Long heterochromatic paths in edge-colored graphs

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Abstract

Let G be an edge-colored graph. A heterochromatic path of G is such a path in which no two edges have the same color. $d^c(v)$ denotes the color degree of a vertex v of G. In a previous paper, we showed that if $d^c(v) \geq k$ for every vertex v of G, then G has a heterochromatic path of length at least $\lceil \frac{k+1}{2} \rceil$. It is easy to see that if $k = 1, 2$, G has a heterochromatic path of length at least k. Saito conjectured that under the color degree condition G has a heterochromatic path of length at least $\lceil \frac{2k+1}{3} \rceil$. Even if this is true, no one knows if it is a best possible lower bound. Although we cannot prove Saito's conjecture, we can show in this paper that if $3 \leq k \leq 7$, G has a heterochromatic path of length at least $k-1$, and if $k \geq 8$, G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$. Actually, we can show that for $1 \leq k \leq 5$ any graph G under the color degree condition has a heterochromatic path of length at least k, with only one exceptional graph K_4 for $k = 3$, one exceptional graph for $k = 4$ and three exceptional graphs for $k = 5$, for which G has a heterochromatic path of length at least $k-1$. Our experience suggests us to conjecture that under the color degree condition G has a heterochromatic path of length at least $k-1$.

1. Introduction

We use Bondy and Murty [3] for terminology and notations not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we will mean a function $C : E \rightarrow \mathbb{N}$, the set of nonnegative integers. If G is assigned such a coloring, then we say that G is an *edge-colored graph.* Denote the colored graph by (G, C), and call $C(e)$ the *color* of the edge $e \in E$ and $C(uv) = \emptyset$ if $uv \notin E(G)$ for any $u, v \in V(G)$. All edges with the same color form a *color class* of the graph. For a subgraph H of G, we let $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) \mid e$ is incident with $v\}$ and the *color degree* is $d^c(v) = |CN(v)|$. A path is called *heterochromatic* if any two edges of it have different colors. If u and v are two vertices on a path P, uPv denotes the segment of P from u to v.

There are many existing literature dealing with the existence of paths and cycles with special properties in edge-colored graphs. In [5], the authors showed that for a 2-edge-colored graph G and three specified vertices x, y and z, to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. The heterochromatic Hamiltonian cycle or path problem was studied by Hahn and Thomassen [9], Rödl and Winkler (see [8]), Frieze and Reed [8], and Albert, Frieze, Reed [1]. For more references, see [2, 6, 7, 10, 11]. Many results in these papers are proved by using probabilistic methods.

In [4], the authors showed that if G is an edge-colored graph with $d^c(v) \geq k$ for every v of G, then G has a heterochromatic path with length at least $\lceil \frac{k+1}{2} \rceil$. It is easy to see that if $k = 1, 2$, G has a heterochromatic path of length at least k. Saito conjectured that under the color degree condition G has a heterochromatic path of length at least $\lceil \frac{2k+1}{3} \rceil$. Even if this is true, no one knows if it is a best possible lower bound in general. Although we cannot prove Saito's conjecture, we can show in this paper that if $3 \leq k \leq 7$, then G has a heterochromatic path of length at least $k-1$, and if $k \geq 8$, then G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$. Actually, we can show that for $1 \leq k \leq 5$ any graph G under the color degree condition has a heterochromatic path of length at least k, with only one exceptional graph K_4 for $k = 3$, one exceptional graphs for $k = 4$ and three exceptional graphs for $k = 5$, for which G has a heterochromatic path of length at least $k - 1$. Our experience suggests us to conjecture that under the color degree condition G has a heterochromatic path of length at least $k - 1$.

2. Long heterochromatic paths for $k \leq 7$

We consider the case when $1 \leq k \leq 7$, first.

For the case when $k = 1$ or 2, it is obvious that there is a heterochromatic path of length k in G. In fact, for $k = 1$ any edge of G is a required heterochromatic path of length k; for $k = 2$, at each vertex there exist two adjacent edges with different colors, and they form a required heterochromatic path of length k. Next, we consider the case when $3 \leq k \leq 7$ and get the following result.

Theorem 2.1 *Let G be an edge-colored graph and $3 \leq k \leq 7$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a heterochromatic path of length at least $k - 1$.*

Proof. (1) $k = 3$. Since $k = 3 > 2$, there is a heterochromatic path of length 2 in G.

(2) $k = 4$. Since $k = 4 > 3$, there is a heterochromatic path of length 2 in G. Let $P = u_1u_2u_3$ be such a path that u_1u_2 has color i_1 and u_2u_3 has color i_2 with $i_1 \neq i_2$. Since $d^c(u_3) \geq k = 4$, there are two vertices $v,\ w \in G$ such that $u_3v,\ u_3w \in E(G)$ have two different colors $i_3,\ i_4 \notin \{i_1, i_2\}$. Let u_4 be a vertex in $\{v, w\} \setminus \{u_1\}$. Then $\{u_4\} \cap \{u_1, u_2, u_3\} = \emptyset$ and $P' = u_1u_2u_3u_4$ is a heterochromatic path of length 3.

(3) $k = 5$. Since $k = 5 > 4$, there is a heterochromatic path of length 3 in G. Let $P = u_1u_2u_3u_4$ be such a path that u_xu_{x+1} has color i_x for $x = 1, 2, 3$. If there exists a $v \notin \{u_1, u_2, u_3, u_4\}$ such that $C(u_4v) \notin \{i_1, i_2, i_3\}$, then $P' = u_1u_2u_3u_4v$ is a heterochromatic path of length 4. Otherwise, since $d^c(u_4) \geq 5$, we have that $|C(\{u_1u_4, u_2u_4\}) - \{i_1, i_2, i_3\}| = 2$ and there exists a $v_1 \in V(G)$ such that $C(u_4v_1) = i_1$. Since $d^c(v_1) \geq 5$, we have that $|CN(v_1) - \{i_1, i_2, i_3\}| \geq 2$. If there exists a $v_2 \in V(G)$ such that $C(v_1v_2) \notin \{i_1, i_2, i_3\}$, then $P' = u_2u_3u_4v_1v_2$ is a heterochromatic path of length 4; if $C(u_1v_1) \notin \{i_1, i_2, i_3\}$, then $P' = v_1u_1u_2u_3u_4$ is a heterochromatic path of length 4. So, the only remaining case is when $|C(\{u_2v_1, u_3v_1\}) - \{i_1, i_2, i_3\}| = 2$, and in this case we have that $P' = u_1u_2v_1u_3u_4$ is a heterochromatic path of length 4.

(4) $k = 6$. Since $k = 6 > 5$, there is a heterochromatic path of length 4 in G. Let $P = u_1u_2u_3u_4u_5$ be such a path that u_xu_{x+1} has color i_x for $x = 1, 2, 3, 4$. If there exists a $v \notin \{u_1, u_2, u_3, u_4, u_5\}$ such that $C(u_5v) \notin \{i_1, \ldots, i_4\}$, then $u_1u_2u_3u_4u_5v$ is a heterochromatic path of length 5. Next we consider the case when there is no such a vertex v, in other words, $|C(\{u_1u_5, u_2u_5, u_3u_5\}) - \{i_1, i_2, i_3, i_4\}| \geq 2$. Since $d^c(u_5) \geq 6$, there is a vertex v_1 such that $C(u_5v_1) = i_1$ or i_2.

(4.1) $C(u_5v_1) = i_1$. Since $d^c(u_5) \geq 6$ and $|C(\{u_1u_5, u_2u_5, u_3u_5\}) - \{i_1, i_2, i_3, i_4\}| \geq 2$, there exists a $u_6 \in V(G)$ such that $C(u_5u_6) = i_2$ or i_3. If there is a vertex $v \notin \{u_2, u_3, u_4, u_5\}$ such that $C(v_1v) \notin \{i_1, \ldots, i_4\}$, then $u_2u_3u_4u_5v_1v$ is a heterochromatic path of length 5. And, since $d^c(v_1) \geq k = 6$, we have that $|C(\{u_2v_1, u_3v_1, u_4v_1\}) - \{i_1, \ldots, i_4\}| \geq 2$. If there exists an $2 \leq i \leq 3$ such that $|C(\{u_iv_1, u_{i+1}v_1\}) - \{i_1, i_2, i_3, i_4\}| = 2$, then $P' = u_1Pu_iv_1u_{i+1}Pu_5$ is a heterochromatic path of length 5. Otherwise, $|C(\{u_2v_1, u_4v_1\}) - \{i_1, i_2, i_3, i_4\}| = 2$ and $u_1u_2v_1u_4u_5u_6$ is a heterochromatic path of length 5.

(4.2) $C(u_5v_1) = i_2$. Since $d^c(u_5) \geq 6$, there exists a $u_6 \in V(G)$ such that $C(u_5u_6) = i_3$. Then we have the following three cases: (i) There is no vertex $v \notin \{u_1, \ldots, u_5\}$ such that $C(v_1v) \notin \{i_1, \ldots, i_4\}$. Since $d^c(v_1) \geq 6$, we have that $|C(\{u_1v_1, u_2v_1, u_3v_1, u_4v_1\}) - \{i_1, i_2, i_3, i_4\}| \geq 2$. If $C(u_1v_1) \notin \{i_1, i_2, i_3, i_4\}$, then $v_1u_1u_2u_3u_4u_5$ is a heterochromatic path of length 5. If there exists an $i = 2$ or 3 such that $|C(\{u_iv_1, u_{i+1}v_1\}) - \{i_1, i_2, i_3, i_4\}| = 2$, then $u_1Pu_iv_1u_{i+1}Pu_5$ is a heterochromatic path of length 5. Otherwise, $|C(\{u_2v_1, u_4v_1\}) - \{i_1, i_2, i_3, i_4\}| = 2$, and then $u_1u_2v_1u_4u_5u_6$ is a heterochromatic path of length 5. (ii) There is a $v_2 \notin \{u_1, \ldots, u_5\}$ such that $C(v_1v_2) - \{i_1, \ldots, i_4\} \neq \emptyset$ and there is no vertex $v \notin \{u_1, \ldots, u_5\}$ such that $|C(\{v_1v_2, v_2v\}) - \{i_1, \ldots, i_4\}| = 2$. Let $i_5 = C(v_1v_2)$. Since $d^c(v_2) \geq 6$, we have that $C(\{u_1v_2, u_2v_2, u_3v_2, u_4v_2\}) - \{i_1, \ldots, i_5\} \neq \emptyset$. If $C(u_iv_2) - \{i_1, \ldots, i_5\} \neq \emptyset$ for $i = 1$ or 2, then $u_3u_4u_5v_1v_2u_i$ is a heterochromatic path of length 5. If $C(u_4v_2) - \{i_1, \ldots, i_5\} \neq \emptyset$, then $u_1u_2u_3u_4v_2v_1$ is a heterochromatic path of length 5. Otherwise, $C(u_3v_2) - \{i_1, \ldots, i_5\} \neq \emptyset$. Since $u_1u_2u_3u_4u_5$ is a heterochromatic path of length 4, we have that $|C(\{u_1u_3, u_1u_4, u_1u_5\}) - \{i_1, i_2, i_3, i_4\}| \geq 2$ which implies that

$C(\{u_1u_3, u_1u_4\}) - \{i_1, i_2, i_3, i_4\} \neq \emptyset$. If $C(u_1u_3) - \{i_1, i_2, i_3, i_4\} \neq \emptyset$, then $u_2u_1u_3u_4u_5v_1$ is a heterochromatic path of length 5. If $C(u_1u_4) \notin \{i_1, i_2, i_3, i_4\}$, then $u_5u_4u_1u_2u_3v_2$ or $u_2u_1u_4u_5v_1v_2$ is a heterochromatic path of length 5. (iii) There exist $v_2, v_3 \notin \{u_1, \dots, u_5\}$ such that $|C(\{v_1v_2, v_2v_3\}) - \{i_1, \dots, i_4\}| = 2$. Then $u_3u_4u_5v_1\ v_2v_3$ is a heterochromatic path of length 5.

(5) $k = 7$. Since $k = 7 > 6$, there is a heterochromatic path of length 5 in G. Let $P = u_1u_2u_3u_4u_5u_6$ be such a path that u_xu_{x+1} has color i_x for $x = 1, \dots, 5$. If there is a vertex $v \notin V(P)$ such that $C(u_6v) \notin C(P)$, then u_1Pu_6v is a heterochromatic path of length 6. Next we consider the case when there is no such v. Since $d^c(u_6) \geq 7$, we have that $|C(\{u_1u_6, u_2u_6, u_3u_6, u_4u_6\}) - \{i_1, \dots, i_5\}| \geq 2$. Let $k_0 = \min\{k |$there is a vertex $v \notin V(P)$ such that $C(u_6v) = i_k\}$, and v_1 be a vertex $\notin V(P)$ such that $C(u_6v) = i_{k_0}$. Then $k_0 = 1$ or 2 or 3.

(5.1) $k_0 = 1$. If there exists a $v_2 \notin V(P)$ such that $C(v_1v_2) \notin C(P)$, then $u_2Pu_6v_1v_2$ is a heterochromatic path of length 6. Or, $|C(\{u_1v_1, \dots, u_5v_1\}) - C(P)| \geq 2$. If there is a vertex $u \notin V(P)$ such that $C(uu_1) \notin C(P)$, then uu_1Pu_6 is a heterochromatic path of length 6. Next we consider the case when $|C(\{u_2v_1, u_3v_1, u_4v_1, u_5v_1\}) - C(P)| \geq 2$ and $|C(\{u_1u_3, \dots, u_1u_6\}) - C(P)| \geq 2$. If $C(u_2v_1) \notin C(P)$, then there is an $3 \leq i \leq 6$ such that $C(u_1u_i) \notin C(P) \cup C(u_2v_1)$, and so $u_{i-1}P^{-1}u_2v_1u_6P^{-1}u_iu_1$ is a heterochromatic path of length 6. If there exists an $i = 3$ or 4 such that $|C(\{u_iv_1, u_{i+1}v_1\}) - C(P)| = 2$, then $u_1Pu_iv_1u_{i+1}Pu_6$ is a heterochromatic path of length 6. In the rest we shall only consider the case when $|C(\{u_3v_1, u_5v_1\}) - C(P)| = 2$, and let $i_6 = C(u_3v_1)$ and $i_7 = C(u_5v_1)$. If there exists a $v \notin V(P)$ such that $C(u_6v) \notin \{i_1, i_2, i_5\}$, i.e., $C(u_6v) \in \{i_3, i_4\}$, then $u_1u_2u_3v_1u_5u_6v$ is a heterochromatic path of length 6. Otherwise, $|C(\{u_1u_6, u_2u_6, u_3u_6, u_4u_6\}) - \{i_1, i_2, i_5\}| = 4$. On the other hand, if $C(u_1u_6) - \{i_1, i_2, i_5\} \neq \emptyset$, then $v_1u_3u_2u_1u_6u_5u_4$ or $v_1u_5u_6u_1u_2u_3u_4$ is a heterochromatic path of length 6, a contradiction.

(5.2) $k_0 = 2$. So, $|C(\{u_1u_6, u_2u_6, u_3u_6, u_4u_6\}) - \{i_2, i_3, i_4, i_5\}| \geq 3$. We have the following three cases:

(i) There is no vertex $v \notin V(P)$ such that $C(v_1v) \notin C(P)$. Since $d^c(v_1) \geq 7$, we have that $|C(\{u_1v_1, u_2v_1, u_3v_1, u_4v_1, u_5v_1\}) - C(P)| \geq 2$. If there exists a $u \notin V(P)$ such that $C(uu_1) \notin C(P)$, then uu_1Pu_6 is a heterochromatic path of length 6. Next we consider the case when $|C(\{u_2v_1, u_3v_1, u_4v_1, u_5v_1\}) - C(P)| \geq 2$ and $|C(\{u_1u_3, \dots, u_1u_6\}) - C(P)| \geq 2$. If $C(u_2v_1) \notin C(P)$, then there is an $3 \leq i \leq 6$ such that $C(u_1u_i) \notin C(P) \cup C(u_2v_1)$, and so $u_3Pu_iu_1u_2v_1u_6P^{-1}u_{i+1}$ is a heterochromatic path of length 6. If there exists an $i = 3$ or 4 such that $|C(\{u_iv_1, u_{i+1}v_1\}) - C(P)| = 2$, then $u_1Pu_iv_1u_{i+1}Pu_6$ is a heterochromatic path of length 6. In the rest we shall only consider the case when $|C(\{u_3v_1, u_5v_1\}) - C(P)| = 2$. Since $|C(\{u_1u_6, u_2u_6, u_3u_6, u_4u_6\}) - \{i_2, i_3, i_4, i_5\}| \geq 3$, there exists a $v \notin V(P)$ such that $C(u_6v) \in \{i_3, i_4\}$, and so $u_1u_2u_3v_1u_5u_6v$ is a heterochromatic path of length 6.

(ii) There exists a $v_2 \notin V(P)$ such that $C(v_1v_2) \notin C(P)$, and there is no $v \notin V(P) \cup \{v_1\}$ such that $C(v_2v) \notin C(P) \cup C(v_1v_2)$. Let $i_6 = C(v_1v_2)$. Then $C(\{u_1v_2, \dots, u_5v_2\}) - \{i_1, \dots, i_6\} \neq \emptyset$. If there exists a $u \notin \{u_1, \dots, u_6\}$ such that $C(uu_1) \notin C(P)$, then uu_1Pu_6 is a heterochromatic path of length 6. If $C(u_2v_2) \notin \{i_1, \dots, i_6\}$, then $v_1v_2u_2Pu_6$ is a heterochromatic path of length 6. If $C(u_5v_2) \notin \{i_1, \dots, i_6\}$, then $u_1Pu_5v_2v_1$ is a heterochromatic path of length 6. So we shall only consider the case when $|C(\{u_1u_3, \ldots, u_1u_6\}) - C(P)| \geq 2$ and $C(u_3v_2) \notin \{i_1, \ldots, i_6\}$ or $C(u_4v_2) \notin \{i_1, \ldots, i_6\}$.

(ii.1) $C(u_3v_2) \notin \{i_1, \ldots, i_6\}$, and let $i_7 = C(u_3v_2)$. Since P is a heterochromatic path of length 5, we have that $C(\{u_1u_3, \ldots, u_1u_6\}) - \{i_1, \ldots, i_5, i_7\} \neq \emptyset$. If $C(u_1u_3) \notin \{i_1, \ldots, i_5, i_7\}$, let $P' = u_2u_1u_3Pu_6v_1$; if $C(u_1u_4) \notin \{i_1, \ldots, i_5, i_7\}$, let $P' = v_2u_3u_2u_1u_4 u_5u_6$; if $C(u_1u_5) \notin \{i_1, \ldots, i_5, i_6, i_7\}$, let $P' = v_1v_2u_3u_2u_1u_5u_6$; if $C(u_1u_6) \notin \{i_1, \ldots, i_5, i_7\}$, let $P' = v_2u_3u_2u_1u_6u_5u_4$. Then, P' is a heterochromatic path of length 6 in all these cases. It remains to show that when $C(u_1u_5) = i_6$, there is a heterochromatic path of length 6. Since $d^c(u_6) \geq 7$ and $|C(\{u_1u_6, \ldots, u_4u_6\}) - \{i_2, i_3, i_4, i_5\}| \geq 3$, we have that $C(u_6v_2) \in \{i_3, i_4\}$ or there exists a $v \notin \{u_1, \ldots, u_6, v_1, v_2\}$ such that $C(u_6v) \in \{i_3, i_4\}$. If $C(u_6v_2) = i_3$, and so $u_1u_2u_3v_2u_6u_5u_4$ is a heterochromatic path of length 6; if $C(u_6v_2) = i_4$, then $u_4u_3u_2u_1u_5u_6v_2$ is a heterochromatic path of length 6. If there is a vertex $v \notin \{u_1, \ldots, u_6, v_1, v_2\}$ such that $C(u_6v) \in \{i_3, i_4\}$, then $v_2u_3u_2u_1u_5u_6v$ is a heterochromatic path of length 6.

