# Long-Horizon Mean-Variance Analysis: A User Guide

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# 1 Introduction

Recent research in empirical finance has documented that expected excess returns on bonds and stocks, real interest rates, and risk shift over time in predictable ways. Furthermore, these shifts tend to persist over long periods of time. Starting at least with the pioneering work of Samuelson (1969) and Merton (1969, 1971, 1973) on portfolio choice, financial economists have argued that asset return predictability can introduce a wedge between the asset allocation strategies of short- and long-term investors. This document examines asset allocation when expected returns and interest rates are time-varying, and investors have mean-variance preferences and long investment horizons.

This paper provide the technical backbone of the empirical results shown in "The Term Structure of the Risk-Return Tradeoff" (Campbell and Viceira, 2004). The paper is organized as follows. Section 2 derives multi-period expected returns, variances and covariances in a model where log (or continuously compounded) return dynamics are described by a homoskedastic VAR(1) process. Section 2 also discusses several important extensions of this VAR(1) model. Section 3 and Section 4 derives conditional mean-variance frontiers at different investment horizons. Section 5 illustrates the general results shown in Sections 2, 3, and 4 using a simple example. The section is particularly useful to understand how return predictability can create nonlinear patterns in variances and covariances across investment horizons. Finally Section 6 concludes.

# 2 Investment Opportunity Set

## 2.1 Dynamics of returns and state variables

We begin describing a stylized model that can capture the expected return and risk of asset returns at different horizons. This model requires specifying the asset classes under consideration, plus any variables that can help forming expectations of future returns on these asset classes—such as price-earnings ratios, interest rates, or yield spreads. We will refer to these variables as "state variables". Our approach allows for any number of asset classes and state variables.

Let $\mathbf{z}_{t+1}$ denote a column vector whose elements are the returns on all asset classes under consideration, and the values of the state variables at time $(t+1)$. Because it is convenient for our subsequent portfolio analysis, we choose to write this vector as

$$\mathbf{z}_{t+1} \equiv \begin{bmatrix} r_{0,t+1} \\ \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota} \\ \mathbf{s}_{t+1} \end{bmatrix} \equiv \begin{bmatrix} r_{0,t+1} \\ \mathbf{x}_{t+1} \\ \mathbf{s}_{t+1} \end{bmatrix}. \tag{1}$$

where $r_{0,t+1}$ denotes the log (or continuously compounded) real return on the asset that we use as a benchmark to compute excess returns on all other asset classes, $\mathbf{x}_{t+1}$ is a vector of log excess returns on all other asset classes with respect to the benchmark, and $\mathbf{s}_{t+1}$ is a vector with the realizations of the state variables. For future reference, we assume there are $n+1$ asset classes, and $m-n-1$ state variables, so that $\mathbf{z}_{t+1}$ has $(m \times 1)$ elements.

Note that all the returns included in $z_{t+1}$ are continuously compounded (or log) returns instead of gross returns. We work with log returns because it is more convenient from a data-modeling perspective. Of course, investors are concerned about gross returns rather than log returns. Thus in our portfolio analysis we reverse the log transformation whenever it is necessary.

The choice of a benchmark asset is arbitrary; we normally choose the benchmark to be a Treasury bill since this is the asset with the smallest short-term risk, but the representation of returns in (1) is perfectly general. We could just as easily write the vector in terms of real returns rather than excess returns. We show how to extract the moments of real returns from this VAR in the next section.

Our key assumption about the dynamics of asset returns and state variables is that they follow a first-order vector autoregressive process, or VAR(1). Each variable $z_{i,t+1}$ included in $\mathbf{z}_{t+1}$ depends linearly on a constant, its own lagged value, the lagged value of all other variables in $\mathbf{z}_{t+1}$, and a contemporaneous random shock $v_{i,t+1}$:

$$z_{i,t+1} = \phi_0 + \phi_1 z_{1,t} + ... + \phi_i z_{i,t} + ... + v_{i,t+1}. \tag{2}$$

Note that this is simply a generalization of a first-order autoregressive process—or AR(1)—to handle multiple forecasting variables. In an AR(1), all coefficients in equation (2) are zero, except $\phi_0$ and $\phi_i$.

Stacking together all these forecasting equations, we can represent the VAR(1) compactly as

$$\mathbf{z}_{t+1} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{z}_t + \mathbf{v}_{t+1}, \tag{3}$$

where $\mathbf{\Phi}_0$ is a vector of intercepts; $\mathbf{\Phi}_1$ is a square matrix that stacks together the slope coefficients; and $\mathbf{v}_{t+1}$ is a vector of zero-mean shocks to realizations of returns and return forecasting variables.

We assume that the matrix of slopes $\mathbf{\Phi}_1$ is well behaved in a statistical sense, by requiring that its determinant is bounded between $-1$ and 1. This is the multivariate equivalent of the stationarity condition in a AR(1), that requires the autoregressive parameter to be bounded between $-1$ and $+1$. This condition ensures that, in the absence of shocks, the variables that enter the VAR(1) converge to their long-run means in a finite number of periods. Thus this condition excludes explosive (or nonstationary) behavior in these variables.

Finally, to complete the description of the return dynamics we need to be more specific about the nature of the vector of shocks to asset returns and return forecasting variables ($\mathbf{v}_{t+1}$). In particular, we assume that the vector of shocks is normally distributed,

$$\mathbf{v}_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}\left(0, \mathbf{\Sigma}_v\right), \tag{4}$$

where $\mathbf{\Sigma}_v$ denotes the matrix of contemporaneous variances and covariances of shocks. This matrix is not necessarily diagonal: We allow unexpected realizations of excess returns on different asset classes to covary with each other, and with shocks to return forecasting variables. For example, it plausible that the excess return on domestic equity is correlated with the excess return on foreign equity, or that unexpected shocks to interest rates (a possible state variable) are correlated with domestic equity returns.

For future reference, we note here that, consistent with our representation of $\mathbf{z}_{t+1}$ in (1), we can write $\mathbf{\Sigma}_v$ as

$$\mathbf{\Sigma}_v \equiv \mathrm{Var}_t\left(\mathbf{v}_{t+1}\right) = \begin{bmatrix} \boldsymbol{\sigma}_0^2 & \boldsymbol{\sigma}_{0x}' & \boldsymbol{\sigma}_{0s}' \\ \boldsymbol{\sigma}_{0x} & \mathbf{\Sigma}_{xx} & \mathbf{\Sigma}_{xs}' \\ \boldsymbol{\sigma}_{0s} & \mathbf{\Sigma}_{xs} & \mathbf{\Sigma}_s \end{bmatrix},$$

where the elements on the main diagonal are the variance of the real return on the benchmark asset ($\boldsymbol{\sigma}_0^2$), the variance-covariance matrix of unexpected excess returns ($\mathbf{\Sigma}_{xx}$), and the variance-covariance matrix of the state variables ($\mathbf{\Sigma}_s$). The off diagonal elements are the covariances of the real return on the benchmark assets with excess returns on all other assets and with shocks to the state variables ($\boldsymbol{\sigma}_{0x}$ and $\boldsymbol{\sigma}_{0s}$), and the covariances of excess returns with shocks to the state variables ($\mathbf{\Sigma}_{xs}$).

To keep our exposition as simple as possible, we assume that these variances and covariances do not vary over time—in other words, that risk does not change over time. However, we want to emphasize here that this assumption, while perhaps not realistic, is nevertheless not constraining from the perspective of long-term portfolio choice. The empirical evidence available suggests that changes in risk are a short-lived phenomenon.[2] In this sense, time varying risk is unlikely to be a major concern to long-term investors.

It is useful to note here that the unconditional mean and variance-covariance matrix of $\mathbf{z}_{t+1}$ is given by

$$\begin{aligned} \boldsymbol{\mu}_z &= \left(\mathbf{I}_m - \mathbf{\Phi}_1\right)^{-1} \mathbf{\Phi}_0, \\ \text{vec}(\mathbf{\Sigma}_{zz}) &= \left(\mathbf{I}_{m^2} - \mathbf{\Phi}_1 \otimes \mathbf{\Phi}_1\right)^{-1} \text{vec}\left(\mathbf{\Sigma}_v\right). \end{aligned} \tag{5}$$

These equations clarify why we require that the determinant of the matrix of slopes $\mathbf{\Phi}_1$ is bounded between $-1$ and 1. Otherwise the unconditional mean and variance of $\mathbf{z}_{t+1}$ would not be defined.

