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Long Range Scattering for Non-Linear Schrödinger and Hartree Equations in Space Dimension $n \geq 2$

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Abstract. We consider the scattering problem for the non-linear Schrödinger (NLS) equation with a power interaction with critical power $p = 1+2/n$ in space dimensions $n = 2$ and 3 and for the Hartree equation with potential $|x|^{-1}$ in space dimension $n \geq 2$. We prove the existence of modified wave operators in the L^2 sense on a dense set of small and sufficiently regular asymptotic states.

1. Introduction

This paper is devoted to the study of the asymptotic behaviour in time of the solutions of the non-linear Schrödinger (NLS) equation and of the Hartree equation

$$i\partial_t u = -(1/2)\Delta u + f(u) \tag{1.1}$$

in the Coulomb like limiting case, in space dimension $n \geq 2$. The non-linear interaction term is

$$f(u) = \lambda |u|^{p-1} u \tag{1.2}$$

with $p - 1 = 2/n$ in the NLS case and

$$f(u) = (V * |u|^2)u = \lambda(|x|^{-1} * |u|^2)u \tag{1.3}$$

in the Hartree case with Coulomb potential $V(x) = \lambda|x|^{-1}$. Here u is a complex function defined in $n + 1$ dimensional space-time, ∂_t denotes the time derivative, Δ denotes the Laplace operator in $\mathbb{R}^n$, and λ is a real constant which matters only through its sign. This paper is the continuation of a previous paper by one of us [17] where the same problem was considered for the NLS equation (1.1), (1.2) in space dimension $n = 1$. We refer to the introduction of [17] for general information on the problem in the 1-dimensional case.

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There is a large amount of literature on the theory of scattering for the equation (1.1), (1.2) and/or (1.3) and we shall briefly summarize some of the available results in order to put the present paper into perspective. We refer to [1–11, 13, 14, 16–23] and references therein quoted for more details. We shall need the weighted Sobolev spaces $H^{m,s}$ defined for $m, s \in \mathbb{R}$ by

$$H^{m,s} = \{v \in \mathscr{S}' : \|v; H^{m,s}\| = \|(1+x^2)^{s/2}(1-\Delta)^{m/2}v\|_2 < \infty\},$$

where $\|\cdot\|_r$ denotes the norm in $L^r \equiv L^r(\mathbb{R}^n)$, and the unitary group associated with the free Schrödinger equation

$$U(t) = \exp[i(t/2)\Delta].$$

The Cauchy problem for the equation (1.1), (1.2) is well understood. In particular the following is known for $n \geq 2$:
(i) Let $0 \leq p-1 < 4/(n-2)$ and $\lambda \geq 0$ if $p-1 \geq 4/n$. Then for any initial data $u(0) \in H^1 \equiv H^{1,0}$ [resp. $u(0) \in \Sigma \equiv H^{0,1} \cap H^{1,0}$], there exists a unique global solution $u \in \mathscr{C}(\mathbb{R}, H^1)$ [resp. $u \in \mathscr{C}(\mathbb{R}, \Sigma)$] [5].
(ii) Let $0 \leq p-1 < 4/n$. Then for any initial data $u(0) \in L^2$, there exists a unique global solution $u \in \mathscr{C}(\mathbb{R}, L^2)$ [21, 22].

The theory of scattering for the equation (1.1), (1.2) (among others) is organized around the following two questions:
(i) Let $v(t) = U(t)u_+$ be a solution of the free Schrödinger equation. Does there exist a solution u of the full equation (1.1) which behaves asymptotically as $v(t)$ when $t \to \infty$, typically in the sense that for X a suitable Banach space

$$\|u(t) - v(t); X\| \to 0 \quad \text{when} \quad t \to \infty. \tag{1.4}$$

This may occur in favourable cases for all $u_+ \in X$, in less favourable cases only for $u_+ \in Y$, where Y is some dense subspace of X. If (1.4) holds, the map $\Omega_+ : u_+ \to u(0)$ is called the wave operator (for positive time) in the sense of the space X. The problem of the existence of u for given u_+ is referred to as the problem of existence of the wave operator. The same problem can be considered for negative times. We once for all restrict our attention to positive times.
(ii) For an "arbitrary" solution u of the full equation (1.1), does there exist an asymptotic state u_+ such that $v(t) = U(t)u_+$ behaves asymptotically as $u(t)$ when $t \to \infty$, typically in the sense that (1.4) holds. If that is the case for any u with initial data $u(0) \in X$ for some $u_+ \in X$, one says that asymptotic completeness holds in X (for positive time).

We now summarize some of the available results.
(i) For $4/n < p-1 < 4/(n-2)$, the wave operators in the H^1 sense are defined at least in a neighborhood of zero in H^1, and if $\lambda \geq 0$, in the whole of H^1. Asymptotic completeness in H^1 holds for $\lambda \geq 0$ [7].
(ii) Let $p_0(n)$ be the positive root of the equation

$$np(p-1) = 2(p+1)$$

[note that $2/n \leq 4/(n+2) < p_0(n) - 1 < 4/n$ for $n \geq 2$]. Then for $4/(n+2) < p-1 < 4/(n-2)$ the wave operators in the Σ sense are defined at least in a neighborhood of zero in Σ, and if either $p-1 < 4/n$ or $\lambda \geq 0$, in the whole of Σ. Actually in that case (1.4) has to be replaced by

$$\|U(-t)u(t) - u_+; \Sigma\| \to 0 \quad \text{when} \quad t \to \infty. \tag{1.4'}$$

Asymptotic completeness in Σ holds if $p \geq p_0(n)$ and $\lambda \geq 0$ [3, 5, 11, 20, 21].
(iii) For $2/n \leq p - 1 \leqq p_0(n) - 1$ and $\lambda \geq 0$, the inverses of the wave operators exist in the L^2 sense on the dense subspace Σ of asymptotic states [23].
(iv) For $0 < p - 1 \leq 2/n$, the wave operators do not exist in the L^2 sense, namely for any $u_+ \in L^2$, (1.4) with $X = L^2$ implies $u = v = 0$ [1, 4, 18].

The limiting case $p - 1 = 2/n$ is the analogue in the present situation of the case of the linear Schrödinger equation with Coulomb potential $V(x) = \lambda|x|^{-1}$ and it is known in that case that the free dynamics $v(t) = U(t)u_+$ is not the correct asymptotic dynamics and has to be replaced by a modified asymptotic dynamics.

The situation for the Hartree equation is similar to that for the NLS equation, the correspondence being roughly that a potential $V = \lambda|x|^{-\gamma}$ in (1.3) corresponds to $p - 1 = 2\gamma/n$ in (1.2). In particular the limiting Coulomb case for the Hartree equation corresponds to the Coulomb potential $V = \lambda|x|^{-1}$ as given in (1.3) [6, 10, 16].

The purpose of the present paper is to prove the existence of modified wave operators in the L^2 sense on a dense subspace of small asymptotic states for the NLS equation (1.1), (1.2) with $p - 1 = 2/n$ in space dimension $n = 2$ and $n = 3$, and for the Hartree equation (1.1), (1.3) in arbitrary space dimension $n \geq 2$. More precisely, for any asymptotic state u_+ in a suitable dense subspace Y of L^2, we shall construct several modified asymptotic dynamics $v(t)$, and for any such u_+ which is small in a suitable sense, we shall prove the existence of a solution $u \in \mathscr{C}(\mathbb{R}, L^2)$ of the full equation (1.1) such that (1.4) holds with $X = L^2$. Actually we shall obtain stronger properties than (1.4) on the comparison between u and v. In particular the left-hand side of (1.4) will exhibit a power law decay as a function of t. The space Y will turn out to be $H^{0,2}$ in most cases, except for the NLS case in space dimension 3 where a smaller space is needed. The precise results appear below as Theorem 3.1 for the NLS equation and Theorem 4.1 for the Hartree equation. A standard property of the wave operators, whether modified or not, is to intertwine the full evolution and the free evolution. That property also holds in the present situation, at least on a suitable subset of asymptotic states. The choice of the modified asymptotic dynamics has been discussed at some length in the introduction of [17]. That discussion will be amplified in Sect. 2 below. The method of proof of the main result will be similar to that in [17] and will mainly consist in recasting the Cauchy problem for (1.1) with prescribed asymptotic state u_+ in the form of an integral equation for the difference $w = u - v$ and solving that equation by a contraction method for large times.

