Long Paths and Cycles in Oriented Graphs

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ABSTRACT

We obtain several sufficient conditions on the degrees of an oriented graph for the existence of long paths and cycles. As corollaries of our results we deduce that a regular tournament contains an edge-disjoint Hamilton cycle and path, and that a regular bipartite tournament is hamiltonian.

An *oriented graph* is a directed simple graph, that is to say, a digraph without loops, mutiple arcs, or cycles of length two. Many authors have obtained various degree conditions which imply that certain families of graphs, or digraphs, contain long paths and cycles. The corresponding literature for oriented graphs, however, is concerned almost entirely with *tournaments*, which are oriented complete graphs. In this paper we consider the problem for other families of oriented graphs. We first work with oriented graphs of fixed minimum in-degree, but an arbitrary number of vertices, and then specialise by also fixing the minimum out-degree. We next obtain some hamiltonian conditions by demanding that the total number of vertices be relatively small. Finally, we prove a result concerning the lengths of cycles in oriented complete bipartite graphs. We give many conjectures throughout, which indicate that the majority of our results are far from being best possible.

All terms not explicitly defined in this paper may be found in [4]. We note, however, that we shall refer to spanning paths, and spanning cycles, as Hamilton paths, and Hamilton cycles, respectively. Let R be an oriented graph and S be a sugraph of R. Denote the set of vertices and arcs of S by $V(S)$ and $A(S)$, respectively. For $v \in V(R)$ and $B \subseteq V(R)$, define $N_B^-(v)$ and $N_B^+(v)$ to be the set of vertices of B which, respectively, dominate, and are dominated by, the vertex v. Put

$$N_B(v) = N_B^-(v) \cup N_B^+(v).$$

To simplify notation, we shall denote $N_{V(S)}^-(v)$, $N_{V(S)}^+(v)$, and $N_{V(S)}(v)$ by $N_S^-(v)$, $N_S^+(v)$, and $N_S(v)$. We shall refer to $|N_R^-(v)|$, $|N_R^+(v)|$, and

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$|N_R(v)|$ as the *in-degree*, the *out-degree*, and the *degree*, of v in R and denote them by $d_R^-(v)$, $d_R^+(v)$, and $d(v)$, respectively. The oriented graph R is said to be *k-diregular*, or more simply *diregular*, if $d_R^-(v) = k = d_R^+(v)$ for all $v \in V(R)$: and *m-disconnected* if, for any distinct pair of vertices u, $v \in V(R)$, there exist m internally disjoint paths from u to v. If R is 1-diconnected, we shall say simply that R is *diconnected*.

Lemma 1. Let $P = v_1 v \cdots v_n$ be a longest path in an oriented graph R. If $d_R^-(v_1) \geq 1$, then R contains a cycle of length at least $d_R^-(v_1) + 2$.

Proof. Since P is a longest path in $R, N_R^-(v_1) \subseteq V(P)$. Let $j = \max \{i \mid v_1 \in N_R^-(v_1)\}$ and put

$$C = v_1 v_2 \cdots v_j v_1.$$

Then $\{v_1, v_2\} \cup N_R^-(v_1) \subseteq V(C)$. Moreover, since R is an oriented graph, $\{v_1, v_2\} \cap N_R^-(v_1) = 0$, and hence

$$|V(C)| \geq d_R^-(v_1) + 2.$$

Lemma 2. Every oriented graph of minimum in-degree k, contains a cycle of length at least $k + 2$.

Proof. Immediate from Lemma 1.

Lemma 2 is, in a sense, best possible since there exist oriented graphs of minimum in-degree k, which contain no cycles of length greater than $k + 2$. To illustrate this we shall construct, recursively, a family of graphs, $\{R_k\}_{k \geq 1}$, with the following properties.

(1) R_k is diconnected.

(2) R_k has a distinguished vertex v_k such that $d_{R_k}^-(v_k) = k + 1$, and $d_{R_k}^-(v) = k$ for all $v \in V(R_k) \setminus \{v_k\}$.

