Logics of essence and accident

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Logic and sermons never convince,
The damp of the night drives deeper into my soul.

(Only what proves itself to every man and woman is so,
Only what nobody denies is so.)

—Walt Whitman, *Leaves of Grass*, Song of Myself,
sec.30 (1855–1881).

Abstract

We say that things happen accidentally when they do indeed happen, but only by chance. In the opposite situation, an essential happening is inescapable, its inevitability being the sine qua non for its very occurrence. This paper will investigate modal logics on a language tailored to talk about essential and accidental statements. Completeness of some among the weakest and the strongest such systems is attained. The weak expressibility of the classical propositional language enriched with the non-normal modal operators of essence and accident is highlighted and illustrated, both with respect to the definability of the more usual modal operators as well as with respect to the characterizability of classes of frames. Several interesting problems and directions are left open for exploration.

Keywords: philosophy of modal logic, non-normal modalities,
formal metaphysics, essence, accident

1 The what-it-is-to-be

A necessary proposition is one whose negation is impossible; a possible proposition is one that is true in some acceptable state-of-affairs. Necessity, $\Box$, and possibility, $\Diamond$, are the modal operators upon which the usual language of normal modal logics is built. We propose here, though, to study some interesting alternative modalities, namely the modalities of *essence* and *accident*. An accidental proposition is one that is the case, but could have been otherwise. An essential proposition is one that, whenever it enjoys a true status, it does it per force. We will write $\bullet\varphi$ to say that "φ is accidental", and $\circ\varphi$ to say that "φ is essential". In formal metaphysics there has often been some confusion between essence and necessity, and between accident and contingency. The present approach contributes to the demarcation of these notions. A quick comparison with the literature on non-contingency logics and some comments on alternative interpretations of the new connectives hereby presented will be postponed to section 5.

Let $\mathcal{P}$ be a denumerable set of sentential letters, and let the set of formulas of classical propositional logic, $\mathsf{S}_{\mathbf{CPL}}$, be inductively defined by:

$$\alpha ::= p \mid \top \mid \bot \mid {\sim}\varphi \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \supset \psi) \mid (\varphi \equiv \psi),$$

where $p \in \mathcal{P}$, and φ and ψ are formulas. The set of formulas of the usual normal modal logics, $\mathsf{S}_{\mathbf{NML}}$, is defined by adding $\Box\varphi \mid \Diamond\varphi$ to the inductive clauses of $\mathsf{S}_{\mathbf{CPL}}$, and the set of formulas of the logics of essence and accident, $\mathsf{S}_{\mathbf{LEA}}$, is defined by adding instead $\circ\varphi \mid \bullet\varphi$ to the clauses of $\mathsf{S}_{\mathbf{CPL}}$.

A *modal frame* $\mathcal{F} = (W, R)$ is a structure containing a set of worlds $W \neq \varnothing$ and an accessibility relation $R \subseteq W \times W$. A *modal model* based on that frame is a structure $\mathcal{M} = (\mathcal{F}, V)$, where $V : \mathcal{P} \longrightarrow \mathsf{Pow}(W)$. The definition of *satisfaction* in a world $x \in W$ of a model $\mathcal{M}$ will be such that:

$\models^{\mathcal{M}}_{x} p$	iff	$x \in V(p)$
$\models^{\mathcal{M}}_{x} {\sim}\varphi$	iff	$\not\models^{\mathcal{M}}_{x} \varphi$
$\models^{\mathcal{M}}_{x} \varphi \vee \psi$	iff	$\models^{\mathcal{M}}_{x} \varphi$ or $\models^{\mathcal{M}}_{x} \psi$
	...	
$\models^{\mathcal{M}}_{x} \bullet\varphi$	iff	$\models^{\mathcal{M}}_{x} \varphi$ and $(\exists y \in W)(xRy \;\&\; \not\models^{\mathcal{M}}_{y} \varphi)$
$\models^{\mathcal{M}}_{x} \circ\varphi$	iff	$\not\models^{\mathcal{M}}_{x} \bullet\varphi$