(ii.2) $C(u_4v_2) \notin \{i_1, \ldots, i_6\}$, and let $i_7 = C(u_4v_2)$. Since P is a heterochromatic path of length 5, we have that $C(\{u_1u_3, \ldots, u_1u_6\}) - \{i_1, \ldots, i_5, i_7\} \neq \emptyset$. If $C(u_1u_3) \notin \{i_1, \ldots, i_5, i_7\}$, let $P' = u_2u_1u_3u_4u_5u_6v_1$; if $C(u_1u_4) \notin \{i_1, \ldots, i_5, i_6, i_7\}$, let $P' = u_2u_1u_4u_5 u_6v_1v_2$; if $C(u_1u_5) \notin \{i_1, \ldots, i_5, i_7\}$, let $P' = v_2u_4u_3u_2u_1u_5u_6$; if $C(u_1u_6) \notin \{i_1, \ldots, i_5, i_7\}$, let $P' = v_2u_4u_3u_2u_1u_6u_5$. Then, P' is a heterochromatic path of length 6 in all these cases. It remains to show that when $C(u_1u_4) = i_6$, there is a heterochromatic path of length 6. Since $u_1u_2u_3u_4v_2v_1$ is a heterochromatic path of length 5, we have that $C(\{u_1u_3, u_1v_1, u_1v_2\}) - \{i_1, i_2, i_3, i_4, i_6, i_7\} \neq \emptyset$. If $C(u_1u_3) \notin \{i_1, i_2, i_3, i_4, i_6, i_7\}$, and so $u_2u_1u_3u_4v_2v_1u_6$ is a heterochromatic path of length 6; if $C(u_1v_1) \notin \{i_1, i_2, i_3, i_4, i_6, i_7\}$, then $v_2v_1u_1u_2u_3u_4u_5$ is a heterochromatic path of length 6; if $C(u_1v_2) \notin \{i_1, i_2, i_3, i_4, i_6, i_7\}$, then $v_1v_2u_1u_2u_3u_4u_5$ is a heterochromatic path of length 6.

(iii) There are vertices $v_2, v_3 \notin \{u_1, \ldots, u_6, v_1\}$ such that $|C(\{v_1v_2, v_2v_3\}) - C(P)| = 2$, and $u_3u_4u_5u_6v_1v_2v_3$ is a heterochromatic path of length 6.

(5.3) $k_0 = 3$. So, $|C(\{u_1u_6, \ldots, u_4u_6\}) - \{i_3, i_4, i_5\}| = 4$. We have the following three cases:

(i) There is no vertex $v \notin V(P)$ such that $C(v_1v) \notin C(P)$, and so $|C(\{u_1v_1, \ldots, u_5v_1\}) - C(P)| \geq 2$. Since $|C(\{u_1u_6, \ldots, u_4u_6\}) - \{i_3, i_4, i_5\}| = 4$, there is a $u_7 \notin V(P)$ such that $C(u_6u_7) = i_4$. If there exists a $u \notin V(P)$ such that $C(uu_1) \notin C(P)$, then uu_1Pu_6 is a heterochromatic path of length 6. If $C(u_3v_1) \notin C(P)$, then $u_1u_2u_3v_1u_6u_5u_4$ is a heterochromatic path of length 6. If there exists an $2 \leq i \leq 4$ such that $|C(\{u_iv_1, u_{i+1}v_1\}) - C(P)| = 2$, then $u_1Pu_iv_1u_{i+1}Pu_6$ is a heterochromatic path of length 6. So we shall only show that there is a heterochromatic path of length 6 when $|C(\{u_1u_3, \ldots, u_1u_6\}) - C(P)| \geq 2$ and $C(u_2v_1) \notin C(P)$. Let $i_6 = C(u_2v_1)$. Then $C(\{u_1u_3, \ldots, u_1u_6\}) - \{i_1, \ldots, i_5, i_6\} \neq \emptyset$. If $C(u_1u_3) \notin \{i_1, \ldots, i_6\}$, let $P' = v_1u_2u_1u_3u_4u_5u_6$; if $C(u_1u_4) \notin \{i_1, \ldots, i_6\}$, let $P' = u_3u_2u_1u_4u_5u_6v_1$; if $C(u_1u_5) \notin \{i_1, \ldots, i_6\}$, let $P' = u_4u_3u_2u_1u_5u_6u_7$; if $C(u_1u_6) \notin \{i_1, \ldots, i_6\}$, let $P' = v_1u_2u_1u_6u_5u_4u_3$. Then, P' is a heterochromatic path of length 6 in all these cases.

(ii) There is a vertex v_2 such that $C(v_1v_2) \notin C(P)$, and there is no vertex $v \notin \{u_1, \ldots, u_6, v_1, v_2\}$ such that $C(v_1v_2) \notin \{i_1, \ldots, i_5\} \cup C(v_1v_2)$. Let $i_6 = C(v_1v_2)$. Then

$C(\{u_1v_2, \dots, u_5v_2\}) - \{i_1, \dots, i_6\} \neq \emptyset$. If $C(u_1v_2) \notin \{i_1, \dots, i_6\}$, then $v_2u_1Pu_6$ is a heterochromatic path of length 6. If $C(u_2v_2) \notin \{i_1, \dots, i_6\}$, then $v_1v_2u_2Pu_6$ is a heterochromatic path of length 6. If $C(u_3v_2) \notin \{i_1, \dots, i_6\}$, then $u_1u_2u_3v_2v_1u_6u_5$ is a heterochromatic path of length 6. If $C(u_5v_2) \notin \{i_1, \dots, i_6\}$, then $u_1Pu_5v_2v_1$ is a heterochromatic path of length 6. Next we shall consider the case when $C(u_4v_2) \notin \{i_1, \dots, i_6\}$. Let $i_7 = C(u_4v_2)$. Since $|C(\{u_1u_6, u_2u_6, u_3u_6, u_4u_6\}) - \{i_3, i_4, i_5\}| = 4$, there is a vertex $u \notin V(P)$ such that $C(u_6u) = i_4$. If $u = v_2$, i.e., $C(u_6v_2) = i_4$, then $u_1u_2u_3u_4v_2u_6u_5$ is a heterochromatic path of length 6. It remains to show that there is a heterochromatic path of length 6 if there is a vertex $u \notin \{u_1, \dots, u_6, v_1, v_2\}$ such that $C(u_6u) = i_4$. In this case, since $|C(\{u_1u_3, \dots, u_1u_6\}) - C(P)| \geq 2$, we have that $C(\{u_1u_4, u_1u_5, u_1u_6\}) - C(P) \neq \emptyset$. If $C(u_1u_4) \notin C(P)$, let $P' = u_3u_2u_1u_4u_5u_6v_1$; if $C(u_1u_5) \notin C(P)$, let $P' = u_4u_3u_2u_1u_5u_6u$; if $C(u_1u_6) \notin \{i_1, \dots, i_5, i_7\}$, let $P' = v_2u_4u_3u_2u_1u_6u_5$. Then, P' is a heterochromatic path of length 6. Last, we consider the case when $C(u_1u_6) = i_7$. Since $u_3u_2u_1u_6v_1v_2$ is a heterochromatic path of length 5, we have $|C(\{u_1v_2, u_2v_2, u_3v_2, u_6v_2\}) - \{i_1, i_2, i_3, i_6, i_7\}| \geq 2$, and so $C(\{u_1v_2, u_3v_2, u_6v_2\}) - \{i_1, i_2, i_3, i_6, i_7\} \neq \emptyset$. If $C(u_1v_2) - \{i_1, i_2, i_3, i_4, i_6, i_7\} \neq \emptyset$, let $P' = v_1v_2u_1Pu_5$; if $C(u_1v_2) = i_4$, let $P' = u_3u_2u_1v_2v_1u_6u_5$; if $C(u_3v_2) - \{i_1, i_2, i_3, i_4, i_6, i_7\} \neq \emptyset$, let $P' = u_1u_2u_3v_2v_1u_6u$; if $C(u_3v_2) = i_4$, let $P' = u_1u_2u_3v_2v_1u_6u_5$; if $C(u_6v_2) - \{i_1, i_2, i_3, i_6, i_7\} \neq \emptyset$, let $P' = u_4u_3u_2u_1u_6v_2v_1$. Then, P' is a heterochromatic path of length 6 in all these cases.

(iii) There are vertices $v_2, v_3 \notin \{u_1, \dots, u_6, v_1\}$ such that $|C(\{v_1v_2, v_2v_3\}) - C(P)| = 2$. Let $i_6 = C(v_1v_2)$ and $i_7 = C(v_2v_3)$. If there exists a $v \notin \{u_4, u_5, v_1\}$ such that $C(v_3v) \notin \{i_3, \dots, i_7\}$, i.e., there exists a $v \notin \{u_4, u_5, u_6, v_1, v_2\}$ such that $C(v_3v) \notin \{i_3, \dots, i_7\}$, then $u_4u_5u_6v_1v_2v_3v$ is a heterochromatic path of length 6. Next we shall only consider the case when $|C(\{u_4v_3, u_5v_3, v_1v_3\}) - \{i_3, \dots, i_7\}| \geq 2$. If $C(u_5v_3) \notin \{i_2, \dots, i_7\}$, then $u_2u_3u_4u_5v_3v_2v_1$ is a heterochromatic path of length 6. If $C(u_5v_3) = i_2$, then $u_3u_4u_5v_3v_2v_1$ is a heterochromatic path of length 5 and $C(v_1u_6) = C(u_3u_4)$, and so there is a heterochromatic path of length 6 from the cases discussed above. If $C(u_4v_3) \notin \{i_1, \dots, i_7\}$, then $v_1v_2v_3u_4u_3u_2u_1$ is a heterochromatic path of length 6. If $C(u_4v_3) = i_1$, then $u_2u_3u_4v_3v_2v_1$ is a heterochromatic path of length 5 and $C(v_1u_6) = C(u_3u_4) = i_3$, and so there is a heterochromatic path of length 6 because of (5.1) and (5.2). Now it remains to show that there is a heterochromatic path of length 6 when $C(u_4v_3) = i_2$ and $C(v_1v_3) \notin \{i_2, \dots, i_7\}$. Since $|C(\{u_1u_6, u_2u_6, u_3u_6, u_4u_6\}) - \{i_3, \dots, i_5\}| = 4$, there is an $1 \leq x \leq 3$ such that $C(u_xu_6) \notin \{i_2, \dots, i_6\}$. If $C(u_xu_6) \notin \{i_2, \dots, i_7\}$, then $v_1v_2v_3u_4u_5u_6u_x$ is a heterochromatic path of length 6; if $C(u_xu_6) = i_7$, then $v_2v_1v_3u_4u_5u_6u_x$ is a heterochromatic path of length 6. The proof is now complete. ∎

Actually, we can show that for $1 \leq k \leq 5$ any graph G under the color degree condition has a heterochromatic path of length at least k, with only one exceptional graph K_4 for $k = 3$, one exceptional graph for $k = 4$ and three exceptional graphs for $k = 5$, for which G has a heterochromatic path of length at least $k - 1$.

3. Long heterochromatic paths for $k \geq 8$

From the above section we know that when $1 \leq k \leq 4$, under the color degree condition G always has a heterochromatic path of length $\lceil \frac{3k}{5} \rceil$, and when $5 \leq k \leq 7$, G always has a heterochromatic path of length $\lceil \frac{3k}{5} \rceil + 1$. In this section we give our main result and do some preparations for its proof. The detailed proof is left in the next section.

Theorem 3.1 *Let G be an edge-colored graph and $k \geq 8$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$.*

Before proving the result, we will do some preparations, first.

Let G be an edge-colored graph and $k \geq 8$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Let $P = u_1u_2u_3 \ldots u_{l-1}u_lu_{l+1}v_1v_2 \ldots v_s$ be a path in G such that

(a) u_1Pu_{l+1} is a longest heterochromatic path in G;

(b) $C(u_{l+1}v_1) = C(u_{k_0}u_{k_0+1})$ and $1 \leq k_0 \leq l$ is as small as possible, subject to (a);

(c) v_1Pv_s is a heterochromatic path in G with $C(u_1Pu_{l+1}) \cap C(v_1Pv_s) = \varnothing$ and v_1Pv_s is as long as possible, subject to (a) and (b).

Let $i_j = C(u_ju_{j+1})$ for $1 \leq j \leq l$ and $i_{l+j} = C(v_jv_{j+1})$ for $1 \leq j \leq s-1$, then $C(u_{l+1}v_1) = i_{k_0}$. There exist $t_1 \geq 0$ and $1 \leq x_1 < x_2 < \ldots < x_{t_1} \leq l$ such that $c(\{u_{x_1}v_s, u_{x_2}v_s, \ldots, u_{x_{t_1}}v_s\}) = t_1$ and $C(\{u_{x_1}v_s, u_{x_2}v_s, \ldots, u_{x_{t_1}}v_s\}) = C(\{u_1v_s, \ldots, u_lv_s\}) - \{i_1, \ldots, i_{l+s-1}\}$. Let $i_{l+s+j-1} = C(u_{x_j}v_s)$ for all $1 \leq j \leq t_1$. There also exist $0 \leq t_2 \leq s-2$ and $1 \leq y_1 < y_2 < \ldots < y_{t_2} \leq s-2$ such that $C(\{v_1v_s, v_2v_s, \ldots, v_{s-2}v_s\}) - \{i_1, i_2, \ldots, i_{l+s+t_1-1}\} = C(\{v_{y_1}v_s, v_{y_2}v_s, \ldots, v_{y_{t_2}v_s}\})$ and $c(\{v_{y_1}v_s,\ v_{y_2}v_s, \ldots, v_{y_{t_2}v_s}\}) = t_2$. Let $i_{l+s+t_1+j-1} = C(v_{y_j}v_s)$ for all $1 \leq j \leq t_2$.

Then it is easy to get the following Lemmas. In these lemmas we assume that $l = \lceil \frac{3k}{5} \rceil$.

Lemma 3.2 *$s \leq k_0 \leq 2l - k$.*

Proof. Since $d^c(u_{l+1}) \geq k$ and u_1Pu_{l+1} is a longest heterochromatic path in G, there are at least $k - l$ different edges in $\{u_1u_{l+1}, u_2u_{l+1}, \ldots u_{l-1}u_{l+1}\}$ that have different colors which are not in $\{i_1, i_2, \ldots, i_l\}$. Then we have that $k_0 \in \{1, 2, \ldots, (l-1) - (k-l) + 1 = 2l - k\}$. On the other hand, if $s > k_0$, then $P' = u_{k_0+1}Pv_{k_0+1}$ is a heterochromatic path of length $l + 1$, a contradiction to the choice of P. So, $s \leq k_0 \leq 2l - k$. ∎

Lemma 3.3 *There are at least $k - l + k_0 - 1$ different colors not in $\{i_{k_0}, \ldots, i_l\}$ that belong to $C(\{u_1u_{l+1}, \ldots, u_{l-1}u_{l+1}\})$, and $C(\{u_1v_s, \ldots, u_sv_s\} \cup \{u_{k_0-s+1}v_s, \ldots, u_{k_0}v_s\} \cup \{u_{l-s+2}v_s, \ldots, u_lv_s\}) \subseteq \{i_1, \ldots, i_{l+s-1}\}$.*

Proof. By the choice of P, we have $CN(u_{l+1}) - C(\{u_1u_{l+1}, \ldots, u_{l-1}u_{l+1}\}) \subseteq \{i_{k_0}, \ldots, i_l\}$. Since $d^c(u_{l+1}) \geq k$, there are at least $k - (l - k_0 + 1) = k - l + k_0 - 1$ different colors not in $\{i_{k_0}, \ldots, i_l\}$ that belong to $C(\{u_1u_{l+1}, \ldots, u_{l-1}u_{l+1}\})$.

If there exists an $x \in \{1, 2, \ldots, s\} \cup \{k_0 - s + 1, \ldots, k_0\} \cup \{l - s + 2, l - s + 3, \ldots, l\}$

such that $u_x v_s$ has a color not in $\{i_1, \dots, i_{l+s-1}\}$, then

$$P' = \begin{cases} v_1 P v_s u_x P u_{l+x-s+1} & \text{if } x \in \{1, 2, \dots, s\}; \\ u_1 P u_x v_s P^{-1} u_{x+s} & \text{if } x \in \{k_0 - s + 1, \dots, k_0\}; \\ u_{x-(l-s+1)} P u_x v_s P^{-1} v_1 & \text{if } x \in \{l - s + 2, l - s + 3, \dots, l\}. \end{cases}$$

is a heterochromatic path of length $l + 1$, a contradiction to the choice of P. So, $C(\{u_1 v_s, \dots, u_s v_s, u_{k_0-s+1} v_s, \dots, u_{k_0} v_s, u_l v_s, \dots, u_{l-s+2} v_s\}) \subseteq \{i_1, \dots, i_{l+s-1}\}$. ∎

Lemma 3.4 *$s < x_1 < x_1 + 1 < x_2 < x_2 + 1 < \ldots < x_{t_1} \le l - s + 1$, $t_1 + t_2 \ge k - (l + s - 1)$ and $\max\{k - l - 2s + 3, 0\} \le t_1 \le \lceil \frac{l-2s+1}{2} \rceil$, $0 \le t_2 \le s - 2$.*

Proof. It is obvious that $t_1 + t_2 \ge k - (l + s - 1)$ and $0 \le t_2 \le s - 2$, and so $t_1 \ge k - (l + s - 1) - (s - 2) = k - l - 2s + 3$. From Lemma 3.3, we have that $s < x_1 < x_2 < \ldots < x_{t_1} \le l - s + 1$. If there exists a j with $1 \le j \le t_1 - 1$ such that $u_{x_j+1} = u_{x_{j+1}}$, let $P' = u_1 P u_{x_j} v_s u_{x_{j+1}} P u_{l+1}$, then P' a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $s \le k_0 < x_1 < x_1 + 1 < x_2 < x_2 + 1 < \ldots < x_{t_1} \le l - s + 1$, $t_1 + t_2 \ge k - (l + s - 1)$ and $\max\{k - l - 2s + 3, 0\} \le t_1 \le \lceil \frac{l-2s+1}{2} \rceil$, $0 \le t_2 \le s - 2$. ∎

Lemma 3.5 *Let $t_1 = 0$. Then $k \equiv 2, 4$ $(mod\ 5)$, $k_0 = s = 2l - k$ and $t_2 = s - 2$ if $k \equiv 4$ $(mod\ 5)$; $t_2 \ge s - 3$ if $k \equiv 2$ $(mod\ 5)$. There are exactly $l - 1$ different colors not in $\{i_{k_0}, i_{k_0+1}, \dots, i_l\}$ that belong to $C(u_1 u_{l+1}, \dots, u_{l-1} u_{l+1})$, and $CN(v_s) - \{u_{s+1} v_s, \dots, u_{l+1} v_s, v_1 v_s, \dots, v_{s-2} v_s\} \subseteq \{i_{k_0}, \dots, i_{l+s-1}\}$.*

Proof. Since $0 = t_1 \ge k - l - 2s + 3$, we have $k - l + 3 \le 2s \le 2(2l - k) = 4l - 2k$. On the other hand, from $(4l - 2k) - (k - l + 3) = 5l - 3k - 3$, we have that $5l - 3k - 3 < 0$ if $k \equiv 0, 1, 3$ $(mod\ 5)$, $5l - 3k - 3 = 0$ if $k \equiv 4$ $(mod\ 5)$ and $5l - 3k - 3 = 1$ if $k \equiv 2$ $(mod\ 5)$, which implies that $k \equiv 2, 4$ $(mod\ 5)$ and $k_0 = s = 2l - k$. Since $t_2 \ge k - l - s + 1$, we have that $t_2 = s - 2$ if $k \equiv 4$ $(mod\ 5)$ and $t_2 \ge s - 3$ if $k \equiv 2$ $(mod\ 5)$. From Lemma 3.3, there are at least $k - l + k_0 - 1 = k - l + s - 1 = k - l + 2l - k - 1 = l - 1$ different colors not in $\{i_{k_0}, i_{k_0+1}, \dots, i_l\}$ that belong to $C(u_1 u_{l+1}, \dots, u_{l-1} u_{l+1})$. If there exists a $v \notin \{u_{s+1}, u_{s+2}, \dots, u_{l+1}, v_1, \dots, v_s\}$ such that $v_s v$ has a color not in $\{i_{k_0}, \dots, i_{l+s-1}\}$, then $P' = u_{s+1} P v_s v$ is a heterochromatic path of length $l + 1$, a contradiction to the choice of P. ∎

Lemma 3.6 *If there exists an $1 \le x \le x_1 - 1$ such that $u_x u_{l+1}$ has a color in $\{i_{l+1}, \dots, i_{l+s-\lceil \frac{x_1-1}{2} \rceil}\} \cup \{i_{l+\lceil \frac{s+x_1-t_2}{2} \rceil - 2}, i_{l+\lceil \frac{s+x_1-t_2}{2} \rceil - 1}, \dots, i_{l+s-1}\}$, then there is a heterochromatic path P' of length $l + 1$ in G.*

Proof. First, note that $(l + \lceil \frac{s+x_1-t_2}{2} \rceil - 2) - (l + s - \lceil \frac{x_1-1}{2} \rceil) = \lceil \frac{s+x_1-t_2}{2} \rceil - 2 - s + \lceil \frac{x_1-1}{2} \rceil \ge \lceil \frac{x_1+2}{2} \rceil + \lceil \frac{x_1-1}{2} \rceil - s - 2 = x_1 + 1 - s - 2 \ge 0$.