In order to focus our attention on the main issue, we have restricted our study to the case where f is a single power interaction with critical power for the NLS equation and where the potential is $V = \lambda|x|^{-1}$ for the Hartree equation. There is no difficulty in extending the present results to more general interactions, for instance by adding a short range interaction of the type (1.2) with a larger power $p - 1 > 2/n$ in the NLS case or by adding to V a short range potential in the Hartree case. Such an extension has been considered in [17] in the NLS case in space dimension $n = 1$. We have also restricted our attention to the study of the wave operators in the L^2 sense, namely to the situation where $X = L^2$ in (1.4). One could also ask for similar results in a stronger sense, namely with $X = H^k$ for some $k \geq 1$, for asymptotic states in a smaller space $Y = H^{m,s}$ for suitable m and s. This question has been considered in [17] in the NLS case in space dimension $n = 1$ with $X = H^1$ and $Y = H^{0,3} \cap H^{1,2}$. A similar extension for $n \geq 2$ is possible in the Hartree case at the cost of additional estimates, since the interaction term is a polynomial in u and

$\bar{u}$. It is much less obvious that a similar extension can be made in the NLS case for $n \geq 2$. Actually for $n = 3$, where $f(u)$ is a fractional power, there arises already some difficulty due to the lack of smoothness of $f(u)$ near zero at the level of the L^2 wave operators (see especially the comments preceding Lemma 3.2 below).

This paper is organized as follows. Section 2 is devoted to preliminaries. We first recall some basic properties of the free Schrödinger equation and then discuss and motivate the choice of asymptotic dynamics, starting with the linear case with time dependent potential and ending with the non-linear case. We propose three modified asymptotic dynamics [see (2.27), (2.28), and (2.34)]. Section 3 is devoted to the study of the NLS equation. We first derive a general existence and uniqueness result using only two basic estimates on the asymptotic dynamics (Proposition 3.1), we then prove that one of the possible asymptotic dynamics satisfies those estimates in dimensions $n = 2$ (Lemma 3.1) and $n = 3$ (Lemma 3.2), we establish some comparison results between the various asymptotic dynamics (Lemma 3.3 and 3.4) and we finally derive the main result (Theorem 3.1) and the intertwining property (Proposition 3.2). Section 4 is devoted to the study of the Hartree equation and follows the same pattern. We first derive a general existence and uniqueness result based on two estimates (Proposition 4.1), we show that those estimates are satisfied by one of the asymptotic dynamics (Lemma 4.1) and we derive the main result (Theorem 4.1) and the intertwining property (Proposition 4.2).

We conclude this introduction by giving some additional notation freely used in this paper. For any r, $1 \leq r \leq \infty$, we denote by $\bar{r}$ the conjugate exponent defined by $1/r + 1/\bar{r} = 1$ and we define the variable $\delta(r) = n/2 - n/r$. The argument r will often be omitted in δ when no confusion can arise. For any interval $I \subset \mathbb{R}$, for any Banach space B, for any q, $1 \leq q \leq \infty$, we denote by $L^q(I, B)$ [respectively $L^q_{\text{loc}}(I, B)$] the space of measurable functions v from I to B such that $\|v(\cdot); B\| \in L^q(I)$ [resp. $\|v(\cdot); B\| \in L^q_{\text{loc}}(I)$]. Finally we shall use the notation $\varrho_+ = \text{Max}(\varrho, 0)$ for $\varrho \in \mathbb{R}$.

2. Preliminaries

In this section we recall some elementary and/or well known facts on the Schrödinger equation, which will be used in the following sections. We begin with the free Schrödinger equation

$$(i\partial_t - H_0)u \equiv (i\partial_t + (1/2)\Delta)u = 0 \tag{2.1}$$

which is solved by the use of the unitary group $U(t) = \exp[i(t/2)\Delta]$. That group can be written as

$$U(t) = M(t)D(t)FM(t)\,, \tag{2.2}$$

where

$$M(t) = \exp(ix^2/2t)\,, \tag{2.3}$$

$$D(t)\psi(x) = (it)^{-n/2}\psi(x/t)\ , \tag{2.4}$$

and F is the Fourier transform. We shall restrict our attention to positive t and the variable t will often be omitted when no confusion can arise. The following commutations, if not well known, are obtained by elementary computations

$$UxU^{-1} = x + it\nabla = Mit\nabla M^{-1}\,, \tag{2.5}$$

$$(x/t)UM^{-1} = UM^{-1}(-i\nabla),$$

$$-i\nabla MU^{-1} = MU^{-1}(x/t)\,, \tag{2.6}$$

$$\begin{aligned}(i\partial_t + (1/2)\Delta)U &\equiv (i\partial_t + (1/2)\Delta)MDFM \\ &= M(i\partial_t + (1/2)\Delta + i(n/2t) + i(x/t)\cdot\nabla)DFM \\ &= MD(i\partial_t + (1/2t^2)\Delta)FM \\ &= MDF(i\partial_t - x^2/2t^2)M \\ &\equiv UM^{-1}(i\partial_t - x^2/2t^2)M = Ui\partial_t \end{aligned} \tag{2.7}$$

(we use the same notation for functions of x and for the associated multiplication operators in L^2). An immediate consequence of those relations and of the Sobolev inequality

$$\|u\|_r \le C_r \|(-\Delta)^{\delta(r)/2} u\|_2 \,, \tag{2.8}$$

where $2 \le r < \infty$ is the following well known decay estimate.

Lemma 2.1. *Let* $2 \le r < \infty$ *and* $\delta = \delta(r)$. *Then*

$$\|U(t)u\|_r \le C_r |t|^{-\delta} \| \, |x|^\delta u\|_2 \,. \tag{2.9}$$

Proof.

$$\begin{aligned}\|Uu\|_r &= \|M^{-1}Uu\|_r \le C_r\|(-\Delta)^{\delta/2}M^{-1}Uu\|_2 \\ &= C_r\|M^{-1}Ut^{-\delta}|x|^\delta u\|_2 = C_r|t|^{-\delta}\|\,|x|^\delta u\|_2 \,. \quad \text{Q.E.D.}\end{aligned}$$

We shall make an essential use of the following well known estimates on U [7, 13, 24].

Lemma 2.2. (1) *Let* r *and* q *satisfy* $0 \le 2/q = \delta(r) < 1$ *and let* $u \in L^2$. *Then* $U(\cdot)u \in L^q(\mathbb{R}, L^r)$ *and*

$$\|U(\cdot)u; L^q(\mathbb{R}, L^r)\| \le C\|u\|_2 \,. \tag{2.10}$$

(2) *Let* r, q *and* r', q' *satisfy* $0 \le 2/q = \delta(r) < 1$, $0 \le 2/q' = \delta(r') < 1$. *Then for any interval* $I \subset \mathbb{R}$ *and any* $s \in I$, *the operator*

$$f \to \int_s^t d\tau U(t-\tau)f(\tau)$$

is bounded from $L^{\bar{q}'}(I, L^{\bar{r}'})$ *to* $L^q(I, L^r)$ *with norm uniformly bounded with respect to* I *and* s.