The other classical operators are evaluated as expected. As usual, a formula φ will be said to be *valid* with respect to a class of frames $\mathbb{C}$, in symbols $\models^{\mathbb{C}} \varphi$, if $\models^{\mathcal{M}}_{x} \varphi$ holds good in every world x of every model $\mathcal{M}$ based on some frame in $\mathbb{C}$. We will write simply $\models$ for $\models^{\mathbb{C}}$ whenever the class of frames $\mathbb{C}$ can be read from the context. We say that a logic $\mathbf{L}$ given by some set of axioms Ax is *determined* by a class of frames $\mathbb{C}$ in case the provable formulas of the former coincide with the valid formulas of the latter.

Given a normal modal logic $\mathbf{L}$ determined by some class of frames $\mathbb{C}$, an *EA-logic* (of essence and accident) $(\mathbf{L})_{EA}$ is obtained by selecting all the formulas and the inferences in the language of **LEA** that are valid in $\mathbb{C}$. Notice that, in general, there is no reason why two logics $\mathbf{L}_1 \neq \mathbf{L}_2$ should imply $(\mathbf{L}_1)_{EA} \neq (\mathbf{L}_2)_{EA}$.

Recall that K, the minimal normal modal logic in the language of **NML**, determined by the class of all frames, can be axiomatized by:

All axioms and rules of **CPL**, plus

(0) $\vdash \varphi \supset \psi \quad \Rightarrow \quad \vdash \Box\varphi \supset \Box\psi$
(1) $\vdash (\Box\varphi \wedge \Box\psi) \supset \Box(\varphi \wedge \psi)$
(2) $\vdash \Box\top$

Sometimes it does not make much difference to work with $\mathsf{S}_{\mathbf{NML}}$ or with $\mathsf{S}_{\mathbf{LEA}}$, given that the modal connectives might turn out interdefinable. Indeed:

Proposition 1.1 Inside extensions of the modal logic K one can:

(i) take $\Box$ as primitive and define $\circ\varphi \overset{\text{def}}{=} \varphi \supset \Box\varphi$, $\bullet\varphi \overset{\text{def}}{=} \varphi \wedge \Diamond{\sim}\varphi$.

Inside extensions of KT, the modal logic axiomatized by $K + \vdash \Box\varphi \supset \varphi$ and determined by the class of all reflexive frames, one can:

(ii) take $\circ$ as primitive and define $\Box\varphi \overset{\text{def}}{=} \varphi \wedge \circ\varphi$.

2 The minimal logic of essence and accident

This section will prove that the axiomatization of $(K)_{EA}$, the minimal EA-logic of essence and accident (that is, the EA-logic determined by the class of all frames), can be given by the axioms Ax_K:

All axioms and rules of **CPL**, plus

(K0.1) $\vdash \varphi \equiv \psi \;\;\Rightarrow\;\; \vdash \circ\varphi \equiv \circ\psi$

(K0.2) $\vdash \varphi \;\;\Rightarrow\;\; \vdash \circ\varphi$

(K1.1) $\vdash (\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \wedge \psi)$

(K1.2) $\vdash ((\varphi \wedge \circ\varphi) \vee (\psi \wedge \circ\psi)) \supset \circ(\varphi \vee \psi)$

(K1.3) $\vdash \bullet\varphi \supset \varphi$

(K1.4) $\vdash \bullet\varphi \equiv {\sim}\circ\varphi$

In particular, notice that:

Proposition 2.1 Here are some consequences of the above axiomatization:

(K2.0) Replacement holds irrestrictedly

(K2.1) $\vdash \circ\top$

(K2.2) $\vdash \varphi \supset (\circ(\varphi \supset \psi) \supset (\circ\varphi \supset \circ\psi))$

(K2.3) $\vdash \varphi \vee \circ\varphi$

(K2.4) $\vdash \circ\bot$

Proposition 2.2 Here are some alternatives to the previous axioms and rules:

(EAd) $\bullet\varphi \overset{\text{def}}{=} {\sim}\circ\varphi$ can be used instead of (K1.4)

(K2.3) instead of (K1.3)