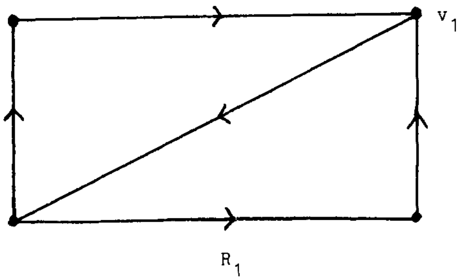

FIGURE 1

(3) All paths having initial vertex v_k, have length at most $k + 1$.
The graph R_1 with distinguished vertex v_1 is given below.
The graph R_k is constructed from $k + 1$ disjoint copies of R_{k-1} and a new distinguished vertex v_k by letting v_k be dominated by the vertex v_{k-1}, and dominate every other vertex, of each R_{k-1}. The fact that R_k contains no cycles of length greater than $k + 2$ follows from properties (1) and (3).

Noting that the graphs R_k are not 2-diconnected, and indeed have many vertices of out-degree one, we make the following conjectures.

Conjecture 1. Every 2-diconnected oriented graph of minimum in-degree k contains either a Hamilton cycle, or else a cycle of length at least $2k + 2$.

Conjecture 2. Every diconnected oriented graph of minimum in-degree and out-degree k, contains a cycle of length at least $2k + 1$.

Both conjectures would be, in a sense, best possible. The following example shows that the hypothesis of diconnectivity is necessary in Conjecture 2. Let $\tilde{R}_k$ be the oriented graph obtained by reversing all arcs in R_k. Then the oriented graph consisting of one copy of R_k and one copy of $\tilde{R}_k$, such that each vertex of R_k dominates every vertex of $\tilde{R}_k$, has minimum in-degree and out-degree k, but no cycles of length greater than $k + 2$.

Conjecture 2 is of some interest in connection with a result of Dirac, [2, Theorem 2], which implies that, if G is a graph of minimum degree $2k$, then G contains a cycle of length at least $2k + 1$. Conjecture 2 would imply that if G were oriented in such a way that the resulting oriented graph $R(G)$ was diconnected and each vertex had in-degree and out-degree at least k, then $R(G)$ would still contain a cycle of length at least $2k + 1$.

Some support for Conjecture 2 may be deduced from the following result.

Theorem 3. Let R be an oriented graph of minimum degree n, such that whenever $uv \notin A(R)$,

$$d_R^+(u) + d_R^-(v) \geq n - 1.$$

Then R contains a path of length n.

Proof. Let $P = v_1 v_2 \cdots v_m$ be a path of maximum length in R and suppose $m \leq n$. Since P is a longest path in $R, N_R^-(v_1) \subseteq V(P)$, and $N_R^+(v_m) \subseteq V(P)$. If $v_m v_1 \in A(R)$, then

$$P_1 = v_2 v_3 \cdots v_m v_1$$

is a path of maximum length in R, and hence $N_R^+(v_1) \subseteq V(P_1)$. If follows that

$$\{v_1\} \cup N_R^-(v_1) \cup N_R^+(v_1) \subseteq V(P),$$

and hence

$$m = |V(P)| \geq d_R(v_1) + 1 \geq n + 1.$$

We may thus assume that $v_m v_1 \notin A(R)$ and hence, by an hypothesis of the theorem