We now present some elementary remarks on the choice of a modified asymptotic dynamics for linear Schrödinger equations

$$(i\partial_t - H(t))u \equiv (i\partial_t - H_0 - V)u \,, \tag{2.11}$$

where $V = V(t, x)$ is a long range time dependent potential, multiplicative in the x variable. Those remarks are of a less general character than the treatment given in [12, 15] since they are concerned only with the special case $H_0 = -(1/2)\Delta$ and with the first order modification of the asymptotic dynamics, but they are useful to understand the non-linear case. Let $U(t, s)$ be the unitary two parameters group solving (2.11) in the sense that

$$\begin{cases} i\partial_t U(t,s) = H(t)U(t,s) \,, \\ i\partial_s U(t,s) = -U(t,s)H(s) \,. \end{cases} \tag{2.12}$$

If one wants to prove the existence of (possibly modified) wave operators in L^2 by the method of Cook, one has to show that

$$i\partial_t U(0,t)U_1(t) \equiv U(0,t)(i\partial_t - H(t))U_1(t)u_+ \in L^1(\mathbb{R}^+, L^2), \tag{2.13}$$

where $u_+ \in L^2$ is the asymptotic state and $U_1(t)$ is the asymptotic dynamics. A simple choice for U_1 consists in taking

$$U_1(t) = U(t)\exp[-iS] = \exp[-itH_0 - iS] \tag{2.14}$$

for a suitable choice of $S \equiv S(t, -i\nabla)$, so that U_1 is simply a multiplication operator in Fourier transformed variables. With that choice, one obtains by using (2.6)

$$\begin{aligned}(i\partial_t - H(t))U_1(t)u_+ &= (\partial_t S(-i\nabla) - V(x))U_1(t)u_+ \\ &= U(t)(\partial_t S(-i\nabla) - M^{-1}V(-it\nabla)M)\exp[-iS]u_+ \,. \end{aligned} \tag{2.15}$$

Since the operator M tends strongly to $\mathbb{1}$ in L^2 when $t \to \infty$, one is tempted to define S by

$$\partial_t S(t,\xi) = V(t, t\xi) \tag{2.16}$$

which is the original choice made by Dollard (see [12]). The factors M^{-1}. M however prevent one from exploiting the cancellation in a direct way. This suggests to introduce a second asymptotic dynamics

$$U_2(t) = U(t)M(t)^{-1}\exp[-iS], \tag{2.17}$$

a choice that is also naturally suggested by the commutation relations (2.7). With that choice, one obtains

$$\begin{aligned}&(i\partial_t - H(t))U_2(t)u_+ \\ &= U(t)\{M^{-1}(\partial_t S(-i\nabla) - V(-it\nabla)) + i\partial_t M^{-1}\}\exp[-iS]u_+ \end{aligned} \tag{2.18}$$

and the cancellation appears explicitly, at the cost of introducing an additional term $\partial_t M^{-1}$. In order to complete the proof of (2.13) with U_1 replaced by U_2, it is then sufficient to prove that

$$\|(\partial_t S(-i\nabla) - V(-it\nabla))u_+\|_2 \in L^1(\mathbb{R}^+) \tag{2.19}$$

and

$$\|(\partial_t M^{-1})\exp[-iS]u_+\|_2 \equiv (1/2)t^{-2}\|x^2\exp[-iS]u_+\|_2 \in L^1(\mathbb{R}^+). \tag{2.20}$$

In the best cases (but not always) one can make the choice (2.16) for S thereby making (2.19) trivial.

The dynamics U_2 can be computed in a very explicit way by using (2.2) and (2.6). In fact let $v_2(t) = U_2(t)u_+$. Then

$$\begin{aligned} v_2(t) &= UM^{-1}\exp[-iS]u_+ \\ &= \exp[-iS(t,x/t)]MDFu_+ \\ &= (it)^{-n/2}\exp[-iS(t,x/t) + ix^2/2t]\hat{u}_+(x/t)\,. \end{aligned} \tag{2.21}$$

In particular $|v_2|$ is independent of the choice of S:

$$|v_2(t,x)| = t^{-n/2}|\hat{u}_+(x/t)|\,. \tag{2.22}$$

It is interesting to ascertain whether the asymptotic dynamics U_1 and U_2 are equivalent in L^2. For that purpose it is sufficient to show that

$$\|(U_1(t) - U_2(t))u_+\|_2 = \|(M - \mathbb{1})\exp[-iS]u_+\|_2 \to 0 \tag{2.23}$$

when $t \to \infty$. In view of the obvious estimate

$$\|(M - \mathbb{1})\exp[-iS]u_+\|_2 \le t^{-\sigma}\| \, |x|^{2\sigma} \exp[-iS]u_+\|_2 , \tag{2.24}$$

where $0 \le \sigma \le 1$, that property follows in practice from estimates similar to, but simpler than, those needed for the proof of (2.20).

We now turn to the non-linear equation

$$(i\partial_t + (1/2)\Delta)u = f(u) , \tag{2.25}$$

where f is a non-linear interaction (possibly non-local and actually so in the case of the Hartree equation). We assume f to be gauge invariant, namely

$$f(\omega u) = \omega f(u) \quad \text{for all } \omega \in \mathbb{C},\ |\omega| = 1 , \tag{2.26}$$

so that $f(u)$ takes the form $f(u) = ug(|u|^2)$.

In the next two sections we shall study the existence of modified wave operators for the equation (2.25) by the same method as in [17]. That method fits nicely in the framework of the previous remarks. We shall use as above two asymptotic dynamics defined by

$$v_1(t) = U(t)\exp[-iS(t, -i\nabla)]u_+ , \tag{2.27}$$

$$v_2(t) = U(t)M(t)^{-1}\exp[-iS(t, -i\nabla)]u_+ . \tag{2.28}$$

Those dynamics are both non-linear since now $S \equiv S(t, -i\nabla)$ will depend on u_+. The equivalence of those dynamics in L^2 reduces as previously to the fact that

$$\|v_1(t) - v_2(t)\|_2 = \|(M - \mathbb{1})\exp[-iS]u_+\|_2 \to 0 \tag{2.29}$$

when $t \to \infty$. The dynamics v_2 will be used for direct comparison with the original one (2.25). Of special interest will be the fact that v_2 approximately satisfies the equation (2.25) in the sense that

$$(i\partial_t + (1/2)\Delta)v_2 - f(v_2) \equiv \tilde{f} \tag{2.30}$$

is suitably small. Now the left-hand side of (2.30) is that of a Schrödinger equation for v_2 with time dependent potential $g(|v_2|^2)$. That potential is independent of the choice of S, since $|v_2|$ is [cf. (2.22)]. By exactly the same computation as that leading to (2.18), we then find

$$\begin{aligned}\tilde{f} &= (i\partial_t + (1/2)\Delta - g(|D\hat{u}_+|^2))v_2 \\ &= UM^{-1}\{\partial_t S(-i\nabla) - g(t^{-n}|\hat{u}_+(-i\nabla)|^2) - x^2/2t^2\}\exp[-iS]u_+ .\end{aligned} \tag{2.31}$$

As in the linear case, we shall need that $\tilde{f} \in L^1(\mathbb{R}^+, L^2)$ and for that purpose that

$$\|\{\partial_t S(-i\nabla) - g(t^{-n}|\hat{u}_+(-i\nabla)|^2)\}u_+\|_2 \in L^1(\mathbb{R}^+) \tag{2.32}$$

$$t^{-2}\|x^2 \exp[-iS]u_+\|_2 \in L^1(\mathbb{R}^+) . \tag{2.33}$$

Actually we shall need a stronger property, see (3.3) and (4.2) below.

Another possible choice of asymptotic dynamics, both in the linear and nonlinear case, is

$$v_3(t) = \exp[-iS(t, x/t)]U(t)u_+ . \tag{2.34}$$

That choice has the virtue that it differs from the free dynamics $U(t)u_+$ only by a phase (in the x space representation). It has been considered in [17] where it is actually used in the statement of the results. That choice is easily seen to be equivalent to the choice v_2 in L^2 since by (2.6) v_2 can be rewritten as

$$v_2(t) = \exp[-iS(t, x/t)]U(t)M(t)^{-1}u_+ \tag{2.35}$$

so that

$$\|v_2(t) - v_3(t)\|_2 = \|(M - \mathbb{1})u_+\|_2 \to 0$$

when $t \to \infty$ under the only assumption that $u_+ \in L^2$ since M tends strongly to $\mathbb{1}$ in L^2 when $t \to \infty$. We shall also make use of v_3 in the present paper.

For completeness, we finally recall that the modified wave operators will be constructed as in [17] by solving the integral equation for u

$$u(t) - v(t) = i \int_t^\infty d\tau U(t - \tau) \{f(u(\tau)) - f(v(\tau)) - \tilde{f}(\tau)\} \tag{2.36}$$

with $v = v_2$. In fact, if u is a solution of (2.25) and if $\tilde{f}$ is defined by (2.30), the following identity holds for $t < t_0 < \infty$ and $v = v_2$:

$$\begin{aligned} u(t) - v(t) = U(t - t_0)(u(t_0) - v(t_0)) - i \int_t^{t_0} d\tau U(t - \tau)\tilde{f}(\tau) \\ + i \int_t^{t_0} d\tau U(t - \tau) \{f(u(\tau)) - f(v(\tau))\} . \end{aligned} \tag{2.37}$$

If $u(t) - v(t) \to 0$ in L^2 when $t \to \infty$ and if $\tilde{f} \in L^1(\mathbb{R}^+, L^2)$, the first two terms in the right-hand side of (2.37) converge in L^2 to the obvious limits when $t_0 \to \infty$ and u is expected to satisfy (2.36).

