always holds, and

$$(3) \quad \begin{array}{lll}
\langle \mathfrak{F}, \beta, w\rangle \vDash p_i & :\Leftrightarrow & w \in \beta(p_i); \\
\langle \mathfrak{F}, \beta, w\rangle \vDash c & :\Leftrightarrow & w \in \mathfrak{I}(c); \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \neg\varphi & :\Leftrightarrow & \langle \mathfrak{F}, \beta, w\rangle \nvDash \varphi; \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \varphi \wedge \chi & :\Leftrightarrow & \langle \mathfrak{F}, \beta, w\rangle \vDash \varphi \text{ and } \langle \mathfrak{F}, \beta, w\rangle \vDash \chi; \\
\langle \mathfrak{F}, \beta, w\rangle \vDash \Box\varphi & :\Leftrightarrow & \text{for all } v \text{ such that } w\ R\ v\text{: } \langle \mathfrak{F}, \beta, v\rangle \vDash \varphi.
\end{array}$$

[9] Whenever propositional constants are left out of the language, a Kripke-frame is just a pair $\langle F, R\rangle$.

Write $\mathfrak{F} \vDash \varphi$ if $\langle \mathfrak{F}, \beta, w \rangle \vDash \varphi$ for all β and w. Given a class $\mathcal{K}$ of frames, let

$$\text{Th}(\mathcal{K}) := \{\varphi : \text{for all } \mathfrak{F} \in \mathcal{K} \ : \ \mathfrak{F} \vDash \varphi\}. \tag{4}$$

$\text{Th}(\mathcal{K})$ is called the **logic** of $\mathcal{K}$, for, whenever $\mathcal{K}$ is some class of Kripke-frames, $\text{Th}(\mathcal{K})$ is a normal modal logic. Conversely, if L is a normal modal logic, we may define

$$\text{Frm}(L) := \{\mathfrak{F} : \ \text{for all } \varphi \in L: \ \mathfrak{F} \vDash \varphi\}. \tag{5}$$

These definitions can be applied to any type of semantics. (In that case, if $\mathcal{X}$ is the class of structures, we index Frm with $\mathcal{X}$ to avoid confusion.) These definitions technically equate a modal logic with its set of axioms (or tautologies), and leaves the set of rules invariant.

§2. Translation into Classical Logic. In and of itself, modal logic does not embody the commitment to possible worlds of any sort—rather, the doctrines of modal realism and anti-realism are subject to considerable philosophical debate. For instance, while Charles Chihara's book on modal realism [31] contains an elaborate attempt at constructing a viable position of modal anti-realism based on the idea to explain modalities not in terms of possible worlds but rather in terms of '*how the world could have been*', the bulk of David Lewis's 'Plurality of worlds' [120] consists of an extended argument of why we cannot legitimately employ 'possible worlds talk' and enjoy its benefits for instance in the philosophy of language without giving it a realistic interpretation [84].

However, both in technical philosophy and in mathematical logic, possible worlds have become a de facto indispensable tool. It was observed that modal laws can be made plausible by using the notion of a possible world and explicating modal operators using quantification over possible worlds. This led to the well-known **standard translation** of modal logic into classical logic (see [14]). By rendering the truth conditions given in (3) into formal logic as well, we arrive at a translation into monadic second-order logic, which runs as follows. For each proposition letter p_i we introduce a monadic second-order predicate letter P_i (which is a variable).

$$\begin{aligned} p_i^\dagger &:= P_i(w) \\ (\neg\varphi)^\dagger &:= \neg\varphi^\dagger \\ (\varphi \wedge \chi)^\dagger &:= \varphi^\dagger \wedge \chi^\dagger \\ (\Box\varphi)^\dagger &:= (\forall v)(w \ R \ v \to \varphi^\dagger(v/w)) \end{aligned} \tag{6}$$

A formula φ is satisfiable in a Kripke-frame $\langle F, R \rangle$ if and only if $\varphi^\dagger$ is satisfiable in $\langle F, R \rangle$ (viewed as a model for second-order predicate logic) iff $(\exists\vec{P})\varphi^\dagger$ holds in it, where $\vec{P}$ contains at least the monadic second-order variables free in $\varphi^\dagger$. Hence $\mathfrak{F} \vDash \varphi$ iff $(\forall\vec{P})\varphi^\dagger$ is valid in $\mathfrak{F}$. Thus frame validity is Π^1_1. If this formula has a first-order equivalent then the propositional logic is complete with respect to Kripke frames, by a result of Kit Fine [46].

Second-order logic adds quantifiers to modal logic that it otherwise does not have. Notice that the formulae on the right hand side of (6) contain only two variables. Thus, the translation is into the two-variable fragment of first-order logic whose satisfiability problem is known to be decidable [137]. (Of course, this does not imply the decidability of modal logics involving additional axioms—this is a science on its own, compare [140].) From a philosophical point of view this translation raises questions. It can be used to explain modal talk as talk of possible worlds in disguise; this is a reduction from right to left. Modal logic is just talk about possible worlds in a syntactic disguise. It cannot, however, be used to explain talk about possible worlds through modal operators, because the latter is less expressive. There are first-order formulae which have no equivalent in modal logic. This has been observed in [117] already, noting that this makes modal logic less expressive than it should be if it is to render natural language statements. This point has been picked up by [74], who extends modal logic by an actuality operator, and further by [57], who allows to index possible worlds in order to be able to refer back to them. The most expressive language is however that of nominals, introduced by Arthur Prior, see also [25]. A comprehensive study of this language has been made in [18]. Nominals are propositional variables that have to be true at precisely one world—they can serve as *names* for worlds. This essentially reifies the worlds inside modal logic and allows the backward pointing devices argued for in [57] (possibly with the help of the **universal modality**, whose interpretation is basically that of the first-order quantifier over worlds).

§3. Ontology and Duality Theory. Ludwig Wittgenstein says in the Tractatus that the world is everything which is the case, or, in modern terminology, a complete state of affairs. A view that identifies worlds with complete (or partial) states-of-affairs denies them their own ontological status, a position for which David Lewis coined the term '**linguistic ersatzism**' [120].[10] Kripke-frames, on the other hand, treat worlds as primitive objects. Prima facie this seems justified in the case of time points or locations. They provide examples of entities that can in principle serve as worlds in the sense of containers which are inhabited by objects and relations (see [57]). If worlds exist and are not simply states-of-affairs then the possibility arises of there being two different worlds in

[10]Compare also Sider's [174] for a more recent proposal in this direction.

which the same facts are true (call them **twin worlds**). Again, whether this is a welcome situation is debated (see below). In duality theory, one studies the construction of frames from algebras and algebras from frames. It turns out that the two model structures are not identical. In passing from frames to algebras, the distinction between twin worlds is lost. On the other hand, the construction of a frame from an algebra may now introduce new worlds to that frame. (They instantiate a complete state-of-affairs that is only finitely satisfied in the original frame.)

A **Boolean algebra with an operator** is a structure of the form $\mathfrak{B} = \langle B, 1, -, \cap, \blacksquare\rangle$, where $\langle B, 1, -, \cap\rangle$ is a Boolean algebra. $\mathfrak{B}$ is a **modal algebra** if in addition

$$\blacksquare 1 = 1, \qquad \blacksquare(a \cap b) = \blacksquare a \cap \blacksquare b \tag{7}$$

Given an assignment $v : V \to A$, there is a unique homomorphism $\overline{v}$ which assigns to every formula over V a value in A. We write $\mathfrak{A} \vDash \varphi$ if $\overline{v}(\varphi) = 1$ for all assignments v and $\mathrm{Th}(\mathfrak{A}) := \{\varphi : \mathfrak{A} \vDash \varphi\}$. $\mathrm{Th}(\mathfrak{A})$ is classical; it is normal iff $\mathfrak{A}$ is a modal algebra. [89] have initiated the study of such algebras and shown that they can be embedded in complete Boolean algebras. The atoms of these algebras can be identified with the worlds. Given a logic, modal algebras can be constructed as follows. Let $\mathfrak{T}(V)$ be the algebra of formulae over the set of variables V. Put

$$\varphi \sim_L \chi :\Leftrightarrow \varphi \leftrightarrow \chi \in L \tag{8}$$

L is classical iff $\sim_L$ is a congruence on $\mathfrak{T}(V)$. $\mathfrak{F}_L(V) := \mathfrak{T}(V)$ is an algebra, called the **canonical** or **Lindenbaum-algebra**. It turns out that for countably infinite V,

$$L = \{\varphi : \mathfrak{F}_L(V) \vDash \varphi\} \tag{9}$$

The construction of the algebra uses the language itself. Propositions are formulae modulo equivalence, but there is no assumption on worlds nor states-of-affairs. Propositions form an algebra. From this algebra it is possible to construct worlds as follows. Let $U(\mathfrak{A})$ be the set of ultrafilters of $\mathfrak{A}$. Put $U\ R^*\ V$ if and only if for all $\blacksquare a \in U$ we have $a \in V$. This yields a frame. Unfortunately, the logic of this frame might be different from the logic of the algebra. It therefore becomes necessary to define the following. For $a \in A$ let

$$\widehat{a} := \{U \in U(\mathfrak{A}) : a \in U\} \tag{10}$$

The map $a \mapsto \widehat{a}$ is an isomorphism from $\mathfrak{A}$ onto a subalgebra of the powerset algebra over $\langle U(\mathfrak{A}), R^*\rangle$, with the additional operator

$$\blacksquare a := \{U : \forall V \in U(\mathfrak{A}) : \text{if } U\ R^*\ V \text{ then } V \in a\} \tag{11}$$

The subalgebra can be proper, and can have a different logic. The reason is that the semantics of Kripke-frames is incomplete in the following sense.

Definition 1. *Let $\mathcal{L}$ be a class of logics, and $\mathcal{K}$ a class of structures. $\mathcal{K}$ is* ***complete for*** *$\mathcal{L}$ if for every logic $L \in \mathcal{L}$, $L = \mathrm{Th}(\mathrm{Frm}_{\mathcal{K}}(L))$.*

If $\mathcal{K}$ is complete for $\mathcal{L}$ then for any two different $L, L' \in \mathcal{L}$ the class of associated structures from $\mathcal{K}$ must be different. Typically, one sets $\mathcal{L}$ to be the extensions of some system (**S4** for example). We shall take here $\mathcal{L}$ to be the class of all normal modal logics. (9) shows that algebraic semantics is complete (for the set of classical logics even). It has been shown in the early 1970s that there are logics which are not completely characterised by their Kripke-frames. In [191], Thomason has proposed to refine the class of structures as follows. A **general frame** is a triple $\mathfrak{F} = \langle F, R, \mathbb{B}\rangle$, where $\langle F, R\rangle$ is a Kripke-frame and $\mathbb{B} \subseteq \wp(F)$ a system of sets closed under intersection, complement and the operation

(12) $$\blacksquare a := \{v : \text{for all } w\text{: if } v\ R\ w \text{ then } w \in a\}$$

So, put

(13) $$\mathfrak{F}_+ := \langle \mathbb{F}, F, -, \cap, \blacksquare\rangle$$

This is a modal algebra. A subset of F is called **internal** if it is in $\mathbb{B}$. Valuations into general frames may take only internal sets as values. Put

(14) $$\mathfrak{A}^+ := \langle U(\mathfrak{A}), R^*, \{\widehat{a} : a \in A\}\rangle.$$

This is a general frame. A Kripke-frame $\langle F, R\rangle$ can be viewed as the general frame $\langle F, R, \wp(W)\rangle$. From a general frame $\mathfrak{F}$ we defined the algebra $\mathfrak{F}_+$, and from an algebra $\mathfrak{A}$ the general frame $\mathfrak{A}^+$. In general, $(\mathfrak{A}^+)_+ \cong \mathfrak{A}$, but $(\mathfrak{F}_+)^+$ is generally not isomorphic to $\mathfrak{F}$. Completeness with respect to general frames is straightforward to show, using the standard completeness results for algebraic logic and duality. For a logic is always complete for its Lindenbaum algebra. General frames have both worlds and an algebra of internal sets in them. We may view internal sets as propositions. Then, a general frame is **differentiated** if from $v \neq w$ it follows that there is an a such that $v \in a$ and $w \notin a$; it is **tight** if, whenever $\blacksquare a \in v$ also $a \in w$, then $v\ R\ w$; it is **compact** if $\bigcap U \neq \varnothing$ for every ultrafilter of internal sets. $\mathfrak{F}$ is **descriptive** if is differentiated, tight and compact. $\mathfrak{A}^+$ always is descriptive. A world w defines a **state-of-affairs** $H(w)$. v and w are twin worlds if $v \neq w$ and $H(v) = H(w)$. $\mathfrak{F}$ is differentiated iff there are no twin worlds; it is tight if accessibility is definable from possibility; and it is compact if every finitely satisfiable set is contained in some $H(w)$. Together with differentiatedness this means that S is an ultrafilter iff $S = H(w)$ for some w. The internal sets may be seen as the clopen (= closed and open) sets of a topological space; its closed sets are the (possibly infinite) intersections of internal sets. In a descriptive frame, every singleton $\{w\}$ is closed.

3.1. Logical Consequence. The semantics is required to be adequate not only in the sense that it handles the tautologies correctly, but that it identifies the deductive rules properly. Again, algebraic semantics takes the lead (see [199], [37] for the general theory of algebraic logic, and [95] for a survey of modal consequence). A **logical matrix** is a pair $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ consisting of an algebra of appropriate type and a set $D \subseteq A$, the set of **designated elements**. A valuation is a function v from the set of variables to the domain A; we can also think of it as a homomorphism from the term algebra into $\mathfrak{A}$. φ is **true in $\mathfrak{M}$ under** v if $\overline{v}(\varphi) \in D$. Also, $\Gamma \vDash_{\mathfrak{M}} \varphi$ if every v which makes all formulae of Γ true also makes φ true. $\mathfrak{M}$ is a **matrix for** L if D is closed under the rules of L. Since the only rule of inference is (MP), this comes down to requiring that D is a Boolean filter and that all tautologies map to an element of D under every v. Filters correspond in the dual frame to topologically closed sets of points. We can however specialise to **ultrafilters**, which in the dual frame correspond to single worlds (more exactly, singleton sets). For discussion see [93].

Theorem 2. *For every normal modal propositional logic L there is a class $\mathcal{V}_L$ of matrices such that if $\Gamma \nvdash_L \varphi$ then there exists a $\mathfrak{M} \in \mathcal{V}_L$ such that $\Gamma \nvDash_{\mathfrak{M}} \varphi$.*

In fact, we can always choose the matrices to be of the form $\langle \mathfrak{F}_L(V), U \rangle$, U an ultrafilter. This creates the necessary structures from the language itself. Duality theory can be applied to the ontology as follows. Either we look at worlds as primitive objects and define facts as sets of worlds, or we consider facts as primitive and define worlds as maximally consistent sets of facts (see [21] for a discussion).

3.2. Situations and Possibilities. In the philosophical and linguistic literature, often an object quite like a possible worlds is used, namely a *situation*, see [149] and references therein. In contrast to worlds, situations are only partial. Moreover, situations can be 'small' in that their description may require only a single sentence. Situations can be accommodated as follows. Let $\mathfrak{A}$ be a modal algebra. The members of the underlying sets are called **situations**. Given a valuation v and a situation a, write

$$\langle \mathfrak{A}, a, \beta \rangle \vDash \varphi :\Leftrightarrow \beta(\varphi) \geq a \tag{15}$$

Technically, the definitions in [149] amount to taking as matrices only those of the form $\langle \mathfrak{A}, F \rangle$, where $F = \{b : b \geq a\}$ is a principal Boolean filter. If that is the case, however, they are technically just a generalised version of worlds. The use of situations has been defended in [9, 10]. The reasons for pressing for situations against possible worlds has been in addition to the reasons just stated also the claim that possible worlds semantics carries an ontological commitment of their existence. For a defense

Table 1. Worlds and Situations

entity	algebra	language
situation	filter	deductively closed set
world	ultrafilter	maximally consistent set

of possible worlds against this criticism see [180]. Notice that while the above matrices sacrifice bivalence, they do support the same tautologies as the ones based on ultrafilters. Thus, they spell out a supervaluationist account of truth. A related approach is that of [86]. A **possibility** is a partial function from propositional letters to truth-values. The interpretation (15) is not truth-functional. Moreover, it is not classical; both φ and $\neg\varphi$ can fail to hold in a. Much has been made of this fact in situation theory. In the present set-up, one can either declare situations to be primitive and construct worlds out of situations, or do the converse. The latter strategy will then treat the truth of a formula in a situation just like supervaluations. Supervaluations are however not compositional. Compositional accounts can of course be given, but they support less formulae than the supervaluation account (for example the law of excluded middle). This is a dilemma also for **quantum logic**. Standardly, quantum logic is supposed to be non-distributive since the algebras are orthomodular lattices. It is however possible to use a supervaluation approach, simply allowing only special filters as sets of designated elements (see [17]).

The correspondences are summarised in Table 1. Notice that from the standpoint of duality there is no ontologically primary object. An element of the algebra can be construed either as a set of worlds, or as a set of ultrafilters. Also, an element corresponds to a finitely generated Boolean filter, hence a particular kind of situation. We add that a general frame contains not only worlds and an accessibility relation but also an algebra of sets, whose members we may actually identify as the situations.

§4. Possible Worlds as an Analytic Tool. Modal logic has proven to be a general tool to model nonclassical logic. This was already apparent in the work of C. I. Lewis from the 1920s [114, 142]. Lewis' systems **S1**–**S5** were systems of **strict implication**, which were attempts to capture the notion of relevant connection between premiss and conclusion of ordinary language implications. Originally conceived as alternatives to classical logic, it has been observed that one can reduce strict implication to material implication by

$$\varphi \twoheadrightarrow \psi := \Box(\varphi \to \psi) \tag{16}$$

Thus, the properties of $\twoheadrightarrow$ derive straight from the properties of $\Box$. This has fostered the development of modal logic, because the attention turned to modal systems that had a classical background logic rather than to weakenings of classical logics. (Linear logics has brought weakenings back into the game, since the reduction techniques do not work for resource conscious logics.) This method was most successful with the reduction of intuitionistic logic to modal logic. Kurt Gödel gave the following translation from intuitionistic language into monomodal logic [68, 69].

$$\begin{aligned}\tau(p) &:= \Box p \\ \tau(\varphi \wedge \chi) &:= \tau(\varphi) \wedge \tau(\chi) \\ \tau(\varphi \vee \chi) &:= \Box(\tau(\varphi) \vee \tau(\chi)) \\ \tau(\varphi \to \chi) &:= \Box(\tau(\varphi) \to \tau(\chi))\end{aligned} \tag{17}$$

Falsum is mapped to falsum, and $\neg\chi := \chi \to \bot$. This translation reduces intuitionistic logic to modal logic. In the background, however, we have a second reduction, that of modal logic to first-order logic, the standard translation. The possible worlds interpretation that results from this cascaded reduction is actually not dissimilar to the interpretation that Brouwer gave for his logic in the first place; the worlds exemplify stages of knowledge, and it is assumed that knowledge grows as we move to an accessible state (which also happens to be a later stage). Although Brouwer would have opposed a reduction of intuitionism to classical logic (he did oppose Heyting's interpretation), the exactness of the reduction shows that the interpretation is at least viable. Several decades of research have culminated in the following result. Let **Int** be intuitionistic logic. An extension of this logic is of the form $\mathbf{Int} + \Delta$, where Δ is some set of axioms. The system **Grz** is defined by

$$\mathbf{Grz} := \mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to \Box p \tag{18}$$

Here, $\mathbf{Grz} \oplus \Delta$ denotes the least normal logic containing both **Grz** and Δ.