If there exists an $1 \le x \le x_1 - 1$ such that $u_x u_{l+1}$ has a color in $\{i_{l+1}, \dots, i_{l+s-\lceil \frac{x_1-1}{2} \rceil}\}$, then let

$$P' = \begin{cases} v_{s-\lceil \frac{x_1-1}{2} \rceil + 1} P v_s u_{x_1} P u_{l+1} u_x P^{-1} u_{x-(\lfloor \frac{x_1-1}{2} \rfloor + 1) + 1} & \text{if } \lfloor \frac{x_1-1}{2} \rfloor + 1 \le x \le x_1 - 1; \\ v_{s-\lceil \frac{x_1-1}{2} \rceil + 1} P v_s u_{x_1} P u_{l+1} u_x P u_{x+(\lfloor \frac{x_1-1}{2} \rfloor + 1) - 1} & \text{if } 1 \le x \le \lfloor \frac{x_1-1}{2} \rfloor. \end{cases}$$

Since $\lfloor\frac{x_1-1}{2}\rfloor+(\lfloor\frac{x_1-1}{2}\rfloor+1)-1=2\lfloor\frac{x_1-1}{2}\rfloor\le x_1-1$, P' is a heterochromatic path of length $l+1$.

If there exists an $1\le x\le x_1-1$ such that u_xu_{l+1} has a color $i_{l+y}\in\{i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-1},\dots,i_{l+s-1}\}$, then since

$$\begin{aligned} t_2-[(s-2)-(\lceil\tfrac{s+x_1-t_2}{2}\rceil-2)] &= t_2-s+\lceil\tfrac{s+x_1-t_2}{2}\rceil\ge t_2-s+\tfrac{s+x_1-t_2}{2}\\ &= \tfrac{t_2-s+x_1}{2}\ge\tfrac{k-l-s+1-t_1-s+s+1}{2}\\ &\ge \tfrac{k-l-s+2}{2}-\tfrac{1}{2}\lceil\tfrac{l-2s+1}{2}\rceil\\ &\ge \tfrac{k-l-s+2}{2}-\tfrac{l-2s+2}{4}\\ &= \tfrac{2k-3l+2}{4}>0 \end{aligned}$$

and $y_{t_2-[(s-2)-(\lceil\frac{s+x_1-t_2}{2}\rceil-2)]}\le\lceil\frac{s+x_1-t_2}{2}\rceil-2$, there are some $y'\in\{1,2,\dots,\lceil\frac{s-t_2-1}{2}\rceil\}\cup\{\lceil\frac{s+x_1-t_2}{2}\rceil-2,\lceil\frac{s+x_1-t_2}{2}\rceil-3,\dots,\lceil\frac{s+x_1-t_2}{2}\rceil-\lfloor\frac{s-t_2-1}{2}\rfloor-1\}$ such that $y'\in\{y_1,y_2,\dots,y_{t_2}\}$. Let

$$P_1=\begin{cases} v_{y'+\lceil\frac{s+x_1-t_2}{2}\rceil-\lceil\frac{s-t_2-1}{2}\rceil-2}P^{-1}v_{y'}v_s & \text{if } y'\in\{1,2,\dots,\lceil\frac{s-t_2-1}{2}\rceil\};\\ v_{y'-\lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil+2}Pv_{y'}v_s & \text{if } y'\in\{\lceil\frac{s+x_1-t_2}{2}\rceil-2,\dots,\\ & \lceil\frac{s+x_1-t_2}{2}\rceil-\lfloor\frac{s-t_2-1}{2}\rfloor-1\}.\end{cases}$$

Note that if $y'\in\{1,2,\dots,\lceil\frac{s-t_2-1}{2}\rceil\}$, we have that $y'+\lceil\frac{s+x_1-t_2}{2}\rceil-\lceil\frac{s-t_2-1}{2}\rceil-2\le\lceil\frac{s-t_2-1}{2}\rceil+\lceil\frac{s+x_1-t_2}{2}\rceil-\lceil\frac{s-t_2-1}{2}\rceil-2=\lceil\frac{s+x_1-t_2}{2}\rceil-2$, and if $y'\in\{\lceil\frac{s+x_1-t_2}{2}\rceil-2,\dots,\lceil\frac{s+i_1-t_2}{2}\rceil-\lfloor\frac{s-t_2-1}{2}\rfloor-1\}$, we have that $y'-\lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil+2\ge\lceil\frac{s+x_1-t_2}{2}\rceil-\lfloor\frac{s-t_2-1}{2}\rfloor-1-\lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil+2\ge 1$. Then, P_1 is a heterochromatic path of length $\lceil\frac{s+x_1-t_2}{2}\rceil-\lceil\frac{s-t_2-1}{2}\rceil-1$ with colors not in $\{i_1,i_2,\dots,i_l,i_{l+y},i_{l+s}\}$. Let

$$P'=\begin{cases} P_1u_{x_1}Pu_{l+1}u_xP^{-1}u_{x-(x_1-\lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil)+1} & \text{if } x_1-\lceil\frac{s+x_1-t_2}{2}\rceil+\\ & \lceil\frac{s-t_2-1}{2}\rceil\le x\le x_1-1;\\ P_1u_{x_1}Pu_{l+1}u_xPu_{x+(x_1-\lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil)-1} & \text{if } 1\le x\le x_1-\\ & \lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil-1.\end{cases}$$

Since $2(x_1-\lceil\frac{s+x_1-t_2}{2}\rceil+\lceil\frac{s-t_2-1}{2}\rceil-1)=2x_1-2\lceil\frac{s+x_1-t_2}{2}\rceil+2\lceil\frac{s-t_2-1}{2}\rceil-2\le 2x_1-(s+x_1-t_2)+(s-t_2)-2=x_1-2$, P' is a heterochromatic path of length $l+1$. ∎

Lemma 3.7 *Let $t_1=0$, $k\ge 8$, $s+1\le x\le l-s+2$ and $C(u_xv_s)=i_1$. If there exists an $2\le x'\le x-1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1},\dots,i_{l+s-\lceil\frac{x}{2}\rceil},i_{l+\lceil\frac{x}{2}\rceil-1},\dots,i_{l+s-1}\}$ then there is a heterochromatic path P' of length $l+1$ in G.*

Proof. Since $t_1=0$, we have that $k\equiv 2,4$ $(mod\ 5)$ and $s=2l-k$, and that $t_2=s-2$ if $k\equiv 4$ $(mod\ 5)$; $t_2\ge s-3$ if $k\equiv 2$ $(mod\ 5)$ from Lemma 3.5.

If there exists an $2\le x'\le x-1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1},\dots,i_{l+s-\lceil\frac{x}{2}\rceil}\}$, then let

$$P'=\begin{cases} v_{s-\lceil\frac{x}{2}\rceil+1}Pv_su_xPu_{l+1}u'_xP^{-1}u_{x'-\lfloor\frac{x}{2}\rfloor+1} & \text{if } \lfloor\frac{x}{2}\rfloor+1\le x'\le x-1;\\ v_{s-\lceil\frac{x}{2}\rceil+1}Pv_su_xPu_{l+1}u'_xPu_{x'+\lfloor\frac{x}{2}\rfloor-1} & \text{if } 2\le x'\le\lfloor\frac{x}{2}\rfloor.\end{cases}$$

Note that if $2 \leq x' \leq \lfloor \frac{x}{2} \rfloor$, then $x' + \lfloor \frac{x}{2} \rfloor - 1 \leq 2\lfloor \frac{x}{2} \rfloor - 1 \leq x - 1$, and so P' is a heterochromatic path of length $l + 1$.

If there exists an $2 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+\lceil \frac{x}{2} \rceil - 1}, \ldots, i_{l+s-1}\}$, since $t_2 = s - 2$ if $k \equiv 4 \ (mod\ 5)$ and $t_2 \geq s - 3$ if $k \equiv 2 \ (mod\ 5)$, and $k \geq 8$, we have $t_2 \geq 1$. On the other hand, if $k \equiv 2 \ (mod\ 5)$, then $2 \leq \lceil \frac{s+1}{2} \rceil - 1 \leq \lceil \frac{x}{2} \rceil - 1 \leq \lceil \frac{l-s+2}{2} \rceil - 1 = \lceil \frac{k-l+2}{2} \rceil - 1 = \lceil \frac{2s+(k-l-2(2l-k)+2)}{2} \rceil - 1 = s - 2$, and so $C(\{v_1v_s, v_{\lceil \frac{x}{2} \rceil - 1}v_s\}) \nsubseteq \{i_1, i_2, \ldots, i_{l+s-1}\}$. Let

$$P_1 = \begin{cases} v_1Pv_{\lceil \frac{x}{2} \rceil - 1}v_s & \text{if } v_{\lceil \frac{x}{2} \rceil - 1}v_s \text{ has a color not in } \{i_1, i_2, \ldots, i_{l+s-1}\}; \\ v_{\lceil \frac{x}{2} \rceil - 1}P^{-1}v_1v_s & \text{if } v_1v_s \text{ has a color not in } \{i_1, i_2, \ldots, i_{l+s-1}\}. \end{cases}$$

$$P' = \begin{cases} P_1u_xPu_{l+1}u_{x'}P^{-1}u_{x'-\lfloor \frac{x}{2} \rfloor + 1} & \text{if } \lfloor \frac{x}{2} \rfloor + 1 \leq x' \leq x - 1; \\ P_1u_xPu_{l+1}u_{x'}Pu_{x'+\lfloor \frac{x}{2} \rfloor - 1} & \text{if } 2 \leq x' \leq \lfloor \frac{x}{2} \rfloor. \end{cases}$$

Note that if $2 \leq x' \leq \lfloor \frac{x}{2} \rfloor$, then $x' + \lfloor \frac{x}{2} \rfloor - 1 \leq 2\lfloor \frac{x}{2} \rfloor - 1 \leq x - 1$, and so P' is a heterochromatic path of length $l + 1$. ∎

Lemma 3.8 *Let $t_1 = 0$, $|C(v_1v_s, \ldots, v_{s-2}v_s) - \{i_2, \ldots, i_{l+s-1}\}| = s - 2$, $k \geq 8$ and $C(u_xv_s) = i_2$. If there exists an $3 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1}, \ldots, i_{l+s-\lceil \frac{x+1}{2} \rceil}, i_{l+\lceil \frac{x+1}{2} \rceil - 1}, \ldots, i_{l+s-1}\}$, then there is a heterochromatic path of length $l + 1$.*

Proof. If there exists an $3 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1}, \ldots, i_{l+s-\lceil \frac{x+1}{2} \rceil}\}$, then let

$$P' = \begin{cases} v_{s-\lceil \frac{x+1}{2} \rceil + 1}Pv_su_xPu_{l+1}u_{x'}P^{-1}u_{x'-\lfloor \frac{x+1}{2} \rfloor + 2} & \text{if } \lfloor \frac{x+1}{2} \rfloor + 1 \leq x' \leq x - 1; \\ v_{s-\lceil \frac{x+1}{2} \rceil + 1}Pv_su_xPu_{l+1}u_{x'}Pu_{x'+\lfloor \frac{x+1}{2} \rfloor - 2} & \text{if } 3 \leq x' \leq \lfloor \frac{x+1}{2} \rfloor. \end{cases}$$

Note that if $3 \leq x' \leq \lfloor \frac{x+1}{2} \rfloor$, then $x' + \lfloor \frac{x+1}{2} \rfloor - 2 \leq 2\lfloor \frac{x+1}{2} \rfloor - 2 \leq x - 1$, and so P' is a heterochromatic path of length $l + 1$.

If there exists an x' with $3 \leq x' \leq x - 1$ such that u_xu_{l+1} has a color in $\{i_{l+\lceil \frac{x+1}{2} \rceil - 1}, \ldots, i_{l+s-1}\}$, then let

$$P' = \begin{cases} v_{\lceil \frac{x+1}{2} \rceil - 1}P^{-1}v_1v_su_xPu_{l+1}u_{x'}P^{-1}u_{x'-\lfloor \frac{x+1}{2} \rfloor + 2} & \text{if } \lfloor \frac{x+1}{2} \rfloor + 1 \leq x' \leq x - 1; \\ v_{\lceil \frac{x+1}{2} \rceil - 1}P^{-1}v_1v_su_xPu_{l+1}u_{x'}Pu_{x'+\lfloor \frac{x+1}{2} \rfloor - 2} & \text{if } 3 \leq x' \leq \lfloor \frac{x+1}{2} \rfloor, \end{cases}$$

and so P' is a heterochromatic path of length $l + 1$. ∎

Lemma 3.9 *Let $t_1 \geq 1$, $k \geq 8$ and $k \equiv 1, 2, 4 \ (mod\ 5)$. Then there is no x_j $(1 \leq j \leq t_1)$ such that $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{x_j-1}u_{l+1}\}) - \{i_1, i_2, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s+j-1}\} \neq \emptyset$.*

Proof. Suppose there is some x_j $(1 \leq j \leq t_1)$ such that $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{x_j-1}u_{l+1}\}) - \{i_1, i_2, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s+j-1}\} \neq \emptyset$. Let x_{j_0} be the one of such x_j with the smallest subscript, and $u_{x'}u_{l+1}(1 \leq x' \leq x_{j_0} - 1)$ has a color not in $\{i_1, i_2, \ldots, i_l, i_{l+1}, \ldots,$

$i_{l+s-1}, i_{l+s+j_0-1}\}$. We distinguish the following three cases.

Case 1 $j_0 = 1$. Let

$$P' = \begin{cases} v_1 P v_s u_{x_1} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_1-s)+1} & \text{if } x_1 - s \leq x' \leq x_1 - 1; \\ v_1 P v_s u_{x_1} P u_{l+1} u_{x'} P u_{x'+(x_1-s)-1} & \text{if } 1 \leq x' \leq x_1 - s - 1. \end{cases}$$

Since $t_1 \geq k - l - 2s + 3$ by Lemma 3.4, we have that $(x_1 - s - 1) + (x_1 - s - 1) = x_1 + (x_1 - 2s - 2) \leq x_1 + (l - s + 1 - 2(t_1 - 1) - 2s - 2) = x_1 + (l - 2t_1 - 3s + 1) \leq x_1 + (l - 2k + 2l + 4s - 6 - 3s + 1) = x_1 + (3l - 2k + s - 5) \leq x_1 + (3l - 2k + 2l - k - 5) = x_1 + (5l - 3k - 5) = x_1 + (5\lceil \frac{3k}{5} \rceil - 3k - 5) \leq x_1 - 1$. So, P' is a heterochromatic path of length $l + 1$, a contradiction to the choice of P.

Case 2 $j_0 = 2$. So $t_1 \geq 2$. Since $t_1 \geq k - l - 2s + 3 \geq k - l - 4l + 2k + 3 = 3k - 5l + 3 = 3k - 5\lceil \frac{3k}{5} \rceil + 3$, we have that $k - l - 2s + 3 < 2$ if and only if $s = 2l - k$ when $k \equiv 1, 2, 4$ $(mod\ 5)$, or $s = 2l - k - 1$ when $k \equiv 2$ $(mod\ 5)$. Then $t_1 \geq \max\{k - l - 2s + 3, 2\} = 2$ if $s = 2l - k$ when $k \equiv 1, 2, 4$ $(mod\ 5)$, or $s = 2l - k - 1$ when $k \equiv 2$ $(mod\ 5)$. Otherwise, $t_1 \geq k - l - 2s + 3$.

We first consider the case when $u_{x'} u_{l+1}$ has a color $l + s$. Then $1 \leq x' \leq x_2 - 1$. Let

$$P' = \begin{cases} v_1 P v_s u_{x_2} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_2-s)+1} & \text{if } x_2 - s \leq x' \leq x_2 - 1; \\ v_1 P v_s u_{x_2} P u_{l+1} u_{x'} P u_{x'+(x_2-s)-1} & \text{if } 1 \leq x' \leq x_2 - s - 1. \end{cases}$$

Note that $(x_2 - s - 1) + x_2 - s - 1 = x_2 + (x_2 - 2s - 2) \leq x_2 + (l - s + 1 - 2(t_1 - 2) - 2s - 2) = x_2 + (l - 3s - 2t_1 + 3) \leq x_2 + l - 3s - 2\max\{k - l - 2s + 3, 2\} + 3$.
If $k \equiv 1, 4$ $(mod\ 5)$, then

$$\begin{aligned}
& x_2 + l - 3s - 2\max\{k - l - 2s + 3, 2\} + 3 \\
= & \begin{cases} x_2 + l - 3s - 2k + 2l + 4s - 3 & \text{if } s \leq 2l - k - 1; \\ x_2 + l - 3s - 1 & \text{if } s = 2l - k. \end{cases} \\
\leq & \begin{cases} x_2 + 3l - 2k + 2l - k - 1 - 3 & \text{if } s \leq 2l - k - 1; \\ x_2 + l - 6l + 3k - 1 & \text{if } s = 2l - k. \end{cases} \\
= & \begin{cases} x_2 + 5\lceil \frac{3k}{5} \rceil - 3k - 4 & \text{if } s \leq 2l - k - 1; \\ x_2 + 3k - 5\lceil \frac{3k}{5} \rceil - 1 & \text{if } s = 2l - k. \end{cases} \\
\leq & x_2 - 1.
\end{aligned}$$

If $k \equiv 2$ $(mod\ 5)$, then

$$\begin{aligned}
& x_2 + l - 3s - 2\max\{k - l - 2s + 3, 2\} + 3 \\
= & \begin{cases} x_2 + l - 3s - 2k + 2l + 4s - 3 & \text{if } s \leq 2l - k - 2; \\ x_2 + l - 3s - 1 & \text{if } s = 2l - k, 2l - k - 1. \end{cases} \\
\leq & \begin{cases} x_2 + 3l - 2k + 2l - k - 2 - 3 & \text{if } s \leq 2l - k - 2; \\ x_2 + l - 6l + 3k + 3 - 1 & \text{if } s = 2l - k, 2l - k - 1. \end{cases} \\
= & \begin{cases} x_2 + 5\lceil \frac{3k}{5} \rceil - 3k - 5 & \text{if } s \leq 2l - k - 2; \\ x_2 + 3k - 5\lceil \frac{3k}{5} \rceil + 2 & \text{if } s = 2l - k, 2l - k - 1. \end{cases} \\
\leq & x_2 - 1.
\end{aligned}$$

So, P' is a heterochromatic path of length $l + 1$, a contradiction to the choice of P.