3. The NLS Equation

In this section we prove the existence of modified wave operators for the NLS equation (1.1) with nonlinearity

$$f(u) = \lambda |u|^{p-1} u . \tag{1.2}$$

Our final result will be concerned with the case $p - 1 = 2/n$. Since however some intermediate results are not restricted to this special choice, we shall keep p as a parameter in some parts of the argument. As regards the dimension n, the case $n = 1$ has been considered in [17] and we restrict our attention to the case $n \geq 2$. The final results will apply to $n = 2$ and 3 only, but again some intermediate results are not restricted to that case.

The main step of the argument consists in solving the equation (2.36) for u in a neighborhood of infinity in time by a contraction method under suitable assumptions on v. For that purpose we define the following Banach spaces, which are the natural generalization of those considered in [17]. Let $\theta > 0$, $0 < 2/q = \delta(r) < 1$ and $T > 0$. We define

$$X(T) \equiv X_{\theta,r}(T) = \{w \in \mathscr{C}([T,\infty), L^2) \cap L^q([T,\infty), L^r) : \\ \|w; X(T)\| \equiv \sup_{t\geq T} t^\theta (\|w(t)\|_2 + \|w; L^q([t,\infty), L^r)\|) < \infty\} . \tag{3.1}$$

The spaces $X_{\theta,r}(T)$ depend monotonously on θ and r (or δ), namely $X_{\theta,r} \subset X_{\theta',r'}$ if $\theta \geq \theta'$ and $r \geq r'$.

The basic existence and uniqueness result is the following.

Proposition 3.1. *Let $n \geq 2$ and $p-1 = 2/n$. Let $T_0 > 0$ and let $v \in \mathscr{C}([T_0,\infty), L^2) \cap L^\infty([T_0,\infty), L^\infty)$ satisfy*

$$\|v(t)\|_\infty \leq c_\infty t^{-n/2} \quad \textit{for} \quad t \geq T_0 , \tag{3.2}$$

$$\|\tilde{f}(t)\|_2 \leq Ct^{-(1+\theta_0)} \quad \textit{for} \quad t \geq T_0 . \tag{3.3}$$

for some $\theta_0 > n/4$, where

$$\tilde{f} \equiv (i\partial_t + (1/2)\Delta)v - f(v) . \tag{3.4}$$

Then for c_∞ sufficiently small (depending only on λ and n), for $0 < 2/q = \delta(r) < 1$ and $n/4 < \theta < \theta_0$, the equation (1.1), (1.2) *has a unique solution*

$$u \in \mathscr{C}(\mathbb{R}, L^2) \cap L^q_{\text{loc}}(\mathbb{R}, L^r)$$

such that $u - v \equiv w \in X_{\theta,r}(T_0)$.

Proof. The first and main step consists in solving the equation (2.36) for w by a contraction method in $X(T)$ for $T \geq T_0$, T sufficiently large. We rewrite (2.36) as

$$w(t) = -i\int_t^\infty d\tau U(t-\tau)\tilde{f}(\tau) + i\int_t^\infty d\tau U(t-\tau)\{f(v(\tau)+w(\tau)) - f(v(\tau))\} \\ = w^{(0)}(t) + F(w)(t) . \tag{3.5}$$

We estimate $w^{(0)}$ by the use of Lemma 2.2 part (2) as

$$\|w^{(0)}; L^\infty([t,\infty), L^2)\| + \|w^{(0)}; L^q([t,\infty), L^r)\| \\ \leq C\|\tilde{f}; L^1([t,\infty), L^2)\| \tag{3.6}$$

so that by (3.3) for any $T \geq T_0$

$$\|w^{(0)}; X(T)\| \leq CT^{-(\theta_0-\theta)} . \tag{3.7}$$

We next estimate the integrand in $F(w)$ for general p and for $w \in X(T)$. From the identity

$$f(u_1) - f(u_2) = (\lambda/2)\int_0^1 d\sigma\{(p+1)|\sigma u_1 + (1-\sigma)u_2|^{p-1}(u_1-u_2) \\ + (p-1)|\sigma u_1 + (1-\sigma)u_2|^{p-3}(\sigma u_1 + (1-\sigma)u_2)^2(\bar{u}_1 - \bar{u}_2)\}$$

we obtain

$$|f(u_1) - f(u_2)| \leq |\lambda|\, p\, |u_1 - u_2|\, \text{Max}(|u_1|, |u_2|)^{p-1}$$

so that

$$|f(v+w) - f(v)| \le |\lambda|\, p\, |w| 2^{(p-2)^+} (|v|^{p-1} + |w|^{p-1}) .$$

Using again Lemma 2.2 part (2), we estimate

$$\begin{aligned} &\|F(w); L^\infty([t,\infty), L^2)\| + \|F(w); L^q([t,\infty), L^r)\| \\ &\quad \le C\{\| \, |v|^{p-1} w; L^1([t,\infty), L^2)\| + \| \, |w|^p; L^{\bar{q}'}([t,\infty), L^{\bar{r}'})\|\} \end{aligned} \tag{3.8}$$

for $0 \le 2/q' = \delta(r') < 1$, r' otherwise arbitrary, and similarly

$$\begin{aligned} &\|F(w_1) - F(w_2); L^\infty([t,\infty), L^2)\| + \|F(w_1) - F(w_2); L^q([t,\infty), L^r)\| \\ &\quad \le C\{\| \, |v|^{p-1} w; L^1([t,\infty), L^2)\| \\ &\qquad + \|(|w_1|^{p-1} + |w_2|^{p-1}) w; L^{\bar{q}'}([t,\infty), L^{\bar{r}'})\|\} \end{aligned} \tag{3.9}$$

with $w = w_1 - w_2$. From (3.2) we obtain for all $t \ge T \ge T_0$,

$$\| \, |v|^{p-1} w; L^1([t,\infty), L^2)\| \le C c_\infty^{p-1} \|w; X(T)\| t^{1-(p-1)n/2-\theta} . \tag{3.10}$$

We next estimate the last norm in (3.8). By the Hölder inequality

$$\|w^p\|_{\bar{r}'} \le \|w\|_2^{p-\sigma} \|w\|_r^\sigma \tag{3.11}$$

with

$$(p-1)n/2 = \delta' + \sigma\delta , \tag{3.12}$$

where $\delta = \delta(r)$ and $\delta' = \delta(r')$, provided $0 \le \sigma \le p$. Taking the appropriate norm in time and using the Hölder inequality, we obtain for any $t \ge T$,

$$\begin{aligned} \|w^p; L^{\bar{q}'}([t,\infty), L^{\bar{r}'})\| &\le \|w; X(T)\|^{p-\sigma} \\ &\quad \times \left\{ \int_t^\infty d\tau \|w(\tau)\|_r^{\sigma\bar{q}'} \tau^{-(p-\sigma)\theta\bar{q}'} \right\}^{1/\bar{q}'} \\ &\le \|w; X(T)\|^{p-\sigma} \|w; L^q([t,\infty), L^r)\|^\sigma \\ &\quad \times \left\{ \int_t^\infty d\tau\, \tau^{-(p-\sigma)\theta\bar{q}'\bar{k}} \right\}^{1/\bar{q}'\bar{k}} , \end{aligned} \tag{3.13}$$

where $1/k = \sigma\bar{q}'/q$ and provided the last time integral converges, namely

$$(p-\sigma)\theta > 1/\bar{q}'\bar{k} , \tag{3.14}$$

so that (3.13) can be continued by

$$\ldots \le \|w; X(T)\|^p t^{-(\theta+\varepsilon)} , \tag{3.15}$$

where

$$(p-\sigma)\theta + \sigma\theta - 1/\bar{q}'\bar{k} = \theta + \varepsilon , \tag{3.16}$$

or equivalently

$$(p-1)\theta - \varepsilon = 1/\bar{k}\bar{q}' = 1 - 1/q' - 1/k\bar{q}' = 1 - \tfrac{1}{2}(\delta' + \sigma\delta) ,$$

namely by (3.12)