$$d_R^-(v_1) + d_R^+(v_m) \geq n - 1.$$

Let

$$j = \max\{i \mid v_i \in N_R^-(v_1)\},$$

and

$$h = \min\{i \mid v_i \in N_R^+(v_m)\}.$$

Then $j \geq d_R^-(v_1) + 2$, $h \leq m - (d_R^+(v_m) + 1)$, and hence

$$j - h \geq d_R^-(v_1) + d_R^+(v_m) + 3 - m \geq n + 2 - m \geq 2.$$

Suppose $h = 2$. The path

$$P_2 = v_{j+1}v_{j+2}\cdots v_m v_2 v_3 \cdots v_j v_1$$

is again of maximum length in R. As above, it follows that

$$\{v_1\} \cup N^-(v_1) \cup N^+(v_1) \subseteq V(P_2) = V(P),$$

and hence $|V(P)| \geq n + 1$. Thus we may assume that $h \geq 3$, and by a similar argument, that $j \leq m - 2$.
Put

$$\begin{aligned} B &= \{v_i \mid h < i < j\}, \\ D &= \{v_i \in B \mid v_{i-1} \in N_R^-(v_1)\}, \\ F &= \{v_i \in B \mid v_{i+1} \in N_R^+(v_m)\}. \end{aligned}$$

Then $|D| \geq d_R^-(v_1) - h + 1$, $|F| \geq d_R^+(v_m) - m + j$, and hence

$$|D| + |F| \geq d_R^-(v_1) + d_R^+(v_m) - m + j - h + 1.$$

However, $|D \cup F| \leq |B| = j - h - 1$. Thus,

$$|D \cap F| \geq d_R^-(v_1) + d_R^+(v_m) - m + 2 \geq n - 1 - m + 2 \geq 1.$$

Choose $v_i \in D \cap F$. Then

$$P_3 = v_i v_{i+1} \cdots v_m v_h v_{h+1} \cdots v_{i-1} v_1 v_2 \cdots v_{h-1}$$

is a path containing all the vertices of P. Thus P_3 is also a path of maximum length in R, and hence $N_R^-(v_i) \subseteq V(P_3) = V(P)$. Similarly,

$$P_4 = v_{j+1} v_{j+2} \cdots v_m v_{i+1} v_{i+2} \cdots v_j v_1 v_2 \cdots v_i$$

is a path of maximum length in R and hence,

$$N_R^+(v_i) \subseteq V(P_4) = V(P).$$

Thus $\{v_i\} \cup N_R^-(v_i) \cup N_R^+(v_i) \subseteq V(P)$, and hence

$$m = |V(P)| \geq d_R(v_i) + 1 = n + 1.$$

This contradicts the initial assumption that $m \leq n$, and completes the proof of the theorem. ∎

Theorem 3 has the following immediate corollary.

Corollary 3. Every oriented graph of minimum in-degree and out-degree k, contains a path of length $2k$.

Corollary 3 is, in a sense, best possible because of the existence of k-diregular tournaments on $2k + 1$ vertices. We have, however, been able to slightly improve the bound on the length of a longest path for the special case of diconnected oriented graphs. We use the following lemma.

Lemma 4. Let R be an oriented graph of minimum in-degree and out-degree k and let C be a longest cycle in R. If $R - C$ contains a longest path P, of length at least one, then

$$|V(C) \cup V(P)| \geq 2k + 3.$$

Proof. Let $C = c_1 c_2 \cdots c_n c_1$ and $P = v_1 v_2 \cdots v_m$. Without loss of generality we may assume that

$$| N_C^-(v_1) | \leq | N_C^+(v_m) | . \tag{1}$$

Let B be the set, and b the number, of distinct pairs (c_i, c_j) such that $c_i \in N_C^-(v_1)$, $c_j \in N_C^+(v_m)$, and

$$\{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\} \cap (N_C^-(v_1) \cup N_C^+(v_m)) = \emptyset.$$