We now check that the above proposal of axiomatization for $(K)_{EA}$ is indeed determined by the class of all frames. Soundness, $\vdash \varphi \;\Rightarrow\; \models \varphi$, can be easily checked directly, by verifying the validity of each of the axioms and the preservation of validity by each of the rules in Ax_K. It will be left as an exercise. Next, the standard technique for checking completeness is the construction of a canonical model $\mathcal{M}^* = (W^*, R^*, V^*)$, where:

W^* is the set of all maximally non-trivial sets of **LEA**-formulas
$x \in V^*(p)$ iff $p \in x$
$y \in R^*(x)$ iff $D(x) \subseteq y$

The only really difficult part here is the definition of $D : W \to \mathsf{Pow(S)}$, in order to settle the appropriate accessibility relation for this canonical model. The idea of using the 'desessentialization' of a world, $D(x) = \{\varphi : \circ\varphi \in x\}$, analogously to what is done in normal modal logics for formulas of the form $\Box\varphi$, does not work here, once the modality $\circ$ of essence itself is not normal. A clever solution adapted from [5] is to define $D(x) = \{\varphi : \circ\varphi \in x$, and $\circ\psi \in x$ for every ψ such that $\vdash \varphi \supset \psi\}$. A simpler solution that also works, adapted from [8], is to define $D(x) = \{\varphi :$ for every $\psi, \circ(\varphi \vee \psi) \in x\}$. The latter definition is the one we will adopt here. Using that one can then prove:

Lemma 2.3 (Lindenbaum) Every non-trivial set of **LEA**-formulas can be extended into a maximally non-trivial set of formulas.

Lemma 2.4 In the canonical model:

(P1) $\varphi \in D(x)$ and $\psi \in D(x) \;\Rightarrow\; (\varphi \wedge \psi) \in D(x)$
(P2) $\circ\varphi \in x \;\Leftrightarrow\; \varphi \notin x$ or $\varphi \in D(x)$
(P3) $D(x) \neq \varnothing$
(P4) $\varphi \in D(x)$ and $\vdash \varphi \supset \psi \;\Rightarrow\; \psi \in D(x)$
(P5) $D(x)$ is a closed set, that is, $D(x) \vdash \alpha \;\Rightarrow\; \alpha \in D(x)$
(P6) $\circ\varphi \notin x \;\Rightarrow\; \varphi \in x$ and $(\exists y \in W^*)(xR^*y$ and $\varphi \notin y)$

Proof For (P1), recall from **CPL** that $\vdash ((\varphi \vee \theta) \wedge (\psi \vee \theta)) \equiv ((\varphi \wedge \psi) \vee \theta)$. Thus, by rule (K0.1), we have $\vdash \circ((\varphi\vee\theta)\wedge(\psi\vee\theta)) \equiv \circ((\varphi\wedge\psi)\vee\theta)$. Call that theorem α. Now, from $\varphi \in D(x)$ and $\psi \in D(x)$ we can conclude that $\circ(\varphi \vee \theta) \in x$ and $\circ(\psi \vee \theta) \in x$, for an arbitrary θ. From axiom (K1.1), the theorem α and the maximality of x it then follows that $\circ((\varphi \wedge \psi) \vee \theta) \in x$.

For (P2), suppose first that both $\circ\varphi \in x$ and $\varphi \in x$. Then it follows, by **CPL**, the maximality of x, and axiom (K1.2), that $\circ(\varphi \vee \psi) \in x$, for an arbitrary ψ. For the converse, use axiom (K2.3), maximality, and the property (P1) for the particular case in which ψ is identical to φ.

For (P3), we may just check that any theorem $\top$ (such as, say, $\varphi \supset \varphi$) belongs to $D(x)$. Indeed, by rule (K0.2) we have that $\vdash \circ\top$, thus $\vdash (\top \wedge \circ\top)$. The result then follows from (K1.2) and the maximality of x.