$$(p-1)(\theta + n/4) = 1 + \varepsilon . \tag{3.17}$$

The conditions to be satisfied by the various parameters reduce to the conditions $0 \le \delta' < 1$, the condition $\sigma \ge 0$ or equivalently $(p-1)n/2 \ge \delta'$, the condition $k \ge 1 \Leftrightarrow \sigma/q \le 1/\bar{q}' \Leftrightarrow \delta' + \sigma\delta \le 2 \Leftrightarrow (p-1)n/2 \le 2$, and the time integrability condition (3.14) which implies $\sigma < p$, and which reduces to $\sigma < 1+\varepsilon/\theta$ by (3.16). We enforce that condition by imposing $\varepsilon > 0$ and $\sigma \le 1$ or equivalently $(p-1)n/2 \le \delta+\delta'$ which implies $(p-1)n/2 \le 2$. Finally, under the conditions $\theta > 0$, $0 \le \delta, \delta' < 1$,

$$1 \le (p-1)n/2 \le \delta' + \delta\,, \tag{3.18}$$

$$(p-1)(\theta + n/4) = 1 + \varepsilon > 1\,, \tag{3.19}$$

we obtain from (3.8), (3.10), (3.13), (3.15),

$$\|F(w); X(T)\| \le C\{c_\infty^{p-1} T^{1-(p-1)n/2} \|w; X(T)\| + T^{-\varepsilon} \|w; X(T)\|^p\}\,. \tag{3.20}$$

In exactly the same way, we obtain from (3.9),

$$\|F(w_1) - F(w_2); X(T)\| \le C\Big\{ c_\infty^{p-1} T^{1-(p-1)n/2} \|w; X(T)\| + T^{-\varepsilon} \sum_{i=1,2} \|w_i; X(T)\|^{p-1} \|w; X(T)\| \Big\}. \tag{3.21}$$

We next specialize to the case $(p-1)n/2 = 1$, where (3.18) (3.19) reduce to $\delta'+\delta \ge 1$ and

$$\theta = (n/4)(1 + 2\varepsilon) > n/4\,. \tag{3.22}$$

It follows immediately from (3.7), (3.20), and (3.21) that for c_∞ sufficiently small and T sufficiently large, the right-hand side of (3.5) defines a contraction from the ball $B(R)$ of radius R in $X(T)$ to itself for a suitable R. By standard arguments, this implies that (3.5) has a unique solution w in $X(T)$.

By known results on L^2-solutions of the equation (1.1), (1.2) [22] the solution $u = v + w$ just defined for $t \ge T$ can be continued to all of $\mathbb{R}$ with the properties stated. QED

The previous result calls for the following remarks.

Remark 3.1. From the fact that r and r' are completely decoupled in Lemma 2.2 part (2) and from the estimates in the proof of Proposition 3.1, it follows that any solution w of the equation (2.36) in $X_{\theta,r}(T)$ for a special choice of $T > 0$, $\theta > n/4$ and r with $\theta < \delta < 1$ belongs to $X_{\theta,r_1}(T)$ for the same T, θ and for any r_1 with $0 < \delta(r_1) < 1$. In particular existence and uniqueness for one special choice of r implies uniqueness for any such choice of r and existence in the intersection over r of such spaces under the same assumptions.

Remark 3.2. The smallness condition on the data u_+, namely c_∞ small, comes from the estimate (3.10). When one tracks back the various constants, one finds that the condition is of the type

$$C_\delta \theta^{-1} c_\infty^{p-1} \le C\,,$$

where C_δ comes from Lemma 2.2, is bounded for $\delta = \delta(r)$ in compact subsets of $[0, 1)$ and blows up when $\delta \uparrow 1$. By Remark 3.1, it is sufficient to impose that condition for one single value of δ to obtain the results for all δ. One can for instance choose one such δ close to zero. As regards the dependence on θ, since one needs $\theta > n/4$, one can impose a smallness condition that is valid for all admissible θ by imposing the previous condition for $\theta = n/4$. If one does so, Proposition 3.1 yields uniqueness in $X_{\theta,r}(T)$ for any admissible θ, and since $X_{\theta,r}(T)$ depends monotonously on θ, the solution belongs to the intersection of all such spaces for admissible values of θ.

Remark 3.3. The restriction to small data coming from (3.10) is needed only for $(p-1)n/2 = 1$. For $(p-1)n/2 > 1$, the same result as in Proposition 3.1 holds without such a restriction. Proposition 3.1 can therefore also be used to prove the existence of (ordinary non-modified) wave operators in L^2 on a dense (in L^2) set of initial states for such values of p in so far as the assumptions (3.2), (3.3) are satisfied, namely for $n = 2$ and 3 (see below). For $n = 3$, that result was previously known only for $4/(n+2) = 4/5 < p - 1 < 4/(n-2) = 4$ [3].

Remark 3.4. Proposition 3.1 is not restricted in dimension. The subsequent restriction to $n \leq 3$ comes from the fact that the available approximate dynamics yields $\theta_0 < 1$ in (3.3) for $n = 2$ and a stronger condition in higher dimensions (see Lemmas 3.1 and 3.2 below). Proposition 3.1 also holds in dimension $n = 1$ with however a restriction on r (or δ) of the type $\delta_0 < \delta \leq 1/2$. This comes from the fact that $\delta, \delta' \leq \delta(\infty) = 1/2$. For instance the condition (3.18) holds with $\delta, \delta' = 1/2, p = 3$, the rest of the proof being identical. This result allows in particular to extend the admissible range of θ to $1/4 < \theta < 1$ for $n = 1$, as compared with $1/2 < \theta < 1$ obtained in [17].

We now come back to the main problem. We shall apply Proposition 3.1 by taking for v the function v_2 defined by (2.28). From (2.22) it follows immediately that (3.2) holds with $c_\infty = \|\hat{u}_+\|_\infty$. We now turn to (3.3). The function $\tilde{f}$ is defined by (3.4). The most obvious choice for S consists in taking [cf. (2.31)]

$$\partial_t S = g(t^{-n}|\hat{u}_+|^2), \tag{3.23}$$

which in the special case $g(|u|^2) = \lambda|u|^{p-1}$ yields

$$S \equiv S(t,\xi) = h(t)\,|\hat{u}_+(\xi)|^{p-1} \tag{3.24}$$

with

$$h(t) = \lambda[t^{(p-1)n/2-1} - 1]\,[(p-1)n/2 - 1]^{-1} \tag{3.25}$$

for $(p-1)n/2 \neq 1$, and

$$h(t) = \lambda \ln t \tag{3.26}$$

for $(p-1)n/2 = 1$, the only relevant case. With that choice of S,

$$\|\tilde{f}\|_2 = (1/2)t^{-2}\|\Delta \exp[-iS]\hat{u}_+\|_2\,. \tag{3.27}$$

We note for future reference that

$$\Delta \exp[-iS]\hat{u} = \exp[-iS]\,(\Delta\hat{u} - 2i\nabla\hat{u}\cdot\nabla S - i\hat{u}\Delta S - \hat{u}|\nabla S|^2)\,. \tag{3.28}$$

We shall consider separately the case $n = 2$, where the choice (3.24) is adequate, and the case $n = 3$, where S thereby obtained is too singular and a regularization is needed. The relevant estimate for $n = 2$ is given in the following lemma.

Lemma 3.1. *Let $n = 2$ and $p - 1 = 2/n = 1$. Let $u_+ \in H^{0,2}$ and let S be defined by* (3.24), (3.26). *Then the following estimate holds for all $t > 0$:*

$$\begin{aligned}\|\Delta \exp[-iS]\hat{u}_+\|_2 \leq \|x^2u_+\|_2 + C|\ln t|\,\|u_+\|_2^{1/2}\|x^2u_+\|_2^{3/2} \\ + C(\ln t)^2\|u_+\|_2\|x^2u_+\|_2^2\,.\end{aligned} \tag{3.29}$$

In particular $\tilde{f}$ *defined by* (3.4) *with* $v = v_2$ *defined by* (2.28) *and* S *defined by* (3.24), (3.26) *satisfies the condition* (3.3) *for any* $T_0 > 0$ *and* $\theta_0 < 1$.