If $(c_i, c_j) \in B$ then, since C is a longest cycle of R,

$$| \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\} | \geq m.$$

Putting $D = \{c_i | c_{i+1} \in N_C^+(v_m)\}$ it follows that $N_C^-(v_1) \cap D = \emptyset$, and

$$| V(C) | \geq | N_C^-(v_1) | + | D | + b(m - 1).$$

However, $| N_C^-(v_1)| \geq k - m + 2$ and $| D| = | N_C^+(v_m)| \geq k - m + 2$. Thus, if $b \geq 1$,

$$| V(C) \cup V(P) | \geq k - m + 2 + k - m + 2 + m - 1 + m = 2k + 3.$$

The only remaining alternative is that $b = 0$. Using (1), it follows that $| N_C^-(v_1)| \leq 1$. Hence $| N_P^-(v_1)| \geq k - 1$, and

$$| V(P) | \geq | N_P^-(v_1) | + 2 = k + 1.$$

By Lemma 2, $| V(C)| \geq k + 2$, and hence

$$| V(C) \cup V(P) | \geq 2k + 3.$$

Theorem 5. let R be a diconnected oriented graph of minimum in-degree and out-degree $k \geq 2$. Then R contains either, a Hamilton path, or else a path of length $2k + 2$.

Proof. Suppose the theorem is false. Let R be an oriented graph which satisfies the hypotheses of the theorem, but not the conclusion. Choose a longest cycle C in R, and a longest path P in $R - C$. The proof splits into two cases, depending on the length of P.

Case 1. P has length at least one.

If $| N_C^-(v_1)| \geq 1$, then R clearly contains a path P_1 such that

$$V(C) \cup V(P) \subseteq V(P_1)$$

and, by Lemma 4, $|V(P_1)| \geq 2k+3$.

If, on the other hand, $|N_C^-(v_1)| = 0$, then $|N_P^-(v_1)| \geq k$ and, by Lemma 1, $R - C$ contains a cycle C_1 of length at least $k+2$. Since R is diconnected, it contains a path P_2 such that

$$V(C) \cup V(C_1) \subseteq V(P_2).$$

By Lemma 2, C also has length at least $k+2$ and hence, $|V(P_2)| \geq 2k+4$.

Case 2. $R - C$ consists entirely of isolated vertices.

Choosing v a vertex of $R - C$, it follows that $N_R(v) \subseteq V(C)$. Moreover $N_R(v) \neq V(C)$ for otherwise R would contain a longer cycle than C. Hence $|V(C)| \geq d_R(v) + 1 \geq 2k+1$. Since, however, v and $V(C)$ lie on a path in R, and R contains no path of length $2k+2$, $|V(C)| = 2k+1$. Again, since C is a longest cycle of R, we may label the vertices of C such that

$$C = c_0 c_1 c_2 \cdots c_{2k} c_0,$$
$$N_R^+(v) = \{c_1, c_2, \cdots c_k\}, \text{ and}$$
$$N_R^-(v) = \{c_{k+1}, c_{k+2}, \cdots, c_{2k}\}.$$

By considering the longest cycle $C_2 = vc_1c_2 \cdots c_{2k}v$ of R, and the vertex c_0 of $R - C_2$, it follows that

$$N_R^+(c_0) = \{c_1, c_2, \cdots, c_k\},$$
$$N_R^-(c_0) = \{c_{k+1}, c_{k+2}, \cdots, c_{2k}\}.$$

Put $B = N_R^+(c_0) \setminus \{c_k\}$ and $D = N_R^-(c_0) \setminus \{c_{k+1}\}$.
Clearly, some vertex c_i of B satisfies

$$|N_B^+(c_i)| \leq \tfrac{1}{2}(|B| - 1) = \tfrac{1}{2}(k-2).$$

If c_i dominates a vertex v' of $R - C$ then

$$P_3 = vc_{i+1}c_{i+2} \cdots c_i v'$$

is a path of length $2k+2$ in R. Moreover, if c_i dominates a vertex c_j of D then

$$C_3 = vc_1c_2\cdots c_ic_jc_{j+1}\cdots c_0c_{i+1}\cdots c_{j-1}v$$

is a longer cycle than C. It follows that

$$N_R^+(c_i) \subsetneq B \cup \{c_k, c_{k+1}\}. \tag{2}$$

Hence $d_R^+(c_i) \le \frac{1}{2}(k-2)+2$. Since $d_R^+(c_i) \ge k \ge 2$, we may deduce that, $k = 2$, $C = c_0c_1c_2c_3c_4c_0$, $N_R^+(v_0) = N_R^+(c_0) = \{c_1, c_2\}$, and $N_R^-(v) = N_R^-(c_0) = \{c_3, c_4\}$. In addition, we must have equality in (2). Hence $N_R^+(c_1) = \{c_2, c_3\}$, and, by a similar argument to the above, $N_R^+(c_4) = \{c_2, c_3\}$.