For (P4), given $\varphi \in D(x)$ we know that $(\varphi \vee \pi) \in x$ for an arbitrary π, and in particular for $\pi = (\psi \vee \theta)$. But, from $\vdash \varphi \supset \psi$ we can conclude, using **CPL**, that $\vdash (\varphi \vee (\psi \vee \theta)) \equiv (\psi \vee \theta)$. The result now follows from (K0.1) and the maximality of x.

For (P5), given $D(x) \vdash \alpha$ we can conclude from property (P3), compacity and monotonicity that $\exists\theta_1, \ldots, \theta_n \in D(x)$ such that $\theta_1, \ldots, \theta_n \vdash \alpha$. But then, from property (P1) we have that $(\theta_1 \wedge \ldots \wedge \theta_n) \in D(x)$, and from property (P4), using **CPL** and the maximality of x, we may conclude that $\alpha \in D(x)$.

At last, for (P6), assume $\circ\varphi \notin x$ and use first (K1.4), (K1.3) and the maximality of x to conclude that $\varphi \in x$. For the second part we have to show that such a world y exists, and as a prerequisite for the Lindenbaum Lemma we must be able to prove that $D(x) \cup \{\sim\varphi\}$ is non-trivial. To proceed by absurdity, suppose the contrary. Then, by **CPL** we will have that $D(x) \vdash \varphi$, and by property (P5) we conclude that $\varphi \in D(x)$. From property (P2) we have $\circ\varphi \in x$, contrary to what has been assumed at the start.

Theorem 2.5 (Canonical Model) $\models_x^{\mathcal{M}^*} \varphi \;\Leftrightarrow\; \varphi \in x$.

Proof This is checked by induction on the structure of φ. The cases of the classical connectives is straightforward. Now, consider the case $\varphi = \circ\psi$ (the case $\varphi = \bullet\psi$ is similar). Suppose first that $\circ\psi \in x$. Then, by property (P2) of the previous lemma we conclude that $\psi \notin x$ or $\psi \in D(x)$. By the definition of R^*, we conclude from $\psi \in D(x)$ that $(\forall y \in W^*)(xR^*y \Rightarrow \psi \in y)$. By the induction hypothesis, we have $\not\models_x^{\mathcal{M}^*} \psi$ or $(\forall y \in W^*)(xR^*y \Rightarrow \models_y^{\mathcal{M}^*} \psi)$, which means, by the definition of satisfaction (Section 1), that $\models_x^{\mathcal{M}^*} \circ\psi$. Conversely, suppose now that $\circ\psi \notin x$. By property (P6) we conclude that $\psi \in x$ and $(\exists y \in W^*)(xR^*y$ and $\psi \notin y)$. Again, the result follows from the induction hypotheses and the definition of satisfaction.

Corollary 2.6 (Completeness) $\Gamma \nvdash \varphi \;\Rightarrow\; \Gamma \not\models \varphi$.

3 Extensions of $(K)_{EA}$, and definability of □s and ◇s

In Proposition 1.1 we learned that $\circ$ and $\Box$ are interdefinable in extensions of KT. In general, let $\Rsh\colon \mathsf{S}_{\mathbf{LEA}} \to \mathsf{S}_{\mathbf{NML}}$ be such that $p^{\Rsh} = p$, $(\circ\varphi)^{\Rsh} = \varphi^{\Rsh} \supset \Box\varphi^{\Rsh}$, $(\bullet\varphi)^{\Rsh} = \varphi^{\Rsh} \wedge \Diamond{\sim}\varphi^{\Rsh}$, and $(\star(\varphi_1, \ldots, \varphi_n))^{\Rsh} = \star(\varphi_1^{\Rsh}, \ldots, \varphi_n^{\Rsh})$ for any other n-ary connective $\star$ common to both languages. We can say that $\Box$ is definable in terms of the language of $\circ$'s and $\bullet$'s of the logic $(\mathbf{L})_{EA}$ in case there is some schema $\circledcirc(p) \in \mathsf{S}_{\mathbf{LEA}}$ such that following is a thesis of **L** (i.e. is provable / valid in **L**): $\Box\psi \equiv (\circledcirc(\psi))^{\Rsh}$. As a particular consequence of that, the following can now be proven:

Proposition 3.1 The definition $\Box\varphi \stackrel{\text{def}}{=} \varphi \wedge \circ\varphi$ is *only* possible in extensions of KT.