Proof. With S defined by (3.24) and dropping the subscript + for brevity, we obtain

$$\nabla S = h(t)\,(p-1)\,|\hat{u}|^{p-3}\,\mathrm{Re}\,\bar{\hat{u}}\nabla\hat{u}\,, \tag{3.30}$$

$$\Delta S = h(t)\,(p-1)\,\{|\hat{u}|^{p-3}(\mathrm{Re}\,\bar{\hat{u}}\Delta\hat{u} + |\nabla\hat{u}|^2) + (p-3)\,|\hat{u}|^{p-5}(\mathrm{Re}\,\bar{\hat{u}}\nabla\hat{u})^2\}\,. \tag{3.31}$$

Substituting (3.30), (3.31) into (3.28) yields

$$\begin{aligned}|\Delta \exp[-iS]\hat{u}| \le |\Delta\hat{u}| &+ C|h(t)|\,(|\hat{u}|^{p-2}|\nabla\hat{u}|^2 + |\hat{u}|^{p-1}|\Delta\hat{u}|)\\ &+ Ch(t)^2|\hat{u}|^{2p-3}|\nabla\hat{u}|^2\,.\end{aligned} \tag{3.32}$$

For $n = 2$, $p = 2$, (3.29) follows from (3.32) and Sobolev inequalities. QED

For $n = 3$ and $p - 1 = 2/n = 2/3$, the previous choice (3.24) would yield

$$\begin{aligned}|\Delta \exp[-iS]\hat{u}| \le |\Delta\hat{u}| &+ C|\ln t|\,(|\hat{u}|^{-1/3}|\nabla\hat{u}|^2 + |\hat{u}|^{2/3}|\Delta\hat{u}|)\\ &+ C(\ln t)^2|\hat{u}|^{1/3}|\nabla\hat{u}|^2\,,\end{aligned} \tag{3.33}$$

which is inadequate because of the term with negative power $|\hat{u}|^{-1/3}$. In order to circumvent this difficulty, we choose instead

$$S_\mu(t,\xi) = \lambda(\ln t)\,(t^{-\mu} + |\hat{u}_+(\xi)|^2)^{1/3} \tag{3.34}$$

for some $\mu > 0$ to be chosen later. One can then prove the following estimates.

Lemma 3.2. *Let* $n = 3$ *and* $p - 1 = 2/n = 2/3$. *Let* $u_+ \in H^{0,2}$ *and* $\hat{u}_+ \in L^1$ *(for instance let* $u_+ \in H^{s,0} \cap H^{0,2}$ *for some* $s > 3/2$*). Let* S_μ *be defined by* (3.34) *for some* $\mu > 0$. *Then the following estimates hold for* $t > 0$:

$$\|(\partial_t S_\mu - g(t^{-n}|\hat{u}_+|^2))\hat{u}_+\|_2 \le t^{-1-7\mu/12}|\lambda|\,(1 + (\mu/3)\,|\ln t|)\,\|\hat{u}_+\|_1^{1/2}\,, \tag{3.35}$$

$$\begin{aligned}\|\Delta \exp[-iS_\mu]\hat{u}_+\|_2 &\le \|x^2u_+\|_2\\ &+ C|\ln t|\,(\|xu_+\|_2^{1/3}\|x^2u_+\|_2^{4/3} + t^{\mu/6}\|xu_+\|_2^{1/2}\|x^2u_+\|_2^{3/2})\\ &+ C(\ln t)^2\|xu_+\|_2^{2/3}\|x^2u_+\|_2^{5/3}\,.\end{aligned} \tag{3.36}$$

In particular for $\mu = 4/3$

$$\begin{aligned}\|\tilde{f}(t)\|_2 &\le C(1 + |\ln t|)t^{-16/9}(\|\hat{u}_+\|_1^{1/2} + \|u_+; H^{0,2}\|^2)\\ &+ C(1 + |\ln t|)^2t^{-2}(\|u_+; H^{0,2}\| + \|u_+; H^{0,2}\|^{7/3})\end{aligned} \tag{3.37}$$

and (3.3) *holds for any* $\theta_0 < 7/9$.

Proof. We compute (dropping again the subscript + for brevity)

$$\begin{aligned}\partial_t S_\mu - g(\cdot) = \lambda t^{-1}&\{(t^{-\mu} + |\hat{u}|^2)^{1/3} - |\hat{u}|^{2/3}\}\\ &- (\mu/3)t^{-1-\mu}(\lambda\ln t)\,(t^{-\mu} + |\hat{u}|^2)^{-2/3}\,.\end{aligned} \tag{3.38}$$

From the inequality

$$\begin{aligned}(a+b)^{1/3} - b^{1/3} &\le \mathrm{Min}(a^{1/3}, (1/3)ab^{-2/3})\\ &\le a^{1/3+2\sigma/3}b^{-2\sigma/3}\end{aligned}$$

valid for $0 \le \sigma \le 1$, we obtain with $\sigma = 3/8$, $a = t^{-\mu}$, and $b = |\hat{u}|^2$,

$$|((t^{-\mu} + |\hat{u}|^2)^{1/3} - |\hat{u}|^{2/3})\hat{u}| \le t^{-7\mu/12}|\hat{u}|^{1/2}, \tag{3.39}$$

while obviously

$$(t^{-\mu} + |\hat{u}|^2)^{-2/3}|\hat{u}| \le t^{5\mu/12}|\mu|^{1/2}. \tag{3.40}$$

Taking the L^2-norm of $(\partial_t S_\mu - g(\cdot))\hat{u}$ and using (3.39), (3.40) yields immediately (3.35).

In order to prove (3.36), we compute

$$\begin{aligned}
\nabla S_\mu &= (2/3)\,(\lambda \ln t)\,(t^{-\mu} + |\hat{u}|^2)^{-2/3}(\mathrm{Re}\,\bar{\hat{u}}\nabla\hat{u}),\\
\Delta S_\mu &= (2/3)\,(\lambda \ln t)\,(t^{-\mu} + |\hat{u}|^2)^{-2/3}(\mathrm{Re}\,\bar{\hat{u}}\Delta\hat{u} + |\nabla\hat{u}|^2)\\
&\quad - (8/9)\,(\lambda \ln t)\,(t^{-\mu} + |\hat{u}|^2)^{-5/3}(\mathrm{Re}\,\bar{\hat{u}}\nabla\hat{u})^2.
\end{aligned}$$

Substituting these expressions into (3.28) yields

$$\begin{aligned}
|\Delta \exp[-iS_\mu]\hat{u}| \le |\Delta\hat{u}| &+ (2/3)\,|\lambda \ln t|\,\{|\hat{u}|^2(t^{-\mu} + |\hat{u}|^2)^{-2/3}|\Delta\hat{u}|\\
&+ 3|\hat{u}|\,(t^{-\mu} + |\hat{u}|^2)^{-2/3}|\nabla\hat{u}|^2\\
&+ (4/3)\,(t^{-\mu} + |\hat{u}|^2)^{-5/3}|\hat{u}|^3|\nabla\hat{u}|^2\}\\
&+ (4/9)\,(\lambda \ln t)^2|\hat{u}|^3(t^{-\mu} + |\hat{u}|^2)^{-4/3}|\nabla\hat{u}|^2.
\end{aligned} \tag{3.41}$$

Omitting the regularization $t^{-\mu}$ in all terms where no negative power of $\hat{u}$ arises and using the elementary inequality

$$|\hat{u}|(t^{-\mu} + |\hat{u}|^2)^{-2/3} \le t^{\mu/6}$$

we obtain [compare with (3.33)]

$$\begin{aligned}
|\Delta \exp[-iS_\mu]\hat{u}| \le |\Delta\hat{u}| &+ (2/3)\,|\lambda \ln t|\,\{|\hat{u}|^{2/3}|\Delta\hat{u}|\\
&+ (13/3)t^{\mu/6}|\nabla\hat{u}|^2\} + (4/9)\,(\lambda \ln t)^2|\hat{u}|^{1/3}|\nabla\hat{u}|^2
\end{aligned} \tag{3.42}$$

from which (3.36) follows by taking the L^2-norm and using Sobolev inequalities.

Finally (3.37) follows immediately from (3.35) and (3.36) in the special case $\mu = 4/3$. QED

The previous two lemmas will enable us to apply Proposition 3.1 by choosing for v the asymptotic dynamics v_2 defined by (2.28). We shall also be interested in using the asymptotic dynamics v_1 and v_3 defined by (2.27) and (2.34). For that purpose, it will be necessary to show that the differences $v_i - v_j$ belong to the spaces $X(T)$ defined by (3.1). This will be achieved by the use of the following lemma.