In this final case, the cycle

$$C_4 = vc_1c_3c_0c_2c_4v$$

contradicts the choice of C and completes the proof of the theorem. We feel that Theorem 5 is far from being best possible.

Conjecture 3. Every diconnected oriented graph of minimum in-degree and out-degree k contains either, a Hamilton path, or else a path of length at least $3k$.

Consider the oriented graph $R(b_1, b_2,\cdots, b_n)$ defined as follows. The vertices of $R(b_1, b_2,\cdots, b_n)$ may be partitioned into n independent sets B_1, $B_2,\cdots, B_n$ such that $|B_i| = b_i$ and each vertex of B_i dominates every *vertex of* B_{i+1} for all i, $1 \le i \le n$, where subscripts are to be read modulo n. The graphs $R(k, k, h)$, for h an integer greater than k, illustrate that Conjecture 3 is, in a sense, best possible. Moreover, the graph $R(1, 1, h)$ shows that the conclusion of Theorem 5 is false when $k = 1$.

Bermond, Germa, Heydemann, and Sotteau [1] have obtained a result, similar to Theorem 5, for the more general family of digraphs, by showing that every diconnected digraph of minimum in-degree and out-degree k contains either, a Hamilton path, or else a path of length $2k$. Theorem 4 is also of interest in connection with a conjecture of P. Kelly [4, p. 7, Problem 9], that every diregular tournament is decomposable into Hamilton cycles. Given a k-diregular tournament T, on $2k+1$ vertices, it follows from a result of Thomassen [6, Theorem 4] that T contains c^k Hamilton cycles for some constant $c > 1$. For any Hamilton cycle H_1 of T, $T - E(H_1)$ is a $(k-1)$-diregular diconnected oriented graph on $2k+1$ vertices, and hence, by Theorem 4, contains a Hamilton path H_2^*. We may thus deduce that T contains c^k Hamilton pairs (H_1, H_2^*), where H_1 is a Hamilton cycle and H_2^* a Hamilton path. We had hoped to deduce that T contains c^k Hamilton pairs (H_1, H_2), where H_1 and H_2 are both Hamilton cycles of T, by showing that every $(k-1)$-diregular oriented graph on $2k+1$ vertices is hamiltonian. Our only success to date, however, is the following result.

Theorem 6. Every oriented graph of minimum in-degree and out-degree $k \geq 2$, on at most $2k + 2$ vertices, is hamiltonian.

Proof. Suppose the theorem is false. Let R be a nonhamiltonian graph which satisfies the hypotheses of the theorem, and choose a longest cycle C in R. Using Lemma 4, it follows that $R - C$ contains an isolated vertex v. By a similar argument to that used in the proof of Theorem 5, case 2, we may deduce that R contains a cycle whose vertex set is $V(C) \cup \{v\}$. This contradicts the choice of C and completes the proof of the theorem.

The following conjecture of Thomassen [7] suggest that Theorem 6 is far from being best possible.

Conjecture 4 (Thomassen). Every oriented graph of minimum in-degree and out-degree k, on at most $3k$ vertices, is hamiltonian.

The graph $R(k, k, k + 1)$ illustrates that Conjecture 4 would be, in a sense, best possible. The graph $R(1, 1, 2)$ also shows that the conclusion of Theorem 6 is false when $k = 1$. The reason that the graphs $R(k, k, k + 1)$ are nonhamiltonian is essentially that they are not diregular. For the special case of diregular oriented graphs, perhaps even the following is true.