## 2.2 Extending the basic VAR(1) model

There are several important extensions of the VAR(1) model described in Section 2.1. First, it is straightforward to allow for any number of lags of asset returns and forecasting variables within the VAR framework. It is interesting however to note that one can rewrite a VAR of any order in the form of a VAR(1) by adding more state variables which are simply lagged values of the original vector of variables. Thus all the formal results in this paper and in the companion technical document are valid for any VAR specification, even if for convenience we write them only in terms of a VAR(1).

An important consideration when considering additional lags is whether the parameters of the VAR are unknown and must be estimated from historical data. In that

[2] Authors such as Campbell (1987), Harvey (1989, 1991), and Glosten, Jagannathan, and Runkle (1993) have explored the ability of the state variables used here to predict risk and have found only modest effects that seem to be dominated by the effects of the state variables on expected returns. Chacko and Viceira (1999) show how to include changing risk in a long-term portfolio choice problem, using a continuous-time extension of the methodology of Campbell and Viceira (1999); they find that changes in equity risk are not persistent enough to have large effects on the intertemporal hedging demand for equities. Aït-Sahalia and Brandt (2001) adopt a semiparametric methodology that accommodates both changing expected returns and changing risk.

case, the precision of those estimates worsens as the number of parameters increases relative to the size of the sample. Since the number of parameters in the model increases exponentially with the number of lags, adding lags can reduce significantly the precision of the parameter estimates.[3] Of course, similar concerns arise with respect to the number of asset classes and state variables included in the VAR.

Second, the empirical results show in "The Term Structure of the Risk-Return Tradeoff" (Campbell and Viceira, 2004) suggest that some return forecasting variables are highly persistent. In that case, previous research shows that relying on estimation procedures to infer the parameters of the VAR(1) might lead to biased estimates of the coefficients of these variables (Stambaugh 1999). A standard econometric procedure to correct for these biases is bootstrapping. Bias-corrected estimates typically show that there is less predictability of excess stock returns than sample estimates suggest (Hodrick, 1992; Goetzmann and Jorion, 1993; Nelson and Kim, 1993), but that there is more predictability of excess bond returns (Bekaert, Hodrick, and Marshall, 1997). The reason for the discrepancy is that the evidence on stock market predictability comes from positive regression coefficients of stock returns on the dividend-price ratio, while the evidence on bond market predictability comes from positive regression coefficients of bond returns on yield spreads. Stambaugh (1999) shows that the small-sample bias in such regressions has the opposite sign to the sign of the correlation between innovations in returns and innovations in the predictive variable. In the stock market the log dividend-price ratio is negatively correlated with returns, leading to a positive small-sample bias which helps to explain some apparent predictability; in the bond market, on the other hand, the yield spread is mildly positively correlated with returns, leading to a negative small-sample bias which cannot explain the positive regression coefficient found in the data.

It is important to note that bootstrapping methods are not a panacea when forecasting variables are persistent, because we do not know the true persistence of the data generating process that is to be used for the bootstrap (Elliott and Stock, 1994, Cavanagh, Elliott, and Stock, 1995, Campbell and Yogo, 2003). Although finite-sample bias may well have some effect on the coefficients reported in Table 2, we do not attempt any corrections here. Instead, we take the estimated VAR coefficients as given and known by investors, and explore their implications for optimal long-term

[3] For example, if there are three asset classes and three state variables (as we are doing in our empirical application), the VAR(1) specification requires estimating a total of 63 parameters—6 intercepts, 36 slope coefficients, 6 variances, and 15 covariance terms. Adding an additional lag for all variables would increase this already fairly large number of parameters to 99 (a 57% increase).

portfolios.

Third, we have assumed that the variance-covariance structure of the shocks in the VAR is constant. However, one can allow for time variation in risk. We have argued that this might not be too important a consideration from the perspective of long-term asset allocation, because the empirical evidence available suggests that changes in risk exhibit very low persistence. Nonetheless, time-varying risk can be easily incorporated into the model, for example, along the lines of the models suggested by Bollerslev (1990), Engle (2002), or Rigobon and Sack (2003).

Fourth, even if we rely on historical data to estimate the parameters of the VAR model, one might have prior views about some of these parameters, particularly about the intercepts which determine the long-run expected return on each asset class, or about the slope coefficients which determine the movements in expected returns over time. Using Bayesian methods one can introduce these prior views and combine them with estimates from the data. Hoevenaars, Molenaar, and Steenkamp (2004) examine this extension of the model in great detail.

## 2.3 Conditional k-period moments

We now derive expressions for the conditional mean and variance-covariance matrix of $(\mathbf{z}_{t+1} + ... + \mathbf{z}_{t+k})$ for any horizon $k$. This is important for our subsequent portfolio analysis across investment horizons, because returns are a subset of the vector $\mathbf{z}_{t+1}$, and cumulative $k$-period log returns are obtained by adding one-period log returns over $k$ successive periods.

We start by deriving a set of equations that relate future values of the vector of state variables to its current value $\mathbf{z}_t$ plus a weighted sum of interim shocks:

$$\begin{aligned} z_{t+1} &= \Phi_0 + \Phi_1 z_t + v_{t+1} \\ \\ z_{t+2} &= \Phi_0 + \Phi_1 z_{t+1} + v_{t+2} \\ &= \Phi_0 + \Phi_1\Phi_0 + \Phi_1\Phi_1 z_t + \Phi_1 v_{t+1} + v_{t+2} \\ \\ z_{t+3} &= \Phi_0 + \Phi_1 z_{t+2} + v_{t+3} \\ &= \Phi_0 + \Phi_1\Phi_0 + \Phi_1\Phi_1\Phi_0 + \Phi_1\Phi_1\Phi_1 z_t + \Phi_1\Phi_1 v_{t+1} + \Phi_1 v_{t+2} + v_{t+3} \\ \\ z_{t+k} &= \Phi_0 + \Phi_1\Phi_0 + \Phi_1^2\Phi_0 + ... + \Phi_1^{k-1}\Phi_0 + \Phi_1^k z_t \\ & +\Phi_1^{k-1} v_{t+1} + \Phi_1^{k-2} v_{t+2} + ... + \Phi_1 v_{t+k-1} + v_{t+k} \end{aligned}$$

These expressions follow immediately from forward recursion of equation (3).

Adding the expressions for $z_{t+1}$, $z_{t+1}$, ..., and reordering terms yields:

$$\begin{aligned} z_{t+1} + ... + z_{t+k} &= [k + (k-1)\Phi_1 + (k-2)\Phi_1^2 + ... + \Phi_1^{k-1}]\Phi_0 \\ & +(\Phi_1^k + \Phi_1^{k-1} + ... + \Phi_1) z_t \\ & +(1 + \Phi_1 + ... + \Phi_1^{k-1}) v_{t+1} \\ & +(1 + \Phi_1 + ... + \Phi_1^{k-2}) v_{t+2} \\ & +... \\ & +(I + \Phi_1) v_{t+k-1} + v_{t+k}. \end{aligned}$$

We can write this expression more compactly as

$$z_{t+1} + ... + z_{t+k} = \left[\sum_{i=0}^{k-1}(k-i)\Phi_1^i\right]\Phi_0 + \left[\sum_{j=1}^{k}\Phi_1^j\right] z_t + \sum_{q=1}^{k}\left[\sum_{p=0}^{k-q}\Phi_1^p v_{t+q}\right].$$

Now we are ready to compute conditional k-period moments of the state vector. The conditional mean is given by

$$\mathrm{E}_t(z_{t+1} + ... + z_{t+k}) = \left[\sum_{i=0}^{k-1}(k-i)\Phi_1^i\right]\Phi_0 + \left[\sum_{j=1}^{k}\Phi_i^j\right] z_t,$$

since the shocks $v_{t+q}$ have zero mean.