Proof To check that, one might just observe that in the minimal normal modal logic K the formula $\Box\psi \supset \psi$ can be inferred from $\Box\psi \supset (\psi \wedge (\psi \supset \Box\psi))$.

Recall that we have proved in the last section the completeness of $(K)_{EA}$, but the following still remains as an open problem:

Open 3.2 Provide a natural axiomatization for the logic $(KT)_{EA}$.

Given a frame $\langle W, R\rangle$, call a world $x \in W$ *autistic* (also known as *dead end*) in case there is no world accessible to it according to R, i.e. there is no $y \in W$ such that xRy. Call x *narcissistic* in case it can only access itself. Consider the axioms (V) $\vdash \Box\bot$ and (T_c) $\vdash \varphi \supset \Box\varphi$. The maximal normal modal logic $Ver = K + $ (V) is determined by the class of all autistic frames (i.e., frames whose worlds are all autistic), and the maximal normal modal logic $Triv = K + $ (T) $+$ (T_c) is determined by the class of all narcissistic frames. Exactly midway in between Ver and $Triv$ lies the logic $TV = K + $ (T_c), determined by the class the class of frames whose worlds are all either autistic or narcissistic. It is easy to check that:

Proposition 3.3 (i) In $(Ver)_{EA}$, $\Box\varphi$ can be defined as $\top$. (ii) In $(Triv)_{EA}$, $\Box\varphi$ can be defined as φ. (iii) The logic $(TV)_{EA}$ can be axiomatized by $(K)_{EA} + \vdash \circ\varphi$.

Which other logics can be axiomatized and which logics can define $\Box$ in the language of **LEA**? A few related results, questions and conjectures will close this section.

Conjecture 3.4 $(K4)_{EA} = (K)_{EA} + \vdash \varphi \supset \circ\circ\varphi$, where $K4$ is the logic determined by the class of transitive frames.

Open 3.5 Find an example of a normal modal logic **L** distinct from TV and not extending the logic KT such that $\Box$ is definable in $(\mathbf{L})_{EA}$.

As in [3], the usual technique for non-definability results consists in showing that the geometry of the canonical model of $(\mathbf{L})_{EA}$ does not allow for the definition of $\Box$ in terms of the language of $\mathsf{S}_{\mathbf{LEA}}$.

Theorem 3.6 Let **L** be some normal modal logic. Then, $\Box$ is *not* definable in $(\mathbf{L})_{EA}$ if the canonical model of this logic contains at least one autistic world and one non-autistic world.

Proof Observe first that the formula $\Box\bot$ is satisfied by every autistic world, but it cannot be satisfied by any non-autistic world. On the other hand, we can check by induction on the construction of $\circledcirc(\bot)$ in the language of $\mathsf{S}_{\mathbf{LEA}}$ that such formula must have the same value in all worlds of the canonical model. Indeed, both the atomic case and the case of the classical connectives are straightforward. Moreover, if the values of the formulas $\theta_1, \ldots, \theta_n$ are the same in all worlds, so are the values of $\circ\theta_1, \ldots, \circ\theta_n$ (as they are all true). Thus, $\Box$ cannot in such circumstances be defined in terms of $\circledcirc$.

Notice that any logic that satisfies the conditions from the previous theorem is a fragment of Ver and also a fragment of KD, the modal logic axiomatized by $K + \vdash \Diamond\top$ and determined by the class of all serial frames. That result was but a shy start. We are still left with the tough brain-teaser:

Open 3.7 Provide a full description of the class of all EA-logics in which $\Box$ is definable.

4 Characterizability of classes of frames

Another good test for the expressibility of a modal language consists in checking whether it can individualize many different classes of frames. A class $\mathbb{C}$ of frames will be said to be **LEA**-*characterized* in case there is some $\Gamma \subseteq \mathsf{S}_{\mathbf{LEA}}$ such that $\mathcal{F} \in \mathbb{C}$ iff $\models^{\mathcal{F}} \gamma$, for every $\gamma \in \Gamma$. Obviously, the class of *all* frames is **LEA**-characterizable (just take $\Gamma = \{\top\}$).