Lemma 3.3. *Let v_i $(i = 1, 2, 3)$ be defined by (2.27), (2.28), (2.34) with $u_+ \in H^{0,2}$ and S sufficiently regular. Then for all q, r with $0 \le 2q = \delta(r) \equiv \delta < n/2$, $\delta \le 2$, the following estimates hold for all $t > 0$:*

$$\|v_2 - v_3; L^q([t,\infty), L^r)\| \le Ct^{-1}\|x^2 u_+\|_2, \tag{3.43}$$

$$\|v_1 - v_2; L^q([t,\infty), L^r)\| \le Ct^{-\theta}\|t^{\theta-1}x^2\exp[-iS]u_+; L^\infty([t,\infty), L^2)\| \tag{3.44}$$

for all $\theta > 0$ for which the last norm is finite.

Proof. For $2 \le r < \infty$ we estimate

$$\|v_2(t) - v_3(t)\|_r = \|U(M^{-1} - \mathbb{1})u_+\|_r = \|M^{-1}U(M^{-1} - \mathbb{1})u_+\|_r$$
$$\le C\|(-\Delta)^{\delta/2}M^{-1}U(M^{-1} - \mathbb{1})u_+\|_2$$

by the Sobolev inequality (2.8)

$$\ldots = Ct^{-\delta}\| \, |x|^{\delta}(M^{-1} - \mathbb{1})u_+\|_2$$

by (2.6),

$$\ldots \le Ct^{-(\delta+\sigma)}\| \, |x|^{\delta+2\sigma} u_+\|_2$$

for $0 \le \sigma \le 1$. Taking the norm in $L^q([t, \infty))$ we obtain

$$\|v_2 - v_3; L^q([t, \infty), L^r)\| \le Ct^{-(\delta/2+\sigma)}\| \, |x|^{\delta+2\sigma} u_+\|_2$$

provided $\delta + 2\sigma > 0$, from which (3.43) follows by taking $\delta + 2\sigma = 2$. Similarly we estimate

$$\begin{aligned}\|v_1(t) - v_2(t)\|_r &= \|U(M^{-1} - \mathbb{1})\exp[-iS]u_+\|_r \\ &\le C\|(-\Delta)^{\delta/2}M^{-1}U(M^{-1} - \mathbb{1})\exp[-iS]u_+\|_2 \\ &= Ct^{-\delta}\| \, |x|^{\delta}(M^{-1} - \mathbb{1})\exp[-iS]u_+\|_2 \\ &\le Ct^{-(\delta+\sigma)}\| \, |x|^{\delta+2\sigma}\exp[-iS]u_+\|_2 \\ &\le Ct^{-(\delta/2+\theta)}t^{\theta-1}\|x^2\exp[-iS]u_+\|_2\end{aligned}$$

by taking $\delta + 2\sigma = 2$. Taking the norm in $L^q([t, \infty))$ yields (3.44). QED

It follows in particular from (3.43) that $v_2 - v_3 \in X_{\theta,r}(T)$ for all $T > 0$, all $\theta \le 1$ and all admissible r. Furthermore for $\theta \le 1$,

$$\|v_2 - v_3; X_{\theta,r}(T)\| \le CT^{\theta-1}. \tag{3.45}$$

Similarly if in $\tilde{f}$ as given by (2.31) the contribution of the term $x^2\exp[-iS]u_+$ satisfies the condition (3.3), then $v_1 - v_2 \in X_{\theta,r}(T)$ for all $T \ge T_0$, all $\theta \le \theta_0$ and all admissible r. Furthermore for $\theta \le \theta_0$,

$$\|v_1 - v_2; X_{\theta,r}(T)\| \le CT^{\theta-\theta_0}. \tag{3.46}$$

In space dimension $n = 3$, Lemma 3.2 will enable us to apply Proposition 3.1 by choosing for v the asymptotic dynamics v_2 defined by (2.28) with S replaced by the regularized function S_μ defined by (3.34) with $\mu = 4/3$. We shall also be interested in using the more natural unregularized dynamics v_i $(i = 1, 2, 3)$ associated with the function

$$S(t, \xi) = \lambda(\ln t)\,|\hat{u}_+(\xi)|^{2/3}. \tag{3.47}$$

For that purpose, it will again be necessary to show that their differences with the regularized ones belong to the spaces $X(T)$ defined by (3.1). This will be achieved for $i = 1, 2$ by the use of the following lemma.

Lemma 3.4. *Let $n = 3$ and let $v_{i\mu}$ and v_i $(i = 1, 2)$ be defined by* (2.27), (2.28) *with S defined by* (3.34) *and* (3.47) *respectively, with $u_+ \in H^{0,2}$ and $\hat{u}_+ \in L^1$. Then the following estimates hold for $i = 1, 2$ and all $t > 0$:*

$$\|v_{i\mu} - v_i\|_2 \le |\lambda \ln t|\, t^{-7\mu/12}\|\hat{u}_+\|_1^{1/2}, \tag{3.48}$$

$$\|v_{i\mu} - v_i\|_6 \le Ct^{-1-\mu/3}\{|\lambda \ln t|\, \|xu_+\|_2 + (\lambda \ln t)^2\|xu_+\|_2^{4/3}\|x^2u_+\|_2^{1/3}\}. \tag{3.49}$$

In particular for $0 \le 2/q = \delta(r) \equiv \delta \le 1$,

$$\|v_{i\mu} - v_i; L^q([t,\infty), L^r)\| \le C|\lambda \ln t|\,(1 + |\lambda \ln t|^\delta) t^{-(\mu/3+\delta/2+(1-\delta)\mu/4)}, \tag{3.50}$$

where C *depends only on* $\|\hat{u}_+\|_1$ *and* $\|u_+; H^{0,2}\|$. *In particular for* $\mu = 4/3$, $v_{i\mu} - v_i \in X_{\theta,r}(T)$ *for all* $\theta < 7/9$, *all* r *with* $0 < \delta \le 1$ *and all* $T > 0$.

Proof. For all r with $2 \le r < \infty$, we estimate

$$\begin{aligned}\|v_2\mu - v_2\|_r &= \|UM^{-1}(\exp[-iS_\mu] - \exp[-iS])u_+\| \\ &\le C\|(-\Delta)^{\delta/2} M^{-1} U M^{-1}(\exp[-iS_\mu] - \exp[-iS])u_+\|_2\end{aligned}$$

by adding a unitary factor M^{-1} and using the Sobolev inequality (2.8)

$$\ldots = Ct^{-\delta}\|\,|x|^\delta(\exp[-iS_\mu] - \exp[-iS])u_+\|_2 \tag{3.51}$$

by (2.6). The same estimate holds for v_1 with exactly the same proof. For $r = 2$ (where actually $C = 1$), we estimate

$$\begin{aligned}\|(\exp[-iS_\mu] - \exp[-iS])u_+\|_2 &\le |\lambda \ln t|\,\|((t^{-\mu} + |\hat{u}_+|^2)^{1/3} - |\hat{u}_+|^{2/3})\hat{u}_+\|_2 \\ &\le |\lambda \ln t|\, t^{-7\mu/12}\|\hat{u}_+\|_1^{1/2}\end{aligned}$$

by (3.39). This proves (3.48). For $r = 6$ or equivalently $\delta = 1$, we have to estimate in L^2 (in Fourier space variables)

$$\begin{aligned}\nabla(\exp[-iS_\mu] - \exp[-iS])\hat{u}_+ &= (\exp[-iS_\mu] - \exp[-iS])\,(\nabla\hat{u}_+ - i(\nabla S_\mu)\hat{u}_+) \\ &\quad - i\exp[-iS]\,(\nabla S_\mu - \nabla S)\hat{u}_+\,. \end{aligned}\tag{3.52}$$