Next we consider the case when $u_{x'} u_{l+1}$ has a color not in $\{i_1, i_2, \ldots, i_l, i_{l+1}, \ldots,$

$i_{l+s-1}, i_{l+s}, i_{l+s+1}\}$. Then $x_1 \le x' \le x_2 - 1$. Let $P' = v_1 P v_s u_{x_2} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_2-s)+1}$. Note that $x' - (x_2 - s) + 1 \ge x_1 - x_2 + s + 1 \ge s + 1 - (l - s + 1 - 2(t_1 - 2)) + s + 1$ $= 3s + 2t_1 - l - 3 \ge 3s + 2\max\{k - l - 2s + 3, 2\} - l - 3$.
If $k \equiv 1, 4$ $(mod\ 5)$, then

$$\begin{aligned}
& 3s + 2\max\{k-l-2s+3,2\} - l - 3 \\
= & \begin{cases} 3s + 2k - 2l - 4s + 6 - l - 3 & \text{if } s \le 2l-k-1; \\ 3s + 4 - l - 3 & \text{if } s = 2l-k. \end{cases} \\
\ge & \begin{cases} 2k - 3l - 2l + k + 1 + 3 & \text{if } s \le 2l-k-1; \\ 6l - 3k - l + 1 & \text{if } s = 2l-k. \end{cases} \\
= & \begin{cases} 3k - 5\lceil \frac{3k}{5} \rceil + 4 & \text{if } s \le 2l-k-1; \\ 5\lceil \frac{3k}{5} \rceil - 3k + 1 & \text{if } s = 2l-k. \end{cases} \\
\ge & 1.
\end{aligned}$$

If $k \equiv 2$ $(mod\ 5)$, then

$$\begin{aligned}
& 3s + 2\max\{k-l-2s+3,2\} - l - 3 \\
= & \begin{cases} 3s + 2k - 2l - 4s + 6 - l - 3 & \text{if } s \le 2l-k-2; \\ 3s + 4 - l - 3 & \text{if } s = 2l-k, 2l-k-1. \end{cases} \\
\ge & \begin{cases} 2k - 3l - 2l + k + 2 + 3 & \text{if } s \le 2l-k-2; \\ 6l - 3k - 3 - l + 1 & \text{if } s = 2l-k, 2l-k-1. \end{cases} \\
= & \begin{cases} 3k - 5\lceil \frac{3k}{5} \rceil + 5 & \text{if } s \le 2l-k-2; \\ 5\lceil \frac{3k}{5} \rceil - 3k - 2 & \text{if } s = 2l-k, 2l-k-1. \end{cases} \\
\ge & 1.
\end{aligned}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3 $3 \le j_0 \le t_1$. So $t_1 \ge 3$. Since $t_1 \ge k - l - 2s + 3 \ge 3k - 5\lceil \frac{3k}{5} \rceil + 3$, we have that $k - l - 2s + 3 < 3$ if and only if $s = 2l - k$ when $k \equiv 1, 2, 4$ $(mod\ 5)$, or $s = 2l - k - 1$ when $k \equiv 2, 4$ $(mod\ 5)$. Then $t_1 \ge \max\{k - l - 2s + 3, 3\} = 3$ if $s = 2l - k$ when $k \equiv 1, 2, 4$ $(mod\ 5)$ or $s = 2l - k - 1$ when $k \equiv 2, 4$ $(mod\ 5)$; otherwise, $t_1 \ge k - l - 2s + 3$.

Case 3.1 $C(u_{x'} u_{l+1}) = i_{l+s+j_0-2}$. Then $x_{j_0-2} \le x' \le x_{j_0} - 1$. Note that $x_{j_0} - x_{j_0-2} \le (l - s + 1 - 2(t_1 - j_0)) - (x_1 + 2(j_0 - 3)) = l - s - x_1 - 2t_1 + 7 \le l - s - (s+1) - 2\max\{k - l - 2s + 3, 3\} + 7 = l - 2s - 2\max\{k - l - 2s + 3, 3\} + 6$.
If $k \equiv 1$ $(mod\ 5)$, then

$$\begin{aligned}
& x_{j_0} - x_{j_0-2} \\
\le & l - 2s - 2\max\{k-l-2s+3,3\} + 6 \\
= & \begin{cases} l - 2s - 2k + 2l + 4s - 6 + 6 & \text{if } s \le 2l-k-1; \\ l - 2s - 6 + 6 & \text{if } s = 2l-k. \end{cases} \\
\le & \begin{cases} s + (3l - 2k + 2l - k - 1) & \text{if } s \le 2l-k-1; \\ s + (l - 6l + 3k) & \text{if } s = 2l-k. \end{cases} \\
= & \begin{cases} s + (5\lceil \frac{3k}{5} \rceil - 3k - 1) & \text{if } s \le 2l-k-1; \\ s + (3k - 5\lceil \frac{3k}{5} \rceil) & \text{if } s = 2l-k. \end{cases} \\
= & \begin{cases} s + 1 & \text{if } s \le 2l-k-1; \\ s - 2 & \text{if } s = 2l-k. \end{cases}
\end{aligned}$$

If $k \equiv 2, 4\ (mod\ 5)$, then

$$\begin{array}{rl} & x_{j_0} - x_{j_0-2} \\ \leq & l - 2s - 2\max\{k-l-2s+3, 3\} + 6 \\ = & \begin{cases} l-2s-2k+2l+4s-6+6 & \text{if } s \leq 2l-k-2; \\ l-2s-6+6 & \text{if } s = 2l-k, 2l-k-1. \end{cases} \\ \leq & \begin{cases} s+(3l-2k+2l-k-2) & \text{if } s \leq 2l-k-2; \\ s+(l-6l+3k+3) & \text{if } s = 2l-k, 2l-k-1. \end{cases} \\ = & \begin{cases} s+(5\lceil\frac{3k}{5}\rceil-3k-2) & \text{if } s \leq 2l-k-2; \\ s+(3k-5\lceil\frac{3k}{5}\rceil+3) & \text{if } s = 2l-k, 2l-k-1. \end{cases} \\ \leq & \begin{cases} s+2 & \text{if } s \leq 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ s+1 & \text{if } s \leq 2l-k-2 \text{ and } k \equiv 4\ (mod\ 5); \\ s & \text{if } s = 2l-k, 2l-k-1. \end{cases} \end{array}$$

If $x_{j_0} - x' \leq s$, let $P' = v_1 P v_s u_{x_{j_0}} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_{j_0}-s)+1}$, then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $x_{j_0} - x_{j_0-2} \geq x_{j_0} - x' \geq s+1$. Then we have the following cases:

Case 3.1.1 $k \equiv 1\ (mod\ 5)$. Then $x_{j_0-2} = s+1+2(j_0-3) = s+2j_0-5$, $x_{j_0} = l-s+1-2(t_1-j_0) = l-s-2t_1+2j_0+1$, $x' = x_{j_0-2} = x_{j_0}-s-1$, $s = 2l-k-1$ and $t_1 = k-l-2s+3 = k-l-4l+2k+2+3 = 3k-5l+5 = 3k-5\lceil\frac{3k}{5}\rceil+5 = 3$. So, $t_2 \geq k-(l+s-1)-t_1 = s+(k-l-2s+1)-3 = s-2$, and then $t_2 = s-2$. Hence, $j_0 = t_1 = 3$, $x_1 = s+1$, $x_3 = l-s+1$, and $4 \leq x_3 - x_1 = l-2s = l-4l+2k+2 = 2k-3l+2 = s+(2k-3l-2l+k+3) = s+(3k-5l+3) = s+1$ and so $s \geq 3$. On the other hand, we have that $s \leq k_0 \leq 2l-k = s+1 = x_1$ which implies that $k_0 - s + 1 \leq 2 < s$. Then, $k_0 < x = s+1$ from Lemma 3.3, and we get that $k_0 = s$.

If there exists an x with $1 \leq x \leq s$ such that $u_x u_{l+1}$ has a color not in $\{i_{k_0}, i_{k_0+1}, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s}\}$, let $P' = v_1 P v_s u_{s+1} P u_{l+1} u_x$, then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Otherwise, there are at least $s-((l-1)-(k-l+k_0-1)) = s-2l+k+k_0 = s-s-1+s = s-1$ colors in $\{i_{l+1}, i_{l+2}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_su_{l+1}\})$. So, there is an x with $1 \leq x \leq s$ such that $u_x u_{l+1}$ has a color i_{l+s-1}. Since $s \geq 3$ and $t_2 = s-2 \geq 1$, we have that $v_1 v_s \in E(G)$ and $v_1 v_s$ has color i_{l+s+3}. So, $P' = v_{s-1} P^{-1} v_1 v_s u_{s+1} P u_{l+1} u_x$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3.1.2 $k \equiv 2\ (mod\ 5)$. So, we have $s = 2l-k-3$ or $s = 2l-k-2$.

Case 3.1.2.1 If $s = 2l-k-3$, then $t_1 = k-l-2s+3 = k-l-4l+2k+6+3 = 3k-5l+6+3 = 5$, and so $s-2 \geq t_2 \geq k-(l+s-1)-t_1 = s+(k-l-2s+1)-5 = s+(k-l-4l+2k+7-5) = s+(3k-5l+2) = s-2$. Then $x_{j_0-2} = s+1+2(j_0-3) = s+2j_0-5$, $x_{j_0} = l-s+1-2(t_1-j_0) = l-s-2t_1+2j_0+1 = l-s+2j_0-9$ and $x' = x_{j_0-2} = x_j-s-1$. Hence, $4 \leq x_{j_0} - x_{j_0-2} = s+1$, which implies $s \geq 3$. Since $s \leq k_0 \leq 2l-k = s+3 = x_1+2$ and $k_0 - s + 1 \leq 4 \leq s+1$, we have $k_0 < x_1 = s+1$ by Lemma 3.3, then $k_0 = s$.

If there exists an x with $1 \leq x \leq x_1 - 1$ such that $u_x u_{l+1}$ has a color not in $\{i_{k_0}, i_{k_0+1}, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s}\}$, let $P' = v_1 P v_s u_{s+1} P u_{l+1} u_x$, then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Otherwise, there are at least $s-((l-1)-(k-l+k_0-1)) = s-2l+k+k_0 = s-s-3+s = s-3$ colors in $\{i_{l+1}, i_{l+2}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_su_{l+1}\})$.

If $s \geq 4$, then there exists an x with $1 \leq x \leq s$ such that u_xu_{l+1} has a color in $\{i_{l+1}, i_{l+s-2}, i_{l+s-1}\}$. Let

$$P_1 = \begin{cases} v_2Pv_s & \text{if } u_xu_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_s & \text{if } u_xu_{l+1} \text{ has color } i_{l+s-2} \text{ or } i_{l+s-1}. \end{cases}$$
$$P' = \begin{cases} P_1u_{s+1}Pu_{l+1}u_xu_{x+1} & \text{if } 1 \leq x \leq s-1\ ; \\ P_1u_{s+1}Pu_{l+1}u_su_{s-1} & \text{if } x = s. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s = 3$, then there are at least $x_2 - 1 - 1 - ((l-1) - (k-l+k_0-1)) = x_2 - 2l + k + k_0 - 2 \geq s + 3 - s - 3 + s - 2 = s - 2$ colors in $\{i_{l+1}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, \ldots, u_{x_2-1}u_{l+2}\})$. So, there exists an edge u_xu_{l+1} $(1 \leq x \leq x_2 - 1)$ such that u_xu_{l+1} has a color i_{l+s-2} or i_{l+s-1}. Let

$$P' = \begin{cases} v_1v_su_{s+1}Pu_{l+1}u_xu_{x-1} & \text{if } 2 \leq x \leq s; \\ v_1v_su_{s+1}Pu_{l+1}u_1u_2 & \text{if } x = 1; \\ v_1v_su_{x_2}Pu_{l+1}u_xP^{-1}u_{x-(x_2-s)} & \text{if } s+1 \leq x \leq x_2 - 1. \end{cases}$$

If $s+1 \leq x \leq x_2 - 1$, then $x - (x_2 - s) \geq 2s - x_2 + 1 \geq 2s - (l - s + 1 - 2(t_1 - 2)) + 1 = 3s - l + 6 = 6l - 3k - 9 - l + 6 = 5l - 3k - 3 = 5\lceil \frac{3k}{5} \rceil - 3k - 3 = 1$. So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3.1.2.2 If $s = 2l - k - 2$, then $t_1 = k - l - 2s + 3 = k - l - 4l + 2k + 4 + 3 = 3k - 5l + 7 = 3$, $s - 2 \geq t_2 \geq k - (l + s - 1) - t_1 = s + (k - l - 2s + 1) - 3 = s + (k - l - 4l + 2k + 2) = s + (3k - 5l + 2) = s - 2$, and so $t_2 = s - 2$. Then, $j_0 = t_1 = 3$ and $s + 1 \leq x_3 - x_1 \leq s + 2$ which implies $s \geq 2$. So, we have that $x_1 = s + 1$ and $x_3 = x_1 + s + 2 = 2s + 3$ or $x_3 = x_1 + s + 1 = 2s + 2$; or $x_1 = s + 2$ and $x_3 = x_1 + s + 1 = 2s + 3$.

We first consider the case when $x_1 = s + 1$. Since $s \leq k_0 \leq 2l - k = s + 2 = x_1 + 1$ and $k_0 - s + 1 \leq 3 \leq s + 1$, we have that $k_0 < x_1 = s + 1$ by Lemma 3.3, and so $k_0 = s$. If there exists an x with $1 \leq x \leq s$ such that u_xu_{l+1} has a color not in $\{i_{k_0}, i_{k_0+1}, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}\}$, let $P' = v_1Pv_su_{s+1}Pu_{l+1}u_x$, then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, there are at least $s - ((l-1) - (k - l + k_0 - 1)) = s - 2l + k + k_0 = s - s - 2 + k_0 = s - 2$ colors in $\{i_{l+1}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, \ldots, u_su_{l+1}\})$.

If $s \geq 3$, then there is an edge u_xu_{l+1} $(1 \leq x \leq s)$ such that u_xu_{l+1} has a color in $\{i_{l+s-2}, i_{l+s-1}\}$. Let

$$P' = \begin{cases} v_{s-2}P^{-1}v_1u_{s+1}Pu_{l+1}u_xu_{x-1} & \text{if } 2 \leq x \leq s; \\ v_{s-2}P^{-1}v_1u_{s+1}Pu_{l+1}u_1u_2 & \text{if } x = 1. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s = 2$, then we have $4 \leq x_3 - x_1 \leq s + 2 = 4$ which implies that $x_1 = x_3 - 4$, $x' = x_1$ or $x_1 + 1$. There are at least $x_2 - 1 - ((l-1) - (k - l + k_0 - 1)) - 1 = x_2 - 2l + k + k_0 - 2 = x_2 - s - 2 + k_0 - 2 \geq s + 3 - s - 2 + s - 2 = s - 1$ colors in $\{i_{l+1}, \ldots, i_{l+s-1}\}$ that belong to

$C(\{u_1u_{l+1}, \ldots, u_{t_2-1}u_{l+1}\})$. So, there is some $1 \leq x \leq x_2 - 1$ such that u_xu_{l+1} has color i_{l+1}. If $x' = x_1 = s+1$, then $x \neq s+1$. Let

$$P' = \begin{cases} v_su_{s+1}Pu_{l+1}u_xu_{x-1} & \text{if } x = 2; \\ v_su_{s+1}Pu_{l+1}u_1u_2 & \text{if } x = 1; \\ v_su_{x_2}Pu_{l+1}u_xP^{-1}u_{x-(x_2-s)} & \text{if } s+2 \leq x \leq x_2 - 1. \end{cases}$$

Note that if $s+2 \leq x \leq x_2-1$, then $x-(x_2-s) = x+s-x_2 \geq 2s+2-x_2 = 2s+2-x_1-2 = 2s-x_1 = s-1 = 1$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. If $x' = x_1+1 = s+2$, let $P' = u_1Pu_{x_1}v_su_{x_3}Pu_{l+1}u_{x_1+1}Pu_{x_3-1}$, then P' is a heterochromatic path of length $l+1$.

Now we consider the case when $x' = x_1 = s+2$ and $x_3 = 2s+3$. Since $s+1 = x_3 - x_1 \geq 4$, we have $s \geq 3$. Then $s \leq k_0 \leq 2l-k = s+2 = x_1$ and $k_0 - s + 1 \leq 3 \leq s$, and so $s \leq k_0 \leq x_1 - 1 = s+1$.

If there exists an $2 \leq x \leq s$ such that u_xu_{l+1} has a color not in $\{i_{k_0}, i_{k_0+1}, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}\}$, let

$$P' = \begin{cases} v_1Pv_su_{s+2}Pu_{l+1}u_xu_{x-1} & \text{if } u_xu_{l+1} \text{ has a color different from } i_{x-1}; \\ v_1Pv_su_{s+2}Pu_{l+1}u_xu_{x+1} & \text{if } u_xu_{l+1} \text{ has color } i_{x-1}, \end{cases}$$

then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Otherwise, there are at least $(s-1) - ((l-1) - (k-l+k_0-1)) = s - 2l + k + k_0 - 1 = s - s - 2 + k_0 - 1 = k_0 - 3 \geq s - 3$ colors in $\{i_{l+1}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_2u_{l+1}, \ldots, u_su_{l+1}\})$.

If $s \geq 4$, then there is an $2 \leq x \leq s$ such that u_xu_{l+1} has color i_{l+1}, i_{l+s-2} or i_{l+s-1}. Let

$$P_1 = \begin{cases} v_2Pv_s & \text{if } u_xu_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_s & \text{if } u_xu_{l+1} \text{ has color } i_{l+s-2} \text{ or } i_{l+s-1}. \end{cases}$$

$$P' = \begin{cases} P_1u_{s+2}Pu_{l+1}u_xu_{x-1}u_{x-2} & \text{if } 3 \leq x \leq s; \\ P_1u_{s+2}Pu_{l+1}u_2u_3u_4 & \text{if } x = 2. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s = 3$ and $k_0 = s+1 = 4$, then there are at least $k_0 - 3 = 1$ color of $\{i_{l+1}, i_{l+2} = i_{l+s-1}\}$ that belong to $C(\{u_2u_{l+1}, u_3u_{l+1}\})$. Let

$$P' = \begin{cases} v_2v_3u_5Pu_{l+1}u_2u_3u_4 & \text{if } u_2u_{l+1} \text{ has color } i_{l+1}; \\ v_2v_3u_5Pu_{l+1}u_3u_2u_1 & \text{if } u_3u_{l+1} \text{ has color } i_{l+1}; \\ v_2v_1v_3u_5Pu_{l+1}u_xu_{x-1} & \text{if } u_xu_{l+1}\ (x = 2, 3) \text{ has color } i_{l+2}. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s = 3$ and $k_0 = s = 3$, then $x_1 = s+2 = 5$ and $x_3 = 2s+3 = 9$, and so $x_2 = 7$, and then $x' = x_1 = 5$. There are at least $k_0 - 3 + x_2 - x_1 - 1 = x_2 - x_1 - 1 = 1$ colors in $\{i_{l+1}, i_{l+2} = i_{l+s-1}\}$ that belong to $C(\{u_2u_{l+1}, u_3u_{l+1}, u_6u_{l+1}\})$. Let

$$P' = \begin{cases} v_2v_3u_5Pu_{l+1}u_2u_3u_4 & \text{if } u_2u_{l+1} \text{ has color } i_{l+1}; \\ v_2v_3u_5Pu_{l+1}u_3u_2u_1 & \text{if } u_3u_{l+1} \text{ has color } i_{l+1}; \\ v_2v_3u_7Pu_{l+1}u_6P^{-1}u_2 & \text{if } u_6u_{l+1} \text{ has color } i_{l+1}; \\ v_2v_1v_3u_5Pu_{l+1}u_xu_{x-1} & \text{if } u_xu_{l+1}(x = 2, 3) \text{ has color } i_{l+2}; \\ v_2v_1v_3u_7Pu_{l+1}u_6P^{-1}u_3 & \text{if } u_6u_{l+1} \text{ has color } i_{l+2}. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3.1.3 $k \equiv 4 \ (mod\ 5)$ and $s = 2l-k-2$. Then, $t_1 = k-l-2s+3 = k-l-4l+2k+4+3 = 3k-5l+7 = 4$, $s-2 \geq t_2 \geq k-(l+s-1)-t_1 = s+(k-l-2s+1)-4 = s+(k-l-4l+2k+1) = s+(3k-5l+1) = s-2$ which implies $t_2 = s-2$, $x_{j_0-2} = s+1+2(j_0-3) = s+2j_0-5$, $x_{j_0} = l-s+1-2(t_1-j_0) = l-s+2j_0-7$ and $x_{j_0-2} = x_{j_0}-s-1$. So, $6 \leq x_4-x_1 = l-s+1-s-1 = l-2s = s+(l-3s) = s+(l-6l+3k+6) = s+(3k-5l+6) = s+3$ which implies that $s \geq 3$. On the other hand, $s \leq k_0 \leq 2l-k = s+2 = x_1+1$ and $k_0-s+1 \leq 3 \leq s$, and then $k_0 < x_1$, and so $k_0 = s$.