THEOREM 3 (Blok). *The map $\boldsymbol{Int} + \Delta \mapsto \boldsymbol{Grz} \oplus \{\tau(\delta) : \delta \in \Delta\}$ is an isomorphism from the lattice of intermediate logics onto the lattice of consistent normal extensions of **Grz**.*

For details see [30]. This much should be enough to demonstrate that modal logic is an exact tool to study intermediate logics. What is more, the original intuitions supplied by Brouwer himself suggest that the formal apparatus actually reflects the analytic notions rather well. Brouwer has spoken about mathematics as a developing structure, where knowledge grows, so that once a statement is accepted, it is accepted for good, while it may be rejected and later come to be accepted. However, its negation is accepted at w only if the formula is not accepted at any v later than w. In intuitionism, facts are always positive; negative facts are in a sense absence of certain positive facts. This asymmetry of positive and negative facts has been removed in the **constructive logic N** of Nelson [138]. In constructive logic, a formula can be true or false or neither. It is accepted if it is always true and it is rejected if it always false. This is used under the name 'vivid logic' in computer science (see [147, 148]). There is a general method to embed intermediate logics into extensions of **N**, which allows to study intermediate logics as special kinds of constructive logics; see [194], and for general results [169, 170].

There are also other alternatives to classical logic that cannot be interpreted in terms of modal logic since they are based on the use of the *resources*. This is exemplified in **relevant logic** ([2, 3]) and **linear logic** of [67], see also [193]. Linear logic, for example, has a connective $\multimap$; $\varphi \multimap \chi$ allows to deduce χ, but the use of this inference means destroying the truth of the antecedent formula, so that it cannot be used again. Actually, it is not impossible to interpret $\multimap$ in classical logic. One can give the translation:

$$\text{if } \varphi \text{ is true at } t \text{ then } \chi \text{ will be true at } u > t.$$

$\multimap$ is hiding the time parameter. It is clear in this interpretation that if φ and $\varphi \multimap \chi$ hold, then the inference to χ means that χ holds at a later time point, but then the truth of φ is not guaranteed any more.

§5. Accessibility. In the earliest models, which interpreted necessity as logical necessity, the universe was just seen as a set of worlds, with one world singled out as the actual one. However, different interpretations of necessity require to put structure on this set.

5.1. Accessibility Relations. For example, if $\Box$ is read as 'it will always be the case that' then the set of worlds must be ordered by temporal precedence. Thus, in addition to worlds we also need an **accessibility relation**. The nature of this relation may be an empirical matter, once the interpretation is decided upon. For example, it is an empirical question how time is structured: it may be discrete, continuous, branching in the future, branching in the past, endless; perhaps we ought to model it as part of relativistic space time rather than just time by itself. These questions are nowadays no longer pure speculation; the structure of space-time is a domain of physics, and thus subject to exact inquiry. Nevertheless, the matter is far from settled. The notion of 'time' continues to be elusive. We may define time either **objectively** or **operationally** ([196] prefers the word **phenomenologically**). An objective definition derives the temporal relation from a manifold, typically the space–time continuum, the phenomenological definition instead uses operational criteria in establishing the structure of time. The nature and dimensionality of the space-time manifold is debated. An extreme example is the theory by Hugh Everett ([43]) according to which the universe develops deterministically by splitting itself into as many worlds as there are possible outcomes for the 'next' moment—the so-called **many worlds interpretation** of quantum mechanics. For such an (objective!) view, probably the most faithful model is based on histories (see §5.3). A more standard approach is to take the n–dimensional Minkowski space. The logic of Minkowski spacetime has been identified in all dimensions by Robert Goldblatt in [71] to be **S4.2** for the reflexive relation, and $\mathbf{D4.2} \oplus \Diamond p_1 \wedge \Diamond p_2 \to \Diamond(\Diamond p_1 \wedge \Diamond p_2)$ [171] for the irreflexive relation. It follows that, in spacetime, the future and past operators do not allow to discriminate different dimensions.

An objective deterministic interpretation is quite compatible with phenomenological nondeterminism. The future may be open simply because we cannot have access to enough facts that will exclude all but one possibility. Many philosophers have been tempted to conclude that this does not apply to the past. Events of the past leave traces ('records') and facts about the past can be answered by looking at these records. However, as [195] points out, these records are necessarily partial. Also, the records exist here and now and are used to 'unravel' the present state into a development that has brought it about (see also [121] on cognitive aspects of this). This unravelling uses the laws of nature, the very laws that we can only deduce by relying on past experience. This creates an abyss for the foundations of physics that has troubled no lesser minds than Einstein himself, see [196]. Objectively time may branch into the future while it does not branch into the past; yet, technically it is not easy to explain why that should be so. The only nonsymmetric law is the second law of thermodynamics, which says that order decreases in direction of the future.

On the other hand, notice also that sometimes special interpretations of temporal logic are used which do not depend on the physical nature of time, such as the **temporal logic of programs**. Here, the fact that time is discrete is a design feature of computers.

While temporal interpretations allow the independent study of the accessibility relation, this is not so for the epistemic modalities. Belief worlds cannot be claimed to exist in the same way as future worlds, and their structure is most likely derived from the belief structure of the agent(s). If A believes δ in v then every belief-alternative w must make δ true. Here, the belief in δ comes first, and the accessibility is constructed. Perhaps this is the reason why epistemic modalities generally fail most common logical laws, in contrast to temporal modalities. Indeed, it has been argued among others by Quine that there is a difference between temporal modalities and epistemic modalities [156]. While it makes sense to attribute de re in temporal logic this is meaningless for belief because we lack a notion of sameness of objects across worlds.

5.2. Neighbourhoods. There are alternatives to accessibility relations and the standard conception of truth. [168] introduces the notion of a **neighbourhood frame**, which is a pair $\langle W, N\rangle$, where W is a set of worlds and N a function from W to $\wp(\wp(W))$. As before, valuations are functions $\beta : V \to \wp(W)$, and formulae are true in a frame at a world relative to a valuation.

(19) $\langle\langle W, N\rangle, \beta, w\rangle \vDash \Box\varphi :\Leftrightarrow \{v : \langle\langle W, N\rangle, \beta, v\rangle \vDash \varphi\} \in N(w)$

Neighbourhood frames are also called **Scott-Montague** frames, after [167] and [134]. The theory of a class of neighbourhood frames is a classical modal logic. Evidently, a Kripke-frame can be turned into a neighbourhood frame (put $N(w) := \{S : S \supseteq \{v : w\ R\ v\}\}$). The converse is not true, however. [64] has provided examples of neighbourhood-incomplete logics, and [63] has shown that there are logics containing **S4** which are determined by their neighbourhood frames but not by their Kripke-frames.

The **logic of conditionals** is an example of a logic for which standard relational semantics is inadequate. The semantics for **counterfactuals** is therefore more complex. [182] for example use choice functions, [60] introduces a relation that depends also on two propositions ([58] argues that conditional logic has ‘hidden variables’ in the sense argued for in quantum mechanics).

David Lewis defines structures in which every world has a set of sets of worlds (set of ‘spheres’) around it [119]. The structures are special neighbourhood frames, for it is assumed that the set of spheres around w are concentric; that is, if S and T are spheres, then either $S \subseteq T$ or $T \subseteq S$. Worlds within the same sphere of w are thought of to be similar to the same degree to w. Without having to specify numerical values for how similar one world is to the next, it is possible to define that a world v is more similar to w than a world v'. This is the case if all spheres around w which contain v also contain v'. Lewis also considers alternative structures, for example with a comparative similarity relation, but shows that they are interreducible [118]. Now, $\varphi > \psi$ is taken to be true at w if in all worlds u most similar to w that make φ true, also ψ is true.

The notion of **similarity** between worlds is widely used. For instance, an idea developed in [184] is to relativise accessibility by attaching the relations with fuzzy numbers, thus talking about ‘degrees of accessibility’, an idea that eventually turned into logics where accessibility is regimented by metric spaces or similarity measures, compare e.g. [110]. Quite often, one imagines a world that is as similar as possible to ours, while something specific is different. So we consider worlds in which φ is true, despite the fact that it is not. The spheres around the worlds are assumed by Lewis to be given. Epistemologically, one would like to be able to say how to find this structure, that is, to construe it from properties of the worlds themselves. A primitive solution is that the spheres around w are the worlds of Hamming-distance $\leq n$ for every given n. (This means that at most n values of the primitive letters can be changed in going from w to a world in that sphere.) We could also say that the worlds in the spheres result in the revision of the theory at w. Revision has been studied extensively in recent decades, but this topic is outside the scope of this article.[11]

A related phenomenon that requires more than basic modal logic is the **imperfective paradox**, exemplified in 'The dog was crossing the road when it got hit by a car.' If the dog was hit by a car how can we sensibly attribute to the dog that it was about to cross the road? David Dowty, in an attempt to explain the semantics behind the imperfective paradox, takes recourse to **inertia worlds** [41]. These are worlds which are very much like ours, in particular normal laws of physics and everyday life hold. Angelika Kratzer develops another idea, whereby the actual world lumps propositions together such that they cannot be given up independently [100] . Kratzer makes this possible by introducing situations and a refinement relation, the maximal members of which are the worlds. All these are attempts at defining accessibility from notions that are verifiable hic-et-nunc. They are only gradually different from causality in the sense that all of these notions are derived from directly observed facts of the world. Causality is different only inasmuch as it is believed to be the source of the regularity, not the other way around.

The clause (19) contains two quantifiers, an existential and a universal. Moreover, the function N is type-theoretically second-order, therefore allowing for greater expressivity. The idea of modal operators with complex interpretive clauses of any level has been pursued by Mark Brown, with applications to various interpretations (group knowledge, ability, among other, see [24]). His interpretations are different from the one above, though. It can be shown that modal logic of the ordinary sort can interpret these complex ones [99].

5.3. Histories. It has been argued that propositions are not true at a world simpliciter but only at a history. This conception, called **Ockhamist**, has been revitalised by Arthur Prior in [153] and formalised in [27]. Call a **history** a maximal set of worlds linearly ordered by time (this definition presupposes a particular structure of the frame to begin with). A formula is evaluated in a frame at a pair $\langle h, v\rangle$ where h is a history and $v \in h$. Then

(20) $$\begin{aligned}&\langle\langle W, R\rangle, h, v, \beta\rangle \vDash [F]\varphi \\ &\quad :\Leftrightarrow \text{ for all } w \in h \text{ such that } v\ R\ w : \langle\langle W, R\rangle, h, w, \beta\rangle \vDash \varphi\end{aligned}$$

[11] But compare [1] for the proposal now referred to as the AGM theory of revision (named after its inventors), and [122] for an overview.

(21) $\langle\langle W, R\rangle, h, v, \beta\rangle \vDash [P]\varphi$
$$:\Leftrightarrow \text{ for all } w \in h \text{ such that } w\ R\ v : \langle\langle W, R\rangle, h, w, \beta\rangle \vDash \varphi$$

(22) $\langle\langle W, R\rangle, h, v, \beta\rangle \vDash \Box\varphi$
$$:\Leftrightarrow \text{ for all } h' \text{ such that } v \in h' : \langle\langle W, R\rangle, h', v, \beta\rangle \vDash \varphi$$

One may additionally complicate these frames by allowing only certain bundles of histories to be values of propositions. In computer science these systems have received growing attention. It has been shown only fairly recently that Ockhamist temporal logic is more expressive [158] and complete axiomatisations have been given among others in [201] for discrete time and (for real lines and branching in the future) in [159]. A different interpretation, the **Peircean** interpretation (see [26]) accepts a future statement as true if and only if for for every history through the given world, it will be true at a later point. This means that φ is certain to happen in the future. This can be rendered in an Ockhamist interpretation as $\Box\langle F\rangle\varphi$.

Part 2. The World of Objects

Combining modality with quantification is a subtle affair. According to Alonzo Church [32], the year 1946 should be considered to be the birth of the discipline as a modern logical enterprise, with Ruth C. Barcan publishing 'A Functional Calculus of First Order Based on Strict Implication' [8] and, almost simultaneously, Rudolf Carnap publishing 'Modalities and Quantification' [28], these two papers being the first modern systematic treatments mixing *unrestricted* quantification as introduced to logic by Gottlob Frege with modalities, taking up a subject that Aristotle left more than two millennia ago.

Barcan's paper also introduced the now famous Barcan formulae which mix **de re** modalities attributing a necessity to a *thing* (as in 'Everything necessarily exists') with **de dicto** modalities, attributing necessity to *what is said* (as in 'Necessarily, everything exists'); a distinction which essentially goes back to Aristotle's distinction between *composition* and *division* [145, 176].

§6. Modal Predicate Logic. The languages of modal predicate logic that we shall consider first differ from the language of modal propositional logic as follows. First, there are neither propositional variables nor propositional constants. Second, we shall distinguish between **first-order** and **second-order** languages of modal predicate logic which differ syntactically as well as with respect to the **substitution principles** assumed, and which are defined thus. In the first-order case, we have a set $U = \{x_i : i \in \mathbb{N}\}$ of **object variables** (we postpone the discussion of object constants to Section 7) and, instead of the propositional variables, a set of **predicate constant letters** together with a map Ω assigning to each letter P its **arity** $\Omega(P)$. Though it is possible to add function symbols, we shall not do so, as they are technically eliminable. In that case, terms are just variables. If $\Omega(P) = n$, and the t_i, $1 \leq i \leq n$, are terms, then $P(t_1, \ldots, t_n)$ is a proposition. A set L is a **normal first-order modal predicate logic** (**first-order MPL**) if

1. L contains all instances of the axioms of predicate logic.
2. L contains the first-order instantiations of the tautologies of **K**.
3. L is closed under (MP) and (MN).

Since L comprises standard predicate logic, it is closed under the usual substitution of terms for variables; call this closure under **first-order substitution**. For the treatment of equality, see below, §6.3.

In the second-order case, we replace the propositional variables with sets $\{P_i^n : i \in \mathbb{N}\}$ of **predicate variables** of arity n for each n. A set L of formulae in this language, then, is called a **normal second-order modal predicate logic** if it is closed under substitution of any formula $\varphi(y_1, \cdots, y_n)$ for an occurrence of $P_i^n(x_1, \cdots, x_n)$ in a formula χ; call this closure under **second-order substitution**.[12]

Notice that the second-order MPLs thus defined are not truly second-order since there are no second-order quantifiers. In general, the terms 'modal predicate logic' or 'first-order modal logic' are typically taken to mean second-order modal predicate logic in the sense above, for instance in [85] and [55].[13]

Much of the previous discussion of modal propositional logic transfers to the predicate setting. It does, however, introduce complications of its own. In what is to follow we shall review some of the proposals for semantics of modal predicate logic that have been made. The standard Kripke-style semantics can obviously never be complete for the reason that already not all propositional logics are complete. The best one can hope for is therefore that a modal predicate logic is complete on condition that its propositional counterpart is. Below, we shall exhibit such a semantics.

While the notion of a predicate is generally taken to be unproblematic, the notion of an object has been the subject of considerable controversy. Part of the controversy can be retraced in type theory as follows. In addition to the type t of truth values and the type of **individuals** e

[12] This substitution must be formulated with care to prevent accidental capture of the y_i. The best way to do this is to rename any bound occurrence of y_i, $1 \leq i \leq n$, in χ prior to substitution. For a detailed discussion consult [97].

[13] Other types of first- or higher-order intensional logics that we will not discuss here include Alonzo Church's attempt to formalise Frege's logic of sense and denotation [32], and, growing out of this, the work of Richard Montague [135], Daniel Gallin [61], and Melvin Fitting [53]; compare [122] and [20] for more details.

(for predicate logic) we introduce a new type, s, that of possible worlds. Propositions have type $s \to t$. The question is: what type do we associate with object variables? Kripke assumes they have type e simpliciter; others have argued they should be seen as having type $s \to e$ (**individual concepts**). In the latter case, predicates may either be regarded as relations of individuals or as relations of individuals under a concept. In the first case, a binary predicate letter is interpreted as $s \to (e \to (e \to t))$, in the second case as $s \to ((s \to e) \to ((s \to e) \to t))$. Once the types are fixed, one can actually infer from Henkin's completeness proof for **Simple Type Theory** [76] a general completeness theorem for various semantics. However, the resulting structures are rather unintuitive.

6.1. The Classical View. Models for modal predicate logic are defined by Saul Kripke as follows. A frame is once again a pair $\langle F, R\rangle$, with F a set of worlds, and $R \subseteq F^2$. Each world is however now a pair $w = \langle D_w, \mathfrak{I}_w\rangle$, where D_w is a set (the domain of w) and $\mathfrak{I}_w$ an interpretation of the predicates and constants in D_w. If $w\ R\ v$ then $D_w \subseteq D_v$. This means every valuation into D_w is a valuation into D_v as well. This allows for a straightforward definition of the truth of $\Box\varphi$:

$$(23) \qquad \langle\langle F, R\rangle, \beta, w\rangle \vDash \Box\varphi :\Leftrightarrow \text{ for all } v \rhd w : \langle\langle F, R\rangle, \beta, v\rangle \vDash \varphi$$

The condition on growing domains corresponds to the validity of the **converse Barcan formula** $\Box\forall x\varphi(x) \to \forall x\Box\varphi(x)$. The converse of this implication, the **Barcan formula**, $\forall x\Box\varphi(x) \to \Box\forall x\varphi(x)$, is not generally valid.[14] It is, however, if we have $D_v = D_w$ for all $v\ R\ w$ so that domains are effectively constant. Another principle generally true in these frames is the **necessity of identity**

$$(24) \qquad (x = y) \to \Box(x = y)$$

which is defended by Saul Kripke in the context of arguing for his **causal theory of reference** [104].[15]

It follows from the strong version of Leibniz' Principle (see §6.3 below):

$$(25) \qquad s = t \to (\varphi(s/x) \leftrightarrow \varphi(t/x))$$

In this interpretation, objects are transcendental, and so the question of trans-world identity does not arise. This fact is responsible for the necessity both of identity and nonidentity. Also, domains have to grow; objects cannot be removed once they have been introduced.

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[14] This appeared first in 1946 as Axiom 11 in Ruth C. Barcan's [8], although as the notational variant $\Diamond\exists x\varphi(x) \twoheadrightarrow \exists x\Diamond\varphi(x)$ involving the existential diamond $\Diamond$ and strict implication.