The conditional variance is given by

$$\begin{aligned}
\mathrm{Var}_t(z_{t+1} + ... + z_{t+k}) &= \mathrm{Var}_t\left[\left[\sum_{i=0}^{k-1}(k-i)\Phi_1^i\right]\Phi_0 + \left[\sum_{j=1}^{k}\Phi_1^j\right]z_t + \sum_{q=1}^{k}\left[\sum_{p=0}^{k-q}\Phi_1^p v_{t+q}\right]\right] \\
&= \mathrm{Var}_t\left[\sum_{q=1}^{k}\left[\sum_{p=0}^{k-q}\Phi_1^p v_{t+q}\right]\right],
\end{aligned}$$

since all other terms are constant or known at time t. Expanding this expression we obtain:

$$\begin{aligned}
\mathrm{Var}_t(z_{t+1} + ... + z_{t+k}) &= \mathrm{Var}_t[\left(I + \Phi_1 + ... + \Phi_1^{k-1}\right)v_{t+1} + \left(I + \Phi_1 + ... + \Phi_1^{k-2}\right)v_{t+2} \\
&\quad + ... + (I + \Phi_1)v_{t+k-1} + v_{t+k}] \\
&= \Sigma_v + (I + \Phi_1)\Sigma_v(I + \Phi_1)' \\
&\quad + (I + \Phi_1 + \Phi_1\Phi_1)\Sigma_v(I + \Phi_1 + \Phi_1\Phi_1)' \\
&\quad + ... \\
&\quad + (I + \Phi_1 + ... + \Phi_1^{k-1})\Sigma_v(I + \Phi_1' + ... + \Phi_1^{k-1})',
\end{aligned}$$

which follows from reordering terms and noting that the conditional variance-covariance matrix of $v_{t+i}$ is the same $(\Sigma_v)$ at all leads $i$.

## 2.4 Conditional k-period moments of returns

We are only interested in extracting conditional moments per period from the portion of the VAR that contains returns. We can do so by using selector matrices. For example, if we want to recover the annualized (or per period) $k$-period conditional moments of the benchmark asset return and excess returns, we can use the following matrix:

$$\mathbf{H}_r = \left[\begin{array}{cc} \mathbf{I}_{(n+1)\times(n+1)} & \mathbf{0}_{(m-n)\times(m-n)} \end{array}\right].$$

It is straightforward to check that this matrix selects the $k$-period expected return and variance-covariance matrix of returns when applied to the expressions for the conditional expectation and variance-covarince matrix of $(\mathbf{z}_{t+1} + ... + \mathbf{z}_{t+k})$.

Thus we can extract the vector of expected $k$-period returns per period as follows:

$$\frac{1}{k}\left[\begin{array}{c} \mathrm{E}_t\left(r_{0,t+1}^{(k)}\right) \\ \mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota}\right) \end{array}\right] = \frac{1}{k}\mathbf{H}_r\,\mathrm{E}_t(z_{t+1} + ... + z_{t+k}),$$

and the $k$-period return variance-covariance matrix per period as

$$\frac{1}{k}\,\mathrm{Var}_t\left[\begin{array}{c} r_{0,t+1}^{(k)} \\ \mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota} \end{array}\right] = \frac{1}{k}\left[\begin{array}{cc} \sigma_0^2\,(k) & \boldsymbol{\sigma}_{0x}\,(k)' \\ \boldsymbol{\sigma}_{0x}\,(k) & \boldsymbol{\Sigma}_{xx}\,(k) \end{array}\right] = \frac{1}{k}\mathbf{H}_r\,\mathrm{Var}_t(z_{t+1} + ... + z_{t+k})\mathbf{H}_r'.$$

Similarly, the following selector matrix extracts per-period, $k$-period conditional moments of log real returns:

$$\mathbf{M}_r = \left[\begin{array}{ccc} 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times(m-n-1)} \\ \boldsymbol{\iota}_{n\times n} & \mathbf{I}_{n\times n} & \mathbf{0}_{(m-n-1)\times(m-n-1)} \end{array}\right], \tag{6}$$

which implies

$$\frac{1}{k}\left[\begin{array}{c} \mathrm{E}_t\left(r_{0,t+1}^{(k)}\right) \\ \mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)}\right) \end{array}\right] = \frac{1}{k}\mathbf{M}_r\,\mathrm{E}_t(z_{t+1} + ... + z_{t+k})$$

and

$$\frac{1}{k}\,\mathrm{Var}_t\left[\begin{array}{c} r_{0,t+1}^{(k)} \\ \mathbf{r}_{t+1}^{(k)} \end{array}\right] = \frac{1}{k}\boldsymbol{\Sigma}_{rr}\,(k) = \frac{1}{k}\mathbf{M}_r\,\mathrm{Var}_t(z_{t+1} + ... + z_{t+k})\mathbf{M}_r'.$$

# 3 One-Period Mean-Variance Analysis

This section derives the standard mean-variance for one-period log returns. The next section extends these results to a multi-period environment. We start by noting that the gross return on the wealth portfolio is given by

$$\begin{aligned} R_{p,t+1} &= \sum_{i=1}^{n}\alpha_{i,t}\,(R_{i,t+1} - R_{0,t+1}) + R_{0,t+1} \\ &= \boldsymbol{\alpha}_t'\,(\mathbf{R}_{t+1} - R_{0,t+1}\boldsymbol{\iota}) + R_{0,t+1}, \end{aligned} \tag{7}$$

where $R_i$ is the gross return on asset $i$, $R_0$ is the gross return on the benchmark asset, and $\alpha_{i,t}$ is the portfolio weight on asset $i$; $\boldsymbol{\alpha}_t$ is is a $(n \times 1)$ vector containing the portfolio weights $\alpha_{i,t}$'s, and $\boldsymbol{\iota}$ is a $(n \times 1)$ vector of 1's.

The return on the portfolio in equation (7) is expressed in terms of the gross returns on the assets. Since we are interested in working with log returns, we need to derive an expression for the log return on the portfolio. Campbell and Viceira (1999, 2001) and Campbell, Chan and Viceira (2003) suggest the following approximation to the log return on the portfolio:

$$r_{p,t+1} = r_{0,t+1} + \boldsymbol{\alpha}_t' \left( \mathbf{r}_{t+1} - r_{0,t+1} \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\alpha}_t' \left( \boldsymbol{\sigma}_x^2 - \boldsymbol{\Sigma}_{xx} \boldsymbol{\alpha}_t \right), \tag{8}$$

where

$$\boldsymbol{\Sigma}_{xx} \equiv \mathrm{Var}_t \left( \mathbf{r}_{t+1} - r_{0,t+1} \boldsymbol{\iota} \right),$$

and

$$\boldsymbol{\sigma}_x^2 \equiv \mathrm{diag} \left( \boldsymbol{\Sigma}_{xx} \right).$$

We now derive the one-period mean-variance frontier with and without a riskless asset, and the weights for the "tangency portfolio," i.e. the portfolio containing only risky assets that also belongs simultaneously to both the mean-variance frontier of risky assets, and the mean-variance frontier with a riskless asset.

## 3.1 One-Period Mean-Variance Frontier with Log Returns and No Riskless Asset

This section derives the mean-variance frontier when none of the assets available for investment is unconditionally riskless in real terms at a one-period horizon. This happens, for example, when there is significant short-term inflation risk, and the only short-term assets available for investment are nominal money market instruments, such as nominal T-bills. In practice short-term inflation risk is small, and one can probably ignore it. However, we still think it is worth analyzing the case with no one-period riskless asset, to gain intuition for the subsequent results for long horizons, where inflation risk is significant, and nominal bonds can be highly risky in real terms.

### 3.1.1 Problem

We state the mean-variance portfolio optimization problem with log returns as follows:

$$\min_{\alpha} \frac{1}{2} \mathrm{Var}_t \left( r_{p,t+1} \right), \tag{9}$$

subject to

$$\mathrm{E}_t\left(r_{p,t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(r_{p,t+1}\right)=\mu_p,$$

where $r_{p,t+1}$ is given in equation (8). That is, we want to find the vector of portfolio weights $\boldsymbol{\alpha}_t$ that minimizes the variance of the portfolio log return when the required expected gross return on the portfolio is $\mu_p$.