If there is an $1 \leq x \leq s$ such that u_xu_{l+1} has a color not in $\{i_{k_0}, i_{k_0+1}, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s}\}$, let $P' = v_1Pv_su_{s+1}Pu_{l+1}u_x$, then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. Otherwise, there are at least $s-((l-1)-(k-l+k_0-1)) = s-2l+k+k_0 = s-s-2+s = s-2$ colors in $\{i_{l+1}, i_{l+2}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_su_{l+1}\})$. In other words, there is some $1 \leq x \leq s$ such that u_xu_{l+1} has color i_{l+s-2} or i_{l+s-1}. Let

$$P' = \begin{cases} v_{s-2}P^{-1}v_1v_su_{s+1}Pu_{l+1}u_xu_{x-1} & \text{if } 2 \leq x \leq s; \\ v_{s-2}P^{-1}v_1v_su_{s+1}Pu_{l+1}u_1u_2 & \text{if } x = 1. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3.2 $C(u_{x'}u_{l+1}) \notin \{i_1, i_2, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s+j_0-2}, i_{l+s+j_0-1}\}$. Then $x_{j_0-1} \leq x' \leq x_{j_0}-1$. Let $P' = v_1Pv_su_{x_{j_0}}Pu_{l+1}u_{x'}P^{-1}u_{x'-(x_{j_0}-s)+1}$. Note that

$$\begin{aligned}
& x'-(x_{j_0}-s)+1 \geq x_{j_0-1}-x_{j_0}+s+1 \\
\geq\ & (s+1+2(j_0-2))-(l-s+1-2(t_1-j_0))+s+1 \\
=\ & 3s-l+2t_1-3 \\
\geq\ & 3s-l+2\max\{k-l-2s+3, 3\}-3 \\
=\ & \begin{cases} 2k-3l-s+3 & \text{if } s \leq 2l-k-2 \text{ and } k \equiv 2,4 \ (mod\ 5); \\ 2k-3l-s+3 & \text{if } s \leq 2l-k-1 \text{ and } k \equiv 1 \ (mod\ 5); \\ 3s-l+3 & \text{if } s = 2l-k \text{ or } s = 2l-k-1 \text{ and } k \equiv 2,4 \ (mod\ 5); \\ 3s-l+3 & \text{if } s = 2l-k \text{ and } k \equiv 1 \ (mod\ 5). \end{cases} \\
\geq\ & \begin{cases} 3k-5l+5 & \text{if } s \leq 2l-k-2 \text{ and } k \equiv 2,4 \ (mod\ 5); \\ 3k-5l+4 & \text{if } s \leq 2l-k-1 \text{ and } k \equiv 1 \ (mod\ 5); \\ 5l-3k & \text{if } s = 2l-k \text{ or } s = 2l-k-1 \text{ and } k \equiv 2,4 \ (mod\ 5); \\ 5l-3k+3 & \text{if } s = 2l-k \text{ and } k \equiv 1 \ (mod\ 5). \end{cases} \\
\geq\ & 1
\end{aligned}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Lemma 3.10 *Let $k \geq 8$ and $k \equiv 1,2,4 \ (mod\ 5)$, $t_1 \geq 2$, $C(\{u_1u_{l+1}, \ldots, u_{x_{t_1}-1}u_{l+1}\}) \subseteq \{i_1, i_2, \ldots, i_{l+s-1}, i_{l+s+t_1-1}\}$. Then there is no $x_1 \leq x' \leq x_{t_1}-1$ such that $C(u_{x'}u_{l+1}) \in \{i_{l+1}, \ldots, i_{l+s-x_{t_1}+x_1+2t_1-4}\} \cup \{i_{l+\lceil\frac{s-t_2-1}{2}\rceil+x_{t_1}-x_1-2t_1+2}, \ldots i_{l+s-1}\}$.*

Proof. Since $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{x_{t_1}-1}u_{l+1}\}) \subseteq \{i_1, i_2, \ldots, i_{l+s-1}, i_{l+s+t_1-1}\}$, we have that $k-(l+s) \leq l-1-x_{t_1}+1$ which implies that $x_{t_1} \leq 2l-k+s$, and $x_1+2(t_1-1) \leq x_{t_1}$

which implies that $2t_1 \le x_{t_1} - x_1 + 2 \le 2l - k + s - s - 1 + 2 = 2l - k + 1$.

If there exists an $x_1 \le x' \le x_{t_1} - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1}, \ldots, i_{l+s-x_{t_1}+x_1+2t_1-4}\}$, then there exists a $2 \le j_0 \le t_1$ such that $x_{j_0-1} \le x' \le x_{j_0} - 1$. Let $P' = v_{s-x_{t_1}+x_1+2t_1-3}Pv_su_{x_{j_0}}Pu_{l+1}u_{x'}P^{-1}u_{x'-(x_{j_0}-x_{t_1}+x_1+2t_1-4)+1}$. Since $k-l-2s+3 \ge k-l-4l+2k+3 = 3k-5l+3 = 3k-5\lceil\frac{3k}{5}\rceil+3$, we have that $k-l-2s+3 < 2$ if and only if $s = 2l-k$ when $k \equiv 1,2,4$ $(mod\ 5)$, or $s = 2l-k-1$ when $k \equiv 2$ $(mod\ 5)$. So,

$$\begin{aligned}
& s - x_{t_1} + x_1 + 2t_1 - 3 \\
\ge\ & s - 2l + k - s + s + 1 + 2\max\{k-l-2s+3, 2\} - 3 \\
=\ & s - 2l + k + 2\max\{k-l-2s+3, 2\} - 2 \\
=\ & \begin{cases} 3k-4l-3s+4 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ 3k-4l-3s+4 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ k-2l+s+2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ k-2l+s+2 & \text{if } s = 2l-k \text{ or } s = 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
\ge\ & \begin{cases} 6k-10l+7 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ 6k-10l+10 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ 2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ 1 & \text{if } s = 2l-k \text{ or } s = 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5), \end{cases} \\
\ge\ & 1
\end{aligned}$$

and $x' - (x_{j_0} - x_{t_1} + x_1 + 2t_1 - 4) + 1 \ge x_{j_0-1} - x_{j_0} + x_{t_1} - x_1 - 2t_1 + 5 = (x_{j_0-1} - x_1) + (x_{t_1} - x_{j_0}) - 2t_1 + 5 \ge 2(j_0-2) + 2(t_1-j_0) - 2t_1 + 5 = 1$. Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Next we consider the following two cases:

Case 1 $s - x_{t_1} + x_1 + 2t_1 - 4 \le \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 1$.

If there exists an $x_1 \le x' \le x_{t_1} - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+\lceil\frac{s-t_2-1}{2}\rceil+x_{t_1}-x_1-2t_1+2}, \ldots, i_{l+s-1}\}$, then there exists a $2 \le j_0 \le t_1$ such that $x_{j_0-1} \le x' \le x_{j_0} - 1$. Since

$$\begin{aligned}
& t_2 - [(s-2) - (\lceil\tfrac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 2)] \\
\ge\ & t_2 - s + 2 + s - x_{t_1} + x_1 + 2t_1 - 3 \\
=\ & t_2 - x_{t_1} + x_1 + 2t_1 - 1 \\
\ge\ & k - l - s + 1 - t_1 - (l - s + 1) + (s + 1) + 2t_1 - 1 \\
=\ & k - 2l + s + t_1 \\
\ge\ & k - 2l + s + \max\{k-l-2s+3, 2\} \\
=\ & \begin{cases} 2k-3l-s+3 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ 2k-3l-s+3 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ k-2l+s+2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ k-2l+s+2 & \text{if } s = 2l-k \text{ or } s = 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
\ge\ & \begin{cases} 3k-5l+4 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ 3k-5l+5 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ 2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ 1 & \text{if } s = 2l-k \text{ or } s = 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
\ge\ & 1
\end{aligned}$$

and $y_{t_2-[(s-2)-(\lceil\frac{s-t_2-1}{2}\rceil+x_{t_1}-x_1-2t_1+2)]} \leq \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x - 1 - 2t_1 + 2$, there exists a $y' \in \{1, 2, \ldots, \lceil\frac{s-t_2-1}{2}\rceil\} \cup \{\lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 2, \ldots, x_{t_1} - x_1 - 2t_1 + \lceil\frac{s-t_2-1}{2}\rceil - \lfloor\frac{s-t_2-1}{2}\rfloor + 3\}$ such that $y' \in \{y_1, y_2, \ldots, y_{t_2}\}$. If $y' \in \{1, 2, \ldots, \lceil\frac{s-t_2-1}{2}\rceil\}$, then let $P' = v_{y'+(x_{t_1}-x_1-2t_1+2)}P^{-1}v_{y'}v_s u_{x_{j_0}} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_{j_0}-x_{t_1}+x_1+2t_1-4)+1}$. Since $y' + (x_{t_1} - x_1 - 2t_1 + 2) \leq \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 2$, and $x' - (x_{j_0} - x_{t_1} + x_1 + 2t_1 - 4) + 1 \geq x_{j_0-1} - x_{j_0} + x_{t_1} - x_1 - 2t_1 + 5 = (x_{j_0-1} - x_1) + (x_{t_1} - x_{j_0}) - 2t_1 + s \geq 2(j_0 - 2) + 2(t_1 - j_0) - 2t_1 + 5 = 1$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. Otherwise, if $y' \in \{\lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 2, \ldots, x_{t_1} - x_1 - 2t_1 + \lceil\frac{s-t_2-1}{2}\rceil - \lfloor\frac{s-t_2-1}{2}\rfloor + 3\}$, then let $P' = v_{y'-(x_{t_1}-x_1-2t_1+2)} P v_{y'} v_s u_{x_{j_0}} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_{j_0}-x_{t_1}+x_1+2t_1-4)+1}$. Since $y' - (x_{t_1} - x_1 - 2t_1 + 2) \geq x_{t_1} - x_1 - 2t_1 + \lceil\frac{s-t_2-1}{2}\rceil - \lfloor\frac{s-t_2-1}{2}\rfloor + 3 - x_{t_1} + x_1 + 2t_1 - 2 \geq 1$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 2 $s - x_{t_1} + x_1 + 2t_1 - 4 \geq \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 2$.

If there exists an $x_1 \leq x' \leq x_{t_1} - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{l + s - x_{t_1} + x_1 + 2t_1 - 3, l + s - x_{t_1} + x_1 + 2t_1 - 2, \ldots, l + s - 1\}$, then there exists a $2 \leq j_0 \leq t_1$ such that $x_{j_0-1} \leq x' \leq x_{j_0} - 1$. Since

$$\begin{aligned}
& t_2 - [(s-2) - (s - x_{t_1} + x_1 + 2t_1 - 3)] \\
= \;& t_2 - x_{t_1} + x_1 + 2t_1 - 1 \\
\geq \;& (k - l - s + 1 - t_1) - (l - s + 1) + (s + 1) + 2t_1 - 1 \\
= \;& k - 2l + s + t_1 \\
\geq \;& k - 2l + s + \max\{k - l - 2s + 3, 2\} \\
\geq \;& 1
\end{aligned}$$

and $y_{t_2-[(s-2)-(s-x_{t_1}+x_1+2t_1-3)]} \leq s - x_{t_1} + x_1 + 2t_1 - 3$, there exists a $y' \in \{1, 2, \ldots, \lceil\frac{s-t_2-1}{2}\rceil\} \cup \{s - x_{t_1} + x_1 + 2t_1 - 3, \ldots, s - x_{t_1} + x_1 + 2t_1 - \lfloor\frac{s-t_2-1}{2}\rfloor - 2\}$ such that $y' \in \{y_1, y_2, \ldots, y_{t_2}\}$. If $y' \in \{1, 2, \ldots, \lceil\frac{s-t_2-1}{2}\rceil\}$, then let $P' = v_{y'+(x_{t_1}-x_1-2t_1+2)} P^{-1} v_{y'} v_s u_{x_{j_0}} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_{j_0}-x_{t_1}+x_1+2t_1-4)+1}$. Since $y' + (x_{t_1} - x_1 - 2t_1 + 2) \leq \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 2 \leq s - x_{t_1} + x_1 + 2t_1 - 4 = (s - x_{t_1} + x_1 + 2t_1 - 3) - 1$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. Otherwise, if $y' \in \{s - x_{t_1} + x_1 + 2t_1 - 3, \ldots, s - x_{t_1} + x_1 + 2t_1 - \lfloor\frac{s-t_2-1}{2}\rfloor - 2\}$, then let $P' = v_{y'-(x_{t_1}-x_1-2t_1+2)} P v_{y'} v_s u_{x_{j_0}} P u_{l+1} u_{x'} P^{-1} u_{x'-(x_{j_0}-x_{t_1}+x_1+2t_1-4)+1}$. Since $y' - (x_{t_1} - x_1 - 2t_1 + 2) \geq s - x_{t_1} + 2t_1 - 2 - \lfloor\frac{s-t_2-1}{2}\rfloor - (x_{t_1} - x_1 - 2t_1 + 2) \geq \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1} - x_1 - 2t_1 + 4 - \lfloor\frac{s-t_2-1}{2}\rfloor - (x_{t_1} - x_1 - 2t_1 + 2) = \lceil\frac{s-t_2-1}{2}\rceil - \lfloor\frac{s-t_2-1}{2}\rfloor + 2 \geq 2$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. ∎

Lemma 3.11 *Let $k \geq 8$, $k \equiv 1, 2, 4$ (mod 5) and $t_1 = 0$. Then $i_1 \notin C(\{u_1v_s, u_2v_s, \ldots, u_lv_s\})$.*

Proof. Since $t_1 = 0$, we have that $k_0 = s = 2l - k$, $k \equiv 2, 4$ (mod 5), $t_2 = s - 2$ if $k \equiv 4$ (mod 5) and $t_2 \geq s - 3$ if $k \equiv 2$ (mod 5), and $|C(\{u_1u_{l+1}, \ldots, u_{l-1}u_{l+1}\}) - \{i_{k_0}, i_{k_0+1}, \ldots, i_l\}| = l - 1$ by Lemma 3.5.

If $i_1 \in C(\{u_1v_s, u_2v_s, \ldots, u_lv_s\})$, then by Lemma 3.5 we have $i_1 \in C(\{u_{s+1}v_s, \ldots, u_lv_s\})$. On the other hand, if there exists an $x \in \{l-s+3, l-s+4, \ldots, l\}$ such that u_xu_{l+1}

has color i_1, then let $P' = v_1Pv_su_xP^{-1}u_{x-(l+2-s)+1}$. Note that when $l-s+3 \le x \le l$, we have that $x-(l+2-s)+1 \ge l-s+3-l-2+s+1 = 2$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, there exists an $s+1 \le x \le l-s+2$ such that u_xu_{l+1} has color i_1.

If there exists an $2 \le x' \le x-1$ such that $u_{x'}u_{l+1}$ has a color not in $\{i_1, i_2, \dots, i_{l+s-1}\}$, then let

$$P' = \begin{cases} v_1Pv_su_xPu_{l+1}u_{x'}P^{-1}u_{x'-(x-s)+1} & \text{if } x-s+1 \le x' \le x-1; \\ v_1Pv_su_xPu_{l+1}u_{x'}Pu_{x'+(x-s)-1} & \text{if } 2 \le x' \le x-s. \end{cases}$$

Note that if $2 \le x' \le x-s$, then $x'+(x-s)-1 \le x+(x-2s-1) \le x+(l-s+2-2s-1) = x+(l-3s+1) = x+(l-6l+3k+1) = x+(3k-5l+1) \le x-2$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $C(\{u_2u_{l+1}, \dots, u_{x-1}u_{l+1}\}) \subseteq \{i_1, i_2, \dots, i_{l+s-1}\}$.

We distinguish the following two cases:

Case 1 $x = s+1$.

If there exists an $1 \le x' \le x-1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_2, \dots, i_{k_0-1}\}$, then let $P' = v_1Pv_su_{s+1}Pu_{l+1}u_{x'}$, which is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, there are $x-2-1 = x-3 = s-2$ colors in $\{i_{l+1}, \dots, i_{l+s-1}\}$ that belong to $C(\{u_2u_{l+1}, \dots, u_{x-1}u_{l+1}\})$. Then, there exists an $2 \le x' \le s$ such that $u_{x'}u_{l+1}$ has color i_{l+1} or i_{l+s-1}. Since $k \ge 8$, $s = 2l-k$ and $t_2 \ge s-3$ if $s \equiv 2$ $(mod\ 5)$ and $t_2 = s-2$ if $s \equiv 4$ $(mod\ 5)$, we have that $\{1, 2\} \cap \{y_1, y_2, \dots, y_{t_2}\} \ne \emptyset$. Let

$$P_1 = \begin{cases} v_2Pv_s & \text{if } u_{x'}u_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_s & \text{if } 1 \in \{y_1, \dots, y_{t_2}\} \text{ and } u_{x'}u_{l+1} \text{ has color } i_{l+s-1}; \\ v_{s-1}P^{-1}v_2v_s & \text{if } 2 \in \{y_1, \dots, y_{t_2}\} \text{ and } u_{x'}u_{l+1} \text{ has color } i_{l+s-1}. \end{cases}$$
$$P' = \begin{cases} P_1u_{s+1}Pu_{l+1}u_{x'}u_{x'-1} & \text{if } 3 \le x' \le s; \\ P_1u_{s+1}Pu_{l+1}u_2u_3 & \text{if } x' = 2; \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 2 $x \ge s+2$.

There are at least $(x-2)-(k_0-1) = x-k_0-1$ different colors in $\{i_{l+1}, \dots, i_{l+s-1}\}$ that belong to $C(\{u_2u_{l+1}, \dots, u_{x-1}u_{l+1}\})$. Since $|\{l+s-\lceil \frac{x}{2} \rceil+1, \dots, l+\lceil \frac{x}{2} \rceil-2\}| = 2\lceil \frac{x}{2} \rceil - s - 2 \le x+1-s-2 = x-k_0-1$, if there exists an $2 \le x' \le x-1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1}, \dots, i_{l+s-\lceil \frac{x}{2} \rceil}, i_{l+\lceil \frac{x}{2} \rceil-1}, \dots, i_{l+s-1}\}$, then by Lemma 3.7 there is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $\lceil \frac{x}{2} \rceil = \frac{x+1}{2}$, and since $l+s-\lceil \frac{x}{2} \rceil+1 = l+1+\frac{2s-x-1}{2} \ge l+1+\frac{2s-l+s-3}{2} = l+1+\frac{5l-3k-3}{2} \ge l+1$, $l+\lceil \frac{x}{2} \rceil-2 \le l+\frac{l-s+1}{2}-1 = l+s-\frac{l-3s-1}{2} = l+s-\frac{3k-5l-1}{2} \le l+s-1$, there exists an $x' \in \{2, \dots, \lfloor \frac{x}{2} \rfloor, \lfloor \frac{x}{2} \rfloor+2, \dots, x-1\}$ such that $u_{x'}u_{l+1}$ has color $i_{l+s-\lceil \frac{x}{2} \rceil+1}$ or $i_{l+\lceil \frac{x}{2} \rceil-2}$. On the other hand, it is not hard to check that $\{1, \lceil \frac{x}{2} \rceil-2\} \cap \{y_1, \dots, y_{t_2}\} \ne \emptyset$ since $t_2 \ge s-3$ if $k \equiv 2$ $(mod\ 5)$ and $t_2 = s-2$ if $k \equiv 4$ $(mod\ 5)$. Let

$$P_1 = \begin{cases} v_{s-\lceil \frac{x}{2} \rceil+2}Pv_s & \text{if } u_{x'}u_{l+1} \text{ has color } i_{l+s-\lceil \frac{x}{2} \rceil+1}; \\ v_{\lceil \frac{x}{2} \rceil-2}P^{-1}v_1v_s & \text{if } 1 \in \{y_1, \dots, y_{t_2}\} \text{ and } u_{x'}u_{l+1} \text{ has color } i_{l+\lceil \frac{x}{2} \rceil-2}; \\ v_1Pv_{\lceil \frac{x}{2} \rceil-2}v_s & \text{if } \lceil \frac{x}{2} \rceil-2 \in \{y_1, \dots, y_{t_2}\} \text{ and } u_{x'}u_{l+1} \text{ has color } i_{l+\lceil \frac{x}{2} \rceil-2}. \end{cases}$$

$$P' = \begin{cases} P_1 u_x P u_{l+1} u_{x'} P^{-1} u_{x'-\lfloor \frac{x}{2} \rfloor} & \text{if } \lfloor \frac{x}{2} \rfloor + 2 \le x' \le x-1; \\ P_1 u_x P u_{l+1} u_{x'} P u_{x'+\lfloor \frac{x}{2} \rfloor} & \text{if } 2 \le x' \le \lfloor \frac{x}{2} \rfloor. \end{cases}$$

Note that if $2 \le x' \le \lfloor \frac{x}{2} \rfloor$, then $x' + \lfloor \frac{x}{2} \rfloor \le 2\lfloor \frac{x}{2} \rfloor = x-1$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

So, we get that $i_1 \notin C(\{u_1v_s, u_2v_s, \ldots, u_lv_s\})$. ∎

Now we turn to proving our main theorem.