[15] Which is closely related to Ruth Barcan Marcus' theory of 'contentless directly referential tags' account of proper names developed in her [127], a famous discussion of which appeared in [129], listing R. B Marcus, W. V. Quine, S. Kripke, J. McCarthy, and D. Føllesdal as discussants. For a historical account, compare [179].

The logic of these structures, **QK**, is characterised over the languages introduced above by the following additional axioms.

1. The converse Barcan formula.
2. $x = y \to \Box(x = y)$.
3. $x \neq y \to \Box(x \neq y)$.

If a strengthening is obtained by adding only purely modal propositional axioms, we refer to it as **quantified** L, where L is the modal counterpart. Often, the symbol $\mathbf{Q}L$ is used. Other strengthenings involve interactions between quantifiers, identity, and modal operators. These are the most popular logics under investigation. The remaining option, to just strengthen the predicate logical axioms, is rarely considered, partly because these axioms can easily be eliminated by conditionalising the other axioms.

6.2. Free Logic. There are many problems with the converse Barcan-formula. Obviously, objects that exist in this world need not exist in others. One way to fix this is to introduce **free logic** (compare [112] for an overview). Free logic adds to predicate logic an existence predicate, denoted here by E, which is interpreted like an ordinary 1-place predicate. The set $\mathfrak{I}(E)$ is called the **domain of existence**, and its members are said to **exist**. Objects outside of $\mathfrak{I}(E)$ exist only in a weaker sense. The standard quantifiers range only over existing objects. Therefore, some laws of predicate logic, notably the law of **universal instantiation**, have to be weakened.[16] The axiomatisation is as follows. In addition to the axioms and rules of **K** we assume (where $\mathrm{fv}(\varphi)$ denotes the variables free in φ)

1. If $x \notin \mathrm{fv}(\varphi)$, then $\varphi \to (\forall x)\varphi$ is a tautology.
2. (a) $(\forall x)E(x)$.
   (b) If y is free for x in $\varphi(x, \vec{z})$ and $y \notin \vec{z}$, then $(\forall x)\varphi(x, \vec{z}) \to (E(y) \to \varphi(y, \vec{z}))$ is a tautology.
3. The tautologies are closed under (UG): $\varphi/(\forall x)\varphi$.

If equality is a symbol of the language, we add the following.

1. $x = x$ is a tautology.
2. $x = y \to (\varphi(x) \to \varphi(y//x))$ is a tautology, where $\varphi(y//x)$ denotes the result of replacing one or more free occurrences of x by y, where y does not become accidentally bound in that occurrence.

The last is the strong Leibniz' Principle, see the discussion below. [80] has observed that existence can be defined by

$$E(x) \leftrightarrow (\exists y)(y = x) \tag{26}$$

[16] Historically, this weakening of the rule of universal instantiation goes back to the work of Saul Kripke [103] and Karel Lambert [111].

Alternatively, one can introduce quantifiers $\bigwedge$ and $\bigvee$, which range over the entire universe, and define

$$(\forall x)\varphi := \bigwedge_x E(x) \to \varphi \tag{27}$$

(See for example [83].) The difference between these two types of quantifiers is often labelled **possibilist** ($\bigwedge$) versus **actualist** ($\forall$), *cf.* [152, 85].

Using free logic one can eliminate a problem that besets the standard semantics. If domains have to grow, there is no way in which some object may cease to exist at some point. However, we do want to say that Aristotle does not exist now, even though at some point he did. Rather than trying to accommodate for this using different model structures, it seems better to use free logic. (The advice to use free logic in the context of modal logic was explicitly given in [167], compare also [62].) Once free logic is introduced we may even assume that $D_w = D_v$ if $w\ R\ v$, so that one may effectively assume that all models have the same domain. However, notice that free logic is more flexible; neither the Barcan formula nor its converse hold any more because objects can freely pass in and out of existence. The converse Barcan formula holds only if $\mathfrak{I}_w(E) \subseteq \mathfrak{I}_v(E)$ for $w\ R\ v$.

There are two ways to go from here. [57] allows predicates to take as values in a given world any set of n-tuples drawn from the domain, regardless of their existence, thus contradicting the **Falsehood Principle** of Kit Fine [48] which states that the extension of a predicate at a given world is a set of existing n-tuples. This principle can be reinstated in the form of the axioms

$$\bigwedge_{x_1} \cdots \bigwedge_{x_n} (P(x_1, \ldots, x_n) \to E(x_1) \wedge \cdots \wedge E(x_n)) \tag{28}$$

for every primitive n-ary predicate letter P.

It should be clear, however, that the semantical problems related to nonexistents or fictional discourse in general (or talk of 'impossible' objects such as the infamous 'round square' for that matter) are rather complex and not simply solved by moving to free logic, as witnessed, for instance, by Gideon Rosen's 'Modal Fictionalism' [160] (which builds on David Lewis's counterpart theory while trying to avoid its ontological commitments) or Terence Parsons's 'Nonexistent Objects' [144].

6.3. Identity, Substitution and Leibniz' Law. Leibniz' Law asserts that identicals are substitutable 'salva veritate'. In modal predicate logics, this applies both to propositions and to objects. We shall look at propositions first. Standardly, Leibniz' Principle is rendered as follows, which we call the **strong Leibniz' Law**:

$$p \leftrightarrow q \vDash_L \varphi(p/r) \leftrightarrow \varphi(q/r) \tag{29}$$

Boolean logic satisfies this. A modal logic satisfies (29) iff it satisfies

$$p \leftrightarrow q \vDash \Box p \leftrightarrow \Box q \tag{30}$$

Since the only rule of inference is (MP), this is equivalent to $p \to \Box p \in L$. The notion is therefore trivialised. [94] argues that Leibniz' Principle is not about accidental truth but about meaning. From this perspective, the principle actually amounts to the admissibility of the rule (RE), repeated here for convenience.

$$\frac{\varphi \leftrightarrow \chi}{\Box\varphi \leftrightarrow \Box\chi} \tag{31}$$

Leibniz' Law in this form is valid in every classical, and hence in every normal modal logic (as opposed to the strengthened version (29), which is valid only if $p \to \Box p \in L$).

When talking about identity and Leibniz' law with respect to objects, typically Leibniz' Principle is spelled out in the strong version. The **weak** version is this.

$$\frac{s = t}{\varphi(s/x) \leftrightarrow \varphi(t/x)} \tag{32}$$

Notice that such a substitution principle is actually derivable in the case of classical first-order logic and more generally for any logic that is axiomatised by *unrestricted* schemata. Nevertheless, by assuming unrestricted second-order substitution for a given logic L one automatically extends the underlying modal theory of identity. *E.g.*, given that $(x = y) \to (P(x, x) \to P(x, y))$ is an admissible instance of Leibniz' Law, second-order substitution yields $(x = y) \to (\Box(x = x) \to \Box(x = y))$ and hence $(x = y) \to \Box(x = y)$. Actually, this situation is one of the reasons for introducing a weaker base logic than the usual **QK**. [96] worked with a system called **FK** which is a combination of propositional modal logic **K** and **positive free logic**, **PFL**.[17]

If equality is introduced, the base logic is enriched by a weak form of Leibniz' Law, which we called the **Modal Leibniz' Law**. This basically results from the usual Leibniz' Law by restricting the Quinean principle of the 'substitutability of identicals' to those instances that do not entail 'transworld-identifications' of individuals of any kind. Briefly, if $x = y$ and the variable x appears free within the scope of a modal operator, then either *all* or *no* occurrence of x may be replaced by y. Hence, $(x = y) \to (\Box(x = x) \to \Box(x = y))$ is not an admissible substitution, which blocks the provability of the necessity of identity.[18]

[17] For a more detailed discussion and an argumentation why free logic is not only useful but necessary, *cf.* [96] or [106].

[18] Thus, the modal operators behave quite similar to what is known as an *unselective binder* in linguistics.

A natural solution to the above problem of generating possibly unintended theorems involving equality is therefore to deal with second-order logics without identity and to add a modal theory of identity, or, alternatively, to incorporate the theory of identity into the logic while restricting substitution in an appropriate way.

Since closure under second-order substitutions has a quite different flavour in a propositional as opposed to a predicate logic setting, there are a number of reasons to be interested in first-order MPLs and to treat them as *genuine* logics. We list just a few of them. First, if atomic propositions/predicates enjoy a special status—like in certain logics of time—then substitution of complex formulae for atoms may not be admissible. Actually, this was one of the reasons for Robert Goldblatt to introduce a similar distinction in the propositional case and to call it a 'significant conceptual change' (compare his [72]). Similarly for the case where basic predicates may be *intensional*. Second, if one works with a weak *logic of identity*, then a restriction of substitution is unavoidable. Last but not least, if generalised semantics are considered, there are naturally defined frame classes whose logic is only first-order closed. However, one can also argue in favour of closure under second-order substitution as a *defining property* of the general concept of a 'logic', which has been attempted for the case of MPL in [12].[19]

To make this discussion less theoretical, let us have a brief look at arguably the most prominent logic for which general substitution fails: Rudolf Carnap's logic of **logical truth** (L-truth) [28, 29]. Let us call the propositional version of Carnap's logic **C**, following Schurz [165] where a detailed discussion can be found. **C** as well as the quantified version **QC** of Carnap's logic are often mistaken to be versions of (quantified) **S5**, but they are in fact much stronger and have rather unusual logical properties.

The source of the differences between **S5** and **C** is, however, easily located. While Carnap considers a sentence to be logically true if it is valid in a *fixed* set $\mathbf{W}$ of all possible interpretations of the language, in Kripke's account [101], a sentence is logically true if it is valid in each variable subspace $V \subset \mathbf{W}$.[20] Thus, while in Carnap's logic we have for every satisfiable sentence A the logical truth $\Diamond A$ as a theorem of **C**, Kripke's **S5** does not contain any non-trivial possibility theorems at all. Whenever $\Diamond A$ is in **S5**, so is A. And exactly these non-trivial possibility theorems of **C** account for the fact that substitution fails in general, as should

[19]According to Alfred Tarski [185], the idea to define *logical truth* syntactically as the requirement of closure under all substitution instances goes back to Bernhard Bolzano, compare also [164].

[20]This was further analysed by Nino Cocchiarella [33] with his distinction between *primary* and *secondary* semantics. Notice also the analogy to the frame/general frame distinction.

be obvious.[21] As [165] argues, if one identifies necessity with **logical necessity** then **C** is the *only* complete modal logic, compare also [34].

In fact, as Carnap [28] and Thomason [192] independently showed, **S5** corresponds exactly to the substitutionally closed fragment of **C**.

6.4. Completion. Propositional modal logics are not always complete with respect to Kripke-semantics. The same holds a fortiori for modal predicate logics. There is a general procedure for turning an algebraic model for the propositional language into a model for the predicate language, which goes back to [113], and ultimately to the completeness proof by Henkin for **Simple Type Theory**. This will be discussed in Section 11.

There is an idea analogous to the construction of a general frame: we start with a Kripke-frame and restrict the interpretation of predicates. This is a purely linguistic restriction; there is nothing in the ontology itself that suggests why such a restriction is warranted. The technical details are as follows. We introduce a 'tower' of sets that represent the possible values for predicates. Before we can give a precise definition, some technical preliminaries are necessary. If $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ is a map, and $\vec{a}$ is an n-tuple, put $\sigma(\vec{a}) := \langle a_{\sigma(1)}, \dots, a_{\sigma(m)} \rangle$. This defines a map from n-tuples to m-tuples; it is lifted to sets as follows.

$$\sigma(c) := \{\sigma(\vec{a}) : \vec{a} \in c\} \tag{33}$$

Also, the **cylindrification** E_m for $m \leq n$ is defined by

$$E_m(c) := \{\langle a_1, \dots, a_n \rangle : \exists d. \langle a_1, \dots, a_{m-1}, d, a_{m+1}, \dots, a_n \rangle \in c\} \tag{34}$$

A **generalised Kripke-frame** is a structure $\langle W, R, \mathcal{C} \rangle$, where $\langle W, R \rangle$ is a Kripke-frame and $\mathcal{C} = \{\mathbb{C}_n : n \in \mathbb{N}\}$ is a sequence of sets $\mathbb{C}_n \subseteq \wp(D^n)$ satisfying the following postulates.

1. $\mathbb{C}_n$ is a modal algebra for every n.
2. If $c \in \mathbb{C}_n$ and $\sigma : m \to n$ then $\sigma(c) \in \mathbb{C}_m$.
3. If $c \in \mathbb{C}_n$ and $n \geq m$ then $E_m(c) \in \mathbb{C}_n$.

We call such a system a **tower**. Now, the condition is that for a predicate P, the interpretation $\mathfrak{I}_w(P)$ is a member of $\mathbb{C}_n$. The following clause is added.

$$\langle W, R, \mathcal{C}, \beta, w \rangle \vDash P(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \Leftrightarrow \langle x_1, \dots, x_m \rangle \in \sigma(\mathfrak{I}_w(P)) \tag{35}$$

By assumption, the interpretation of a formula is a member of the tower. The sets of the tower do not change from world to world.

So far, we have discussed the syntactic definition of modal predicate logics, some of the standard semantical approaches including Kripkean possible worlds semantics and free logic, as well as related problems, as

[21] Clearly, $\bot$ cannot be substituted for A in $\Diamond A$. It also follows that, unlike **QS5**, the theorems of **QC** are not even recursively enumerable.

for instance the validity of the Barcan formulae in standard semantics and the problems with substitution and identity. Many of the more general semantics that have been proposed are designed to deal with one or the other of these problems, with the eventual goal of very general completeness theorems encapsulating a minimum of ontological presuppositions, or—as Johan van Benthem has formulated it—referring to the merits of completeness proofs in general, that 'embody conditions of adequacy on empirical theories in semantics' [15].

The term **Kripke-type semantics**, as introduced in [177], refers to a family of semantics that, rather than introducing purely algebraic structures, tries to keep as much as possible of the basic intuitions of 'possible worlds semantics' (the 'geometric approach') while transcending its scope [107].

The main differences to the standard semantics are twofold: first, instead of taking a Kripke frame, that is, a set of possible worlds together with an accessibility relation, and to enrich it by assigning domains to worlds, one starts with a family of first-order domains and adds some set of functions or relations between the domains, which in turn *define* accessibility between worlds. Hence, a *plain* accessibility relation is no longer a primitive of the frame but rather depends on (can be defined by) the functions/relations being present. This leads to the second fundamental difference, namely that there may indeed be many distinct functions/relations between two given worlds, a feature that we will label 'modes of transgression' in the context of counterpart semantics that we will investigate in the next section. Figure 1 shows some of the different proposed Kripke-type semantics and their interdependencies. An arrow from A to B indicates that the semantics A is a special case of (can be simulated by) semantics B.

§7. Counterpart Semantics. David Lewis [117] provided the first formal theory of counterparts. It is a two-sorted first-order theory, whose sorts are objects and worlds, and which has four predicates: $W(x)$ says that x is a world, $I(x, y)$ that x is in the world y, $A(x)$ that x is an actual object, and $C(x, y)$ that x is a counterpart of y. The postulates codify that every object is in one and only one world, that counterparts of objects are objects, that no two different objects of the same world can be counterparts of each other, any object is a counterpart of itself, and that there is a world that contains all and only the actual objects and which is non-empty.

The connection to the standard modal language is obtained by translating sentence of MPL to the counterpart theory via the so-called **Lewis translation** (for details compare [117]).

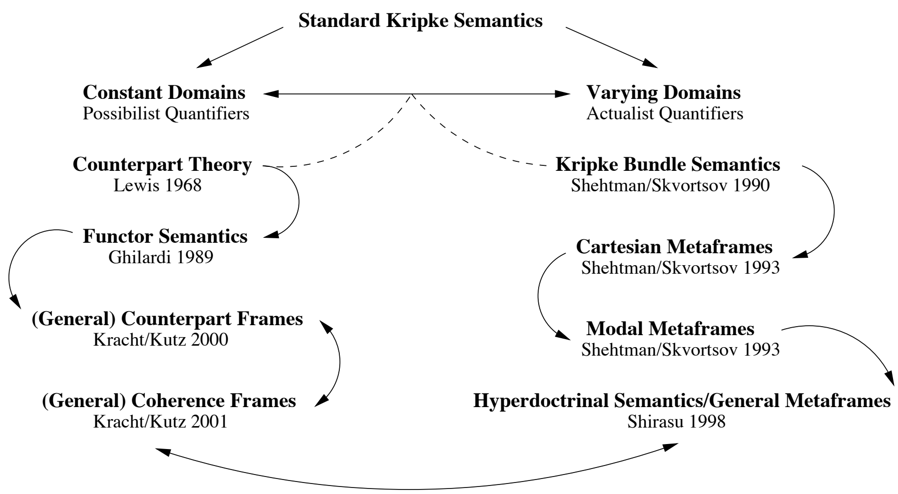

FIGURE 1. An overview of various semantics for modal predicate logics ordered by relative generality.

For instance, an expression of the form $[\Diamond\varphi(t_1, \ldots, t_n)]^v$ (‘$\Diamond\varphi(t_1, \ldots, t_n)$ holds in world v’) is recursively translated to

(36) $$\exists w \exists s_1 \ldots \exists s_n (W(w) \wedge I(s_1, w) \wedge C(s_1, t_1) \wedge \ldots \\ \ldots \wedge I(s_n, w) \wedge C(s_n, t_n) \wedge [\varphi(s_1, \ldots, s_n]^w)$$

However, there are several problems related to this translation approach. First, in its original form, counterpart theory does not validate all theorems of the smallest quantified modal logic **QK** since it fails to generally support the principle of **Box distribution**

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi),$$

which is valid in all normal modal logics, compare [85]. In fact, the simultaneous quantification over both worlds and individuals in counterpart theory obscures the notion of accessibility between worlds and thus leads to the semantic refutability of certain **K**-theorems [106].

Further, it has often been argued that the standard modal languages are not expressive enough. For instance, the natural language sentence

‘There could be something that doesn’t actually exist’

can only be rendered in a language comprising an **actuality operator** **A** as well as an existence predicate E [81, 82], namely as $\Diamond\exists x \neg \mathbf{A} E(x)$.[22]

[22]This example is related to an ontological objection raised by Plantinga [150] as well as Konyndyk [92], namely that the standard interpretation of modal language implies the existence of fictional, non-existent objects, which is at odds with several varieties of modal actualism. Compare also [31].

Extensions of the modal language, however, require modification to the Lewis translation scheme, as has been attempted for instance by Graeme Forbes [56] and Murali Ramachandran [157].

But these translations are only partially faithful, as Fara and Williamson argue [44] who translate inconsistent sentences of modal predicate logic involving an actuality operator (interpreted in standard Kripke semantics) to satisfiable sentences of counterpart theory via different translation schemes. (Yet, this is not too surprising given that the semantics of counterpart theory is more flexible than the standard Kripkean semantics of modal predicate logic.)

We will not follow this line of research here any further. Rather, we are interested in the ideas underlying counterpart theory as a semantical framework for quantified modal logic in general, an approach that was initiated by Allen Hazen [75]. From this perspective, counterpart theory can be reformulated as follows.