Equation (8) implies that the variance of the log portfolio return is given by

$$\mathrm{Var}_t\left(r_{p,t+1}\right)=\boldsymbol{\alpha}_t'\boldsymbol{\Sigma}_{xx}\boldsymbol{\alpha}_t+\sigma_0^2+2\boldsymbol{\alpha}_t'\boldsymbol{\sigma}_{0x}$$

and the log expected portfolio return is given by

$$\mathrm{E}_t\left(r_{p,t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(r_{p,t+1}\right)=\left(\mathrm{E}_t\, r_{0,t+1}+\frac{1}{2}\sigma_0^2\right)+\boldsymbol{\alpha}_t'\left(\mathrm{E}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right)+\frac{1}{2}\boldsymbol{\sigma}_x^2+\boldsymbol{\sigma}_{0x}\right). \tag{10}$$

It is useful to note here that the VAR(1) model for returns given in equation (3) implies that

$$\mathrm{E}_t\, r_{0,t+1}=\mathbf{H}_0'\left(\boldsymbol{\Phi}_0+\boldsymbol{\Phi}_1\mathbf{z}_t\right)$$

and

$$\begin{aligned}\mathrm{E}_t\left(\mathbf{r}_{t+1}-r_{0,t+1}\boldsymbol{\iota}\right) &= \mathrm{E}_t\left(\mathbf{x}_{t+1}\right)\\ &= \mathbf{H}_x\left(\boldsymbol{\Phi}_0+\boldsymbol{\Phi}_1\mathbf{z}_t\right),\end{aligned}$$

where $\mathbf{H}_0'$ is a $(1\times m)$ selection vector $(1,0,...,0)$ that selects the first element of the matrix which it multiplies, and $\mathbf{H}_x$ is a $(n\times m)$ selection matrix that selects rows 2 through $n+1$ of the matrix which it multiplies:

$$\mathbf{H}_x=\left[\begin{array}{ccc}\mathbf{0}_{n\times 1} & \mathbf{I}_{n\times n} & \mathbf{0}_{(m-n-1)\times(m-n-1)}\end{array}\right],$$

where $\mathbf{I}_{n\times n}$ is an identity matrix.

The Lagrangian for this problem is

$$\mathcal{L}=\frac{1}{2}\mathrm{Var}_t\left(r_{p,t+1}\right)+\lambda\left(\mu_P-\left(\mathrm{E}_t\left(r_{p,t+1}\right)+\frac{1}{2}\mathrm{Var}_t\left(r_{p,t+1}\right)\right)\right),$$

where $\lambda$ is the Lagrange multiplier on the expected portfolio return constraint. Note that our definition of the portfolio return already imposes the constraint that portfolio weights must add up to one.

### 3.1.2 Mean-variance portfolio rule

The solution to the mean-variance program (9) is

$$\boldsymbol{\alpha}_t = \lambda \boldsymbol{\Sigma}_{xx}^{-1} \left[ \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_{0,t+1} \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_x^2 + \boldsymbol{\sigma}_{0x} \right] - \boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\sigma}_{0x}. \tag{11}$$

We can rewrite $\boldsymbol{\alpha}_t$ as a portfolio combining two distinct portfolios:

$$\boldsymbol{\alpha}_t = \lambda \boldsymbol{\Sigma}_{xx}^{-1} \left[ \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_{0,t+1} \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_x^2 \right] + (1 - \lambda) \left( -\boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\sigma}_{0x} \right).$$

The first portfolio is proportional to the expected gross excess return on all assets, weighted by the inverse of the variance-covariance matrix of excess returns. If asset returns are independent of each other, so $\boldsymbol{\Sigma}_{xx}$ is a diagonal matrix containing the variances of asset returns, this portfolio is simply the vector of Sharpe ratios for each asset class—each one divided by its return standard deviation.

The second portfolio, with weights $\boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\sigma}_{0x}$, is the portfolio of assets with minimum absolute variance, or global minimum-variance portfolio. That is, this portfolio is the solution to:

$$-\boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\sigma}_{0x} = \min_{\alpha} \frac{1}{2} \mathrm{Var}_t \left( r_{p,t+1} \right).$$

In the literature on dynamic optimal portfolio choice, the mean-variance portfolio is also known as the "myopic portfolio." This name derives from the fact that it takes into consideration only expected returns next period, ignoring other considerations that might affect the portfolio decisions of risk averse investors, such as the ability that certain asset classes might have to protect the spending needs of these investors from a sudden deterioration in investment opportunities.

### 3.1.3 Equation for mean-variance frontier

Equation (11) implies the following relation between $\sigma_p^2 \equiv \mathrm{Var}_t(r_{p,t+1})$ and $\mu_p$:[4]

$$\sigma_p^2 = \frac{1}{A} \left( e\mu_p^2 + 2Be\mu_p + B^2 \right) + \sigma_0^2 - C, \tag{12}$$

[4] To get this equation, note that at the optimum either $\lambda = 0$, or $\mu_P = \left( \mathrm{E}_t \left( r_{p,t+1} \right) + \frac{1}{2} \sigma_p^2 \right)$.

where $e\mu_p$ is the expected excess return on the portfolio over the benchmark asset,

$$e\mu_p = \mu_p - \left( \mathrm{E}_t \, r_{0,t+1} + \frac{1}{2}\sigma_0^2 \right),$$

and

$$A = \left( \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota} \right) + \frac{1}{2}\boldsymbol{\sigma}_x^2 + \boldsymbol{\sigma}_{0x} \right)' \boldsymbol{\Sigma}_{xx}^{-1} \left( \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota} \right) + \frac{1}{2}\boldsymbol{\sigma}_x^2 + \boldsymbol{\sigma}_{0x} \right),$$

$$B = \boldsymbol{\sigma}_{0x}' \boldsymbol{\Sigma}_{xx}^{-1} \left( \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_{0,t+1}\boldsymbol{\iota} \right) + \frac{1}{2}\boldsymbol{\sigma}_x^2 + \boldsymbol{\sigma}_{0x} \right),$$

$$C = \boldsymbol{\sigma}_{0x}' \boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\sigma}_{0x}.$$

## 3.2 One-Period Mean-Variance Frontier with Log Returns and a Riskless Asset

In this section we consider mean-variance analysis with log returns when investors have an additional asset which is truly riskless in real terms over one period. This asset is an inflation-indexed T-Bill. We denote its return as $r_f$. Thus the investor has now $n+2$ assets available for investment. This derivation is useful to compute the tangency portfolio.

### 3.2.1 Problem

$$\min_{\alpha} \frac{1}{2} \mathrm{Var}_t \left( r_{p,t+1} - r_f \right) \tag{13}$$

$$\text{subject to } \mathrm{E}_t \left( r_{p,t+1} - r_f \right) + \frac{1}{2} \mathrm{Var}_t \left( r_{p,t+1} - r_f \right) = \mu_p - r_f,$$

where

$$r_{p,t+1} - r_f = \boldsymbol{\omega}_t' \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\omega}_t' \left( \boldsymbol{\sigma}_r^2 - \boldsymbol{\Sigma}_{rr} \boldsymbol{\omega}_t \right),$$

$\boldsymbol{\omega}_t$ is a $(n+1) \times 1$ vector of portfolio weights,

$$\boldsymbol{\Sigma}_{rr} \equiv \mathrm{Var}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right),$$

and

$$\boldsymbol{\sigma}_r^2 \equiv \operatorname{diag}\left(\boldsymbol{\Sigma}_{rr}\right).$$

The variance of the log excess portfolio return is given by

$$\operatorname{Var}_t\left(r_{p,t+1} - r_f\right) = \boldsymbol{\omega}_t' \boldsymbol{\Sigma}_{rr} \boldsymbol{\omega}_t \tag{14}$$

and the log excess expected portfolio return is given by:

$$\mathrm{E}_t\left(r_{p,t+1} - r_f\right) + \frac{1}{2}\operatorname{Var}_t\left(r_{p,t+1} - r_f\right) = \boldsymbol{\omega}_t'\left(\mathrm{E}_t\left(\mathbf{r}_{t+1} - r_f\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2\right). \tag{15}$$

Note that:

$$\mathrm{E}_t\left(\mathbf{r}_{t+1} - r_f\boldsymbol{\iota}\right) = \mathbf{M}_r\left(\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1\mathbf{z}_t\right) - r_f\boldsymbol{\iota},$$

where $\mathbf{M}_r$ is the $(n + 1 \times m)$ selection matrix defined in (6).

The Lagrangian for this problem is

$$\mathcal{L} = \frac{1}{2}\operatorname{Var}_t\left(r_{p,t+1} - r_f\right) + \lambda_f\left(\mu_P - r_f - \left(\mathrm{E}_t\left(r_{p,t+1} - r_f\right) + \frac{1}{2}\operatorname{Var}_t\left(r_{p,t+1} - r_f\right)\right)\right),$$

where $\lambda$ is the Lagrange multiplier on the expected portfolio return constraint.