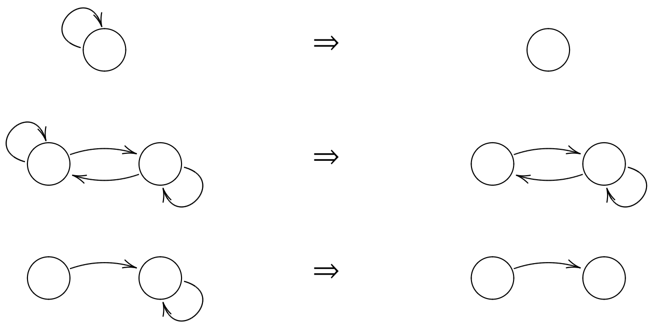
Say that a frame $\mathcal{F}^m = (W, R^m)$ is a *mirror reduction* of a frame $\mathcal{F} = (W, R)$ in case $\mathcal{F}^m$ is obtainable from $\mathcal{F}$ simply by erasing some or all reflexive arrows that appear in the latter, that is, in case $R \setminus \{(x, x) : x \in W\} \subseteq R^m \subseteq R$. Two frames are said to be *mirror-related* in case they are mirror reductions of some common frame.

Example 4.1 Here are some examples of mirror reduction:

(E1)

(E2)

(E3)



One can now immediately prove the following Reduction Lemma:

Lemma 4.2

(RL1) If $\mathcal{F}^m = (W, R^m)$ is a mirror reduction of $\mathcal{F} = (W, R)$, then $\models^{\mathcal{F}^m} \varphi \Leftrightarrow \models^{\mathcal{F}} \varphi$.

(RL2) If two frames are mirror-related then they validate the same formulas.

Proof Part (RL1) can in fact be strengthened. Where $x \in W$, $\mathcal{M}^m$ is a model of $\mathcal{F}^m$ and $\mathcal{M}$ a model of $\mathcal{F}$, then an easy induction can prove that $\models_x^{\mathcal{M}^m} \varphi \Leftrightarrow \models_x^{\mathcal{M}} \varphi$. An interesting case is that of $\varphi = \circ\psi$ (or similarly, that of $\varphi = \bullet\psi$). First, note that $\models_x^{\mathcal{M}^m} \circ\psi$ iff $\not\models_x^{\mathcal{M}^m} \psi$ or $(\forall y \in W)(xR^m y \Rightarrow \models_y^{\mathcal{M}^m} \psi)$. Using the induction hypotheses, this reduces to $\not\models_x^{\mathcal{M}} \psi$ or $(\forall y \in W)(xR^m y \Rightarrow \models_y^{\mathcal{M}} \psi)$. In case $\models_x^{\mathcal{M}} \psi$ and xRx we obviously obtain $(\forall y \in W)(xRy \Rightarrow \models_y^{\mathcal{M}} \psi)$. The converse is straightforward.

Part (RL2) follows from (RL1).

As a consequence of the previous lemma, any **LEA**-characterizable class of frames must be closed under mirror-relatedness. In particular, note that:

Corollary 4.3 The following classes of frames are *not* **LEA**-characterizable:

(i) reflexive frames
(ii) serial frames
(iii) transitive frames
(iv) euclidean frames
(v) convergent frames

Proof Recall Example 4.1. The frame at the left-hand side of (E1) is both reflexive and serial, the frame at the l.h.s. of (E2) is transitive, and at the l.h.s. of (E3) we find a frame that is both euclidean and convergent. None of those properties is satisfied after mirror-reduction, as we can see at the right-hand sides of each example.

Compare the above with the more well-known situation of **NML**-characterizability (check for instance ch. 3 of [1]). The class of serial frames, for example, is **NML**-characterized by taking $\Gamma = \{\Diamond\top\}$.