We assume that a frame is a collection of first-order structures over pairwise disjoint domains, and add a relation C on the union of the domains. If x and y are from the same domain, then $C(x, y)$ if and only if $x = y$. Thus, a **counterpart frame** is a pair $\langle W, C\rangle$, where W is a set of worlds such that

$$D_v \cap D_w = \varnothing \text{ iff } v \neq w, \tag{37}$$

and $C \subseteq (\bigcup_{w\in W} D_w)^2$ a relation such that, for each v

$$C \cap D_v^2 = \{\langle x, x\rangle : x \in D_v\}. \tag{38}$$

Instead of specifying actual objects we make use of the actual world. The formula $\Diamond\varphi(\vec{a})$ is true at w if there is a world v and counterparts c_i for a_i in v, i.e. $\langle a_i, c_i\rangle \in C(w, v)$ for all i, such that $\varphi(\vec{c})$ is true at v. The following are *not* theorems in counterpart frames, though they are valid in standard constant-domain Kripke-frames.

1. $(x = y) \to \Box(x = y)$
2. $(x \neq y) \to \Box(x \neq y)$
3. $\forall x\Box\varphi(x) \to \Box\forall x\varphi(x)$

The necessity of identity and distinctness fail because counterpart relations need not be functional nor do they need to be injective. Also, the Barcan formula is not valid for there could be individuals in an accessible world w that are not the counterpart of any individual in v (counterpart relations are not assumed to be 'surjective'). The converse Barcan formula $\Box\forall x\varphi(x) \to \forall x\Box\varphi(x)$, however, is also a theorem of counterpart theory, for if *everything* in w makes φ true so does a fortiori everything that is a counterpart of something in v.

Notice that Lewis considers only counterpart relation from one world to another world. However, there may be many 'different ways' to move from one world to another. This distinguishes Kripke-type semantics also significantly from standard counterpart theory (*cf.* [117]) and its derived possible worlds semantics (*cf.* [75]).

That this feature of multiple functions or relations is not eliminable is due to the fact that there are second-order closed MPLs that are complete only with respect to frames having at least two counterpart relations between worlds, *cf.* [97].

Counterparts need not be unique, an object can have several counterparts in another world. Neither are counterpart relations assumed in general to be symmetric or transitive. If counterparts were unique both ways, the theory would be exactly like that of Kripke's, only that we have changed the ontology. Objects in the Kripke-frames become equivalence classes of the counterparts relation. Although that seems to be just a technical move, Lewis did try to identify criteria for establishing whether an object is a counterpart of another. One principle may be explicated as follows (see [57]).

> For $c \in D_u$ and $d \in D_v$, d is a counterpart of c only if nothing in v is more similar to c as it is in u than is d as it is in v.

This is only a necessary condition and it is not clear what would be sufficient. Furthermore, interpretations of counterparthood based on similarity face several problems in general. For instance, as Feldman discusses in [45], how could we possibly translate a sentence such as 'I could have been quite unlike what I in fact am' when counterparthood is understood solely as a relation of similarity?

Another problem is that the counterparts are defined for each object individually. This creates technical problems. Suppose we only have equality, and axioms that say 'there are exactly three individuals' and 'there is exactly one world'. Then the counterpart relations must be functions, despite the fact that the above definition would predict that any object can serve as a counterpart for any given object. Another problem is this. If similarity is defined in terms of how many of the properties the objects share, we can only use 1-place properties to determine similarity and must exclude properties such as 'a is father of b'. However, once we have chosen a counterpart for b in the next world there is only one way to choose a counterpart for a, despite the fact that prior to the choice of b, we might have had more than one counterpart.

Thus, we see that we must actually specify the counterparts for all objects at once. This leads to the **presheaf semantics** outlined in [65]. The original definitions worked only for extensions for **S4**, but they can be generalised easily. Consider a category $\mathcal{C}$. A category has objects, which are here considered as *worlds*, and **morphisms** or **maps** between objects, which define what we call **modes of transgression**. The idea is that from a world u there are several ways to transgress into the world

v. Now, a presheaf is a functor F from $\mathcal{C}$ to the category of sets and functions. This means that every world u is mapped to a set $F(u)$, and every map $f : u \to v$ is mapped to a function $F(f) : F(u) \to F(v)$. There are two further conditions: $F(f \circ g) = F(f) \circ F(g)$, and for the identity id_u we have $F(\mathrm{id}_u) = \mathrm{id}_{F(u)}$. We consider for simplicity valuations at a single world. If u is a world, then a valuation β at u is a function from the object variables to $F(u)$. The clause for the modal operator is as follows.

(39) $\langle F, u, \beta\rangle \vDash \Diamond\varphi :\Leftrightarrow$ there is $f : u \to v$ such that $\langle F, v, F(f) \circ \beta\rangle \vDash \varphi$

Thus, one thinks of $F(f)$ as the function that takes each object in u into its counterpart in v under the mode of transgression f. If β is a valuation at u, and $f : u \to v$ a mode of transgression, then $F(f) : F(u) \to F(v)$ can be composed with β to give a valuation at v. The following is valid in presheaf semantics:

1. All axioms of **QS4** are tautologies.
2. $(x = y) \to \Box(x = y)$.
3. The tautologies are closed under (MP), Universal Generalisation and Leibniz' Principle.

[96] take one more step of generalisation. Rather than using functions, they employ once again relations, but allow a set of them. We give a slight adaptation of the definitions of that paper to take care of the fact that we now deal with possibilist quantifiers plus an existence predicate as opposed to 'proper' free-logical quantifiers. Furthermore, we shall make more explicit the world dependence of the universes.

Call a relation $R \subseteq M \times N$ a **CE-relation** (CE stands for 'counterpart existence') if for all $x \in M$ there exists a $y \in N$ such that $x\ R\ y$ and, likewise, for all $y \in N$ there exists an $x \in M$ such that $x\ R\ y$. Since we assume free logic, this is a harmless complication (it does not entail claims about existence, but is needed to avoid truth-value gaps).

Definition 4. *A* ***poly-counterpart frame*** *is a quadruple* $\langle W, T, \mathcal{U}, \mathcal{C}\rangle$*, where* $W, T \neq \varnothing$ *are non-empty sets,* $\mathcal{U}$ *a function assigning to each* $v \in W$ *a non-empty subset* $\mathcal{U}_v$ *of* T *(its* ***domain****) and, finally,* $\mathcal{C}$ *a function assigning to each pair of worlds* v*,* w *a set* $\mathcal{C}(v, w)$ *of CE-relations from* $\mathcal{U}_v$ *to* $\mathcal{U}_w$*. A pair* $\langle\mathfrak{W}, \mathfrak{I}\rangle$ *is called a* ***counterpart structure*** *if* $\mathfrak{W}$ *is a counterpart frame and* $\mathfrak{I}$ *an interpretation, that is, a function assigning to each* $w \in W$ *and to each* n*-ary predicate letter a subset of* $\mathcal{U}_w^n$*.*

We say that v **sees** w in $\mathfrak{F}$ if $\mathcal{C}(v, w) \neq \varnothing$—thus the notion of 'accessibility' is completely reduced to the existence of a counterpart relation. A **valuation** is a function η which assigns to every possible world v and every variable an element from the universe $\mathcal{U}_v$ of v. We write η_v for the valuation η at v. A **counterpart model** is a quadruple $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{I}, \eta, w\rangle$, where $\mathfrak{F}$ is a counterpart frame, $\mathfrak{I}$ an interpretation, η a valuation and

$w \in W$. If $\rho \in \mathcal{C}(v, w)$, write $\eta \xrightarrow{\rho} \widetilde{\eta}$ if for all $x \in V$: $\langle \eta_v(x), \widetilde{\eta}_w(x)\rangle \in \rho$. Truth in a counterpart model is defined as follows. Let $v \in W$ and η be a valuation.

$$(40)\quad \langle \mathfrak{F}, \mathfrak{I}, \eta, v\rangle \vDash \Diamond\varphi(\vec{y}) \text{ there are } w \in W, \rho \in \mathcal{C}(v, w) \text{ and } \widetilde{\eta} \text{ such that } \eta \xrightarrow{\rho} \widetilde{\eta} \text{ and } \langle \mathfrak{F}, \mathfrak{I}, \widetilde{\eta}, w\rangle \vDash \varphi(\vec{y})$$

This semantics can be enriched by towers in the sense above to obtain general structures. The minimal logic of these structures can be axiomatised as follows. On top of free logic and modal logic, we require the truth of the following principles for identity.

1. $x = x$.
2. If y is free for $\varphi(x)$, then $x = y \to (\varphi(x) \leftrightarrow \varphi(y//x))$, where $\varphi(y//x)$ denotes the result of replacing one or more free occurrences of x not in the scope of a modal operator by y.
3. $x = y \wedge \Diamond\top \to \Diamond(x = y)$.

The last postulate is true generally of counterpart theory. However, it can hardly be considered valid when one thinks of the quantifiers as quantifying over intensional rather than extensional (trace-like) objects (see the discussion of constants below). The necessity of identity, $(x = y) \to \Box(x = y)$, as well as the necessity of difference are both invalid. To see the first, let u be a world containing just a with two counterparts b and c in v. Then we can choose for x the counterpart b and for y the counterpart c.

Notice that for constants the situation is different. For suppose that the language contains constants and that they are interpreted by individual concepts. Then neither principle is valid. For let in the same model the interpretation of $\underline{d}$ be a in u and b in v, but for $\underline{e}$ it is a in u and c in v. Then $(\underline{d} = \underline{e}) \wedge \Diamond\top \to \Diamond(\underline{e} = \underline{e})$ as well as $(\underline{d} = \underline{e}) \to \Box(\underline{d} = \underline{e})$ are both invalid. This disparity of constants and variables is a rather worrying aspect of counterpart semantics. Presheaf semantics does not have this defect since counterparts are unique. On the other hand, in presheaf semantics the necessity of identity is valid.

The modal Leibniz' law of [96] allows for simultaneous substitution of all free occurrences of x by y in $\Diamond\chi$ (denoted by $\Diamond\chi(y//x)$, provided that $x = y$ is true.

$$(41)\qquad \bigwedge x.\bigwedge y.x = y. \to .\Diamond\chi(x) \to \Diamond\chi(y//x)$$

Now, notice that in Kripke-semantics the rule of replacing constants for universally quantified variables is valid. In counterpart frames this creates unexpected difficulties.

For suppose that constants are present and may be substituted for variables. Then we may derive from (41), using the substitution of c for

x and d for y:

$$(c = d \wedge \Diamond\top) \to \Diamond(c = d) \tag{42}$$

Since a constant has a fixed interpretation in each world, this means that if two constants are equal in a world and there exists some accessible world, then there will also be some accessible world in which they are equal. This is not generally valid. What is happening here is a shift from a *de re* to a *de dicto* interpretation. If we follow the traces of the objects, the formula is valid, but if we substitute intensional objects, namely constants, it becomes refutable. Notice that this situation is also reflected in the way non-rigid constants are treated in [55]. There, the two possible readings of the above formula, the de dicto and de re reading, are distinguished by actually binding the interpretation of the constants to the respective worlds by using the term-binding λ-operator.[23]

Applied to Hesperus and Phosphorus, this means that if they are equal, then there is a belief world of George's in which they are equal. However, if George believes that they are different, this cannot be the case. So, the counterpart semantics cannot handle constants correctly—at least not in a straightforward way, i.e., without restricting the possible values of constants in accessible worlds. This paradox is avoided in Kracht and Kutz [96] by assuming that the language actually has *no* constants.

Counterpart theory as well as modal predicate logic were originally intended to formalise natural language statements involving modalities, but their expressive richness make them also obvious candidates for knowledge representation in Artificial Intelligence. Thus, let us make a few comments about their computational properties.

Counterpart theory is formulated as a first-order theory. While the **monadic fragment** [125] (containing only unary predicates) or the **two-variable fragment** [166, 137] of classical predicate logic are decidable, the fragment of predicate logic with binary predicates and three variables is already undecidable [183], which means that only a very limited fragment of counterpart theory is decidable.

When we move to modal predicate logic, though, the situation does not improve. Kripke already showed in 1962 that the monadic fragment of MPL is undecidable [102]. Later, it was shown that also the two-variable fragment of practically all standard MPLs based on Kripkean constant domain semantics are undecidable [40], which seemed to imply that the

[23] λ-abstraction was introduced to modal logic by Robert Stalnaker and Richmond H. Thomason in [181, 190] and corresponds to Bertrand Russell's scoping device used to analyse **definite descriptions**, originating in [161] and systematically used in Principia Mathematica [198]. More recently, λ-abstraction was studied by Melvin Fitting [51], who also investigated modal languages that comprise λ-abstraction for constants but without allowing quantification, thus defining languages that are situated between modal propositional and predicate logics [52]. For a discussion compare also [20].

decision problem for modal predicate logic is almost hopelessly difficult. Only fairly recently has the search for decidable fragments of MPL been successful, namely with the definition of so-called **monodic fragments**[24] [200]. These are languages of MPL where at most one free variable is allowed in the scope of a modal operator; and basically any extension of a decidable fragment of classical predicate calculus to its monodic modal language yields a decidable formalism [59].

In general, more subtle combinations of languages have been studied, such as **fusions** [98, 50] or **products** [59] of modal logics (where modal predicate logic can be understood as the product of first-order predicate logic and modal propositional logic). But again, while fusions behave very nicely both computationally and logically, they are rather inexpressive, and, on the other hand, products, being quite expressive, are once again computationally very difficult in general [59].

Interestingly, then, the general idea of counterpart relations being based on a notion of similarity also gives rise to a framework of knowledge representation languages that is rather expressive, natural from a semantical point of view, and which is very well-behaved computationally, namely the theory of **$\mathcal{E}$-connections** [108, 109]. In $\mathcal{E}$-connections, a finite number of formalisms talking about disjoint domains are 'connected' by relations relating entities in different domains, intended to capture different aspects or representations of the 'same object'. For instance, an 'abstract' object o of a description logic L_1 can be related via a relation R to its life-span in a temporal logic L_2 (a set of time points) as well as to its spatial extension in a spatial logic L_3 (a set of points in a topological space, for instance). As with poly-counterpart frames, the presence of multiple relations between domains is essential for the versatility of this framework, the expressiveness of which can be varied by allowing different language constructs to be applied to the connecting relations. $\mathcal{E}$-connections approximate the expressivity of products of logics 'from below' and could, perhaps, be considered a more 'cognitively adequate' version of counterpart theory.

§8. Individual Concepts. We have noticed above that in many analyses there is an asymmetry between variables and constants. This asymmetry is easily explained. A constant is standardly interpreted by an **individual concept**, while variables are often used to refer to objects. In counterpart theory, objects are world bound, so the notion of an object is different from that of an individual concept. There are proposals which argue that the values of variables should likewise be individual concepts.

[24] A 'monody' is a kind of music distinguished by having a single melodic line and accompaniment.

Recent proposals to this effect are the worldline frames of [163] and the coherence frames of [97].

A **coherence frame** is a quintuple $\langle W, R, O, T, \tau\rangle$, where $\langle W, R, O\rangle$ is a predicate Kripke-frame, consisting of a set W of worlds, an accessibility relation R, and a set O, the set of **objects**, and where T is a set, the set of **things**, and $\tau : U \times W \to T$ a surjective function. We call τ the **trace function** and $\tau(o, w)$ the **trace of** o **in** w. The set $T_w = \{t \in T : t = \tau(o, w), o \in O\}$ is the set of things constituting the world w, i.e. the set of things which can bear properties. In [163], a **worldline** is defined to be a function $W \to T$. It turns out that objects can be identified with worldlines, and we shall simplify our exposition accordingly.

An **interpretation** is a function $\mathfrak{I}$ mapping each predicate letter P of arity $\Omega(P)$ to a function from W to $T_w^{\Omega(P)}$ and each constant symbol c to a member of U. Let us call an interpretation $\mathfrak{I}$ **equivalential** if for all $\vec{a}, \vec{c} \in U^{\Omega(P)}$ and $w \in W$, if $\tau(a_i, w) = \tau(c_i, w)$ for all $i < \Omega(P)$ then $\vec{a} \in \mathfrak{I}(P)(w)$ if and only if $\vec{c} \in \mathfrak{I}(P)(w)$. (This condition is enforced only for extensional predicates.) A **coherence structure**, then, is a pair $\langle \mathfrak{W}, \mathfrak{I}\rangle$ where $\mathfrak{W}$ is a coherence frame and $\mathfrak{I}$ an equivalential interpretation. A **coherence model** is a triple $\langle \mathfrak{C}, \beta, w\rangle$, where $\mathfrak{C}$ is a coherence structure, $\beta : V \to U$ a valuation, $w \in W$. Every term t evaluates into a unique object $\varepsilon(t)$ (which depends on the valuation).

$$(43)\quad \begin{array}{lll} \langle \mathfrak{C}, \beta, w\rangle \vDash P(\vec{t}) & :\Leftrightarrow & \varepsilon(\vec{t}) \in \mathfrak{I}_w(P) \\ \langle \mathfrak{C}, \beta, w\rangle \vDash s = t & :\Leftrightarrow & \tau(\varepsilon(s), w) = \tau(\varepsilon(t), w) \\ \langle \mathfrak{C}, \beta, w\rangle \vDash \bigvee x.\varphi & :\Leftrightarrow & \text{for some } \gamma \text{ with } \gamma \sim_x \beta : \langle \mathfrak{C}, \gamma, w\rangle \vDash \varphi \\ \langle \mathfrak{C}, \beta, w\rangle \vDash \Diamond\varphi & :\Leftrightarrow & \text{there is } w' \rhd w \text{ such that } \langle \mathfrak{C}, \beta, w'\rangle \vDash \varphi \end{array}$$

In worldline semantics, predicates are interpreted as sequences of n-tuples of traces. The following is put in place of the first condition:

$$(44)\qquad \langle \mathfrak{C}, \beta, w\rangle \vDash P(\vec{t}) :\Leftrightarrow \langle \tau(\varepsilon(t_1), w), \cdots, \tau(\varepsilon(t_n), w)\rangle \in \mathfrak{I}_w(P)$$

This allows to drop the condition of equivalentiality. However, it removes the flexibility in dealing with intensional predicates. It is a matter of straightforward verification to show that all axioms and rules of the minimal MPL are valid in a coherence frame. Moreover, the set of formulae valid in a coherence structure constitutes a first-order MPL. Notice that the fourth postulate for equality holds in virtue of the special clause for equality and the condition that the interpretation must be equivalential. For if $\langle \mathfrak{C}, \beta, w\rangle \vDash y_i = y_n$, then $\tau(\beta(y_i), w) = \tau(\beta(y_n), w)$. So, if $\langle \mathfrak{C}, \beta, w\rangle \vDash P(y_0, \ldots, y_{n-1})$ for $P \in \Pi$, then $\langle \beta(y_i) : i < n\rangle \in \mathfrak{I}(P)(w)$. Let $\beta' \sim_{y_i} \beta$ be such that $\beta'(y_i) = \beta(y_n)$. By equivalentiality, $\langle \beta'(y_i) : i < n\rangle \in \mathfrak{I}(P)(w)$. This means that $\langle \mathfrak{C}, \beta', w\rangle \vDash P(y_0, \ldots, y_{n-1})$, and so $\langle \mathfrak{C}, \beta, w\rangle \vDash [y_n/y_i]P(y_0, \ldots, y_{n-1})$. If $\mathfrak{F}$ is a coherence frame, put $\mathfrak{F} \vDash \varphi$ if $\langle \mathfrak{F}, \mathfrak{I}\rangle \vDash \varphi$ for all equivalential interpretations $\mathfrak{I}$. Evidently, $\{\varphi : \mathfrak{F} \vDash \varphi\}$ is a first-order MPL.