4. Proof of Theorem 3.1

Proof. We will use induction on k to prove the theorem.

For $k = 8$, by Theorem 2.1 there is a heterochromatic path of length $6 = \lceil \frac{3k}{5} \rceil + 1$ in G.

Suppose now $k \ge 9$, and the theorem is true for all graphs G such that $d^c(v) \ge k'$ for every vertex v in G with $8 \le k' \le k-1$. In the following, all the notations are the same as in Section 3. Since $d^c(v) \ge k > k-1$ for every vertex v of G, G has a heterochromatic path of length $\lceil \frac{3(k-1)}{5} \rceil + 1$. It remains to show that $l \ge \lceil \frac{3k}{5} \rceil + 1$, which implies that $P' = u_1 P u_{\lceil \frac{3k}{5} \rceil + 2}$ is a heterochromatic path of length $\lceil \frac{3k}{5} \rceil + 1$. We will proceed by contradictions. Suppose that $l \le \lceil \frac{3k}{5} \rceil$. On the other hand, by induction hypothesis we have that $l \ge \lceil \frac{3(k-1)}{5} \rceil + 1 \ge \lceil \frac{3k}{5} \rceil$. Then $l = \lceil \frac{3k}{5} \rceil$ and $k \equiv 1, 2, 4$ $(mod\ 5)$. We will distinguish three cases: $t_1 \ge 2$, $t_1 = 1$ and $t_1 = 0$.

Case 1 $t_1 \ge 2$.

By Lemma 3.9, there is no x_j $(1 \le j \le t_1)$ such that $C(\{u_1u_{l+1}, u_2u_{l+1} \ldots, u_{x_j-1}u_{l+1}\}) - \{i_1, i_2, \ldots, i_l, i_{l+1}, \ldots, i_{l+s-1}, i_{l+s+j-1}\} \ne \emptyset$. So, $C(\{u_1u_{l+1}, \ldots, u_{x_{t_1}-1}u_{l+1}\}) \subseteq \{i_1, i_2, \ldots, i_{l+s-1}, i_{l+s+t_1-1}\}$. Then, by Lemma 3.10 there is no $x_1 \le x' \le x_{t_1} - 1$ such that $C(u_{x'}u_{l+1}) \in \{i_{l+1}, \ldots, i_{l+s-x_{t_1}+x_1+2t_1-4}\} \cup \{i_{l+\lceil \frac{s-t_2-1}{2} \rceil + x_{t_1} - x_1 - 2t_1 + 2}, \ldots i_{l+s-1}\}$. Now we will distinguish the following two cases:

Case 1.1 $s - x_{t_1} + x_1 + 2t_1 - 4 \le \lceil \frac{s-t_2-1}{2} \rceil + x_{t_1} - x_1 - 2t_1 + 1$. Since $|\{s - x_{t_1} + x_1 + 2t_1 - 3, \ldots, \lceil \frac{s-t_2-1}{2} \rceil + x_{t_1} - x_1 - 2t_1 + 1\}| = \lceil \frac{s-t_2-1}{2} \rceil - s + 2(x_{t_1} - x_1 - 2t_1) + 5$, there are at least $(l-1) - [(k_0 - 1) + (\lceil \frac{s-t_2-1}{2} \rceil - s + 2(x_{t_1} - x_1 - 2t_1) + 5) + ((l-1) - (k - l + k_0 - 1))] = k - l - \lceil \frac{s-t_2-1}{2} \rceil + s - 2(x_{t_1} - x_1 - 2t_1) - 5$ different colors not in $\{i_1, i_2, \ldots, i_l, i_{l+(s-x_{t_1}+x_1+2t_1-3)}, \ldots, i_{l+(\lceil \frac{s-t_2-1}{2} \rceil + x_{t_1} - x_1 - 2t_1 + 1)}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}\})$. So, there are at least

$$\begin{aligned} & k - l - \lceil \tfrac{s-t_2-1}{2} \rceil + s - 2(x_{t_1} - x_1 - 2t_1) - 5 - (l - 1 - x_{t_1} + 1) - 1 \\ = \; & k - 2l + s - \lceil \tfrac{s-t_2-1}{2} \rceil - x_{t_1} + 2x_1 + 4t_1 - 6 \\ \ge \; & k - 2l + s - \lceil \tfrac{s-(k-l-s+1-t_1)-1}{2} \rceil - (l - s + 1) + 2s + 2 + 4t_1 - 6 \\ \ge \; & (3k - 7l + 6s + 7t_1 - 9)/2 \\ \ge \; & (3k - 7l + 6s + 7\max\{k - l - 2s + 3, 2\} - 9)/2 \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} (10k-14l-8s+12)/2 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ (10k-14l-8s+12)/2 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ (3k-7l+6s+5)/2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ (3k-7l+6s+5)/2 & \text{if } s = 2l-k, 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
&\ge \begin{cases} (18k-30l+20)/2 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ (18k-30l+28)/2 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ (5l-3k+5)/2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ (5l-3k-1)/2 & \text{if } s = 2l-k, 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
&\ge 1
\end{aligned}
$$

colors in $\{i_{l+1}, i_{l+2}, \ldots, i_{l+s-x_{t_1}+x_1+2t_1-4}, i_{l+\lceil\frac{s-t_2-1}{2}\rceil+x_{t_1}-x_1-2t_1+2}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{x_{t_1}-1}u_{l+1}\})$, i.e., they belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{x_1-1}u_{l+1}\})$. Note that

$$
\begin{aligned}
& |\{1,2,\ldots,s-\lceil\tfrac{x_1-1}{2}\rceil\} \cup \{\lceil\tfrac{s+x_1-t_2}{2}\rceil-2, \lceil\tfrac{s+x_1-t_2}{2}\rceil-1, \ldots, s-1\}| \\
= \; & s-\lceil\tfrac{x_1-1}{2}\rceil + s - \lceil\tfrac{s+x_1-t_2}{2}\rceil + 2 = 2s - \lceil\tfrac{x_1-1}{2}\rceil - \lceil\tfrac{s+x_1-t_2}{2}\rceil + 2,
\end{aligned}
$$

$$
\begin{aligned}
& |(\{1,2,\ldots,s-x_{t_1}+x_1+2t_1-4\} \cup \{\lceil\tfrac{s-t_2-1}{2}\rceil + x_{t_1}-x_1-2t_1+2, \\
& \ldots, s-1\}| \\
= \; & s-x_{t_1}+x_1+2t_1-4+s-\lceil\tfrac{s-t_2-1}{2}\rceil - x_{t_1}+x_1+2t_1-2 \\
= \; & 2s-2x_{t_1}+2x_1+4t_1-\lceil\tfrac{s-t_2-1}{2}\rceil-6,
\end{aligned}
$$

$$
\begin{aligned}
& (2s-2x_{t_1}+2x_1+4t_1-\lceil\tfrac{s-t_2-1}{2}\rceil-6)-(2s-\lceil\tfrac{x_1-1}{2}\rceil-\lceil\tfrac{s+x_1-t_2}{2}\rceil+2) \\
= \; & (k-2l+s-\lceil\tfrac{s-t_2-1}{2}\rceil - x_{t_1}+2x_1+4t_1-6) \\
& +(\lceil\tfrac{x_1-1}{2}\rceil+\lceil\tfrac{s+x_1-t_2}{2}\rceil - x_{t_1}-k+2l-s-2)
\end{aligned}
$$

and

$$
\begin{aligned}
& \lceil\tfrac{x_1-1}{2}\rceil+\lceil\tfrac{s+x_1-t_2}{2}\rceil - x_{t_1}-k+2l-s-2 \\
\le \; & \tfrac{x_1+s+x_1-(k-l-s+1-t_1)+1}{2} - x_{t_1}-k+2l-s-2 \\
= \; & x_1-x_{t_1}-k+2l-2+\tfrac{l+t_1-k}{2} = x_1+2(t_1-1)-x_{t_1}-\tfrac{3k-5l+3t_1}{2} \\
\le \; & -\tfrac{6+3k-5l}{2} \le -1.
\end{aligned}
$$

So, $2s-2x_{t_1}+2x_1+4t_1-\lceil\frac{s-t_2-1}{2}\rceil-6)-(2s-\lceil\frac{x_1-1}{2}\rceil-\lceil\frac{s+x_1-t_2}{2}\rceil+2) < k-2l+s-\lceil\frac{s-t_2-1}{2}\rceil-x_{t_1}+2x_1+4t_1-6$, $C(\{u_1u_{l+1},\ldots,u_{x_1-1}u_{l+1}\}) \cap \{i_{l+1},\ldots,i_{l+s-\lceil\frac{x_1-1}{2}\rceil}, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2},\ldots,i_{l+s-1}\} \ne \emptyset$. By Lemma 3.6, there is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 1.2 $s-x_{t_1}+x_1+2t_1-4 \ge \lceil\frac{s-t_2-1}{2}\rceil + x_{t_1}-x_1-2t_1+2$. Then there are at least $(l-1)-[(k_0-1)+(l-1)-(k-l+k_0-1)] = k-l$ colors not in $\{i_1,i_2,\ldots,i_l\}$ that belong to $C(\{u_1u_{l+1},\ldots,u_{l-1}u_{l+1}\})$. So, there are at least

$$
\begin{aligned}
& k-l-(l-1-x_{t_1}+1)-1 \\
= \; & k-2l+x_{t_1}-1 \\
\ge \; & k-2l+s+1+2t_1-3 \\
\ge \; & k-2l+s+2\max\{k-l-2s+3,2\}-2
\end{aligned}
$$

$$
\begin{array}{rl}
= & \begin{cases} 3k-4l-3s+4 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ 3k-4l-3s+4 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ k-2l+s+2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ k-2l+s+2 & \text{if } s = 2l-k, 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
\ge & \begin{cases} 6k-10l+7 & \text{if } s \le 2l-k-1 \text{ and } k \equiv 1,4\ (mod\ 5); \\ 6k-10l+10 & \text{if } s \le 2l-k-2 \text{ and } k \equiv 2\ (mod\ 5); \\ 2 & \text{if } s = 2l-k \text{ and } k \equiv 1,4\ (mod\ 5); \\ 1 & \text{if } s = 2l-k, 2l-k-1 \text{ and } k \equiv 2\ (mod\ 5). \end{cases} \\
\ge & 1
\end{array}
$$

colors in $\{i_{l+1}, i_{l+2}, \dots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \dots, u_{x_{t_1}-1}u_{l+1}\})$, i.e., they belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \dots, u_{x_1-1}u_{l+1}\})$.
Since

$$
\begin{array}{rl}
& (s-1) - (2s - \lceil \frac{x_1-1}{2} \rceil - \lceil \frac{s+x_1-t_2}{2} \rceil + 2) \\
= & \lceil \frac{x_1-1}{2} \rceil + \lceil \frac{s+x_1-t_2}{2} \rceil - s - 3 \\
= & (k-2l+x_{t_1}-1) + (\lceil \frac{x_1-1}{2} \rceil + \lceil \frac{s+x_1-t_2}{2} \rceil - s - k + 2l - x_{t_1} - 2) \\
< & k-2l+x_{t_1}-1,
\end{array}
$$

we have that $C(\{u_1u_{l+1}, u_2u_{l+1}, \dots, u_{x_1-1}u_{l+1}\}) \cap \{i_{l+1}, \dots, i_{l+s-\lceil \frac{x_1-1}{2} \rceil}, i_{l+\lceil \frac{s+x_1-t_2}{2} \rceil - 1}, \dots, i_{l+s-1}\} \neq \emptyset$. By Lemma 3.6, there is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 2 $t_1 = 1$. Then, by Lemma 3.4 we have $1 = t_1 \ge k-l-2s+3$, and so $2s \ge k-l+2$. On the other hand, $s \le 2l-k$. Since $2(2l-k)-(k-l+2) = 5l-3k-2$, we have that $s = k_0 = 2l-k$ if $k \equiv 1,4\ (mod\ 5)$ or $2l-k-1 \le s \le k_0 \le 2l-k$ if $k \equiv 2\ (mod\ 5)$.

By Lemma 3.9, we have that $C(\{u_1u_{l+1}, \dots, u_{x_1-1}u_{l+1}\}) \subseteq \{i_1, i_2, \dots, i_{l+s-1}, i_{l+s}\}$. On the other hand, if there exists an $x_1 - s \le x \le s$ such that u_xu_{l+1} has a color in $\{i_1, i_2, \dots, i_{k_0-1}\}$, let $P_1' = u_xP^{-1}u_{x-(x_1-s)+1}$ and $P_1'' = u_xPu_{x+(x_1-s)-1}$, then there is a $P_1 \in \{P_1', P_1''\}$ such that $C(u_xu_{l+1}) \cap C(P_1) = \emptyset$, and so $P' = v_1Pv_su_{x_1}Pu_{l+1}P_1$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $C(\{u_{x_1-s}u_{l+1}, \dots, u_su_{l+1}\}) \cap \{i_1, i_2, \dots, i_{k_0-1}\} = \emptyset$. We consider the following two cases:

Case 2.1 $s-(x_1-s)+1 \ge (x_1-1)-(k_0-1)+1 \ge 0$, i.e., $2x_1 \le 2s+k_0$. Then, there are at least $[s-(x_1-s)+1]-[(l-1)-(k-l+k_0-1)]-1 > 0$ ($[s-(x_1-s)+1]-[(l-1)-(k-l+k_0-1)]-1 = 2s-x_1+1-l+1+k-l+k_0-1-1 = k-2l+2s+k_0-x_1 \ge k-2l+2x_1-x_1 \ge k-2l+s+1 \ge k-2l+\frac{k-l+2}{2}+1 = \frac{3k-5l+4}{2} \ge 0$, and so $[s-(x_1-s)+1]-[(l-1)-(k-l+k_0-1)]-1 = 0$ if and only if $2(s+1) = 2x_1 = 2s+k_0 \ge 3s$, i.e., $s \le 2$. On the other hand, $k \ge 8$ implies that $s \ge 3$, and so we have $[s-(x_1-s)+1]-[(l-1)-(k-l+k_0-1)]-1 > 0$.) different colors in $\{i_{l+1}, \dots, i_{l+s-1}\}$ that belong to $C(\{u_{x_1-s}u_{l+1}, \dots, u_su_{l+1}\}) \subseteq C(\{u_1u_{l+1}, \dots, u_{x_1-1}u_{l+1}\})$. Since $|\{s - \lceil \frac{x_1-1}{2} \rceil + 1, \dots, \lceil \frac{s+x_1-t_2}{2} \rceil - 3\}| \le \lceil \frac{s+x_1-(k-l-s+1-t_1)}{2} \rceil + \lceil \frac{x_1-1}{2} \rceil - s - 3 = \lceil \frac{2s-k+l+x_1}{2} \rceil + \lceil \frac{x_1-1}{2} \rceil - s - 3 = s + \lceil \frac{l-k+x_1}{2} \rceil + \lceil \frac{x_1-1}{2} \rceil - s - 3 \le \lceil \frac{l-k}{2} \rceil + x_1 - 3 = (k-2l+2s+k_0-x_1) + (\lceil \frac{l-k}{2} \rceil + 2x_1 - 2s + 2l - k - k_0 - 3) \le (k-2l+2s+k_0-x_1) + (\lceil \frac{l-k}{2} \rceil + k_0 - k + 2l - k_0 - 3) \le (k-2l+2s+k_0-x_1) + (\frac{5l-3k+1}{2} - 3) < k-2l+2s+k_0-x_1$, we have that $C(\{u_1u_{l+1}, \dots, u_{x_1-1}u_{l+1}\}) \cap \{i_{l+1}, i_{l+2}, \dots, i_{l+s-\lceil \frac{x_1-1}{2} \rceil}, i_{l+\lceil \frac{s+x_1-t_2}{2} \rceil - 2}, i_{l+\lceil \frac{s+x_1-t_2}{2} \rceil - 1}, \dots, i_{l+s-1}\} \neq \emptyset$. Then, by Lemma

3.6 there is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 2.2 $0 < s-(x_1-s)+1 \le (x_1-1)-(k_0-1)$, i.e., $2x_1 \ge 2s+k_0+1$. Then there are at least $[(x_1-1)-(k_0-1)]-[(l-1)-(k-l+k_0-1)]-1 = x_1-2l+k-1 \ge s+\frac{k_0+1}{2}-2l+k-1 \ge \frac{3s-1}{2}-2l+k \ge \frac{3k-3l+4}{4}-2l+k = \frac{7k-11l+4}{4}$ ($=0$ if $k=12, l=8, s=k_0=3, x_1=5$ and ≥ 1 otherwise) different colors in $\{i_{l+1}, i_{l+2}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{x_1-1}u_{l+1}\})$. We first consider the case when $k=12, l=8, s=k_0=3$ and $x_1=5$. There are at least $x_1-2-[(l-1)-(k-l+k_0-1)]-1 = 5-2-(7-(12-8+3-1))-1 = 1$ colors in $\{i_1, i_2, i_9, i_{10}\}$ that belong to $C(u_2u_9, u_3u_9, u_4u_9)$, i.e., there is an $2 \le x \le 4$ such that $C(u_xu_9) \in \{i_1, i_2, i_9, i_{10}\}$. If $C(u_xu_9) \in \{i_1, i_2\}$, then $v_1v_2v_3u_5Pu_9u_xu_{x-1}$ or $v_1v_2v_3u_5Pu_9u_xu_{x+1}$ is a heterochromatic path of length 9; if $C(u_xu_9) = i_{10}$, then $v_2v_1v_3u_5Pu_9u_xu_{x-1}$ is a heterochromatic path of length 9; if $C(u_xu_9) = i_9$, then $v_2u_5Pu_9u_xu_{x-1}u_{x-2}$ or $v_2u_5Pu_9u_xu_{x+1}u_{x+2}$ is a heterochromatic path of length 9, a contradiction to the choice of P.