### 3.2.2 Mean-variance portfolio rule

The solution to the mean-variance program (13) is

$$\boldsymbol{\omega}_t = \lambda_f \boldsymbol{\Sigma}_{rr}^{-1}\left[\mathrm{E}_t\left(\mathbf{r}_{t+1} - r_f\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_x^2\right]. \tag{16}$$

### 3.2.3 Equation for mean-variance frontier

Equation (16) implies the following relation between $\sigma_p^2 \equiv (\operatorname{Var}_t r_{p,t+1} - r_f)$ and $\mu_p$:

$$\sigma_p^2 = \frac{1}{A} e \mu_p^2, \tag{17}$$

where $e\mu_p$ is the expected excess return on the portfolio over the benchmark asset,

$$e\mu_p = \mu_p - r_f$$

and

$$A = \left( \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_r^2 \right)' \boldsymbol{\Sigma}_{xx}^{-1} \left( \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_r^2 \right).$$

Equation (17) implies that, in a standard deviation-expected return space, the frontier is a line that intersects the expected return axis at $r_f$, and it has a slope equal to $\sqrt{A}$. We show below that this slope is the Sharpe ratio of the tangency portfolio.

### 3.2.4 Tangency portfolio

We can now derive the weights for the tangency portfolio between frontiers (12) and (17). This is a portfolio whose weights $\boldsymbol{\omega}_{T,t}$ satisfy $\boldsymbol{\omega}'_{T,t} \boldsymbol{\iota} = \mathbf{1}$. That is, it is a portfolio with no loading on the risk-free asset.

We can easily solve for the weights of this portfolio by finding the Lagrange multiplier $\lambda_f$ for which $\boldsymbol{\omega}_t$ in (16) satisfies the constraint $\boldsymbol{\omega}'_{T,t} \boldsymbol{\iota} = \mathbf{1}$:

$$1 = \boldsymbol{\omega}'_{T,t} \boldsymbol{\iota} = \lambda_f \left[ \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_r^2 \right]' \left( \boldsymbol{\Sigma}_{rr}^{-1} \right)' \boldsymbol{\iota},$$

which implies

$$\lambda_f = \frac{1}{\left[ \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_r^2 \right]' \boldsymbol{\Sigma}_{rr}^{-1} \boldsymbol{\iota}}.$$

Therefore,

$$\boldsymbol{\omega}_{T,t} = \frac{1}{\left[ \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_r^2 \right]' \left( \boldsymbol{\Sigma}_{rr}^{-1} \right)' \boldsymbol{\iota}} \boldsymbol{\Sigma}_{rr}^{-1} \left[ \mathrm{E}_t \left( \mathbf{r}_{t+1} - r_f \boldsymbol{\iota} \right) + \frac{1}{2} \boldsymbol{\sigma}_r^2 \right]. \tag{18}$$

It is straightforward to show that the Sharpe ratio of the tangency portfolio equals $\sqrt{A}$, i.e., that the Sharpe ratio is the slope of the mean-standard deviation frontier with a riskless asset. To see this, substitute (18) into equations (14) and (15) to compute expressions for the variance and expected excess return on the tangency portfolio, and compute the ratio of the expected excess return to the standard deviation.

# 4 Long-Horizon Mean-Variance Analysis

In this section we extend to a multiperiod horizon the results of Section 3 for mean-variance efficient allocations and frontiers. We assume that the investor has an investment horizon of $k$ periods, and that she chooses portfolio weights only every $k$ periods, i.e., at $t$, $t+k$, $t+2k$, .... We denote these weights as $\boldsymbol{\alpha}_t(k)$. The investor chooses those weights that minimize the variance per period of the log portfolio return subject to a required expected return per period on the portfolio.

To solve this problem we need to compute the log $k$-period portfolio return $r_{p,t+k}^{(k)}$. One possibility is to use, for similarity with the approximation for the log 1-period portfolio return given in (8), the following expression:

$$r_{p,t+k}^{(k)} = r_{0,t+k}^{(k)} + \boldsymbol{\alpha}_t'(k)\left(\mathbf{r}_{t+k}^{(k)} - r_{0,t+k}^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\alpha}_t'(k)\left(\boldsymbol{\sigma}_x^2(k) - \boldsymbol{\Sigma}_{xx}(k)\,\boldsymbol{\alpha}_t(k)\right), \quad (19)$$

where $\boldsymbol{\iota}$ is a $(n \times 1)$ vector of 1's,

$$\boldsymbol{\Sigma}_{xx}(k) \equiv \mathrm{Var}_t\left(\mathbf{r}_{t+k}^{(k)} - r_{0,t+k}^{(k)}\boldsymbol{\iota}\right),$$

$$\boldsymbol{\sigma}_x^2(k) \equiv \mathrm{diag}\left(\boldsymbol{\Sigma}_{xx}(k)\right).$$

and, for any asset (or portfolio of assets) $l$, the cumulative $k$-period log return (or log excess return) is

$$r_{l,t+k}^{(k)} = \sum_{i=1}^{k} r_{l,t+i}. \quad (20)$$

Note that to simplify notation we are ignoring the subscript $t$ when denoting conditional variances and covariances. Our assumption that the dynamics of state variables and returns follow a homoskedastic VAR implies that conditional variances are not a function of time—though they are a function of $k$, the investment horizon.

It is also important to keep in mind that (19) is an approximation to the log return on wealth. This accuracy of this approximation is high at short horizons, but it deteriorates as the horizon $k$ becomes longer.

## 4.1 K-Period Mean-Variance Frontier with Log Returns and No Riskless Asset

### 4.1.1 Problem

The investor chooses those weights that minimize the variance per period of the log portfolio return subject to a required expected return per period on the portfolio. If no riskless asset over horizon $k$ is available, the problem becomes

$$\min_{\alpha} \frac{1}{2} \frac{\operatorname{Var}_t\left(r_{p,t+k}^{(k)}\right)}{k} \tag{21}$$

$$\text{subject to } \frac{\mathrm{E}_t\left(r_{p,t+k}^{(k)}\right) + \frac{1}{2}\operatorname{Var}_t\left(r_{p,t+k}^{(k)}\right)}{k} = \mu_p,$$

where $\mu_p$ is the required portfolio portfolio return per period.

From equation (19), the conditional variance of the cumulative log portfolio return is given by

$$\operatorname{Var}_t\left(r_{p,t+k}^{(k)}\right) = \boldsymbol{\alpha}_t'(k)\,\boldsymbol{\Sigma}_{xx}(k)\,\boldsymbol{\alpha}_t(k) + \sigma_0^2(k) + 2\boldsymbol{\alpha}_t'(k)\,\boldsymbol{\sigma}_{0x}(k),$$

and the cumulative log expected portfolio return is given by

$$\mathrm{E}_t\left(r_{p,t+k}^{(k)}\right) + \frac{1}{2}\sigma_p^2(k) = \left(\mathrm{E}_t\, r_{0,t+1}^{(k)} + \frac{1}{2}\sigma_0^2(k)\right) + \boldsymbol{\alpha}_t'\left(\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_x^2(k) + \boldsymbol{\sigma}_{0x}(k)\right).$$

Note that we already have expressions for the moments in the RHS of these equations based on our VAR specification.

The Lagrangian for this problem is

$$\mathcal{L} = \frac{1}{2}\frac{\operatorname{Var}_t\left(r_{p,t+k}^{(k)}\right)}{k} + \lambda\left(\mu_P - \frac{\mathrm{E}_t\left(r_{p,t+k}^{(k)}\right) + \frac{1}{2}\operatorname{Var}_t\left(r_{p,t+k}^{(k)}\right)}{k}\right),$$

where $\lambda$ is the Lagrange multiplier on the expected portfolio return constraint. Note that we have already imposed the constraint that portfolio weights must add up to one in the way we define the log return on the portfolio.