Conjecture 5. For $k \neq 2$, every k-diregular oriented graph on at most $4k + 1$ vertices is hamiltonian.

Examples of nonhamiltonian, 2-diregular oriented graphs on seven and eight vertices are given in [3]. It follows from a result of Nash-Williams [5, Theorem 3] that, if G is a $2k$-regular graph on at most $4k + 1$ vertices, then G is hamiltonian. Conjecture 5 would imply that if G were given a k-diregular orientation, then it would remain hamiltonian for $k \neq 2$. We note that Conjectures 4 and 5 would also imply that a k-diregular tournament contained, respectively $[\frac{1}{3}(k + 2)]$, and $[\frac{1}{2}k]$, edge-disjoint Hamilton cycles.

Following [4], we define an oriented complete bipartite graph to be a *bipartite tournament*. Using the following lemma, we obtain a bound on the length of a longest cycle in a bipartite tournament.

Lemma 7. If C is a longest cycle in a diconnected bipartite tournament T, then $T - C$ is acyclic.

Proof. Let $V(T) = X \cup Y$ be the bipartition of T and

$$C = x_1 y_1 x_2 y \cdots x_n y_n x_1,$$

where $x_i \in X$ and $y_i \in Y$ for all i, $1 \leq i \leq n$. Suppose $T - C$ contains a cycle C'. Choose $x \in V(C') \cap X, y \in V(C') \cap Y$ and, without loss of generality, assume that x_1 dominates y. Since C' contains a yx-path, and C is a longest cycle of T, x does not dominate y_1. Hence y_1 dominates x. Similarly, since C'

contains an xy-path, y does not dominate x_2, and hence x_2 dominates y. By repeating the above argument, we may deduce that

(i) each vertex of $V(C) \cap X$ dominates every vertex of $V(C') \cap Y$ and

(ii) each vertex of $V(C) \cap Y$ dominates every vertex of $V(C') \cap X$.

Since T is dicönnected, however, T contains a path which passes from $V(C')$ to $V(C)$, and is internally disjoint from $V(C) \cup V(C')$. using (i) and (ii), it easily follows that T contains a longer cycle than C, which contradicts the hypothesis of the lemma, that C is a longest cycle in T. Thus the assumption that $T - C$ contains a cycle is false.

Theorem 8. Let T be a diconnected bipartite tournament such that whenever u and v are vertices of T and $uv \notin A(T)$,

$$d_T^+(u) + d_T^-(v) \geq n.$$

Then T contains a cycle of length at least $2n$.

Proof. Let $V(T) = X \cup Y$ be the bipartition of T and choose x and x' vertices of X. Since neither xx' nor $x'x$ are arcs of T, it follows that

$$2n \leq d_T^+(x) + d_T^-(x') + d_T^+(x') + d_T^-(x) = d_T(x) + d_T(x') = 2\,|\,Y\,|\,.$$

Hence $|\,Y| \geq n$, and similarly $|\,X| \geq n$.

Let $C = x_1y_1x_2y_2\cdots x_my_mx_1$ be a longest cycle of T, and P be a longest path of $T - C$. If P consists of a single vertex y, then $N_T(y) \subseteq V(C)$. Since, however, $N_T(y)$ is equal to either X or Y, it follows that C has length at least $2n$. Hence we may suppose that P is a uv-path for distinct vertices u and v of T. Since, by Lemma 7, $T - C$ is acyclic, it follows that $N_T^-(u) \cap V(P) = 0$ and $N_T^+(v) \cap V(P) = 0$. In particular, $vu \notin A(T)$ and hence

$$d_T^+(u) + d_T^-(v) \geq n.$$

Without loss of generality assume that $u \in X$.