The frames allow two objects to share the entire worldline, just as worlds can support the same propositions. This means that in a coherence frame the objects are not mere constructions, they exist in their own right. We say that $o = o'$ is true at w if and only if $\tau(o)(w) = \tau(o')(w)$. It follows that the formula

$$x = y \wedge \Diamond\top \rightarrow \Diamond(x = y) \tag{45}$$

is not valid in coherence frames, since there is no way to predict how the worldline of an object develops from one world to the next. If we turn to predicates, their interpretation is now a function $\mathfrak{I}(P)$, which assigns to each world w a subset of D_w^n. Thus predicates, like identity, are predicated not of the objects themselves, but of the traces in their worlds. Thus if $\tau(o)(w) = \tau(o')(w)$ then o and o' satisfy the same nonmodal formulae. The semantics is generalised using towers. This leads to the notion of a **generalised coherence frame**. Free logic is assumed as well. [97] show that the semantics of generalised coherence frames is complete for all modal predicate logics. Moreover, similar to the case of standard Kripke semantics, a variant of the well-known Henkin construction yields a general completeness result with respect to structures involving interpretations (rather than with respect to frames). The difference, however, to standard Kripke semantics is that coherence frames also cover various logics of identity, for details compare [97].

This semantics eliminates on the one hand the asymmetry between variables and constants and on the other the possibility of speaking of things ('res') simpliciter. A thing can only be understood as the materialisation of an individual concept, and is not conceptualised itself. This is especially noticeable in the light of the criticism leveled against counterpart theory. It has namely been observed that counterpart theory makes every actual property of an object at a world w a necessary one, for the reason that no object of another world is identical to it (see [150] for the argument and [139] for a rebuttal). Thus, the condition that attributions are exclusively of the individual concept remove that problem. The semantics based on the interpretation of variables as individual concepts is complete and provides a rather elementary completeness proof for modal predicate logic. Also, the notion of an individual concept is not more suspicious than the counterpart theory. Counterpart relations can be seen as the traces of these functions. On the other hand, given a counterpart relation, we can define individual concepts in the following way. Given a counterpart relation C, a compatible worldline is any function f such that if $v \; R \; w$ then $f(v) \; C \; f(w)$. However, if $v \; R \; v$ it may become impossible to choose enough worldlines. However, one can always first unravel the counterpart frame before extracting the worldlines (see [97]).

Note that since trace functions are assumed to be surjective, every trace has to be the trace of some object. This is a natural condition, because objects are considered to be the primary entity, and traces a derived notion. The notion of equivalence is perhaps a curious one. It says that the basic properties of objects cannot discriminate between objects of equal trace. So, if Pierre believes that London is beautiful and Londres is not, while at the same time Londres *is* London [105], we have two objects which happen to have the same trace in this world, though not in any of Pierre's belief worlds. Hence they must share all properties *in this world*. So, London and Londres can only be both beautiful or both ugly. This seems very plausible indeed. From a technical point of view, however, the fact that they cannot simply have different properties is a mere stipulation on our part. On the other hand, it is conceivable that there are basic predicates that are actually intensional, which would mean that they fail the substitution under (extensional) equality.

The difference with the counterpart semantics is that we have disentangled the quantification over objects from the quantification over worlds. Moreover, objects exist independently of worlds. Each object leaves a trace in a given world, though it need not exist there. Furthermore, two objects can have the same trace in any given world without being identical. However, identity of two objects holds in a world if and only if they have the same trace in it. If we also have function symbols, the clauses for basic predicates and equality will have to be generalised in the obvious direction.

§9. Objects in Counterpart Frames. One of the main intuitions behind counterpart frames is that objects do not exist in more than one world, they are, in Lewis' terminology, **worldbound**. We call such objects **individuals**. Variables are interpreted in them as individuals, with no connection between the values at different worlds assumed. A more abstract, transworld notion of an **object** can only be reconstructed by following an individuals's counterparts along the counterpart relations being present. It turns out, however, that counterpart frames may have very few objects in this sense.

DEFINITION 5 (Objects). *Let $\mathfrak{F} = \langle W, T, \mathcal{U}, \mathcal{C}\rangle$ be a counterpart frame. An* ***object*** *is a function $f : W \to T$ such that (i) $f(v) \in \mathcal{U}_v$ for all $v \in W$, (ii) for each pair $v, w \in W$ with $\mathcal{C}(v, w) \neq \varnothing$ there is $\rho \in \mathcal{C}(v, w)$ such that $\langle f(v), f(w)\rangle \in \rho$.*

So, objects are constructed using the counterpart relation. If the trace b in world w is a counterpart of the trace a in world v, then there may be an object leaving trace a in v and trace b in w. If not, then not. However, there are frames which are not empty and possess no objects. Here is an example. Let $W = \{v\}$, $T = \{a, b\}$, $\mathcal{U}_v = \{a, b\}$, and $\mathcal{C}(v, v) = \{\rho\}$ with

$\rho = \{\langle a, b\rangle, \langle b, a\rangle\}$. It is easy to see that this frame has no objects. The crux is that we can only choose one trace per world, but when we pass to an accessible world, we must choose a counterpart. This may become impossible the moment we have cycles in the frame.

Thus, the connection between coherence frames and counterpart frames is not at all straightforward. Since the logic of a counterpart frame is a first-order modal predicate logic, one might expect that for every counterpart frame there is a coherence frame having the same logic. This is only approximately the case. It follows from Theorem 7 below that for every counterpart *structure* there is a coherence structure having the same theory. This is not generally true for frames. However, adopting a modification of coherence frames proposed by Melvin Fitting in [54], namely balanced coherence frames (in [54] the corresponding frames are called **Riemann FOIL frames** in analogy to Riemann surfaces in complex analysis), it can indeed be shown that for every counterpart frame there is a balanced coherence frame validating the same logic (under a translation).

Let us begin by elucidating some of the connections between counterpart and coherence frames. Note again that since counterpart structures as defined above do not interpret constants, we have to assume that the language does not contain any constants.

First, fix a coherence structure $\mathfrak{C} = \langle W, \lhd, U, T, \tau, \mathfrak{I}\rangle$. We put $U_v := \{\tau(o, v) : o \in U\}$. This defines the domains of the world. Next, for $v, w \in W$ we put $\rho(v, w) := \{\langle\tau(o, v), \tau(o, w)\rangle : o \in U\}$ and $\mathcal{C}(v, w) := \varnothing$ if $v \lhd w$ does not obtain; otherwise, $\mathcal{C}(v, w) := \{\rho(v, w)\}$. Finally, $\langle\tau(a_i, w) : i < \Omega(P)\rangle \in \mathfrak{I}'(P)(w)$ iff $\langle a_i : i < \Omega(P)\rangle \in \mathfrak{I}(P)(w)$. Then $\langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{I}'\rangle$ is a counterpart structure. We shall denote it by $CP(\mathfrak{C})$. Notice that there is at most one counterpart relation between any two worlds.

Conversely, let a counterpart structure $\mathfrak{N} = \langle W, T, \mathcal{U}, \mathcal{C}, \mathfrak{I}\rangle$ be given. We put $v \lhd w$ iff $\mathcal{C}(v, w) \neq \varnothing$. $U := T$. Let O be the set of all objects $o : W \to T$. Further, $\tau(o, w) := o(w)$. This defines a coherence frame if the set of objects is nonempty.[25] Finally, $\langle o_i : i < \Omega(P)\rangle \in \mathfrak{I}'(P)(w)$ iff $\langle o_i(w) : i < n\rangle \in \mathfrak{I}(P)(w)$. It is easy to see that this is an equivalential interpretation. So, $\langle W, \lhd, O, U, \tau, \mathfrak{I}'\rangle$ is a coherence structure, which we denote by $CH(\mathfrak{N})$.

Unfortunately, the logical relation between these two types of structures is rather opaque, not the least since the notion of satisfaction in them is different. Moreover, the operations just defined are not inverses of each other. For example, as we have already seen, there exist counterpart structures with nonempty domains which have no objects. In this case

[25] Strictly speaking, we have to reduce the set U of traces to those elements $t \in T$ that actually are the trace of some object o, but this makes no difference semantically.

$CP(CH(\mathfrak{N})) \ncong \mathfrak{N}$. Also let $\mathfrak{C}$ be the following coherence frame. $W := \{v, w, x, y\}$, $T := \{1, 2, 3, 4, 5, 6\}$, $U = \{a, b\}$, $\lhd = \{\langle v, w\rangle, \langle w, x\rangle, \langle x, y\rangle\}$. Finally, $\tau(a, -) : v \mapsto 1, w \mapsto 2, x \mapsto 4, y \mapsto 5$, $\tau(b, -) : v \mapsto 1, w \mapsto 3, x \mapsto 4, y \mapsto 6$. Generating the counterpart frame we find that 2 and 3 are counterparts of 1, and 5 and 6 are counterparts of 4. Hence, there are more objects in the counterpart frame than existed in the coherence frame, for example the function $v \mapsto 1, w \mapsto 2, x \mapsto 4, y \mapsto 6$.

Definition 6 (Threads). *Let $\mathfrak{N}$ be a counterpart frame. A sequence $\langle(w_i, t_i) : i < n\rangle$ is called a* ***thread*** *if (1) for all $i < n$: $w_i \in W$, $t_i \in \mathcal{U}_{w_i}$, and (2) for all $i < n-1$: $w_i \lhd w_{i+1}$ and $\langle t_i, t_{i+1}\rangle \in \rho$ for some $\rho \in \mathcal{C}(w_i, w_{i+1})$. $\mathfrak{N}$ is* ***rich in objects*** *if for all threads there exists an object o such that $o(w_i) = t_i$ for all $i < n$.*

Notice that if $\lhd$ has the property that any path between two worlds is unique then $\mathfrak{N}$ is automatically object rich. Otherwise, when there are two paths leading to the same world, we must be able to choose the same counterpart in it. Using unravelling one can always produce an object rich structure from a given one. Additionally, we can ensure that between any two worlds there is at most one counterpart relation. We call counterpart frames that satisfy the condition $|\mathcal{C}(v, w)| \leq 1$ for all worlds $v, w \in W$ **Lewisian counterpart frames**. The proof of the following theorem makes use of the unravelling technique.

Theorem 7. *For every counterpart structure $\mathfrak{N}$ there exists a Lewisian counterpart structure $\mathfrak{N}'$ rich in objects such that $\mathfrak{N}$ and $\mathfrak{N}'$ have the same first-order modal theory.*

For a proof, let $\mathfrak{N} = \langle\langle W, T, \mathcal{U}, \mathcal{C}\rangle, \mathfrak{I}\rangle$ be a counterpart structure. A **path** in $\mathfrak{N}$ is a sequence $\pi = \langle w_0, \langle\langle w_i, \rho_i\rangle : 0 < i < n\rangle\rangle$ such that $\rho_i \in \mathcal{C}(w_{i-1}, w_i)$ for all $0 < i < n$. We let $e(\pi) := w_{n-1}$ and $r(\pi) = \rho_{n-1}$ and call these, respectively, the **end point** and the **end relation** of π. Let W' be the set of all paths in $\mathfrak{N}$ and $T' := T$. Further, let $\mathcal{U}'_\pi := \mathcal{U}_{e(\pi)}$ and for two paths π and μ put $\mathcal{C}'(\pi, \mu) := r(\mu)$ if $r(\mu) \in \mathcal{C}(e(\pi), e(\mu))$ and empty otherwise. Finally, let P be an n-ary predicate letter. Then $\mathfrak{I}'(P)(\pi) := \mathfrak{I}(P)(e(\pi))$. Now let $\mathfrak{N}' = \langle\langle W', T', \mathcal{U}', \mathfrak{C}'\rangle, \mathfrak{I}'\rangle$. This is a Lewisian counterpart structure and clearly rich in objects. It can be verified by induction that if β is a valuation on $\mathfrak{N}$ and w a world, and if β' is a valuation on $\mathfrak{N}'$ and π a path such that $e(\pi) = w$ and $\beta'_\pi(x_i) = \beta_w(x_i)$, then $\langle\mathfrak{N}, \beta, w\rangle \vDash \varphi$ iff $\langle\mathfrak{N}', \beta', \pi\rangle \vDash \varphi$ for all φ.

Notice by the way that for second-order closed logics this theorem is false. This is so because the interpretation of a predicate in π must be identical to that of μ if the two have identical end points. We will re-encounter this problem in the next section when we will modify Kripkean semantics to match counterpart frames in terms of generality. However, if we are interested in characterising MPLs by means of models, it follows from the above result that we can restrict ourselves in the discussion to Lewisian counterpart structures that are rich in objects.

But we can also strike the following compromise. Let us keep the counterpart semantics as it is, but interpret formulae in a different way. We say that $\mathfrak{M} = \langle\langle \mathfrak{F}, \mathfrak{I}\rangle, \beta, v\rangle$ is an **objectual counterpart model**, if $\mathfrak{F}$ is a counterpart frame as before, $\mathfrak{I}$ is an **objectual interpretation**, that is, a counterpart interpretation that additionally assigns objects to constant symbols, β an **objectual valuation** into $\mathfrak{F}$, i.e., a function that assigns to each variable an object in a given a world. In this context, $\varepsilon_v(o) := \beta_v(o)$ if o is a variable and $\varepsilon_v(o) = \mathfrak{I}(o)(v)$ if o is a constant symbol.

Write $\beta \to^{\vec{y}}_{v,w} \beta$ if for some $\rho \in \mathfrak{C}(v, w)$ we have $\langle \beta_v(x_i), \beta_w(x_i)\rangle \in \rho$ for all $x_i \in \vec{y}$. Furthermore, write $\beta \to^{\vec{y}}_{v,w} \gamma$ if for some $\rho \in \mathfrak{C}(v, w)$ we have $\langle \beta_v(x_i), \gamma_w(x_i)\rangle \in \rho$ for all $x_i \in \vec{y}$, where γ is an objectual valuation. Terms t_i denote either variables or constants, $\vec{y}$ tuples of variables and $\vec{c}$ tuples of constants. The symbol $\vDash^*$ is called the **weak objectual truth-relation** and is defined thus:

(46)

$$\begin{array}{lll}\langle \mathfrak{M}, \beta, v\rangle \vDash^* R(\vec{t}) & :\Leftrightarrow & \langle \varepsilon_v(t_0), \dots, \varepsilon_v(t_{n-1})\rangle \in \mathfrak{I}_v(R) \\ \langle \mathfrak{M}, \beta, v\rangle \vDash^* \Diamond\varphi(\vec{y}, \vec{c}) & :\Leftrightarrow & \text{there is } \beta \to^{\vec{y}}_{v,w} \gamma : \langle \mathfrak{M}, \gamma, w\rangle \vDash^* \varphi(\vec{y}, \vec{c}) \\ \langle \mathfrak{M}, \beta, v\rangle \vDash^* \bigvee y.\varphi(y, \vec{c}) & :\Leftrightarrow & \text{there is } \widetilde{\beta} \sim_y \beta : \langle \mathfrak{M}, \widetilde{\beta}, v\rangle \vDash^* \varphi(y, \vec{c})\end{array}$$

The **strong objectual truth-relation** $\vDash^\dagger$ is like $\vDash^*$ except for the clause for $\Diamond$ which is now

(47)

$$\langle \mathfrak{M}, \beta, v\rangle \vDash^\dagger \Diamond\varphi(\vec{y}, \vec{c}) \quad :\Leftrightarrow \quad \text{there is } \beta \to^{\vec{y}}_{v,w} \beta : \langle \mathfrak{M}, \beta, w\rangle \vDash^\dagger \varphi(\vec{y}, \vec{c})$$

These interpretations remove the asymmetry between variables and constants in the sense that constants and variables are now assigned the same kind of values. However, while the strong objectual interpretation brings us very close to coherence semantics, the weak interpretation still bears essential properties of counterpart semantics, namely that we may move via a counterpart relation to a new object. More precisely we have the following:

Proposition 8. *The rule of substituting constants for universally quantified variables is valid in the strong objectual interpretation. More specifically, for every counterpart frame* $\mathfrak{F}$

$$\mathfrak{F} \vDash^\dagger (\bigwedge x.\varphi) \to [c/x]\varphi. \tag{48}$$

Furthermore, there is an objectual counterpart model $\mathfrak{M}$ *such that*

$$\mathfrak{M} \nvDash^\dagger \bigwedge x. \bigwedge y.(x = y) \to (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z})) \tag{49}$$

Both claims are false for the weak objectual interpretation.

In object rich frames, the weak objectual interpretation gives the same theorems as the standard interpretation:

THEOREM 9. *Let* $\mathfrak{N}$ *be a counterpart structure rich in objects,* v *a world and let* β *be an objectual valuation and* $\widetilde{\beta}$ *a counterpart valuation such that* $\beta_v(x_i) = \widetilde{\beta}_v(x_i)$ *for all variables. Then for all* φ*:*

$$\langle \mathfrak{N}, \beta, v \rangle \vDash^* \varphi \quad \Leftrightarrow \quad \langle \mathfrak{N}, \widetilde{\beta}, v \rangle \vDash \varphi \tag{50}$$

§10. Dual Ontologies, or, The Semantical Impact of Haecceitism. No two philosophers agree on the nature of individuals, even when restricted to the more mundane question of what a material object is. The same, unsurprisingly, holds true for the notion of a **modal individual**—a concept which some would even claim to be nonsensical or inconsistent, but which, nevertheless, is central to the metaphysics of modality.

For instance, the discussion concerning the question of how spatio-temporal objects can **persist** in time, currently divided between adherents of **endurantism** (also called **3-dimensionalism**, the view that an object exists at a time by being *wholly present* at that time, and *persists* by being wholly present at more than one time [87]) versus **perdurantism** (also called **4-dimensionalism**, roughly, the view that a material object has temporal as well as spatial parts, and *persists* by having distinct temporal parts at different times), consists essentially in competing ontologies as concerns the notion of a spatio-temporal object [143, 173]—and these conceptions of object correspond roughly to the different ontologies as represented in coherence frames versus standard semantics.[26] Obviously, this discussion can be specialised to the question whether objects in Minkowski space-time should be considered as perduring or rather as enduring, where, prima facie, special relativity seems to support the four-dimensional ontology, compare [6, 7].