#### 4.1.2 Myopic portfolio rule

The solution to the mean-variance program (9) is

$$\boldsymbol{\alpha}_t(k) = \lambda \boldsymbol{\Sigma}_{xx}^{-1}(k)\left[\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_x^2(k) + \boldsymbol{\sigma}_{0x}(k)\right] - \boldsymbol{\Sigma}_{xx}^{-1}(k)\,\boldsymbol{\sigma}_{0x}(k). \quad (22)$$

Again, it is useful to note that we can write (22) as

$$\boldsymbol{\alpha}_t(k) = \lambda \boldsymbol{\Sigma}_{xx}^{-1}(k)\left[\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_x^2(k)\right] + (1-\lambda)\left(-\boldsymbol{\Sigma}_{xx}^{-1}(k)\,\boldsymbol{\sigma}_{0x}(k)\right),$$

where $\boldsymbol{\Sigma}_{xx}^{-1}(k)\,\boldsymbol{\sigma}_{0x}(k)$ is the $k$-horizon global minimum-variance portfolio:

$$-\boldsymbol{\Sigma}_{xx}^{-1}(k)\,\boldsymbol{\sigma}_{0x}(k) = \min_{\alpha}\frac{1}{2}\,\mathrm{Var}_t\left(r_{p,t+k}^{k}\right).$$

Thus $\boldsymbol{\alpha}_t(k)$ is a linear combination of two portfolios, one of which is the global minimum-variance portfolio.

#### 4.1.3 Equation for k-period mean-variance frontier

Equation (11) implies the following relation between $\sigma_p^2(k)/k \equiv \mathrm{Var}_t\left(r_{p,t+k}^{(k)}\right)/k$ and $\mu_p$:

$$\frac{\sigma_p^2(k)}{k} = \frac{1}{A(k)}\left(e\mu_p^2 + 2B(k)\,e\mu_p + B^2(k)\right) + \sigma_0^2(k) - C(k), \quad (23)$$

where $e\mu_p$ is the required expected excess return per period on the portfolio over the benchmark asset,

$$e\mu_p = \mu_p - \frac{1}{k}\left(\mathrm{E}_t\, r_{0,t+1}^{(k)} + \frac{1}{2}\sigma_0^2(k)\right),$$

and

$$\begin{aligned} A(k) &= \frac{1}{k}\left(\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_x^2(k) + \boldsymbol{\sigma}_{0x}(k)\right)' \\ &\quad \times \boldsymbol{\Sigma}_{xx}^{-1}(k)\left(\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_{0,t+1}^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_x^2(k) + \boldsymbol{\sigma}_{0x}(k)\right), \end{aligned}$$

$$B\left(k\right)=\frac{1}{k}\boldsymbol{\sigma}_{0x}'\left(k\right)\boldsymbol{\Sigma}_{xx}^{-1}\left(k\right)\left(\mathrm{E}_{t}\left(\mathbf{r}_{t+1}^{(k)}-r_{0,t+1}^{(k)}\boldsymbol{\iota}\right)+\frac{1}{2}\boldsymbol{\sigma}_{x}^{2}\left(k\right)+\boldsymbol{\sigma}_{0x}\left(k\right)\right),$$

and

$$C\left(k\right)=\frac{1}{k}\boldsymbol{\sigma}_{0x}'\left(k\right)\boldsymbol{\Sigma}_{xx}^{-1}\left(k\right)\boldsymbol{\sigma}_{0x}\left(k\right).$$

#### 4.1.4 Unconditional mean-variance frontier

The unconditional mean variance frontier simply follows from taking limits in (23) as $k\rightarrow\infty$. That is, we use the unconditional variance-covariance matrix of log returns implied by the VAR.

## 4.2 K-Period Mean-Variance Frontier with Log Returns and a Riskless Asset

In this section we consider mean-variance analysis with log returns when investors have an aditional asset which is truly riskless in real terms over their investment horizon. This asset is an inflation-indexed zero-coupon bond. The investor has now $n+1$ assets available for investment. This derivation is useful to compute the tangency portfolio.

>From the perspective of a $k$-period mean-variance investor—who is a buy-and-hold investor—the return on the riskfree asset is the yield on this bond, which we denote by $r_{f}^{(k)}$. We assume that $r_{f}^{(k)}$ is continuously-compounded, in natural units per period. Thus to compound $k$ periods ahead we need to multiply it by $k$.

#### 4.2.1 Problem

$$\min_{\alpha}\frac{1}{2}\frac{\mathrm{Var}_{t}\left(r_{p,t+k}^{(k)}-kr_{f}^{(k)}\right)}{k} \tag{24}$$

$$\text{subject to}\quad\frac{\mathrm{E}_{t}\left(r_{p,t+k}^{(k)}-r_{f}^{(k)}k\right)+\frac{1}{2}\mathrm{Var}_{t}\left(r_{p,t+k}^{(k)}-kr_{f}^{(k)}\right)}{k}=\mu_{p}-r_{f}^{(k)},$$

where we assume that $r_f^{(k)}$ is given in units per period, and $r_{l,t+k}^{(k)}$ is defined in (20).

To solve this problem we need to compute the log k-period portfolio return $r_{p,t+k}^{(k)}$. We adopt the same convention as in the previous section, and use the k-period ahead conditional covariances for the Jensen's inequality correction in the approximation to the log excess return on wealth k periods ahead:

$$r_{p,t+k}^{(k)} - kr_f^{(k)} = \boldsymbol{\omega}_t'(k)\left(\mathbf{r}_{t+k}^{(k)} - kr_f^{(k)}\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\omega}_t'(k)\left(\boldsymbol{\sigma}_r^2(k) - \boldsymbol{\Sigma}_{rr}(k)\boldsymbol{\omega}_t(k)\right), \quad (25)$$

where

$$\boldsymbol{\Sigma}_{rr}(k) \equiv \mathrm{Var}_t\left(\mathbf{r}_{t+k}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right),$$

$$\boldsymbol{\sigma}_r^2(k) \equiv \mathrm{diag}\left(\boldsymbol{\Sigma}_{rr}(k)\right).$$

Using equation (25), the conditional variance of the cumulative log portfolio return is given by

$$\mathrm{Var}_t\left(r_{p,t+k}^{(k)} - kr_f^{(k)}\right) = \boldsymbol{\omega}_t'(k)\,\boldsymbol{\Sigma}_{rr}(k)\,\boldsymbol{\omega}_t(k), \quad (26)$$

and the cumulative log expected excess portfolio return is given by:

$$\mathrm{E}_t\left(r_{p,t+k}^{(k)} - r_f^{(k)}k\right) + \frac{1}{2}\mathrm{Var}_t\left(r_{p,t+k}^{(k)} - kr_f^{(k)}\right) = \boldsymbol{\omega}_t'\left(\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right). \quad (27)$$

Note that we already have expressions for the moments in the RHS of these equations based on our VAR specification.

The Lagrangian for this problem is

$$\mathcal{L} = \frac{1}{2}\frac{\mathrm{Var}_t\left(r_{p,t+k}^{(k)} - kr_f^{(k)}\right)}{k} + \lambda_f\left(\mu_P - r_f^{(k)} - \frac{1}{k}\left(\mathrm{E}_t\left(r_{p,t+k}^{(k)} - r_f^{(k)}k\right) + \frac{1}{2}\mathrm{Var}_t\left(r_{p,t+k}^{(k)} - kr_f^{(k)}\right)\right)\right),$$

where $\lambda_f$ is the Lagrange multiplier on the expected portfolio return constraint. Note that we have already imposed the constraint that portfolio weights must add up to one in the way we define the log return on the portfolio.

### 4.2.2 Myopic portfolio rule

The solution to the mean-variance program (24) is

$$\boldsymbol{\omega}_t(k) = \lambda_f\boldsymbol{\Sigma}_{rr}^{-1}(k)\left[\mathrm{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right]. \quad (28)$$

#### 4.2.3 Equation for mean-variance frontier

Equation (28) implies the following relation between $\sigma_p^2(k) \equiv \text{Var}_t\left(r_{p,t+k}^{(k)} - r_f^{(k)}k\right)$ and $\mu_p$:

$$\frac{\sigma_p^2(k)}{k} = \frac{1}{A(k)} e\mu_p^2, \tag{29}$$

where $e\mu_p$ is the expected excess return on the portfolio over the riskfree asset,

$$e\mu_p = \mu_p - r_f^{(k)},$$

and

$$A(k) = \frac{1}{k}\left(\text{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right)' \boldsymbol{\Sigma}_{rr}^{-1}(k)\left(\text{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right).$$

Equation (29) implies that, in a standard deviation-expected return space, the frontier is a line that intersects the expected return axis at $r_f^{(k)}$, and whose slope is the Sharpe ratio.