$$\text{If } v \in Y, \text{ put } B = \{y_i \mid x_{i+1} \in N_T^+(v)\}.$$

$$\text{If } v \in X, \text{ put } B = \{y_i \mid y_{i+1} \in N_T^+(v)\}.$$

In both cases, since C is a longest cycle of T,

$$N_C^-(u) \cap B = \emptyset.$$

Moreover, $N_C^-(u) \cup B \subseteq V(C) \cap Y$.
Hence,

$$|V(C)| \geq 2(|N_C^-(u)| + |B|) = 2(d_T^-(u) + d_T^+(v)) \geq 2n.$$

Corollary 8.1. Every diconnected bipartite tournament of minimum in-degree h and minimum out-degree k contains a cycle of length at least $2(h + k)$.

Proof. Immediate.

Corollary 8.2. Every diregular bipartite tournament is hamiltonian.

Proof. Follows immediately from Corollary 8.1, since a k-diregular bipartite tournament has exactly $4k$ vertices.

Corollary 8.1 is, in a sense, best possible since, for m large, we may orient the complete bipartite graph $K_{h+k,m}$ to form a diconnected bipartite tournament, $T(K_{h+k,m})$, of minimum in-degree h and minimum out-degree k. Although the graphs $T(K_{h+k,m})$ contain a cycle which spans one vertex set of the bipartition, we note that there exist infinitely many diconnected bipartite tournaments of minimum in-degree h and minimum out-degree k, whose longest cycle spans neither vertex set of the bipartition, and has length at most $2(h + k + 1)$. This is most easily illustrated for the case $h = k$ by the graphs $R(k, l, m, n,)$, for l, m, and n integers greater than or equal to k; where, in fact, the longest cycle has length $4k$.

We conclude by suggesting that Kelly's conjecture remains valid for diregular bipartite tournaments.

Conjecture 6. Every diregular bipartite tournament is decomposable into Hamilton cycles.

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Note added to proof: We have recently been informed of several further results on oriented graphs.

(1) M. C. Heydemann has proved the following theorems.

Theorem (Heydemann). Let R be a diconnected oriented graph on n vertices such that whenever $uv \notin A(R)$ and $vu \notin A(R)$,

$$d(u) + d(v) \geq 2n - 2h - 1,$$

for $1 < h < n - 1$. Then R contains a cycle of length at least $\lfloor (n-1)/h \rfloor + 1$ and a path of length at least $\lfloor (n-1)/h \rfloor + \lfloor (n-2)/h \rfloor$.

For n and k positive integers, let r be the least non-negative residue of $(n - 1)$ modulo $(k - 2)$, and put

$$f(n, k) = \frac{n^2(k - 3) + 2n - k + 1 - r(k - 2 - r)}{2(k - 2)}.$$

Theorem (Heydemann). Every diconnected oriented graph on n vertices and more than $f(n, k)$ arcs contains a cycle of length at least k.

(2) J. Ayel has generalized Lemma 7 to diconnected m-partite tournaments (oriented complete m-partite graphs). Using this result she has verified Conjecture 2 for the special case of diconnected m-partite tournaments of minimum in-degree and out-degree k.

(3) Lowell W. Beineke and Charles H. C. Little have proved the following.

Theorem. Every hamiltonian bipartite tournament either contains cycles of all possible even lengths, or else is isomorphic to $R(k, k, k, k)$ for some $k \geq 1$. Using this result, it follows that if a bipartite tournament T satisfies the hypotheses of Theorem 8 or its corollaries, then T contains cycles of many different lengths.

(4) Carsten Thomassen has generalized theorem 6 by proving that every oriented graph on n vertices with each in-degree and out-degree at least $n/2 - (n/1000)^{1/2}$ is hamiltonian.

(5) Roland Häggkvist has shown that conjectures 3 and 4 are false by constructing an oriented graph on $8t + 4$ vertices, which does not contain a hamilton path, and each in-degree and out-degree is at least $3t$.

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