Finally, here is a problem whose solution is highly non-trivial already in the analogous case of the language of **NML**:

Open 4.4 Provide a full description of the class of **LEA**-characterizable classes of frames.

5 On essence, and beyond

How much of our intuitions about essence and accident are captured by the new connectives $\circ$ and $\bullet$ studied above? And how do these notions differ from other usual modal notions such as those of *contingency* and *non-contingency*?

Suppose we extend the classical language by adding the unary connectives $\triangledown$ for contingency and $\vartriangle$ for non-contingency. The usual way of interpreting these notions is by extending the notion of satisfaction such that:

$$\models_x^{\mathcal{M}} \triangledown\varphi \quad \text{iff} \quad (\exists y \in W)(xRy \;\&\; \models_y^{\mathcal{M}} \varphi) \text{ and } (\exists z \in W)(xRz \;\&\; \not\models_z^{\mathcal{M}} \varphi)$$
$$\models_x^{\mathcal{M}} \vartriangle\varphi \quad \text{iff} \quad \not\models_x^{\mathcal{M}} \triangledown\varphi$$

The modal base for (non-)contingency was studied sporadically in the literature since the mid-60s (cf. [10]), for several classes of frames, and an axiomatization for the minimal logic of non-contingency was finally offered in [5], and immediately simplified in [8]. In the language of **NML** one could obviously define $\triangledown\varphi$ as $\Diamond\varphi \vee \Diamond{\sim}\varphi$ and $\vartriangle\varphi$ as $\Diamond\varphi \supset \Box\varphi$. One could now also easily consider the languages with both contingency and accidental statements and their duals, and then note for instance that $\models_K (\circ\varphi \wedge \circ{\sim}\varphi) \supset \vartriangle\varphi$ and $\models_{KT} \vartriangle\varphi \supset (\circ\varphi \wedge \circ{\sim}\varphi)$.

In the philosophy of modal logic, every modality has at least two central readings, a metaphysical reading that takes it as qualifying the truth of some statement, and an ontological reading that takes it as qualifying the properties of some object. Necessity, possibility, contingency and non-contingency were all used in the literature either in the metaphysical or in the ontological reading. Traditionally, the philosophical literature has often talked about essential and accidental properties of objects. A somewhat sophisticated way of internalizing that talk at the object-language level was devised by Kit Fine (cf. [4]), with the help of a sort of multimodal language in which there are operators intended to represent truth by reason of the nature of the involved objects, and a further binary predicate intended to represent ontological dependence. The present paper investigated instead a particular rendering of those notions in their naive metaphysical reading, simply by turning essence and accident into new propositional connectives.

Is the above reading solid from a philosophical standpoint? The question is not trivial to resolve. One has to concede that there is complete bedlam in the philosophical literature as potentially different kinds of modality often get conflated without much care. Sometimes one finds an identification between the notion of contingency and the notion of accident, sometimes necessity is opposed to contingency and the corresponding square of oppositions is turned into a triangle (maybe the Reverend has stolen a diamond, as in Stevenson's story?), sometimes the analytic $\times$ synthetic distinction is reformulated in terms of essential $\times$ accidental modes of judgement (somehow perverting Kant's proposal to understand essence as expressing an *a priori* synthetic truth). To be sure, the same terms can indeed receive several (hopefully related) uses in different areas of philosophy. But considerable prudence should be exercised so that the corresponding notions do not confound, and so that they do not get too circumscribed nor too stretched in their meanings.

The grammar of modalities in formal languages can often be mirrored in the grammar of adverbs in natural language (or was it the other way around?). Let's explore this analogy a bit. Adverbs are parts of speech comprised of words that modify a verb, an adjective, or another adverb. The first two cases are of interest here. In case the adverbs modify a verb, they derivatively modify a sentence of which this verb is the main verb. The *assertoric* status of the sentence is then subjected to the mood expressed by the adverb. In case they modify an adjective, they derivatively modify a noun. The *attributes* of the object to be denoted by that noun are then subjected to the revaluation set by the adverb. Most adverbs will allow for assertoric and attributive uses, at different circumstances, and a similar thing happens with modalities.