Another metaphysical doctrine that has been subject to considerable inquiry is (first-order) **Haecceitism**, the view that there might be worlds that are distinguishable only by what individuals play what roles,[27] first introduced by David Kaplan [90], and elaborated on for instance by

[26] There is some controversy on which terminology is more appropriate (or less confusing). David Lewis and Mark Johnstone used the endurantism/perdurantism distinction [120, 88], while many later authors, including Theodore Sider [173], preferred the 3/4-dimensionalism distinction.

[27] As expected, philosophers disagree on this notion. According to [133], David Lewis [120] and David Armstrong [5] should be regarded as anti-haecceitists, while the later Kaplan [90] is a haecceistist. Brian Skyrms, on the other hand, accepts haecceitism for actual entities, but not for merely possible ones [178].

Joseph Melia [131] in connection with determinism and the 'hole argument' of Earman and Norton [42]. In [131], he defines **D-Haecceitism** to be the view that a theory may be indeterministic, even if all the different possible futures open to any world which makes the theory true are qualitatively identical. In [132, 133], Melia also discusses **second-order Haecceitism**, which may be defined as the position that there could be distinct worlds that agree on which second-order properties are occupied, but which disagree on which properties play which roles.[28]

A recurrent theme of this article was the idea that a semantics for modal logic should make as little ontological commitments as possible. It should not be necessary to define a new semantics once you commit to a new theory of identity across worlds, for example. The most flexible semantics in this respect so far (without moving to an algebraic semantics) was the generalised counterpart theory embodied in the notion of a poly-counterpart frame. In the following, we analyse how we can achieve the same kind of semantical generality while sticking to the more conventional Kripkean picture of possible worlds semantics. Curiously, this modification of Kripkean semantics requires to build a notion of Haecceitism into the frames, namely the notion of a **world-mirror**, see below.

By the theorems above we can introduce the notion of an object into counterpart frames, which then makes them rather similar to coherence frames. However, counterpart structures with object valuations are still different from coherence structures. A different approach is to translate $\Diamond$ in order to accommodate the truth relation $\vDash^*$ within the language of counterpart structures.

$$(51)\quad (\Diamond\varphi(y_0,\dots,y_{n-1}))^\gamma := \bigvee z_0.\dots\bigvee z_{n-1}.z_0 = y_0 \wedge z_1 = y_1 \wedge \cdots \wedge z_{n-1} = y_{n-1} \wedge \Diamond\varphi(\vec{z}/\vec{y})^\gamma$$

Here, y_i $(i < n)$ are the free variables of φ and z_i $(i < n)$ distinct variables not occurring in φ. This is actually unique only up to renaming of bound variables. This translation makes explicit the fact that variables inside a $\Diamond$ are on a par with bound variables.

PROPOSITION 10. *Let $\mathfrak{N}$ be a counterpart structure and x a world. Then for any φ:*

$$\langle\mathfrak{N}, x\rangle \vDash \varphi^\gamma \quad\Leftrightarrow\quad \langle\mathfrak{N}, x\rangle \vDash \varphi$$

In object rich structures also $\vDash$ and $\vDash^*$ coincide, which makes all four notions the same. So, while in counterpart structures the formulae φ and φ^γ are equivalent, they are certainly not equivalent when interpreted in coherence structures.

[28] David Lewis and David Armstrong both accept *this* version of Haecceitism.

In [163], it is shown that worldline semantics provides for the same class of frame complete logics in the absence of extra equality axioms as standard constant domain semantics. It follows that the same holds for coherence frames. This means that while coherence frames allow for a more natural treatment of non-rigid designation for example, unlike counterpart frames, they do not enlarge the class of frame complete logics unless one moves to the full second-order semantics involving towers as sketched above. But there is a different approach to this problem. Instead of introducing algebras of admissible interpretations we can assume that certain worlds are isomorphic copies of each other. So, we add to the frames an equivalence relation between worlds and require that predicates are always interpreted in the same way in equivalent worlds. This idea is due to Melvin Fitting (see [54]). Call a relation $\mathcal{E} \subseteq W \times W$ a **world-mirror on** $\mathfrak{F}$ if $\mathcal{E}$ is an equivalence relation and whenever $v \ \mathcal{E} \ w$ and $v \lhd u_1$, there is a u_2 such that $w \lhd u_2$ and $u_1 \ \mathcal{E} \ u_2$. Intuitively, two mirrored worlds v and w may be understood as a situation seen from two different perspectives (because v and w may have 'different histories', but have the 'same future').

Definition 11. *A pair $\langle \mathfrak{F}, \mathcal{E} \rangle$ is called a* ***balanced coherence frame****, if $\mathfrak{F} = \langle W, \lhd, U, T, \tau \rangle$ is a coherence frame and $\mathcal{E}$ is a world-mirror on $\mathfrak{F}$. An interpretation $\mathfrak{I}$ is called* ***balanced****, if it is equivalential and $\langle u_0, \ldots, u_{n-1} \rangle \in \mathfrak{I}_v(P)$ iff $\langle u_0, \ldots, u_{n-1} \rangle \in \mathfrak{I}_w(P)$ for all n-ary relations P and for all worlds v, w such that $v \, \mathcal{E} \, w$. A* ***balanced coherence model*** *is a triple $\langle \langle \mathfrak{B}, \mathfrak{I} \rangle, \beta, w \rangle$, where $\mathfrak{B}$ is a balanced coherence frame, $\mathfrak{I}$ a balanced interpretation, β a valuation and w a world.*

Theorem 12. *For every counterpart frame $\mathfrak{F}$ there exists a balanced coherence frame $\mathfrak{F}^*$ such that for all formulae φ:*

$$\mathfrak{F} \vDash \varphi \quad \Leftrightarrow \quad \mathfrak{F}^* \vDash \varphi^\gamma. \tag{52}$$

The details of the proof are rather technical and can be found in [97]. This result has interesting consequences. For example, since counterpart semantics is frame complete with respect to all first-order extensions $\mathbf{Q}L$ of canonical propositional modal logics L (compare [66]), the same holds true for the translation $\mathbf{Q}L^\gamma$ with respect to balanced coherence frames. Now we noted above that coherence frames per se characterise the same logics as standard constant domain semantics if no extra equality axioms are involved. But it is known that already rather simple canonical propositional logics have frame incomplete predicate extensions. In [36] it is shown that to complete frame incomplete MPLs by adding appropriate axioms, one needs mixed de re formulae rather than substitution instances of purely propositional formulae. So, the above result gives a hint on where the source for frame incompleteness with respect to standard semantics is to be found. In particular, note that the translation $.^\gamma$ leaves propositional formulae untouched, whereas de re formulae of the form $\Diamond\varphi(y_0, \ldots, y_{n-1})$ are transformed into formulae $\bigvee z_0. \ldots \bigvee z_{n-1}. \bigwedge_{i<n} z_i = y_i \wedge \Diamond\varphi(\vec{z}/\vec{y})^\gamma$, which are de re formulae involving equality.

So what we need if we want to use standard possible worlds semantics to characterise a large class of logics via frame completeness are basically three things: firstly, the distinction between trace and object, secondly a different understanding of the modal operator as given by $.^\gamma$, and, thirdly, the assumption that certain worlds are copies of each other.

Let us make this claim more explicit. Given a standard constant domain frame $\langle W, \lhd, U\rangle$, we may add, as before, an equivalence relation $\mathcal{E}$ relating worlds. Furthermore, we technically do not need traces but can add a family of equivalence relations $(\mu_w)_{w\in W}$ interpreting equality at each world. Let us call frames of the form $\mathfrak{F} = \langle W, \lhd, U, (\mu_w)_{w\in W}, \mathcal{E}\rangle$ **balanced standard frames**. An interpretation $\mathfrak{I}$ is called **admissible** if interpretations agree on worlds related by $\mathcal{E}$ and, moreover, they respect the equivalence relations μ_w in the sense that $\vec{a} \in \mathfrak{I}(w)(P)$ iff $\vec{b} \in \mathfrak{I}(w)(P)$ whenever $a_i \mu_w b_i$ for all i. We may think of objects being related by μ_w as indiscriminable with respect to world w and basic extensional predicates. Call a valuation $\widetilde{\gamma}$ a w-**ignorant** $\vec{x}$-**variant** of γ, if $\widetilde{\gamma}(x_i)\mu_w\gamma(x_i)$ for all $x_i \in \vec{x}$. The truth definition for balanced standard frames is as usual except for the equality and modal clauses, which are as follows:

- $\langle\mathfrak{F}, \mathfrak{I}, \gamma, w\rangle \vDash x = y$ iff $\gamma(x)\mu_w\gamma(y)$;
- $\langle\mathfrak{F}, \mathfrak{I}, \gamma, w\rangle \vDash \Diamond\varphi(\vec{x})$ iff there is a w-ignorant $\vec{x}$-variant $\widetilde{\gamma}$ and a world $v \rhd w$ such that $\langle\mathfrak{F}, \mathfrak{I}, \widetilde{\gamma}, v\rangle \vDash \varphi(\vec{x})$;

It should be rather clear that there is a bijective correspondence between balanced coherence frames and balanced interpretations on the one hand and balanced standard frames and admissible interpretations on the other. Furthermore, for every $\langle W, \lhd, U, T, \tau, \mathcal{E}\rangle$ there is a $\langle W, \lhd, U, (\mu_w)_{w\in W}, \mathcal{E}\rangle$ such that for all φ

$$\langle W, \lhd, U, T, \tau, \mathcal{E}\rangle \vDash \varphi^\gamma \text{ iff } \langle W, \lhd, U, (\mu_w)_{w\in W}, \mathcal{E}\rangle \vDash \varphi.$$

Hence, the following theorem is an immediate corollary to Theorem 12.

THEOREM 13. *For every poly-counterpart frame there is a balanced standard frame having the same logic.*

Table 2 gives a comparison of the different ontologies embodied in poly-counterpart frames versus balanced standard frames. It shows that if we want to avoid counterpart theory but still strive for a semantics that is of the same generality, we arrive at a framework where we not only have to give up numerical identity and replace it with equivalence relations, but where we are also committed to a certain form of Haecceitism, namely the one that corresponds to the notion of a world-mirror. (However, similar

TABLE 2. Competing Ontologies

	poly-counterpart	balanced standard
accessibility	relations between individuals	relations between worlds
objects	worldbound individuals	global universe of objects
identity	between individuals	equivalence between objects
Predication	locally	globally (but admissible)
Haecceitism	No	Yes, world-mirrors

assumptions are arrived at in the many-worlds interpretation of quantum mechanics, although, obviously, with entirely different motivations, compare [187].)

§11. Metaframes. The most radical shift is presented in the semantics proposed by Valentin Shehtman and Dmitrij Skvortsov in [177], which we alluded to in §6.4. Once again, it turned out in retrospect that the idea had an algebraic precursor in the hyperdoctrines of [113]. Though hyperdoctrines were originally defined as model structures for **Int**, the generalisation to other logics is straightforward (see [172]). Consider the category Σ of natural numbers and mappings between them. A **modal hyperdoctrine** is a covariant functor H from Σ into the category MA of modal algebras. $H(n)$ may be thought of as the algebra of meanings of formulae containing n free variables. To be well–defined, H must satisfy among other the so–called **Beck-Chevalley-condition**, which ensures that cylindrification has the same meaning on all levels. We shall not spell this out in detail; instead, we shall look at the natural dual $F := (-)^+ \circ H$ of H, where $(-)^+$ is defined as in (14). This maps n into the frame dual of $H(n)$. F is a contravariant functor into the category GFr of generalised frames. The system $\{F(n) : n \in \mathbb{N}\}$ provides a 'tower' of admissible systems of sets of their respective frames. In general, a **metaframe** is simply a contravariant functor from the category Σ to the category of general frames. Here is a recipe to obtain a metaframe which is adequate for the logic L. For every natural number n, let $H(n)$ be the algebra of the formulae which contain only x_1 to x_n free (under logical equivalence); and let $H(\sigma)$ be the result of substituting via σ. This defines a hyperdoctrine whose logic is L. To obtain a metaframe whose logic is L, simply take $M := (-)^+ \circ H$. The world of $M(n)$ are the n-**types** of L, and they constitute the worlds. The maps $M(\sigma)^+$ are the pre-images under the substitutions. They are p-morphisms between these frames. In general, for a metaframe M, $M(0)$ may be thought of as the frame of worlds in the usual sense. The frame of 1-types is different. First, since there is a map $\sigma : 0 \to 1$ (the empty map), there is a p-morphism $M(\sigma) : M(1) \to M(0)$. This map associates a 0-type to every 1-type. We may view each member of $M(1)$ as a world bound individual, but only up to logical equivalence. Similarly, $M(2)$ contains the world bound pairs of individuals, up to equivalence. An interpretation is a function ξ which maps every n-ary predicate letter P to a member of the internal sets of $M(n)$. Let v be in $M(k)$ for some k, and let $\sigma : n \to k$.

(53) $\langle M, \xi, \beta, v\rangle \vDash P(x_{\sigma(1)}, \dots, x_{\sigma(n)}) :\Leftrightarrow \langle \beta(x_1), \dots, \beta(x_k) \in \sigma(\xi(P))$

Notice that worlds of $M(n)$ can support the truth or falsity of formulae only if the free variables of that formula are within $\{x_1, \dots, x_n\}$. Thus, the clause for $\exists$ requires care, while the clause for $\Diamond$ is actually straightforward. The problem with the existential is that it requires to shift to another world v', since eliminating the quantifier makes a variable free that need not be supported by v.

The method yields a metaframe M for each logic L. Moreover, if the propositional counterpart of L is canonical, so is L (simply pass from $M(n)$ to the corresponding full frame). Thus, from a technical point of view this construction is rather well understood. Unfortunately, from a metaphysical point of view it is not. It dispenses completely with the notion of an object and in some sense also with that of an ordinary possible world. What we are left with is possible states-of-affairs relating n individuals. Notice that accessibility relations are defined over these states-of-affairs for every n. So, if a and b are such state-of-affairs at level n, then $a\ R\ b$ means that b is possible for a. Now, a can be expanded to a member of $M(n+1)$, and it can be reduced to a member of $M(n-1)$, and these operations are p-morphisms. This allows to reconstruct some of the classical notions. First, we can identify worlds as members of $M(0)$. Given a world v and $n > 0$, the set $O_n(v) := M(n)^{-1}(v)$ is the set of n-tuples of v. Now, ideally we would like to think of $O_n(v)$ as the n-fold cartesian product of $O_1(v)$. If that is so, [177] call the metaframe **cartesian**. [11] has shown that under mild conditions it is possible to construct a cartesian metaframe with the same n-types for all n out of a given metaframe. It then follows that all modal predicate logics are complete with respect to cartesian metaframes.

§12. Essence and Identity. An object is said to have a property P **essentially** if whenever x fails to have P it also fails to exist.[29] It has P **accidentally** in a world w if it has P. [150] has criticised counterpart theory of eliminating the distinction between accidental and essential properties of an object. [26] paints a different picture. In his view, we can certainly speak of sortal objects and essential properties that these objects have even if there is no identity in the strict sense. The problem is partly due also to the question of what an object actually is. In coherence frames objects are interpreted as individual concepts. Now, we effectively

[29]A discussion of various notions of 'essence' can be found in [49].

have two sorts of identification: two objects can be identical in this world iff they have the same trace, while transworld identity is simply identity as object (identity of the whole worldline). How this changes matter is best seen with the theory of sets. [57], as well as [47], expose a theory of sets, which embodies the idea that the essence of the set X is the fact that it contains the elements that it contains (extensionality). Moreover, if the fact that it has the given objects as members is an essential property, then we should have the following postulate to start, called **membership rigidity**.

$$(54) \qquad x \in X \to \Box(x \in X), \qquad x \notin X \to \Box(x \notin X)$$

(Notice that since we do not work with a distinguished actual world, our principles need not be prefixed by a box.) Additionally, it is assumed that $x \in X$ implies the existence of both x and X (the **falsehood principle**):

$$(55) \qquad (\forall x)(\forall X)(x \in X \to E(x) \wedge E(X))$$

How can finally transworld identity be expressed? In counterpart theory we need to go back and forth between worlds, assuring us that if $x \in X$ in w, say, then this is the case in v (for the respective counterparts) and conversely. In coherence frames we only need to state the principle in its original form.

$$(56) \qquad (\forall X)(\forall Y)(X = Y \leftrightarrow (\forall x)(x \in X \leftrightarrow x \in Y))$$

However, in this reading identity depends on the world we are in. If X is the set containing the morning star and the evening star, and Y the set containing only the evening star, then $X = Y$ is true at w iff, in w, the trace of the evening star is identical to the trace of the morning star. Of course, it is possible to retroactively introduce a stronger notion of identity, as with objects (see [167] on identity).

Coherence frames and worldline frames alike use identity in trace as a criterion of identity at a world. However, from a metaphysical point of view the introduction of traces is a dubious thing. What for example, is the trace of Yuri Gagarin in this world right now? Or that of Clark Kent? How do we decide whether or not Clark Kent is Superman on the basis of traces? If one is troubled by traces, one may do without them. All it takes is to introduce a binary predicate Eq whose interpretation is an equivalence relation in every world. The interpretation of Eq will hardly be that of identity. This has caused great concern; however, it must be borne in mind that Eq is a predicate of objects, not of individuals. All it takes is to consider objects to exist independently of their realisation ('individual'). This move is virtually forced on us when we want to supply metaframe semantics for logics that do not validate the necessity of identity (see [97]).

§13. On the Status of the Modal Language. [130] exposes three views one can have with regards to modal language; one can be a **primitivist**, a **eliminativist** and a **reductionist**. A primitivist holds that talk about modality is not reducible to anything else, and that modal statements have the same status as nonmodal ones. They are meaningful and they do not lack truth values. A **reductionist** believes that modal talk can be eliminated by talk about something else, for example possible worlds. Such a reduction might proceed along the lines of the standard translation. Finally, an **eliminativist** denies the meaningfulness of modal talk altogether. For him, modal talk is meaningless. For example, one might hold that there is no way to attach verification procedures to modal statements in the way that one does with ordinary, nonmodal statements. From a formal point of view, the reductionist works with the presupposition that modal talk is just proxy for talk about possible worlds, using the standard translation (6). But we have seen that modal logic is incomplete; the second-order interpretation in Kripke-frames confuses certain logics (namely the incomplete logics with their completion). The reductionist thus is committed to the rejection of incomplete logics, unless he agrees to yet another entity, the internal sets. It is the task of the reductionist to either show that the need for these sets does not arise (by showing that the logic of the modal operator is complete), or, if he cannot do this, he is committed to a kind of entity that we have shown to be equivalent in character with that of the propositions themselves. This actually strengthens the primitivist position, since the primitivist can point out that these objects he had in the first place. We do not aim at discussing the eliminativist's position here. It is mostly rejected on the grounds that it fails to explain human communication. Suffice it to point out that even Quine did consider certain interpretations of modal operators as unproblematic, for example those relating to time and space. While he rejected an 'intensional ontology' proper, he accepted modal discourse about real-world objects [156]. In his view, this does not constitute transgression into another world. (This is implicit in other work as well.) It is unclear, though, which interpretation of modality requires the addition of possible worlds in this sense, and which one does not. For example, suppose future is indeterminate and you might either toss head or tail in the next moment. Now, if that moment arrives and you have tossed head, does the other outcome in which you tossed tail exist or not? And, if it exists, does it exist in another world or in this one? On the many worlds interpretation of quantum mechanics it is really part of this world (despite the name 'many worlds interpretation'). If so, most counterfactual reasoning is actually perfectly acceptable because it talks about how things are in some other history. If one is inclined to reject this line because it is in principle impossible to know what happens in another history, notice that **verificationism** frequently employs procedures that cannot be carried out even in principle. The formulation of these principles involves modal talk, as [130] points out, so there seems to be something irreducible about modal language. Interestingly, notice that much of set theory is founded around the idea that every potentially infinite chain defines an actual object. The natural numbers exist as soon as we have the infinite ascending chain of numbers 1, 2, 3, $\cdots$. Georg Cantor addressed the distinction between potential and actual infinity by saying that the principle of potential infinity cannot be formulated without conceiving of the totality of numbers in the first place (for example, by saying that *for every number there is a larger number*).