#### 4.2.4 Tangency portfolio

We can now derive the weights for the tangency portfolio between frontiers (23) and (29). This is a portfolio whose weights $\boldsymbol{\omega}_{T,t}(k)$ satisfy $\boldsymbol{\omega}_{T,t}(k)'\boldsymbol{\iota} = \mathbf{1}$. That is, it is a portfolio with no loading on the risk-free asset. We can easily find the weights in this portfolio by finding the Lagrange multiplier $\lambda_f$ for which $\boldsymbol{\omega}_{T,t}(k)$ in (28) satisfies this constraint:

$$1 = \boldsymbol{\omega}_{T,t}(k)'\boldsymbol{\iota} = \lambda_f\left[\text{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right]\left(\boldsymbol{\Sigma}_{rr}^{-1}(k)\right)'\boldsymbol{\iota}$$

or

$$\lambda_f = \frac{1}{\left[\text{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right]'\left(\boldsymbol{\Sigma}_{rr}^{-1}(k)\right)'\boldsymbol{\iota}},$$

so that

$$\boldsymbol{\omega}_{T,t}(k) = \frac{1}{\left[\text{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right]'\left(\boldsymbol{\Sigma}_{rr}^{-1}(k)\right)'\boldsymbol{\iota}}\boldsymbol{\Sigma}_{rr}^{-1}(k)\left[\text{E}_t\left(\mathbf{r}_{t+1}^{(k)} - r_f^{(k)}k\boldsymbol{\iota}\right) + \frac{1}{2}\boldsymbol{\sigma}_r^2(k)\right]. \tag{30}$$

Once again, it is straightforward to show that the Sharpe ratio of the tangency portfolio equals $\sqrt{A(k)}$, i.e., that the Sharpe ratio is the slope of the mean-standard deviation frontier with a riskless asset. To see this, substitute (30) into equations (26) and (27) to compute expressions for the variance and expected excess return on the tangency portfolio, and compute the ratio of the expected excess return to the standard deviation.

# 5 A Simple Example

To illustrate how a VAR(1) works, we will restrict attention in this section to a simple example with only two asset classes (equities and bonds) and one single forecasting variable. Furthermore, we will assume in this example that the predictable component of returns and the state variable depends only on the lagged value of the state variable. We then show that this simplified VAR(1) model is very useful to build intuition about horizon effects on expected return and risk. The empirical application in the paper uses a more complicated VAR(1) model.

In our example, the dynamics of returns and the state variable are given by

$$\begin{aligned} re_{t+1} &= \phi_{es} s_t + v_{e,t+1}, \\ rb_{t+1} &= \phi_{bs} s_t + v_{b,t+1}, \\ s_{t+1} &= \phi_{ss} s_t + v_{s,t+1}, \end{aligned} \tag{31}$$

where $re_{t+1}$ denotes the log return on equities, $rb_{t+1}$ denotes the log return on bonds, and $s_{t+1}$ denotes the state variable.

In the model (31), the predictable component of returns is given by the terms $\phi_{es} s_t$ and $\phi_{bs} s_t$. These terms describe the expected return on equities and bond one period ahead. The unpredictable or unexpected component of realized returns and the return forecasting variable is given by $v_{e,t+1}$, $v_{b,t+1}$, and $v_{s,t+1}$. That is, $v_{e,t+1} = re_{t+1} - \mathrm{E}_t[re_{t+1}]$ and likewise $v_{b,t+1} = rb_{t+1} - \mathrm{E}_t[rb_{t+1}]$, and $v_{s,t+1} = s_{t+1} - \mathrm{E}_t[s_{t+1}]$.

To complete the description of return dynamics we need to be more specific about the nature of the shocks $v_{e,t+1}$, $v_{b,t+1}$, and $v_{s,t+1}$.[5] We make two assumptions. First,

[5] We also need that the slopes $\phi$ are such that the system does not lead to explosive behavior in returns and the forecasting variable as we project their values forward. This is achieved if the absolute value of $\phi_{ss}$ is smaller than 1.

we assume that these shocks are serially uncorrelated, but we pose no constraints on their contemporaneous correlation structure. It is conceptually and empirically plausible that realizations of returns on different asset classes covary with each other, and with shocks to return forecasting variables. For example, we might have that the return on domestic equity is positively correlated with the return on long term bonds, or that unexpected shocks to interest rates (a possible state variable) are correlated with domestic equity returns.

Second, to keep our exposition as simple as possible, we assume that the variances and covariances of shocks do not vary over time—in other words, that risk does not change over time. We discuss in text why this assumption, while this assumption is perhaps not realistic, it is not too constraining from the perspective of long-term portfolio choice. We also discuss in text how to relax this assumption.

These assumptions imply that the contemporaneous variance-covariance structure of the shocks is a matrix of constants, which we denote for future reference as

$$\mathrm{Var}_t \begin{pmatrix} v_{e,t+1} \\ v_{b,t+1} \\ v_{s,t+1} \end{pmatrix} = \begin{bmatrix} \sigma_e^2 & \sigma_{eb} & \sigma_{es} \\ \sigma_{eb} & \sigma_b^2 & \sigma_{bs} \\ \sigma_{es} & \sigma_{bs} & \sigma_s^2 \end{bmatrix} \equiv \mathbf{\Sigma}_v. \tag{32}$$

## 5.1 Risk and horizon in the example

We now examine how asset return predictability generates horizon effects in risk, in the context of the simple example (31). In the spirit of traditional one-period mean-variance analysis, we define risk per period at horizon $k$ as the conditional variance-covariance matrix of $k$-period log asset returns, divided by the length of the horizon:

$$\frac{1}{k}\mathrm{Var}_t \begin{pmatrix} re_{t+k}^{(k)} \\ rb_{t+k}^{(k)} \end{pmatrix} = \frac{1}{k} \begin{bmatrix} \mathrm{Var}_t\left(re_{t+k}^{(k)}\right) & \mathrm{Cov}_t\left(re_{t+k}^{(k)}, rb_{t+k}^{(k)}\right) \\ \mathrm{Cov}_t\left(re_{t+k}^{(k)}, rb_{t+k}^{(k)}\right) & \mathrm{Var}_t\left(rb_{t+k}^{(k)}\right) \end{bmatrix}. \tag{33}$$

Here $re_{t+1}^{(k)}$ and $rb_{t+1}^{(k)}$ denote the $k$-horizon log return on equities and bonds, respectively. It is useful to note for future reference that $k$-period log returns obtain by adding one-period log returns over $k$ successive periods. For example,

$$re_{t+k}^{(k)} = re_{t+1} + ... + re_{t+k}, \tag{34}$$

where $re_{t+1} \equiv re_{t+1}^{(1)}$ is the one-period log return on equities.

When $k = 1$, equation (33) reduces to the variance-covariance matrix of one period returns, which is the standard measure of portfolio risk in mean-variance analysis. This variance-covariance matrix is given by:

$$\mathrm{Var}_t \begin{pmatrix} re_{t+1} \\ rb_{t+1} \end{pmatrix} = \mathrm{Var}_t \begin{pmatrix} v_{e,t+1} \\ v_{b,t+1} \end{pmatrix} = \begin{bmatrix} \sigma_e^2 & \sigma_{eb} \\ \sigma_{eb} & \sigma_b^2 \end{bmatrix}. \quad (35)$$

The first equality in equation (35) follows from the fact that a conditional variance is computed over deviations of the variable of interest with respect to its conditional mean; for next-period equity returns, this deviation is

$$re_{t+1} - \mathrm{E}_t[re_{t+1}] = v_{e,t+1}, \quad (36)$$

and likewise for $rb_{t+1}$. The second equality in equation (35) follows directly from equation (32).

When the investment horizon goes beyond one period, (33) does not generally reduce to (35) except in the special but important case where returns are not predictable. This in turn implies that the risk structure of returns varies across-investment horizons. To illustrate this, we explore the variances and covariances of two-period returns in detail, and note that the results for this case extend to longer horizons.

We start with the variance of two-period equity returns. Using (34), we can decompose this variance in terms of the variances and autocovariances of one-period equity returns as follows:

$$\begin{aligned} \frac{1}{2}\mathrm{Var}_t\left(re_{t+2}^{(2)}\right) &= \frac{1}{2}\mathrm{Var}_t\left(re_{t+1} + re_{t+2}\right) \\ &= \frac{1}{2}\mathrm{Var}_t\left(re_{t+1}\right) + \frac{1}{2}\mathrm{Var}_t\left(re_{t+2}\right) + \mathrm{Cov}_t\left(re_{t+1}, re_{t+2}\right) \end{aligned} \quad (37)$$

The decomposition of the variance of two-period bond returns is identical, except that we replace $re$ with $rb$. Thus the variance per period of two-period returns depends on the variance of one-period returns next period and two periods ahead, and on the serial covariance of returns.