Next we shall only consider the case when $x_1-2l+k-1 \ge 1$. If there exists an $1 \le x \le x_1-1$ such that u_xu_{l+1} has a color in $\{i_{l+1}, i_{l+2}, \ldots, i_{l+s-\lceil\frac{x_1-1}{2}\rceil}, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-1}, \ldots, i_{l+s-1}\}$, then by Lemma 3.6 there is a heterochromatic path of length $l+1$ in G, a contradiction to the choice of P. Otherwise, since $|\{s-\lceil\frac{x_1-1}{2}\rceil+1, \ldots, \lceil\frac{s+x_1-t_2}{2}\rceil-3\}| \le \lceil\frac{s+x_1-(k-l-s+1-t_1)}{2}\rceil+\lceil\frac{x_1-1}{2}\rceil-s-3 = \lceil\frac{2s-k+l+x_1}{2}\rceil+\lceil\frac{x_1-1}{2}\rceil-s-3 = s+\lceil\frac{l-k+x_1}{2}\rceil+\lceil\frac{x_1-1}{2}\rceil-s-3 \le \lceil\frac{l-k}{2}\rceil+x_1-3 = (x_1-2l+k-1)+(\lceil\frac{l-k}{2}\rceil+2l-k-2) = (x_1-2l+k-1)+(\lceil\frac{5l-3k}{2}\rceil-2) \le x_1-2l+k-1$, we have that $\{i_1, i_2, \ldots, i_{k_0-1}, i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-3}\} \subseteq C(\{u_1u_{l+1}, \ldots, u_{x_1-1}u_{l+1}\})$, and $k \equiv 2, 4 \ (mod\ 5)$ as well as $t_2 = k-l-s$ if $k \equiv 4 \ (mod\ 5)$, $t_2 = k-l-s$ or $k-l-s+1$ if $k \equiv 2 \ (mod\ 5)$. We distinguish two subcases:

Case 2.2.1 If $k \equiv 2 \ (mod\ 5)$, then $4l-2k-(k-l+2) = 5l-3k-2 = 2$, and so $s = 2l-k-1$ or $2l-k$.

If $s = 2l-k-1$, then $s-2 \ge t_2 \ge k-l-s+1-1 = s+(k-l-2s) = s+(k-l-4l+2k+2) = s-2$, $x_1-2l+k-1 = x_1-s-2$ and $|\{\lfloor\frac{s}{2}\rfloor+3, \ldots, x_1-\lceil\frac{s}{2}\rceil-2\}| = x_1-\lceil\frac{s}{2}\rceil-\lfloor\frac{s}{2}\rfloor-4 = x_1-s-4$.

Since $k \ge 8$ and $s \ge 3$, we have that $t_2 = s-2 > 0$. If $s \equiv 0 \ (mod\ 2)$, then there exists some $x \in \{1, 2, \ldots, \frac{s}{2}+1\} \cup \{x_1-\frac{s}{2}-1, \ldots, x_1-1\}$ such that u_xu_{l+1} has a color in $\{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-3}\}$. Let

$$P_1 = \begin{cases} v_{\frac{s}{2}}P^{-1}v_1v_s & \text{if } u_xu_{l+1} \text{ has a color in } \{i_{l+\frac{s}{2}}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-3}\}; \\ v_{\frac{s}{2}}Pv_s & \text{if } u_xu_{l+1} \text{ has a color in } \{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\frac{s}{2}-1}\}. \end{cases}$$

$$P_2 = \begin{cases} u_xP^{-1}u_{x-(x_1-\frac{s}{2}-2)} & \text{if } x_1-\frac{s}{2}-1 \le x \le x_1-1; \\ u_xPu_{x+(x_1-\frac{s}{2}-2)} & \text{if } 1 \le x \le \frac{s}{2}+1. \end{cases}$$

Then, $P' = P_1u_{x_1}Pu_{l+1}P_2$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s \equiv 1 \ (mod\ 2)$, then there exists some $x \in \{1, 2, \ldots, \lfloor\frac{s}{2}\rfloor+2\} \cup \{x_1-\lceil\frac{s}{2}\rceil-1, \ldots, x_1-2, x_1-1\}$ such that u_xu_{l+1} has a color in $\{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{s}{2}\rceil-2},$

$i_{l+\lceil\frac{s}{2}\rceil},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-3}\}$. Let

$$P_1 = \begin{cases} v_{\lceil\frac{s}{2}\rceil}P^{-1}v_1v_s & \text{if } u_xu_{l+1} \text{ has a color in } \{i_{l+\lceil\frac{s}{2}\rceil},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-3}\}; \\ v_{\lceil\frac{s}{2}\rceil-1}Pv_s & \text{if } u_xu_{l+1} \text{ has a color in } \{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1},\ldots,i_{l+\lceil\frac{s}{2}\rceil-2}\}. \end{cases}$$
$$P_2 = \begin{cases} u_xP^{-1}u_{x-(x_1-\lceil\frac{s}{2}\rceil-2)} & \text{if } x_1-\lceil\frac{s}{2}\rceil-1\le x\le x_1-1; \\ u_xPu_{x+(x_1-\lceil\frac{s}{2}\rceil-2)} & \text{if } 1\le x\le\lfloor\frac{s}{2}\rfloor+2. \end{cases}$$

Then, $P'=P_1u_{x_1}Pu_{l+1}P_2$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s=2l-k$, then $x_1-2l+k-1=x_1-s-1$ and $t_2\ge k-l-s=s-4\ge 0$.

We first consider the case when $t_2=0$. Then $k=12$, $l=8$, $k_0=s=4$, and $5=s+1\le x_1\le l-s+1=5$ which implies $x_1=5$. If there exists a $v\notin\{u_5,u_6,u_7\}$ such that u_9v has a color not in $\{i_4,\ldots,i_{12}\}$, then $v\notin\{u_5,\ldots,u_8,v_1,\ldots,v_4\}$ and $P'=v_1Pv_4u_5Pu_9v$ is a heterochromatic path of length $l+1$, a contradiction. So, $i_9\in C(\{u_1u_9,u_2u_9,u_3u_9,u_4u_9\})$. Let

$$P'=\begin{cases} v_2Pv_4u_5Pu_9u_1u_2 & \text{if } u_1u_9 \text{ has color } i_9; \\ v_2Pv_4u_5Pu_9u_xu_{x-1} & \text{if } u_xu_9\ (2\le x\le 4) \text{ has color } i_9. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction.

Next we consider the case when $t_2>0$. Since if $s\ge 5$, we have that $2<\lceil\frac{s}{2}\rceil\le\frac{s+1}{2}\le s-2$, and so $\{1,2,\lceil\frac{s}{2}\rceil\}\cap\{y_1,y_2,\ldots,y_{t_2}\}\ne\emptyset$. If $s\equiv 0\ (mod\ 2)$. Then there exists some $x\in\{1,2,\ldots,\frac{s}{2}+1\}\cup\{x_1-\frac{s}{2}-1,\ldots,x_1-1\}$ such that u_xu_{l+1} has a color in $\{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1},\ldots,i_{l+\frac{s}{2}-1},i_{l+\frac{s}{2}+1},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}$. Let

$$P_1=\begin{cases} v_{\frac{s}{2}}P^{-1}v_1v_s & \text{if } 1\in\{y_1,\ldots,y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\frac{s}{2}+1},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_{\frac{s}{2}+1}P^{-1}v_2v_s & \text{if } 2\in\{y_1,\ldots,y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\frac{s}{2}+1},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_1Pv_{\frac{s}{2}}v_s & \text{if } \frac{s}{2}\in\{y_1,\ldots,y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\frac{s}{2}+1},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_{\frac{s}{2}}Pv_s & \text{if } u_xu_{l+1} \text{ has a color in } \{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1},\ldots,i_{l+\frac{s}{2}-1}\}. \end{cases}$$
$$P_2=\begin{cases} u_xP^{-1}u_{x-(x_1-\frac{s}{2}-2)} & \text{if } x_1-\frac{s}{2}-1\le x\le x_1-1; \\ u_xPu_{x+(x_1-\frac{s}{2}-2)} & \text{if } 1\le x\le\frac{s}{2}+1. \end{cases}$$

Then, $P'=P_1u_{x_1}Pu_{l+1}P_2$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

If $s\equiv 1\ (mod\ 2)$, then there exists some $x\in\{1,2,\ldots,\frac{s+3}{2}\}\cup\{x_1-\frac{s+3}{2},\ldots,x_1-1\}$ such that u_xu_{l+1} has a color in $\{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1},\ldots,i_{l+\frac{s-3}{2}},i_{l+\frac{s+3}{2}},\ldots,i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}$. Let

$$P_1 = \begin{cases} v_{\frac{s+1}{2}}P^{-1}v_1v_s & \text{if } 1 \in \{y_1, \ldots, y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\frac{s+3}{2}}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_{\frac{s+3}{2}}P^{-1}v_2v_s & \text{if } 2 \in \{y_1, \ldots, y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\frac{s+3}{2}}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_1Pv_{\frac{s+1}{2}}v_s & \text{if } \frac{s+1}{2} \in \{y_1, \ldots, y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\frac{s+3}{2}}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_{\frac{s-1}{2}}Pv_s & \text{if } u_xu_{l+1} \text{ has a color in } \{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\frac{s-3}{2}}\}. \end{cases}$$

$$P_2 = \begin{cases} u_xP^{-1}u_{x-(x_1-\frac{s+5}{2})} & \text{if } x_1 - \frac{s+3}{2} \le x \le x_1 - 1; \\ u_xPu_{x+(x_1-\frac{s+5}{2})} & 1 \le x \le \frac{s+3}{2}. \end{cases}$$

Then, $P' = P_1u_{x_1}Pu_{l+1}P_2$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 2.2.2 If $k \equiv 4$ $(mod\ 5)$, then $(4l-2k)-(k-l+2) = 5l-3k-2 = 1$ which implies $k_0 = s = 2l-k$, $x_1-2l+k-1 = x_1-s-1$, and $t_2 = k-l-s = s+(k-l-2s) = s+(3k-5l) = s-3 \ge 0$.

We first consider the case when $t_2 = s-3 = 0$, which implies that $k = 9$, $l = 6$, $k_0 = s = 3$ and $4 = s+1 \le x_1 \le l-s+1 = 4$. If there exists a $v \notin \{u_4, u_5\}$ such that u_7v has some color not in $\{i_3, \ldots, i_9\}$, then $v \notin \{u_4, u_5, u_6, v_1, v_2, v_3\}$ and $P' = v_1v_2v_3u_4u_5u_6u_7v$ is a heterochromatic path of length $l+1$, a contradiction. So, $i_7 \in \{u_1u_7, u_2u_7, u_3u_7\}$. Let

$$P' = \begin{cases} v_2v_3u_4u_5u_6u_7u_1u_2 & \text{if } u_1u_7 \text{ has color } i_7; \\ v_2v_2u_4u_5u_6u_7u_xu_{x-1} & \text{if } u_xu_7\ (2 \le x \le 3) \text{ has color } i_7. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction.

Next we consider the case when $t_2 > 0$. Since $s \ge 4$, we have that $1 < \lceil\frac{s}{2}\rceil \le s-2$ which implies that $\{1, \lceil\frac{s}{2}\rceil\} \in \{y_1, \ldots, y_{t_2}\}$. Since $|\{\lfloor\frac{s}{2}\rfloor+2, \ldots, x_1-\lceil\frac{s}{2}\rceil-1\}| = x_1 - \lceil\frac{s}{2}\rceil - \lfloor\frac{s}{2}\rfloor - 2 = x_1-s-2$, there exists some $x \in \{1, 2, \ldots, \lfloor\frac{s}{2}\rfloor+1\} \cup \{x_1-\lceil\frac{s}{2}\rceil, \ldots, x_1-1\}$ such that u_xu_{l+1} has a color in $\{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}$. Let

$$P_1 = \begin{cases} v_{\lceil\frac{s}{2}\rceil}P^{-1}v_1v_s & \text{if } 1 \in \{y_1, \ldots, y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\lceil\frac{s}{2}\rceil}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}; \\ v_1Pv_{\lceil\frac{s}{2}\rceil}v_s & \text{if } \lceil\frac{s}{2}\rceil \in \{y_1, \ldots, y_{t_2}\} \text{ and } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+\lceil\frac{s}{2}\rceil}, \ldots, i_{l+\lceil\frac{s+x_1-t_2}{2}\rceil-2}\}. \\ v_{\lceil\frac{s}{2}\rceil}Pv_s & \text{if } u_xu_{l+1} \text{ has a color in} \\ & \{i_{l+s-\lceil\frac{x_1-1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{s}{2}\rceil-1}\}. \end{cases}$$

$$P_2 = \begin{cases} u_xP^{-1}u_{x-(x_1-\lfloor\frac{s}{2}\rfloor-2)} & x_1 - \lceil\frac{s}{2}\rceil \le x \le x_1-1; \\ u_xPu_{x+(x_1-\lfloor\frac{s}{2}\rfloor-2)} & 1 \le x \le \lfloor\frac{s}{2}\rfloor+1. \end{cases}$$

Then, since $(x_1-\lceil\frac{s}{2}\rceil)-(x_1-\lfloor\frac{s}{2}\rfloor-2) \ge 1$ and $(\lfloor\frac{s}{2}\rfloor+1)+(x_1-\lfloor\frac{s}{2}\rfloor-2) = x_1-1$, $P' = P_1u_{x_1}Pu_{l+1}P_2$ is a heterochromatic path of length at least $l+1$, a contradiction to the choice of P.

Case 3 $t_1 = 0$.

Since $t_1 = 0$, we know by Lemma 3.5 that $k \equiv 2, 4$ $(mod\ 5)$, $k_0 = s = 2l - k$ and $t_2 = s - 2$ if $k \equiv 4$ $(mod\ 5)$; $t_2 \geq s - 3$ if $k \equiv 2$ $(mod\ 5)$, and there are exactly $l - 1$ different colors not in $\{i_{k_0}, i_{k_0+1}, \ldots, i_l\}$ that belong to $C(u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1})$. On the other hand, by Lemma 3.11 we have that $i_1 \notin C(u_1v_s, u_2v_s, \ldots, u_lv_s)$, and by Lemma 3.5 $CN(v_s) - C(\{u_{s+1}v_s, \ldots, u_{l+1}v_s, v_1v_s, \ldots, v_{s-2}v_s\}) \subseteq \{i_{k_0}, \ldots, i_{l+s-1}\}$. Since $d^c(v_s) \geq k$ and if $k \equiv 4$ $(mod\ 5)$, $t_2 = s - 2$; if $k \equiv 2$ $(mod\ 5)$, $t_2 \geq s - 3$, which implies that $k \equiv 2$ $(mod\ 5)$, $t_2 = s - 2$ or $i_1 \in \{v_1v_s, \ldots, v_{s-2}v_s\}$ and $i_2 \in \{u_{s+1}v_s, \ldots, u_lv_s\}$, $i_3 \in \{u_{s+1}v_s, \ldots, u_lv_s\}$ if $s \geq 4$.

Since $k \geq 8$, we consider the case when $s \geq 4$. If there exists an $x \in \{l - s + 4, \ldots, l\}$ such that u_xu_{l+1} has color i_2, then let $P' = v_1Pv_su_xP^{-1}u_{x-(l+2-s)+1}$. Since $l-s+4 \leq x \leq l$, we have $x-(l+2-s)+1 \geq (l-s+4)-(l-s+2)+1 = 3$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $i_2 \in C(\{u_{s+1}v_s, \ldots, u_{l-s+3}v_s\})$.

If there exists an $3 \leq x' \leq x-1$ such that $u_{x'}u_{l+1}$ has a color not in $\{i_2, i_3, \ldots, i_{l+s-1}\}$, then let

$$P' = \begin{cases} v_1Pv_su_xPu_{l+1}u_{x'}P^{-1}u_{x'-(x-s)+1} & \text{if } x - s + 2 \leq x' \leq x - 1; \\ v_1Pv_su_xPu_{l+1}u_{x'}Pu_{x'+(x-s)-1} & \text{if } 3 \leq x' \leq x - s + 1 \end{cases}$$

Note that if $3 \leq x' \leq x-s+1$, then $x'+(x-s)-1 \leq x+(x-2s) \leq x+(l-s+3-2s) = x + (l - 6l + 3k + 3) = x - 1$, and so P' is a heterochromatic path of length $l + 1$, a contradiction to the choice of P. So, $C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\}) \subseteq \{i_2, i_3, \ldots, i_{l+s-1}\}$.

We first consider the case when $x = s + 1$. If there exists an $1 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_3, \ldots, i_{k_0-1}\}$, then $P' = v_1Pv_su_xPu_{l+1}u_{x'}$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, there are $x-1-1 = x-2 = s-1$ colors in $\{i_{l+1}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_1u_{l+1}, \ldots, u_{x-1}u_{l+1}\})$. Then there must exist an $1 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has color i_{l+s-1}, and so $P' = v_{s-1}P^{-1}v_1v_su_{s+1}Pu_{l+1}u_{x'}$ is a heterochromatic path of length $l + 1$, a contradiction to the choice of P.

Next we consider the case when $x \geq s+2$. There are at least $(x-3)-(k_0-2) = x-k_0-1 = x - s - 1 > 0$ colors in $\{i_{l+1}, \ldots, i_{l+s-1}\}$ that belong to $C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\})$. If there exists an $3 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has a color in $\{i_{l+1}, \ldots, i_{l+s-\lceil\frac{x+1}{2}\rceil}, i_{l+\lceil\frac{x+1}{2}\rceil-1}, \ldots, i_{l+s-1}\}$, then by Lemma 3.8 there is a heterochromatic path of length $l + 1$, a contradiction to the choice of P. So, $C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\}) \subseteq \{i_2, \ldots, i_{k_0-1}, i_{l+s-\lceil\frac{x+1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{x+1}{2}\rceil-2}\}$. Then, we consider the color i_3. If there exists a $z \in \{1, \ldots, s, l - s + 5, \ldots, l - 1\}$ such that u_zv_s has color i_3, then let

$$P' = \begin{cases} u_{s+1}Pv_su_z & \text{if } 1 \leq z \leq s; \\ v_1Pv_su_zP^{-1}u_{z-(l+2-s)+1} & \text{if } l - s + 5 \leq z \leq l - 1. \end{cases}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. Since $k_0 = s \geq 4$, we have $i_3 \in C(\{u_1v_s, \ldots, u_lv_s\})$, and so there exists a $s + 1 \leq z \leq l - s + 4$ such that u_zv_s has color i_3. Then we consider the following subcases:

Case 3.1 $i_2, i_3 \in C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\})$.

First we consider the case when there exists an $4 \leq x' \leq x - 1$ such that $u_{x'}u_{l+1}$ has color i_3.

If $x = l-s+3$ and $x' = s+1$, since there are at least $(x-3)-(k_0-2) = x-s-1 = l-2s+2 = s+(l-3s+2) = s+(l-6l+3k+2) = s-2$ colors in $\{i_{l+1},\dots,i_{l+s-1}\}$ that belong to $C(\{u_3u_{l+1},\dots,u_{x-1}u_{l+1}\})$, then there exists an $x'' \in \{3,\dots,s-1,s+2,\dots,x-1\}$ such that $u_{x''}u_{l+1}$ has a color in $\{i_{l+1},i_{l+s-2},i_{l+s-1}\}$. Let

$$\begin{aligned} P_1 &= \begin{cases} v_2Pv_s & \text{if } u_{x''}u_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_s & \text{if } u_{x''}u_{l+1} \text{ has color } i_{l+s-2} \text{ or } i_{l+s-1}. \end{cases} \\ P' &= \begin{cases} P_1u_xPu_{l+1}u_{x''}P^{-1}u_{x''-(x-s)} & \text{if } s+2 \le x'' \le x-1; \\ P_1u_xPu_{l+1}u_{x''}Pu_{x''+(x-s)} & \text{if } 3 \le x'' \le s-1. \end{cases} \end{aligned}$$

Note that if $s+2 \le x'' \le x-1$, then $x''-(x-s) \ge 2s-x+2 = 2s-l+s-3+2 = 3(2l-k)-l-1 = 5l-3k-1 = 3$; if $3 \le x'' \le s-1$, then $x''+(x-s) \le x-1$. So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Otherwise, $x \le l-s+2$ or $x = l-s+3$, $x' \ne s+1$. Let

$$P' = \begin{cases} v_1Pv_su_xPu_{l+1}u_{x'}P^{-1}u_{x'-(x-s)+1} & \text{if } x-s+3 \le x' \le x-1; \\ v_1Pv_su_xPu_{l+1}u_{x'}Pu_{x'+(x-s)-1} & \text{if } 4 \le x' \le x-s+2. \end{cases}$$

Note that if $x \le l-s+2$, then $x-s+2 \le l-s+2-s+2 = l-2s+4 = s+(l-3s+4) = s+(3k-5l+4) = s$, and if $x = l-s+3$ and $x' \ne s+1$, then $x-s+2 = l-s+3-s-2 = l-2s+5 = s+1$. So, if $4 \le x' \le x-s+2$, then $x' \le s$ and hence $x'+(x-s)-1 \le x-1$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Now we consider the case when u_3u_{l+1} has color i_3. Then there exists an $4 \le x' \le x-1$ such that $u_{x'}u_{l+1}$ has color i_2. Let

$$P' = \begin{cases} v_1Pv_su_zPu_{l+1}u_{x'}P^{-1}u_{x'-(z-s)+1} & \text{if } z-s+3 \le x' \le x-1; \\ v_1Pv_su_zPu_{l+1}u_{x'}Pu_{x'+(z-s)-1} & \text{if } 4 \le x' \le \min\{z-s+2, s\}. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $s+1 \le x' \le z-s+2$, which implies $z \ge 2s-1$. On the other hand, $z \le l-s+4 = 2s+(l-3s+4) = 2s$, $2s-1 \le z \le 2s$. So we distinguish two cases: $z = 2s-1$, then $s+1 \le x' \le z-s+2 = s+1$; and $z = 2s$, then $s+1 \le x' \le z-s+2 = s+2$.