It appears that the notions of essence and accident have been more widely used attributively, at least in recent years. They have been often applied to predications, qualities, and properties. But in formal metaphysics one can also find those notions in their assertoric use. In [11], for instance, Gödel's modal reconstruction of the Ontological Argument is presented with an understanding of 'accidental truth' that is identical to the one that is adopted here. But, despite the relative infrequency of its employment in our times, the assertoric use of essence and accident is also not new. Indeed, in [6], a reasonably influential logic textbook from the XIX Century, John Neville Keynes (the father of John Maynard) already talked freely about essential and accidental propositions, as opposed to essential and accidental predications. Suspending for a while the final judgement about the soundness of the attributive use of such adverbs of essence and accident, this paper has tackled the investigation the technicalities involved in the choice of a modal language obtained simply by adding connectives for essence and for accident to the language of classical logic.

A few more technical objections could still be raised against the above modal renderings of essence and accident. One of them runs as follows. According to the present interpretation of 'essence', a formula is said to have an essentially true status in case it is simply false, and, indeed, in the Proposition 2.1, (K2.4) showed $\circ\bot$ to be a theorem of $(K)_{EA}$. What is that supposed to *mean*? Recall from the modal definition of satisfaction, in section 1, that a statement was defined to be 'accidentally true' in case it *is* true, but could have been false, had the world been different. An antilogical statement obviously cannot be accidentally true, thus it must be essentially so. A similar phenomenon happens in the logics of non-contingency, in which $\vartriangle\bot$ is always provable: a statement that is false in all worlds cannot be contingently true, thus it must be non-contingently so. If, notwithstanding the above explanation, the circumstance of an antilogical statement having an essential (that is, a non-accidental) status still upsets one's modal intuitions, a way of modifying the definition of essence in order to avoid this would be by exchanging the material conditional in the definition of $\circ\varphi$ as $\varphi \supset \Box\varphi$ for some stronger connective conveying the sense of strict implication (defining $\circ\varphi$ as $\varphi \prec \Box\varphi$ or more simply $\Box(\varphi \supset \Box\varphi)$). A related intuitive objection points to the fact that in the present formalization the notion of essence is still too local: a statement could be essentially true in a world, but fail to be essentially true in another world that can access or be accessed from the former world. Again, one way of fixing that might be by way of the use of some sort of strict implication in the definition of essence, but a more direct solution might be just to make use of some heredity condition on the models, in order to guarantee that statements that are essentially true in a world have the same essential status in all other worlds that belong to its accessibility class. All such alternative formalizations of the notion of essence seem worth exploring.

Finally, for some more positive remarks on the present notion of essence and its possible uses, one might notice for instance that the received modal semantics of intuitionistic logic by way of a translation into the modal logic $S4$ already assumes (through the heredity condition) that all atomic sentences are essentially true in all worlds, so that any eventual truth is preserved into the future (monotonic proofs do not become false as more things get proven). The traditional ontological argument, as proposed by Anselm, discussed by Leibniz, or formalized by Gödel, also involves an appeal to propositions about essence: The sentence positing God's existence would be shown to express a non-accidental truth. Another immediate use for the present notion of essence is in formalizing Saul Kripke's notion of 'rigid designation', and understanding how some truths could be simultaneously necessary and *a posteriori* (cf. [7]): From a physicalist *a priori* true statement according to which "Water is essentially H_2O" (based on the presupposition that any chemical component of water is an essential component of it) and from an empirical verification of the statement that “Water is H_2O” it would arguably follow that “Water is necessarily H_2O” is an *a posteriori* truth. Yet another promising use of the notion of essence is in expressing the consistency of a formula in situations in which negation is non-explosive, allowing for paraconsistent phenomena to appear. With that idea in mind, any non-degenerate normal modal logic could be easily recast as a *logic of formal inconsistency* (cf. [2]), a paraconsistent logic that is rich enough as to be able internalize the very notion of consistency. From that point of view, an inconsistency is interpreted simply as an accident. This idea is explored in detail in another paper (cf. [9]).

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