Direct examination of equation (37) for the two-period case reveals under which conditions there are no horizon effects in risk: When the variance of single period returns is the same at all forecasting horizons, and when returns are not autocorrelated. This guarantees that the variance per period of $k$-period returns remains the same for all horizons.

It turns out these conditions do not hold when returns are predictable, so horizon matters for risk. This is easy to see in the context of the VAR(1) model given in (31). For this model we have that

$$re_{t+2} - \mathrm{E}_t[re_{t+2}] = v_{e,t+2} + \phi_{es} v_{s,t+1}. \tag{38}$$

A similar expression obtains for bond returns. Equation (38) says that, from today's perspective, future unexpected returns depend not only on their own shocks, but also on lagged shocks to the forecasting variable.

Plugging (36) and (38) into (37) and computing the moments we obtain an explicit expression for (37):

$$\frac{1}{2}\,\mathrm{Var}_t\left(re^{(2)}_{t+2}\right) = \frac{1}{2}\sigma_e^2 + \frac{1}{2}\left(\sigma_e^2 + \phi_{es}^2\sigma_s^2\right) + \phi_{es}\sigma_{es}. \tag{39}$$

In general, this expression does not reduce to $\mathrm{Var}_t\left(re_{t+1}\right) = \sigma_e^2$ unless returns are not predictable—i.e. unless $\phi_{es} = 0$.

The terms in equation (39) correspond one for one to those in equation (37). They show that return predictability has two effects on the variance of multiperiod returns. First, it increases the conditional variance of future single period returns, relative to the variance of next period returns, because future returns depend on past shocks to the forecasting variable. The second term of the expression captures this effect.

Second, it induces autocorrelation in single period returns, because future single period returns depend on past shocks to the forecasting variable, which in turn are contemporaneously correlated with returns. The third term of the equation captures this effect. The sign of this autocorrelation is equal to the sign of the product $\phi_{es}\sigma_{es}$. For example, when $s_t$ forecasts returns positively ($\phi_{es} > 0$), and the contemporaneous correlation between unexpected returns and the shocks to $s_{t+1}$ is negative ($\sigma_{es}$), return predictability generated by $s_t$ induces negative first-order autocovariance in one period returns. This is known as "mean-reversion" in returns.

These effects are present at any horizon. Equation (39) generalizes to the $k$-horizon case in the sense that the expression for $\mathrm{Var}_t(re^{(k)}_{t+k})/k$ is equal to $\sigma_e^2$ plus two terms: a term in $\sigma_s^2$ which is always positive, and a term in $\phi_{es}\sigma_{es}$ which may be positive or negative, depending on the sign of this product. It is possible to show that the marginal contribution of these two terms to the overall variance per period increases at rates proportional to $\phi_{ss}^{2k}/k$ and $\phi_{ss}^{k}/k$, respectively. Since $|\phi_{ss}| < 1$, this marginal contribution becomes negligible at long horizons, with the contribution of the term in $\sigma_s^2$ becoming negligible sooner. However it is important to note that the closer is $|\phi_{ss}|$ to one, that is, the more persistent is the forecasting variable, the longer it takes for its marginal contribution to the variance to disappear. Taken together, all of this implies that the variance per period of multiperiod returns, when plotted as a function of the horizon $k$, may be increasing, decreasing or hump-shaped at intermediate horizons, and it eventually converges to a constant—the unconditional variance of returns—at long enough horizons.

Similarly, we can decompose the covariance per period of two-period bond and equity returns as

$$\begin{aligned}\frac{1}{2}\,\mathrm{Cov}_t\left(re_{t+2}^{(2)}, rb_{t+2}^{(2)}\right) &= \frac{1}{2}\,\mathrm{Cov}_t\left(re_{t+1}+re_{t+2}, rb_{t+1}+rb_{t+2}\right)\\ &= \frac{1}{2}\,\mathrm{Cov}_t\left(re_{t+1}, rb_{t+1}\right)+\frac{1}{2}\,\mathrm{Cov}_t\left(re_{t+2}, rb_{t+2}\right)\\ &\quad+\frac{1}{2}\left[\mathrm{Cov}_t\left(re_{t+2}, rb_{t+1}\right)+\mathrm{Cov}_t\left(re_{t+1}, rb_{t+2}\right)\right],\end{aligned} \tag{40}$$

which, for the VAR(1) model (31), specializes to

$$\frac{1}{2}\,\mathrm{Cov}_t\left(re_{t+2}^{(2)}, rb_{t+2}^{(2)}\right)=\frac{1}{2}\sigma_{eb}+\frac{1}{2}\left(\sigma_{eb}+\phi_{es}\phi_{bs}\sigma_s^2\right)+\frac{1}{2}\left(\phi_{es}\sigma_{bs}+\phi_{bs}\sigma_{es}\right). \tag{41}$$

Equation (41) obtains after plugging (36) and (38) into (40) and computing the moments Once again, we find that the covariance per period of long-horizon returns does not equal the covariance of one-period returns ($\sigma_{eb}$) unless returns are not predictable—i.e., unless $\phi_{es} = 0$ and $\phi_{bs} = 0$.

Asset return predictability has two effects on the covariance of multiperiod returns. These effects are similar to those operating on the variance of multiperiod returns. The first one is the effect of past shocks to the forecasting variable on the contemporaneous covariance of future one-period stock and bond returns. This corresponds to the second term in (40) and (41). As equation (38) shows, a shock to $s_{t+1}$ feeds to future stock and bond returns through the dependence of stock and bond returns on $s_{t+1}$. If $s_{t+1}$ forecasts future stock and bond returns with the same sign (i.e., $\phi_{es}\phi_{bs} > 0$), a shock to $s_{t+1}$ implies that bond and stock returns will move in the same direction in future periods; if $s_{t+1}$ forecasts future stock and bond returns with different signs, a shock to $s_{t+1}$ implies that bond and stock returns will move in opposite directions in future periods. Thus, asset return predictability implies that $\text{Cov}_t\left(re_{t+2}, rb_{t+2}\right)$ may be larger or smaller than $\text{Cov}_t\left(re_{t+1}, rb_{t+1}\right)$, depending on the sign and magnitude of $\phi_{es}\phi_{bs}\sigma_s^2$.

Asset return predictability also induces cross autocorrelation in returns. Future stock (bond) returns are correlated with lagged bond (stock) returns through their dependence on past shocks to the forecasting variable, and their contemporaneous correlation with shocks to this variable. This effect is captured by the third term in equations (40) and (41). For example, suppose that $s_{t+1}$ forecasts negatively stock returns (i.e., $\phi_{es} < 0$) and that realized bond returns and shocks to $s_{t+1}$ are negatively correlated ($\sigma_{bs} < 0$). Then equity returns two period ahead will be positively correlated with bond returns one period ahead: A positive bond return one period ahead will tend to coincide with a negative shocks to $s_{t+1}$, which in turn forecasts a positive stock return two periods ahead.

These effects are present at any horizon, and can lead to different shapes of the covariance of multiperiod returns at intermediate horizons, as one effect reinforces the other, or offsets it. Of course, at long enough horizons, the covariance per period eventually converges to a constant—the unconditional covariance of returns.

# 6 Conclusion

This user guide has shown how to calculate the conditional means and covariances of returns, at any investment horizon, that are implied by a vector autoregression for returns and state variables. These means and covariances can be used in a number of ways. One straightforward application is to calculate the conditional mean-variance efficient frontier for any investment horizon. An alternative approach, which may have some appeal in practice, is to combine unconditional means with conditional second moments to find the portfolios that are mean-variance optimal for "average" market conditions. Such portfolios might be sensible choices for policy portfolios of long-term institutional investors.

An important open question is the relation between long-horizon mean-variance analysis and the intertemporal hedging analysis of Merton (1973), as implemented for example by Campbell, Chan, and Viceira (2003). The calculation of optimal intertemporal portfolios is notoriously difficult, and it would be good to know how well such portfolios are approximated by long-horizon mean-variance optimal portfolios. In future research we plan to explore this issue.

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