Case 3.1.1 $z = 2s-1$, $x' = s+1$. If there exists an $4 \le x'' \le z-1$ such that $u_{x''}u_{l+1}$ has a color not in $\{i_3,\dots,i_{k_0-1},i_{l+1},\dots,i_{l+s-1}\}$, then $x'' \ne s+1$. Let

$$P' = \begin{cases} v_1Pv_su_zPu_{l+1}u_{x''}P^{-1}u_{x''-(z-s)+1} & \text{if } s+2 \le x'' \le z-1; \\ v_1Pv_su_zPu_{l+1}u_{x''}Pu_{x''+(z-s)-1} & \text{if } 4 \le x'' \le s. \end{cases}$$

Note that if $s+2 \le x'' \le z-1$, then $x''-(z-s)+1 \ge s+2-(z-s)+1 = 4$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, there are at least $(z-4)-(k_0-3) = z-s-1 = s-2$ colors in $\{i_{l+1},\dots,i_{l+s-1}\}$ that belong to $C(\{u_4u_{l+1},\dots,u_{z-1}u_{l+1}\})$, and there are at least 2 colors in $\{i_{l+1},i_{l+s-2},i_{l+s-1}\}$ that belong to $C(\{u_4u_{l+1},\ldots,u_su_{l+1},u_{s+2}u_{l+1},\ldots,u_{z-1}u_{l+1}\})$. If there exists an $4\le x''\le z-1$ such that $u_{x''}u_{l+1}$ has color i_{l+s-1}, then $x''\neq s+1$. Let

$$P'=\begin{cases} v_{s-1}P^{-1}v_1v_su_zPu_{l+1}u_{x''}P^{-1}u_{x''-(z-s)+1} & \text{if } s+2\le x''\le z-1;\\ v_{s-1}P^{-1}v_1v_su_zPu_{l+1}u_{x''}Pu_{x''+(z-s)-1} & \text{if } 4\le x''\le s.\end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $i_{l+s-1}\notin C(\{u_4u_{l+1},\ldots,u_{z-1}u_{l+1}\})$ and $i_{l+1},i_{l+s-2}\in\{u_4u_{l+1},\ldots,u_{z-1}u_{l+1}\}$. If there exists an $4\le x''\le z-1$, $x''\neq s,s+1,s+2$ such that $u_{x''}u_{l+1}$ has color i_{l+1} or i_{l+s-2}, then let

$$\begin{aligned} P_1 &= \begin{cases} v_2Pv_s & \text{if } u_{x''}u_{l+1} \text{ has color } i_{l+1};\\ v_{s-2}P^{-1}v_1v_s & \text{if } u_{x''}u_{l+1} \text{ has color } i_{l+s-2}.\end{cases}\\ P' &= \begin{cases} P_1u_zPu_{l+1}u_{x''}P^{-1}u_{x''-(z-s)} & \text{if } s+3\le x''\le z-1;\\ P_1u_zPu_{l+1}u_{x''}Pu_{x''+(z-s)} & \text{if } 4\le x''\le s-1.\end{cases}\end{aligned}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $C(\{u_su_{l+1},u_{s+2}u_{l+1}\})=\{i_{l+1},i_{l+s-2}\}$. Since $x\ge s+2$, we consider the case when $x\ge s+3$, first. Let

$$P'=\begin{cases} v_2Pv_su_xPu_{l+1}u_{s+2}P^{-1}u_{2s-x+2} & \text{if } u_{s+2}u_{l+1} \text{ has color } i_{l+1};\\ v_{s-2}P^{-1}v_1v_su_xPu_{l+1}u_{s+2}P^{-1}u_{2s-x+2} & \text{if } u_{s+2}u_{l+1} \text{ has color } i_{l+s-2}.\end{cases}$$

Since $2s-x+2\ge 2s-(l-s+3)+2=6l-3k-l-1=3$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. Then $x=s+2$. Let

$$P'=\begin{cases} v_2Pv_su_{s+2}Pu_{l+1}u_su_{s-1}u_{s-2} & \text{if } u_su_{l+1} \text{ has color } i_{l+1};\\ v_{s-2}P^{-1}v_1v_su_{s+2}Pu_{l+1}u_su_{s-1}u_{s-2} & \text{if } u_su_{l+1} \text{ has color } i_{l+s-2}.\end{cases}$$

If $s\ge 5$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $s=4$. Since there are exactly $l-1$ different colors not in $\{i_{k_0},i_{k_0+1},\ldots,i_l\}$ that belong to $C(\{u_1u_{l+1},\ldots,u_{l-1}u_{l+1}\})$, $v_1Pv_su_{s+2}Pu_{l+1}u_1u_2$ is a heterochromatic path of length $l+1$ if $C(u_1u_{l+1})\notin\{i_1,\ldots,i_{l+s-1}\}$; $v_1Pv_su_{s+2}Pu_{l+1}u_2u_1$ is a heterochromatic path of length $l+1$ if $C(u_2u_{l+1})\notin\{i_1,\ldots,i_{l+s-1}\}$. Then u_1u_{l+1} or u_2u_{l+1} has color i_{l+s-1} and $\{1,2\}\cap\{y_1,\ldots,y_{t_2}\}\neq\emptyset$. If u_1u_{l+1} has color i_{l+s-1}, then let

$$P'=\begin{cases} u_{s-1}P^{-1}u_1u_{l+1}v_1v_2v_su_zP^{-1}u_{s+1} & \text{if } 2\in\{y_1,\ldots,y_{t_2}\};\\ v_{s-1}P^{-1}v_1v_su_{s+2}Pu_{l+1}u_1u_2 & \text{if } 1\in\{y_1,\ldots,y_{t_2}\}.\end{cases}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, u_2u_{l+1} has color i_{l+s-1}. Let

$$P'=\begin{cases} v_3v_2v_1v_su_{s+2}Pu_{l+1}u_2u_1 & \text{if } 1\in\{y_1,\ldots,y_{t_2}\};\\ u_1u_2u_{l+1}u_sPu_zv_sv_2v_3 & \text{if } 2\in\{y_1,\ldots,y_{t_2}\} \text{ and } u_su_{l+1} \text{ has color } i_{l+1};\\ u_1u_2u_{l+1}u_sPu_zv_sv_2v_1 & \text{if } 2\in\{y_1,\ldots,y_{t_2}\} \text{ and } u_su_{l+1} \text{ has color } i_{l+s-2};\end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3.1.2 $z=2s$, $x'=s+1$ or $s+2$. If there exists an $4\le x''\le z-1$,$x''\neq s+1,s+2$ such that $u_{x''}u_{l+1}$ has a color not in $\{i_3,\ldots,i_{l+s-1}\}$, then let

$$P'=\begin{cases} v_1Pv_su_zPu_{l+1}u_{x''}P^{-1}u_{x''-(z-s)+1} & \text{if } s+3\le x''\le z-1;\\ v_1Pv_su_zPu_{l+1}u_{x''}Pu_{x''+(z-s)-1} & \text{if } 4\le x''\le s.\end{cases}$$

Note that if $s+3 \leq x'' \leq z-1$, then $x''-(z-s)+1 \geq s+3-(z-s)+1=4$, and so P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, there are at least $(z-4-1)-(k_0-3)=z-s-2=s-2$ colors in $\{i_{l+1},\ldots,i_{l+s-1}\}$ that belong to $C(\{u_4u_{l+1},\ldots,u_su_{l+1},u_{s+3}u_{l+1},\ldots,u_{z-1}u_{l+1}\})$, and there are at least 2 colors in $\{i_{l+1},i_{l+s-2},i_{l+s-1}\}$ that belong to $C(\{u_4u_{l+1},\ldots,u_su_{l+1},u_{s+3}u_{l+1},\ldots,u_{z-1}u_{l+1}\})$. If there exists an $4 \leq x'' \leq z-1$, $x'' \neq s+1, s+2$ such that $u_{x''}u_{l+1}$ has color i_{l+s-1}, then let

$$P' = \begin{cases} v_{s-1}P^{-1}v_1v_su_zPu_{l+1}u_{x''}P^{-1}u_{x''-(z-s)+1} & \text{if } s+3 \leq x'' \leq z-1; \\ v_{s-1}P^{-1}v_1v_su_zPu_{l+1}u_{x''}Pu_{x''+(z-s)-1} & \text{if } 4 \leq x'' \leq s. \end{cases}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $i_{l+s-1} \notin C(\{u_4u_{l+1},\ldots,u_su_{l+1},u_{s+3}u_{l+1},\ldots,u_{z-1}u_{l+1}\})$, $i_{l+1},i_{l+s-2} \in \{u_4u_{l+1},\ldots, u_su_{l+1},u_{s+3}u_{l+1},\ldots,u_{z-1}u_{l+1}\}$. If there exists an $4 \leq x'' \leq z-1$, $x'' \neq s, s+1, s+2, s+3$ such that $u_{x''}u_{l+1}$ has color i_{l+1} or i_{l+s-2}, then let

$$P_1 = \begin{cases} v_2Pv_s & \text{if } u_{x''}u_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_s & \text{if } u_{x''}u_{l+1} \text{ has color } i_{l+s-2}. \end{cases}$$
$$P' = \begin{cases} P_1u_zPu_{l+1}u_{x''}P^{-1}u_{x''-(z-s)} & \text{if } s+4 \leq x'' \leq z-1; \\ P_1u_zPu_{l+1}u_{x''}Pu_{x''+(z-s)} & \text{if } 4 \leq x'' \leq s-1. \end{cases}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $C(\{u_su_{l+1},u_{s+3}u_{l+1}\})=\{i_{l+1},i_{l+s-2}\}$. Since $x \geq s+2$, we consider the case when $x \geq s+4$, first. Let

$$P' = \begin{cases} v_2Pv_su_xPu_{l+1}u_{s+3}P^{-1}u_{2s-x+3} & \text{if } u_{s+3}u_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_su_xPu_{l+1}u_{s+3}P^{-1}u_{2s-x+3} & \text{if } u_{s+3}u_{l+1} \text{ has color } i_{l+s-2}. \end{cases}$$

Since $2s-x+3 \geq 2s-(l-s+3)+3=6l-3k-l=4$, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. Then we consider the case when $x=s+2$. Let

$$P' = \begin{cases} v_2Pv_su_{s+2}Pu_{l+1}u_su_{s-1}u_{s-2} & \text{if } u_su_{l+1} \text{ has color } i_{l+1}; \\ v_{s-2}P^{-1}v_1v_su_{s+2}Pu_{l+1}u_su_{s-1}u_{s-2} & \text{if } u_su_{l+1} \text{ has color } i_{l+s-1}. \end{cases}$$

If $s \geq 5$, then P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P, and so $s=4$. Since $C(\{u_3u_{l+1},\ldots,u_{x-1}u_{l+1}\}) \subseteq \{i_2,\ldots,i_{l+s-1}\}$, u_3u_{l+1} has color i_3, and if u_1u_{l+1} has a color not in $\{i_1,i_2,\ldots,i_{l+s-1}\}$, then $P'=v_1Pv_su_{s+2}Pu_{l+1}u_1u_2$ is a heterochromatic path of length $l+1$; if u_2u_{l+1} has a color not in $\{i_1,i_2,\ldots,i_{l+s-1}\}$, then $P'=v_1Pv_su_{s+2}Pu_{l+1}u_2u_1$ is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, $u_{s+2}u_{l+1}$ has a color not in $\{i_1,i_2,\ldots,i_{l+s-1}\}$, $u_{s+1}u_{l+1}$ has color i_2, u_1u_{l+1} or u_2u_{l+1} has color i_{l+s-1} and $\{1,2\} \cap \{y_1,\ldots,y_{t_2}\} \neq \emptyset$. If u_1u_{l+1} has color i_{l+s-1}, then let

$$P' = \begin{cases} u_{s-1}P^{-1}u_1u_{l+1}v_1v_2v_su_zP^{-1}u_{s+2} & \text{if } 2 \in \{y_1,y_2,\ldots,y_{t_2}\}; \\ v_{s-1}P^{-1}v_1v_su_{s+2}Pu_{l+1}u_1u_2 & \text{if } 1 \in \{y_1,\ldots,y_{t_2}\}. \end{cases}$$

So, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P. So, u_2u_{l+1} has color i_{l+s-1}. Let

$$P' = \begin{cases} u_1u_2u_{l+1}u_sPu_zv_sv_2 & \text{if } 2 \in \{y_1,\ldots,y_{t_2}\}; \\ v_{s-1}P^{-1}v_1v_su_{s+2}Pu_{l+1}u_2u_1 & \text{if } 1 \in \{y_1,\ldots,y_{t_2}\}. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Last, we consider the case when $x = s+3$. Since $C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\}) \subseteq \{i_2, \ldots, i_{l+s-1}\}$, we have that $C(\{u_{s+1}u_{l+1}, u_{s+2}u_{l+1}\}) = \{i_2, i_{l+s-1}\}$. Let

$$P' = \begin{cases} v_{s-1}P^{-1}v_1v_su_{s+3}Pu_{l+1}u_{s+2}u_{s+1}u_s & \text{if } u_{s+2}u_{l+1} \text{ has color } i_{l+s-1}; \\ v_{s-1}P^{-1}v_1v_su_{s+3}Pu_{l+1}u_{s+1}u_su_{s-1} & \text{if } u_{s+1}u_{l+1} \text{ has color } i_{l+s-1}. \end{cases}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

Case 3.2 $\{i_2, i_3\} \nsubseteq C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\})$. Since $|(\{i_4, \ldots, i_{k_0-1}, i_{l+s-\lceil\frac{x+1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{x+1}{2}\rceil-2}\}| + 1 = 2\lceil\frac{x+1}{2}\rceil - 5 \leq x+2-5 = x-3$, we have that $\lceil\frac{x+1}{2}\rceil = \frac{x+2}{2}$ and $\{i_{l+s-\lceil\frac{x+1}{2}\rceil+1}, \ldots, i_{l+\lceil\frac{x+1}{2}\rceil-2}\} \subseteq C(\{u_3u_{l+1}, \ldots, u_{x-1}u_{l+1}\})$. Then, there exists an $x' \in \{3, \ldots, \frac{x}{2}, \frac{x}{2}+2, \ldots, x-1\}$ such that $u_{x'}u_{l+1}$ has color $i_{l+s-\lceil\frac{x+1}{2}\rceil+1}$ or $i_{l+\lceil\frac{x+1}{2}\rceil-2}$. Let

$$\begin{aligned} P_1 &= \begin{cases} v_{s-\lceil\frac{x+1}{2}\rceil+2}Pv_s & \text{if } u_{x'}u_{l+1} \text{ has color } i_{l+s-\lceil\frac{x+1}{2}\rceil+1}; \\ v_{\lceil\frac{x+1}{2}\rceil-2}P^{-1}v_1v_s & \text{if } u_{x'}u_{l+1} \text{ has color } i_{l+\lceil\frac{x+1}{2}\rceil-2}. \end{cases} \\ P' &= \begin{cases} P_1u_xPu_{l+1}u_{x'}P^{-1}u_{x'-\frac{x}{2}+1} & \text{if } \frac{x}{2}+2 \leq x' \leq x-1; \\ P_1u_xPu_{l+1}u_{x'}Pu_{x'+\frac{x}{2}-1} & \text{if } 3 \leq x' \leq \frac{x}{2}; \end{cases} \end{aligned}$$

Then, P' is a heterochromatic path of length $l+1$, a contradiction to the choice of P.

The proof is now complete. ∎

5. Concluding remarks

Finally, we consider whether our lower bound is best possible. It is obvious that when $k = 1, 2$, the bound is best possible. Next we consider the case when $k \geq 3$.

Remark 5.1 *For any integer $k \geq 3$, there is an edge-colored graph G_k with $d^c(v) \geq k$ for all the vertices v in G such that any longest heterochromatic path of G is of length $k-1$.*

In fact, let G_k be an edge-colored graph whose vertices are the ordered $(k-1)$-tuples of 0's and 1's; two vertices are joined by an edge if and only if they differ in exactly one coordinate or they differ in all coordinates. An edge is in color j $(1 \leq j \leq k-1)$ if and only if its two ends differ in exactly the jth coordinate, or in color k if and only if its two ends differ in all the coordinates. Then it is not difficult to check that G_k is a graph we want.

Another class of graphs is given as follows. Since K_k is $(k-1)$-edge-colorable when k is even, we can get a proper k-edge coloring for K_{k+1} when k is odd. Denote it by G'_k. Then, it is obvious that the longest heterochromatic path in G'_k is of length $k-1$ when k is odd.

Remark 5.2 *Actually, we can show that for $1 \leq k \leq 5$ any graph G under the color degree condition has a heterochromatic path of length at least k, with only one exceptional graph K_4 for $k = 3$, one exceptional graph G_4 for $k = 4$ and three exceptional graphs for $k = 5$, for which G has a heterochromatic path of length at least $k-1$.*

To show this, let us construct an edge-colored graph H, first. Let H have the vertex set $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$. The edges of H and their colors are given as follows: $C(\{u_1u_2, u_3u_7, u_4u_8, u_5u_6\}) = 1$, $C(\{u_1u_4, u_2u_3, u_5u_8, u_6u_7\}) = 2$, $C(\{u_1u_5, u_2u_6, u_3u_4, u_7u_8\}) = 3$, $C(\{u_1u_6, u_2u_5, u_3u_8, u_4u_7\}) = 4$, and $C(\{u_1u_7, u_2u_8, u_3u_5, u_4u_6\}) = 5$. Then, $d^c(v) = 5$ for every vertex $v \in V(H)$ and the longest heterochromatic path in H is of length 4.

Then, by exhausting all the possible adjacency and colorings of edges for $1 \leq k \leq 5$, we can get that any graph G under the color degree condition has a heterochromatic path of length at least k, with only one exceptional graph K_4 for $k = 3$, one exceptional graph G_4 for $k = 4$ and three exceptional graphs G_5, G'_5 and H for $k = 5$, for which G has a heterochromatic path of length at least $k - 1$, where G_k and G'_k are given in the proof of Remark 5.1. When k becomes larger, there might be more such exceptional graphs. The tedious details have to be omitted.

Now we know that our lower bound is best possible when $1 \leq k \leq 7$. But we still do not know whether it is best possible when $k \geq 8$. We have tried all the possible cases when $k = 8$ in order to find a graph to show that our bound is best possible, but failed. To end this paper, we propose the following conjecture:

Conjecture 5.3 *Let G be an edge-colored graph and $k \geq 3$ an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a heterochromatic path of length at least $k - 1$.*

From the examples above we know that if this conjecture is true, then it would be best possible. T. Jiang once told us that they showed that the conjecture is true for complete graphs.

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