Still, the interpretations of modality seem to present different degrees of ontological commitment. Also, they might operate on different lines. Epistemic modalities seem to be very different from what we may call 'objective modalities' in that the former may be at odds with Leibniz' Principle of substituting equivalent descriptions. Belief, for example, is highly problematic in this respect. Even if the axioms of the natural numbers imply that Fermat's conjecture is true, one may believe the first but not the second. On these grounds it seems that the attempt to model belief via possible worlds is highly problematic. Jaakko Hintikka [78] confronts the problem with the notion of *implicit belief*, but that is not helpful since it fails to analyse the notion we are after, namely that of belief, and substitutes it with a different notion.

In a similar vein, talk of objects and counterparts is not simply problematic as such. If we talk about time and space, the notion of an object *seems* to be clear even to those who reject the notion of modality. However, even if there is (seemingly or not) no talk of modality, the questions about the nature of objects do not disappear, they just take a different form. For example, why is this cat today the same as that cat yesterday? The answer might go like this. There is a continuous function from the location of that cat yesterday to the location of this cat today, and at every instance between these time points, this function yields the location of an object at that time point, and this object was a cat. (See [156] for a discussion.) However implausible it is to expect from anyone to really go by this definition (rather than just checking whether this cat looks or acts the same as that cat yesterday, supposing that our memory and perceptions are infallible etc.), the question is whether this answer always works. Consider, for example, artefacts, such as cars or bicycles. They can be the same even though some parts have been exchanged. It is not immediately clear how to apply the previous definition in the case where you give your bicycle to a mechanic and get it back tomorrow with some parts being exchanged. (Suppose your bicycle gets totally disassembled—you have to say that it disappeared at that moment.) If we accept that the bicycle you get back today is the same as the one you handed in yesterday, you are additionally in for a Sorites type argument (see [57]). However this matter can be resolved, the answers to the questions of identity across time do not seem to be less challenging than those of identity across worlds in general [175, 7, 173]. Additionally, the verificationist answer we gave before does not even work in principle. Notice that if quantum physics is correct, there is no way to check the continuity of the movement of any object, not just elementary particles (compare also [38]). There is, moreover, no object as such: there is just a vector in an infinite Hilbert space, of which the object is but one aspect (and the de Broglie wave another). Below the uncertainty threshold just about any monstrosity can happen (creating and destroying particles, for example). This means, at least in theory, that it is impossible in principle to track the exact movement of that cat yesterday until it becomes this cat today.

REFERENCES

[1] Carlos Alchourrón, Peter Gärdenfors, and David Makinson, *On the logic of theory change*, ***The Journal of Symbolic Logic***, vol. 50 (1985), pp. 510–530.

[2] Allan Ross Anderson and Nuel D. Belnap, ***Entailment: The Logic of Relevance and Necessity Vol 1***, Princeton University Press, 1975.

[3] Allan Ross Anderson, Nuel D. Belnap, and Michael J. Dunn, ***Entailment: The Logic of Relevance and Necessity Vol 2***, Princeton University Press, 1992.

[4] Aristotle, ***Prior Analytics***, Hackett Publishing Company, Indianapolis/Cambridge, 1989, translated, with introduction, notes, and commentary, by Robin Smith.

[5] David Malet Armstrong, ***A Combinatorial Theory of Possibility***, Cambridge University Press, Cambridge, 1989.

[6] Yuri V. Balashov, *Enduring and Perduring Objects in Minkowski Space-Time*, ***Philosophical Studies***, vol. 99 (2000), pp. 129–166.

[7] ———, *Persistence and Space-Time: Philosophical Lessons of the Pole and Barn*, ***The Monist***, vol. 83 (2000), no. 3, pp. 321–340.

[8] Ruth C. Barcan, *A Functional Calculus of First Order Based on Srict Implication*, ***The Journal of Symbolic Logic***, vol. 11 (1946), pp. 1–16.

[9] John Barwise and John Perry, ***Situations and Attitudes***, MIT Press, Cambridge (Massachussetts), 1983.

[10] ———, *Shifting situations and shaken attiudes*, ***Linguistics and Philosophy***, vol. 8 (1985), pp. 103–161.

[11] Sebastian Bauer, *Metaframes, Typen und Modelle der modalen Prädikatenlogik*, ***Master's thesis***, Fachbereich Mathematik, Humboldt–Universität zu Berlin, 2000.

[12] Sebastian Bauer and Heinrich Wansing, *Consequence, Counterparts and Substitution*, ***The Monist***, vol. 85 (2002), no. 4, pp. 483–497.

[13] Oskar Becker, *Zur Rekonstruktion des Kyrieuon Logos des Diodoros Kronos*, ***Erkenntnis und Verantwortung: Festschrift für Theodor Litt zum 80. Geburtstag*** (Josef Derbolav and Friedhelm Nicolin, editors), Düsseldorf, 1960, pp. 250–263.

[14] Johan van Benthem, ***Modal and Classical Logic***, Bibliopolis, 1983.

[15] ———, ***Essays in Logical Semantics***, D. Reidel Publishing Company, Dordrecht, 1986.

[16] Johan van Benthem, Patrick Blackburn, and Frank Wolter (editors), ***Handbook of Modal Logic***, Elsevier, Amsterdam, 2006.

[17] Tomasz Bigaj, *Three-Valued Logic, Indeterminism, and Quantum Mechanics*, ***Journal of Philosophical Logic***, vol. 30 (2001), pp. 97–119.

[18] Patrick Blackburn, *Nominal tense logic*, ***Notre Dame Journal of Formal Logic***, vol. 39 (1993), pp. 56–83.

[19] Joseph. M. Bocheński, ***Formale Logik***, Verlag Karl Alber, Freiburg, München, 2002, 5th edition, first edition 1956, english translation by Ivo Thomas, A History of Formal Logic, University of Notre Dame Press, 1961.

[20] Torben Braüner and Silvio Ghilardi, *First-Order Modal Logic*, ***Handbook of Modal Logic*** (Johan van Benthem, Patrick Blackburn, and Frank Wolter, editors), Elsevier, Amsterdam, 2006.

[21] Chris Brink and Ingrid Rewitzki, *Three Dual Ontologies*, ***Journal of Philosophical Logic***, vol. 31 (2002), pp. 543–568.

[22] Luitzen Egbertus Jan Brouwer, *Over de Grondslagen der Wiskunde*, ***Ph.D. thesis***, Amsterdam, 1907, Translation: "On the foundation of mathematics" in Brouwer, Collected Works, I, (A. Heyting ed.), 1975, North-Holland, Amsterdam, pp.11–101.

[23] ———, *De onbetrouwbaarheid der logische principes*, ***Tijdschrift voor Wijsbegeerte***, vol. 2 (1908), pp. 152–158, Translation: The unreliability of the logical principles, Ibid. pp. 107–111.

[24] Mark A. Brown, *On the Logic of Ability*, ***Journal of Philosophical Logic***, vol. 17 (1988), pp. 1–26.

[25] Robert A. Bull, *On Possible Worlds in Propositional Calculus*, ***Theoria***, vol. 34 (1968), pp. 171–182.

[26] John P. Burgess, *The unreal future*, ***Theoria***, vol. 44 (1978), pp. 157–179.

[27] ———, *Logic and Time*, ***The Journal of Symbolic Logic***, vol. 44 (1979), pp. 566–582.

[28] Rudolf Carnap, *Modalities and Quantification*, ***The Journal of Symbolic Logic***, vol. 11 (1946), pp. 33–64.

[29] ———, ***Meaning and Necessity. A Study in Semantics and Modal Logic***, The University of Chicago Press, Chicago, 1947.

[30] Alexander Chagrov and Michael Zakharyaschev, ***Modal logic***, Oxford University Press, Oxford, 1997.

[31] Charles Chihara, ***The Worlds of Possibility. Modal Realism and the Semantics of Modal Logic***, Clarendon Press, Oxford, 1998.

[32] Alonzo Church, *A Formulation of the Logic of Sense and Denotation*, ***Structure, Method and Meaning: Essays in Honor of H. M. Sheffer*** (P. Henle, H. M. Kallen, and S. K. Langer, editors), The Liberal Arts Press, New York, 1951, pp. 3–24.

[33] Nino B. Cocchiarella, *On the Primary and Secondary Semantics of Logical Necessity*, ***Journal of Philosophical Logic***, vol. 4 (1975), no. 1, pp. 13–18.

[34] ———, *Logical necessity based on Carnap's criterion of adequacy*, ***Korean Journal of Logic***, vol. 5 (2002), no. 2, pp. 1–21.

[35] Jack Copeland, *The genesis of possible worlds semantics*, ***Journal of Philosophical Logic***, vol. 31 (2002), no. 2, pp. 99–137.

[36] Max J. Cresswell, *How to Complete Some Modal Predicate Logics*, ***Advances in Modal Logic. Volume 2*** (Stanford) (Michael Zakharyaschev et al., editor), CSLI Publications, 2001.

[37] Janusz Czelakowski, ***Protoalgebraic logics***, Trends in Logic, no. 10, Kluwer, 2001.

[38] Newton C. A. Da Costa and Décio Krause, *An Intensional Schrödinger logic*, ***Notre Dame Journal of Formal Logic***, vol. 38 (1997), no. 2, pp. 179–195.

[39] John Divers, ***Possible Worlds***, The Problems of Philosophy, Routledge, London and New York, 2002.

[40] Gabbay Dov and Valentin Shehtman, *Undecidability of modal and intermediate first-order logics with two individual variables*, ***The Journal of Symbolic Logic***, vol. 58 (1993), pp. 800–823.

[41] David Dowty, *Towards a Semantic Analysis of Verb Aspect and the English "imperfective progressive"*, ***Linguistics and Philosophy***, vol. 1 (1977), pp. 54–78.

[42] J. Earman and Norton J., *What Price Space-time Substantivalism? The Hole Story*, ***British Journal for the Philosophy of Science***, vol. 38 (1987), pp. 515–525.

[43] Hugh Everett III, *The theory of the universal wave function*, ***Ph.D. thesis***, Princeton University, 1956.

[44] Michael Fara and Timothy Williamson, *Counterparts and Actuality*, ***Mind***, vol. 114 (2005), pp. 1–30.

[45] Fred Feldman, *Counterparts*, ***The Journal of Philosophy***, vol. 68 (1971), pp. 406–409.

[46] Kit Fine, *Some connections between elementary and modal logic*, ***Proceedings of the third scandinavian logic symposium*** (Stig Kanger, editor), North–Holland, 1975, pp. 15–31.

[47] ———, *Properties, Propositions, and Sets*, ***Journal of Philosophical Logic***, vol. 6 (1977), pp. 135–191.

[48] ———, *Model Theory for Modal Logic. Part III: Existence and Predication*, ***Journal of Philosophical Logic***, vol. 10 (1981), pp. 293–307.

[49] ———, *Senses of Essence*, In Sinnott-Armstrong [176], pp. 53–73.

[50] Kit Fine and Gerhard Schurz, *Transfer theorems for stratified modal logics*, ***Proceedings of the arthur prior memorial conference*** (Christchurch, New Zealand), 1991.

[51] Melvin Fitting, *On Quantified Modal Logic*, ***Fundamenta Informaticae***, vol. 39 (1999), no. 1–2, pp. 105–121.

[52] ———, *Modal Logics Between Propositional and First-Order*, ***Journal of Logic and Computation***, vol. 12 (2002), pp. 1017–1026.

[53] ———, ***Types, Tableaus and Gödel's God***, Trends in Logic, vol. 12, Kluwer Academic Publishers, Dordrecht, 2002.

[54] ———, *First-order intensional logic*, ***Proccedings of the Alfred Tarski Centenary Conference*** (D. Niwinski and J. Wolenski, editors), Journal of Pure and Applied Logic (Special Issue), 2003.

[55] Melvin Fitting and Richard L. Mendelsohn, ***First–Order Modal Logic***, Kluwer Academic Publishers, Dordrecht, 1998.

[56] Graeme Forbes, *Canonical counterpart theory*, ***Analysis***, vol. 42 (1982), pp. 33–37.

[57] ———, ***The Metaphysics of Modality***, Oxford Clarendon Press, Oxford, 1985.

[58] Bas C. van Fraassen, *Hidden variables in conditional logic*, ***Theoria***, vol. 40 (1974), pp. 176–190.

[59] Dov Gabbay, Agnes Kurucz, Frank Wolter, and Michael Zakharyaschev, ***Many-Dimensional Modal Logics: Theory and Applications***, Studies in Logic and the Foundations of Mathematics, no. 148, Elsevier Science Publishers, Amsterdam, 2003.

[60] Dov M. Gabbay, *A general theory of the conditional in terms of a ternary operator*, ***Theoria***, vol. 38 (1972), pp. 97–104.

[61] Daniel Gallin, ***Intensional and Higher-Order Modal Logic***, North-Holland, Amsterdam, 1975.

[62] James W. Garson, *Applications of Free Logic to Quantified Intensional Logic*, ***Philosophical Applications of Free Logic*** (Karel Lambert, editor), Oxford University Press, New York, Oxford, 1991, pp. 111–144.

[63] Martin Gerson, *An extension of S4 complete for neighbourhood semantics but incomplete for relational semantics*, ***Studia Logica***, vol. 34 (1975), pp. 333–342.

[64] ———, *The inadequacy of neighborhood semantics for modal logics*, ***The Journal of Symbolic Logic***, vol. 40 (1975), no. 2, pp. 141–147.

[65] Silvio Ghilardi, *Presheaf semantics and independence results for some non-classical first-order logics*, ***Archiv für mathematische Logik***, vol. 29 (1991), pp. 125–136.

[66] ———, *Quantified Extensions of Canonical Propositional Intermediate Logics*, ***Studia Logica***, vol. 51 (1992), pp. 195–214.

[67] Jean-Yves Girard, *Linear Logic*, ***Theoretical Computer Science***, vol. 50 (1987), pp. 1–102.

[68] Kurt Gödel, *Zum intuitionistischen Aussagenkalkül*, ***Anzeiger der Akademie der Wissenschaften in Wien***, vol. 69 (1932), pp. 65–66.

[69] ———, *Eine Interpretation des intuitionistischen Aussagenkalküls*, ***Ergebnisse eines mathematischen Kolloquiums***, vol. 4 (1933), pp. 39–40.

[70] ———, *Ontological proof*, ***Kurt Gödel Collected Works, Volume III, Unpublished Essays and Lectures*** (Soloman Feferman, Jr. John W. Dawson, Warren Goldfarb, Charles Parsons, and Robert N. Solovay, editors), Oxford University Press, New York, 1995, pp. 403–404.

[71] Robert I. Goldblatt, *Diodorean modality in Minkowski spacetime*, ***Studia Logica***, vol. 39 (1980), pp. 219–236.

[72] ———, ***Logics of Time and Computation***, Studies in Logic, Language and Information, CSLI, Stanford, 1992, second edition.

[73] ———, *Mathematical modal logic: A view of its evolution*, ***Journal of Applied Logic***, vol. 1 (2003), pp. 309–392.

[74] Allen Hazen, *Expressive completeness in modal languages*, ***Journal of Philosophcal Logic***, vol. 5 (1976), pp. 25–46.

[75] ———, *Counterpart-theoretic semantics for modal logic*, ***The Journal of Philosophy***, vol. 76 (1979), pp. 319–338.

[76] Leon Henkin, *Completeness in the Theory of Types*, ***The Journal of Symbolic Logic***, vol. 15 (1950), pp. 81–91.

[77] Arend Heyting, *Die formalen Regeln der intuitionistischen Logik*, ***Sitzungsberichte der Preussischen Akademie der Wissenschaften***, (1930), pp. 42–56.

[78] Jaakko Hintikka, ***Knowledge and Belief. An introduction into the logic of the two notions***, Cornell University Press, Ithaca, 1962.

[79] ———, *Aristotle and the 'Master Argument' of Diodorus*, ***American Philosophical Quarterly***, vol. 1 (1964), pp. 101–114.

[80] ———, *Existential Presuppositions and Their Elimination*, ***Models for Modalities. Selected Essays***, D. Reidel Publishing Company, Dordrecht, 1969, pp. 23–44.

[81] Harold Hodes, *Axioms for Actuality*, ***Journal of Philosophical Logic***, vol. 13 (1984), pp. 27–34.

[82] ———, *Some Theorems on the Expressive Limitations of Modal Languages*, ***Journal of Philosophical Logic***, vol. 13 (1984), pp. 13–26.

[83] ———, *Individual Actualism and Three Valued Modal Logics. Part I: Model-Theoretic Semantics*, ***Journal of Philosophical Logic***, vol. 15 (1986), pp. 369–401.

[84] Richard Holton, *David Lewis's Philosophy of Language*, ***Mind & Language***, vol. 18 (2003), no. 3, pp. 286–295.

[85] George E. Hughes and Max J. Cresswell, ***A New Introduction to Modal Logic***, Routledge, London, 1996.

[86] Lloyd Humberstone, *From Worlds to Possibilities*, ***Journal of Philosophical Logic***, vol. 10 (1981), pp. 313–339.

[87] Frank Jackson, *Metaphysics by possible cases*, ***Mind, Method, and Conditionals: Selected Essays***, Routledge, 1998, pp. 133–153.

[88] Mark Johnston, *Is there a problem about persistence?*, ***Aristotelean Society Supp.***, vol. 61 (1987), pp. 107–135.

[89] Bjarni Jónsson and Alfred Tarski, *Boolean algebras with operators*, ***American Journal of Mathematics***, vol. 73 (1951), pp. 891–939.

[90] David Kaplan, *How to Russell a Frege-Church*, ***Journal of Philosophy***, vol. 72 (1975), pp. 716–729, reprinted in Loux, 1979.

[91] William Kneale and Martha Kneale, ***The Development of Logic***, Clarendon Press, Oxford, 1984, paperback edition, first edition 1962.

[92] Kenneth Konyndyk, ***Introduction to Modal Logic***, University of Notre Dame Press, Notre Dame, Indiana, 1986.

[93] Marcus Kracht, ***Tools and Techniques in Modal Logic***, Studies in Logic, no. 142, Elsevier, Amsterdam, 1999.

[94] ———, *Notes on Substitution in First–Order Logic*, ***First-order logic 75*** (Berlin) (Uwe Scheffler, editor), Logos Verlag, 2004, pp. 155–172.

[95] ———, *Modal Consequence Relations*, ***Handbook of Modal Logic*** (Johan van Benthem, Patrick Blackburn, and Frank Wolter, editors), Elsevier, Amsterdam, 2006.

[96] Marcus Kracht and Oliver Kutz, *The Semantics of Modal Predicate Logic I. Counterpart–Frames.*, ***Advances in Modal Logic. Volume 3*** (Frank Wolter, Heinrich Wansing, Maarten de Rijke, and Michael Zakharyaschev, editors), World Scientific Publishing Company, 2002.

[97] ———, *The Semantics of Modal Predicate Logic II. Modal Individuals Revisited*, ***Intensionality*** (Reinhard Kahle, editor), ASL Lecture Notes in Logic, vol. 22, A. K. Peters, Los Angeles, 2005, pp. 60–97.

[98] Marcus Kracht and Frank Wolter, *Properties of independently axiomatizable bimodal logics*, ***The Journal of Symbolic Logic***, vol. 56 (1991), pp. 1469–1485.

[99] ———, *Normal monomodal logics can simulate all others*, ***The Journal of Symbolic Logic***, vol. 64 (1999), pp. 99–138.

[100] Angelika Kratzer, *An Inverstigation into the Lumps of Thought*, ***Linguistics and Philosophy***, vol. 12 (1989), pp. 607–653.

[101] Saul A. Kripke, *A Completeness Theorem in Modal Logic*, ***The Journal of Symbolic Logic***, vol. 24 (1959), no. 1, pp. 1–14.

[102] ———, *The undecidability of monadic modal quantificational theory*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 8 (1962), pp. 113–116.

[103] ———, *Semantical Considerations on Modal Logic*, ***Acta Philosophica Fennica***, vol. 16 (1963), pp. 83–94.

[104] ———, *Naming and Necessity*, ***Semantics of natural language*** (Gilbert Harman and Donald Davidson, editors), Synthese Library, no. 40, Reidel, Dordrecht, 1972, pp. 243–355.

[105] ———, *A puzzle about belief*, ***Meaning and use*** (A. Margalit, editor), D. Reidel Publishing Co., Dordrecht, Boston, 1979, pp. 239–283.

[106] Oliver Kutz, *Kripke–Typ Semantiken für die modale Prädikatenlogik*, ***Master's thesis***, Fachbereich Mathematik, Humboldt–Universität zu Berlin, 2000.

[107] ———, *New Semantics for Modal Predicate Logics.*, ***Foundations of the Formal Sciences II: Applications of Mathematical Logic in Philosophy and Linguistics*** (Dordrecht, Boston, London) (Benedikt Löwe, Wolfgang Malzkorn, and Thoralf Räsch, editors), Trends in Logic, vol. 17, Kluwer Academic Publishers, 2003, pp. 151–162.

[108] ———, *$\mathcal{E}$-Connections and Logics of Distance*, ***Ph.D. thesis***, The University of Liverpool, 2004.

[109] Oliver Kutz, Carsten Lutz, Frank Wolter, and Michael Zakharyaschev, *$\mathcal{E}$-Connections of Abstract Description Systems*, ***Artificial Intelligence***, vol. 156 (2004), no. 1, pp. 1–73.

[110] Oliver Kutz, Holger Sturm, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, *Logics of metric spaces*, ***ACM Transactions on Computational Logic (TOCL)***, vol. 4 (2003), no. 2, pp. 260–294.

[111] Karel Lambert, *Existential Import Revisited*, ***Notre Dame Journal of Formal Logic***, vol. 4 (1963), pp. 288–292.

[112] ———, ***Free Logics: Their Foundations, Character, and Some Applications Thereof***, Academia Verlag, Sankt Augustin, 1997.

[113] Francis William Lawvere, *Adjointness in Foundations*, ***Dialectica***, vol. 23 (1969), no. 3/4, pp. 281–296.

[114] Clarence I. Lewis and Cooper H. Langford, ***Symbolic Logic***, Appleton-Century-Crofts, New York, 1932.

[115] Clarence Irving Lewis, ***A survey of symbolic logic***, University of California Press, Berkeley, 1918.

[116] ———, *Strict Implication—An Emendation*, ***Journal of Philosophy, Psychology, and Scientific Methods***, vol. 17 (1920), pp. 300–302.

[117] David K. Lewis, *Counterpart Theory and Quantified Modal Logic*, ***Journal of Philosophy, vol. 65***, repr. in *The Possible and the Actual*, ed. Michael J. Loux, Ithaca 1979; also in: David K. Lewis, *Philosophical papers 1*, Oxford 1983., 1968, pp. 113–126.

[118] ———, *Completeness and decidability of three logics of counterfactual conditionals*, ***Theoria***, vol. 36 (1971), pp. 74–85.

[119] ———, ***Counterfactuals***, Blackwell, Oxford, 1973.

[120] ———, ***On the Plurality of Worlds***, Blackwell, 1986.

[121] Michael Leyton, ***Symmetry, Causality, Mind***, MIT Press, Cambridge (Mass.), 1992.

[122] Sten Lindström and Krister Segerberg, *Modal Logic and Philosophy*, ***Handbook of Modal Logic*** (Johan van Benthem, Patrick Blackburn, and Frank Wolter, editors), Elsevier, Amsterdam, 2006.

[123] Leonard Linsky (editor), ***Reference and Modality***, Oxford University Press, Oxford, 1971.

[124] Michael J. Loux (editor), ***The Possible and the Actual: Readings in the Metaphysics of Modality***, Cornell University Press, Ithaca and London, 1979.

[125] Leopold Löwenheim, *Über Möglichkeiten im Relativkalkül*, ***Mathematische Annalen***, vol. 76 (1915), pp. 447–470.

[126] Jan Łukasiewicz, ***Aristotle's Syllogistic from the Standpoint of Modern Formal Logic***, Oxford University Press, Oxford, 1957, 2nd edition.

[127] Ruth Barcan Marcus, *Modalities and Intensional Languages*, ***Synthese***, vol. XIII (1961), no. 4, pp. 303–322, reprinted in [128].

[128] ———, ***Modalities: Philosophical Essays***, Oxford University Press, New York and Oxford, 1993.

[129] Ruth Barcan Marcus, W. V. Quine, S. Kripke, D. Føllesdal, and J. McCarthy, *Discussion of the paper of Ruth B. Marcus*, ***Synthese***, vol. XIV (1962), pp. 132–143, reprinted in [128].

[130] Stephen McLeod, ***Modality and Anti-Metaphysics***, Ashgate, Aldershot, 2001.

[131] Joseph Melia, *Holes, Haecceitism and Two Conceptions of Determinism*, ***British Journal for the Philosophy of Science***, vol. 50 (1999), pp. 639–664.

[132] ———, *Reducing possibilities to language*, ***Analysis***, vol. 61 (2001), no. 1, pp. 19–29.

[133] ———, ***Modality***, Central Problems of Philosophy, Acumen, 2003.

[134] Richard Montague, *Pragmatics*, ***Contemporary Philosophy: A Survey*** (B. Klibansky, editor), La Nuova Italia Editrice, 1968, pp. 102–122.

[135] ———, *On the Nature of Certain Philosophical Entities*, ***Monist***, vol. 53 (1969), pp. 159–194, reprinted in [136], Chapter 5.

[136] ———, ***Formal Philosophy: Selected Papers of Richard Montague***, Yale University Press, New Haven and London, 1974, edited by Richmond H. Thomason.

[137] M. Mortimer, *On languages with two variables*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 21 (1975), pp. 135–140.

[138] Nels David Nelson, *Constructible Falsity*, ***The Journal of Symbolic Logic***, vol. 14 (1949), no. 1, pp. 16–26.

[139] Brian Norton, *Is Counterpart Theory Inadequate?*, ***Journal of Philosophical Logic***, vol. 5 (1976), pp. 79–89.

[140] Hans Jürgen Ohlbach, Andreas Nonnengart, Maarten de Rijke, and Dov M. Gabbay, *Encoding Two-Valued Nonclassical Logics in Classical Logic.*, ***Handbook of Automated Reasoning*** (John Alan Robinson and Andrei Voronkov, editors), Elsevier and MIT Press, 2001, pp. 1403–1486.

[141] I. E. Orlov, *The calculus of compatibility of propositions*, ***Mathematics of the USSR, Sbornik***, vol. 35 (1928), pp. 263–286, (Russian).

[142] William Tuthill Parry, *In Memoriam: Clarence Irving Lewis (1883–1964)*, ***Notre Dame Journal of Formal Logic***, vol. 11 (1970), no. 2, pp. 129–140.

[143] Josh Parsons, *Must a Four-Dimensionalist Believe in Temporal Parts?*, ***The Monist***, vol. 83 (2000), no. 3, pp. 399–418.

[144] Terence Parsons, ***Nonexistent Objects***, Yale University Press, New Haven and London, 1980.

[145] ———, *Ruth Barcan Marcus and the Barcan Formula*, In Sinnott-Armstrong [176], pp. 3–11.

[146] Richard Patterson, ***Aristotle's Modal Logic. Essence and Entailment in the Organon***, Cambridge University Press, Cambridge, 1995.

[147] David Pearce and Gerd Wagner, *Reasoning with Negative Information I: Strong Negation in Logic Programs*, ***Language, Knowledge and Intentionality (Acta Philosophica Fennica 49)***, Helsinki, 1990, pp. 405–419.

[148] ———, *Logic Programming with Strong Negation*, ***Extensions of Logic Programming*** (Peter Schroeder-Heister, editor), Lecture Notes in Artificial Intelligence, no. 475, Springer, 1991, pp. 311–326.

[149] John Perry, *From Worlds to Situations*, ***Journal of Philosophical Logic***, vol. 15 (1986), pp. 83–107.

[150] Alvin Plantinga, ***The Nature of Necessity***, Oxford Clarendon Press, 1974.

[151] Arthur N. Prior, *Diodoran modalities*, ***Philosophical Quarterly***, vol. 5 (1955), pp. 205–212.

[152] ———, ***Time and Modality***, Oxford University Press, Oxford, 1957.

[153] ———, ***Past, Present and Future***, Oxford University Press, Oxford, 1967.

[154] Willard van Orman Quine, *The Problem of Interpreting Modal Logic*, ***The Journal of Symbolic Logic***, vol. 12 (1947), no. 2, pp. 43–48.

[155] ———, *Three grades of modal involvement*, ***Proceedings of the XIth international congress of philosophy***, North Holland, Amsterdam, 1953, Reprinted in *The Ways of Paradox*, Random House, New York, 1966.

[156] ———, *Worlds away*, ***Journal of Philosophy***, vol. 73 (1976), pp. 859–863.

[157] Murali Ramachandran, *An alternative translation scheme for counterpart theory*, ***Analysis***, vol. 49 (1989), pp. 131–141.

[158] Mark Reynolds, *Axioms for branching time*, ***Journal of Logic and Computation***, vol. 12 (2002), pp. 679–697.

[159] ———, *An Axiomatization of Prior's Ockhamist Logic of Historical Necessity*, ***Advances in modal logic, vol. 4*** (Philippe Balbiani, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, editors), King's College Publications, London, 2003, pp. 355–370.

[160] Gideon Rosen, *Modal Fictionalism*, ***Mind***, vol. 99 (1990), pp. 327–354.

[161] Bertrand Russell, *On denoting*, ***Mind***, vol. 14 (1905), pp. 479–493.

[162] Klaus J. Schmidt, ***Die modale Syllogistik des Aristoteles***, Mentis Verlag, Paderborn, 2002.

[163] Gerhard Schurz, ***The Is-Ought Problem: An Investigation in Philosophical Logic***, Trends in Logic, vol. 1, Kluwer Academic Publishers, Dordrecht, 1997.

[164] ———, *Tarski and Carnap on Logical Truth—or: What is Genuine Logic*, ***Alfred Tarski and the Vienna Circle—Austro-Polish Connections in Logical Empiricism*** (Jan Wolenski and Eckehart Köhler, editors), Kluwer Academic Publishers, Dordrecht, 1998, pp. 77–94.

[165] ———, *Carnap's Modal Logic*, ***Zwischen traditioneller und moderner Logik. Nichtklassische Ansätze*** (W. Stelzner and M. Stöckler, editors), Mentis Verlag, Paderborn, 2001, pp. 365–380.

[166] Dana Scott, *A decision method for validity of sentences in two variables*, ***The Journal of Symbolic Logic***, vol. 27 (1962), p. 477.

[167] ———, *Advice on Modal Logic*, ***Philosophical Problems in Logic. Some Recent Developments*** (K. Lambert, editor), Reidel, Dordrecht, 1970, pp. 143–174.

[168] Krister Segerberg, *An essay in classical modal logic*, Mimeograph, Uppsala, 1971.

[169] Andrzej Sendlewski, *Some investigations of varieties of N–lattices*, ***Studia Logica***, vol. 43 (1984), pp. 257–280.

[170] ———, *Nelson Algebras Through Heyting Ones*, ***Studia Logica***, vol. 49 (1990), pp. 106–126.

[171] Ilya Shapirowsky and Valentin Shehtman, *Chronological future modality in Minkowski spacetime*, ***Advances in Modal Logic, Vol. 4*** (Philippe Balbiani, Nobu-Yuki Suzuki, Frank Wolter, and Michael Zakharyaschev, editors), King's College Publications, London, 2003, pp. 437–459.

[172] Hiroyuki Shirasu, *Duality in Superintuitionistic and Modal Predicate Logics*, ***Advances in Modal Logic, volume 1*** (Marcus Kracht et. al., editor), CSLI Publications, Stanford, 1998, pp. 223–236.

[173] Theodore Sider, ***Four-Dimensionalism: An Ontology of Persistence and Time***, Clarendon Press, Oxford, 2001.

[174] ———, *The Ersatz Pluriverse*, ***Journal of Philosophy***, vol. 99 (2002), pp. 279–315.

[175] Peter M. Simmons, *On Being Spread Out in Time: Temporal Parts and the Problem of Change*, ***Existence and Explanation*** (Dordrecht) (Wolfgang Spohn et. al., editor), Kluwer Academic Publishers, 1991.

[176] Walter Sinnott-Armstrong (editor), ***Modality, Morality, and Belief: Essays in Honor of Ruth Barcan Marcus***, Cambridge University Press, Cambridge, 1995.

[177] Dmitrij P. Skvortsov and Valentin B. Shehtman, *Maximal Kripke–Type Semantics for Modal and Superintuitionistic Predicate Logics*, ***Annals of Pure and Applied Logic***, vol. 63 (1993), pp. 69–101.

[178] Brian Skyrms, *Tractarian Nominalism*, ***Philosophical Studies***, vol. 40 (1981), pp. 199–206.

[179] Quentin Smith, *Direct, Rigid Designation and A Posteriori Necessity: A History and Critique* , ***The New Theory of Reference: Kripke, Marcus and its Origins*** (James Fetzer and Paul Humphreys, editors), Kluwer Academic Publishers, Dordrecht, 1998, pp. 137–178.

[180] Robert Stalnaker, *Possible Worlds and Situations*, ***Journal of Philosophical Logic***, vol. 15 (1986), pp. 109–123.

[181] Robert Stalnaker and Richmond H. Thomason, *Abstraction in first-order modal logic*, ***Theoria***, vol. 34 (1968), pp. 203–207.

[182] ———, *A semantic analysis of conditional logics*, ***Theoria***, vol. 36 (1970), pp. 23–42.

[183] János Surányi, *Zur Reduktion des Entscheidungsproblems des logischen Funktionenkalküls*, ***Mathematikai és Fizikai Lapok***, vol. 50 (1943), pp. 51–74.

[184] Nobu-Yuki Suzuki, *Kripke frames with graded accessibility and fuzzy possible world semantics*, ***Studia Logica***, vol. 59 (1997), pp. 249–269.

[185] Alfred Tarski, *Über den Begriff der logischen Folgerung*, ***Actes du Congrés International de Philosophie Scientifique***, vol. 7 (1936), pp. 1–11, translated as 'On the Concept of Logical Consequence' in [186], pp. 409–420.

[186] ———, ***Logics, Semantics, Metamathematics***, Clarendon Press, Oxford, 1956.

[187] Max Tegmark, *Parallel Universes*, ***Science and Ultimate Reality: Quantum Theory, Cosmology and Complexity*** (John D. Barrow, Paul C. W. Davies, and Charles L. Harper, editors), Cambridge University Press, Cambridge, 2004, pp. 459–491.

[188] Paul Thom, *Aristotle's Syllogistic*, ***Notre Dame Journal of Formal Logic***, vol. 20 (1979), no. 4, pp. 751–759.

[189] ———, ***The Logic of Essentialism: An Interpretation of Aristotle's Modal Syllogistic***, Kluwer, Dordrecht, 1996.

[190] Richmond H. Thomason and Robert Stalnaker, *Modality and reference*, ***Noûs***, vol. 2 (1968), pp. 359–372.

[191] Steven K. Thomason, *Semantic analysis of tense logic*, ***The Journal of Symbolic Logic***, vol. 37 (1972), pp. 150–158.

[192] ———, *A New Representation of S5*, ***Notre Dame Journal of Formal Logic***, vol. 14 (1973), no. 2, pp. 281–284.

[193] Anne S. Troelstra, ***Lectures on linear logic***, CSLI, Stanford, 1992.

[194] Dimiter Vakarelov, *Notes on N–lattices and constructive logic with strong negation*, ***Studia Logica***, vol. 36 (1977), pp. 109–125.

[195] Carl Friedrich von Weizsäcker, ***Der Aufbau der Physik***, Carl Hanser Verlag, München, 1985.

[196] ———, ***Zeit und Wissen***, Carl Hanser Verlag, München, 1992.

[197] Michael J. White, *The Necessity of the Past and Modal-Tense Logic Incompleteness*, ***Notre Dame Journal of Formal Logic***, vol. 25 (1984), no. 1, pp. 59–71.

[198] Alfred North Whitehead and Bertrand Russell, ***Principia Mathematica***, Cambridge University Press, Cambridge, 1910, 1912, 1913, 3 volumes.

[199] Ryszard Wójcicki, ***Theory of logical calculi***, Kluwer, Dordrecht, 1988.

[200] Frank Wolter and Michael Zakharyaschev, *Decidable fragments of first-order modal logics*, ***The Journal of Symbolic Logic***, vol. 66 (2001), pp. 1415–1438.

[201] Alberto Zanardo, *A finite axiomatization of the set of strongly valid Ockhamist formulas*, ***Journal of Philosophical Logic***, vol. 14 (1985), pp. 447–468.

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