Now

$$\begin{aligned}|\cdot| &\le (S_\mu - S)\,(|\nabla\hat{u}_+| + |\nabla S_\mu|\,|\hat{u}_+|) + |\nabla S_\mu - \nabla S|\,|\hat{u}_+|\,, \\ (S_\mu - S)\,|\nabla\hat{u}_+| &\le |\lambda \ln t|\, t^{-\mu/3}|\nabla\hat{u}_+|\,,\end{aligned}\tag{3.53}$$

$$\begin{aligned}|S_\mu - S|\,|\nabla S_\mu|\,|\hat{u}_+| &= (1/3)\,(\lambda \ln t)^2((t^{-\mu} + |\hat{u}_+|^2)^{1/3} - |\hat{u}_+|^{2/3}) \\ &\quad \times (t^{-\mu} + |\hat{u}_+|^2)k^{-2/3}2|\hat{u}_+|^2|\nabla\hat{u}_+| \\ &\le (2/3)\,(\lambda \ln t)^2 t^{-\mu/3}|\hat{u}_+|^{2/3}|\nabla\hat{u}_+|\,,\end{aligned}\tag{3.54}$$

$$\begin{aligned}|\nabla S_\mu - \nabla S|\,|\hat{u}_+| &= (2/3)\,|\lambda \ln t|\,(|\hat{u}_+|^{-4/3} - (t^{-\mu} + |\hat{u}_+|^2)^{-2/3})\,|\hat{u}_+|^2|\nabla\hat{u}_+| \\ &= (2/3)\,|\lambda \ln t|\,((t^{-\mu} + |\hat{u}_+|^2)^{1/3} - |\hat{u}_+|^{2/3}) \\ &\quad \times ((t^{-\mu} + |\hat{u}_+|^2)^{1/3} + |\hat{u}_+|^{2/3}) \\ &\quad \times (t^{-\mu} + |\hat{u}_+|^2)^{-2/3}|\hat{u}_+|^{2/3}|\nabla\hat{u}_+| \\ &\le (4/3)\,|\lambda \ln t|\, t^{-\mu/3}|\nabla\hat{u}_+|\,.\end{aligned}\tag{3.55}$$

Using (3.53)-(3.55) to estimate (3.52) and substituting the result into (3.51) with $\delta = 1$ immediately yields (3.49).

Taking the norm of (3.49) in $L^2([t,\infty))$ yields (3.50) for $\delta = 1$ (i.e. $r = 6$), from which (3.50) for general δ, $0 \le \delta \le 1$, follows by interpolation with (3.48). QED

Lemma 3.4 does not cover the case of the asymptotic dynamics v_3 for which a direct estimate of $v_{3\mu} - v_3$ would be less convenient to obtain directly. This is however immaterial since v_3 is easily compared to v_2 uniformly in the choice of S by Lemma 3.3, esp. (3.43).

We are now in a position to state the main result of this section.

Theorem 3.1. *Let $n = 2$ or 3 and $p-1 = 2/n$. Let $u_+ \in H^{0,2}$ with $\|\hat{u}_+\|_\infty$ sufficiently small. If $n = 3$, assume in addition that $\hat{u}_+ \in L^1$. For $n = 2$, let S be defined by*

$$S(t, \xi) = \lambda(\ln t)\, |\hat{u}_+(\xi)| \, . \tag{3.56}$$

For $n = 3$, let S be defined either by (3.47) *or by* (3.34) *with $\mu = 4/3$, namely*

$$S(t, \xi) = S_{4/3}(t, \xi) = \lambda \ln t (t^{-4/3} + |\hat{u}_+(\xi)|^2)^{1/3} \, . \tag{3.57}$$

Let v_i $(i = 1, 2, 3)$ be defined by (2.27), (2.28), (2.34). *Then the equation* (1.1), (1.2) *has a unique solution $u \in \mathscr{C}(\mathbb{R}, L^2)$ such that for any r with $0 < 2/q = \delta(r) < 1$, $u \in L^q_{\mathrm{loc}}(\mathbb{R}, L^r)$ and for any such r, any θ with $1/2 < \theta < 1$ for $n = 2$ and $3/4 < \theta < 7/9$ for $n = 3$, any $i = 1, 2, 3$ and any $t > 0$,*

$$\|u(t) - v_i(t)\|_2 \le Ct^{-\theta} \, , \tag{3.58}$$

$$\|u - v_i; L^q([t, \infty), L^r)\| \le Ct^{-\theta} \, . \tag{3.59}$$

A similar result holds for negative times.

Proof. We prove the result first for $i = 2$, and, for $n = 3$, with the choice (3.57) for S. It follows from Lemma 3.1 for $n = 2$ and from Lemma 3.2 for $n = 3$ that $\bar{f}$ defined by (3.4) with $v = v_2$ satisfies the assumption (3.3) for any $T_0 > 0$ and any θ_0 with $\theta_0 < 1$ for $n = 2$ and $\theta_0 < 7/9$ for $n = 3$, while in all cases v_2 satisfies (3.2) with $c_\infty = \|\hat{u}_+\|_\infty$. The result for v_2 [with the choice (3.57) for $n = 3$] is then an immediate application of Proposition 3.1. The result in the other cases follows immediately from the fact that all other asymptotic dynamics differ from the previous one by functions which belong to the spaces $X_{\theta,r}(T)$ used in Proposition 3.1 for all the relevant values of θ and r (actually are small in those spaces for large T). For v_3 [and for $n = 3$, S given by (3.57)] this fact follows from Lemma 3.3, esp. (3.43). For v_1 [and for $n = 3$, S given by (3.57)], it follows from Lemma 3.3 again, esp. (3.44), supplemented by Lemma 3.1, esp. (3.29) for $n = 2$ and by Lemma 3.2, esp. (3.36) for $n = 3$. For $n = 3$ and for S given by (3.47), it follows from Lemma 3.4 for v_1 and v_2, and finally from Lemma 3.3 again, esp. (3.43) for v_3. QED

Remark 3.5. In the same way as for Proposition 3.1, uniqueness of u holds for any special choice of θ and r in the allowed range, and the decay estimates (3.58), (3.59) hold for the common solution u for all θ and r in the allowed range. The smallness condition on c_∞ can be taken independent of θ and r (see Remarks 3.1 and 3.2).

Remark 3.6. As in the one dimensional case [17], one can derive additional decay estimates of u from Theorem 3.1. For instance it follows from the fact that

$$\|v_2(t)\|_r \le Ct^{-\delta(r)} \tag{3.60}$$

for all $t > 0$ and all r, $2 \le r \le \infty$, that

$$\|v_2; L^q([t, \infty), L^r)\| \le Ct^{-\delta(r)/2} \tag{3.61}$$

for $0 \le 2/q = \delta(r) \le n/2$. Combining this result with (3.58), (3.59) and noting that $\delta(r) < 1$ implies $\delta(r)/2 < \theta$, we see that

$$\|u; L^q([t, \infty), L^r)\| \le Ct^{-\delta(r)/2} \tag{3.62}$$

for all $t > 0$ and for $0 \le 2/q = \delta(r) < 1$.

Remark 3.7. Although the free dynamics $v_0(t) \equiv U(t)u_+$ does not yield the correct asymptotic behaviour of the full dynamics $u(t)$ in the present situation, it nevertheless yields the correct behaviour for $|u(t)|$. In fact, from (3.58), (3.59) with $i = 3$, from the fact that $|v_0(t)| = |v_3(t)|$ and therefore that

$$|u(t)| - |v_0(t)| = |u(t)| - |v_3(t)| \leq |u(t) - v_3(t)|$$

it follows that for the solutions obtained in Theorem 3.1 and for all relevant values of θ and r, the following estimates hold:

$$\| \, |u(t)| - |U(t)u_+| \|_2 \leq Ct^{-\theta}, \tag{3.63}$$

$$\| \, |u| - |U(\cdot)u_+|; L^q([t,\infty), L^r)) \| \leq Ct^{-\theta}. \tag{3.64}$$

Similarly, in Fourier transformed variables, from (3.58) with $i = 1$ and from the fact that $|\hat{v}_0(t)| = |\hat{v}_1(t)|$, it follows that

$$\| \, |\hat{u}(t)| - |\widehat{U(t)u_+}| \|_2 \leq \|\hat{u}(t) - \hat{v}_1(t)\|_2 \leq Ct^{-\theta} \tag{3.65}$$

for the solutions of Theorem 3.1 and for all relevant values of θ.

In the end of this section, we prove that the wave operator $\Omega_+ : u_+ \to u(0)$ defined through Theorem 3.1 satifies the intertwining property, at least on a suitable subset of asymptotic states. For that purpose, we define

$$Y(\varrho) = \{u_+ \in H^{2,0} \cap H^{0,2} : \|\hat{u}_+\|_\infty < \varrho\}. \tag{3.66}$$

Clearly $Y(\varrho)$ is invariant under the free evolution for any $\varrho > 0$, namely

$$U(t)Y(\varrho) = Y(\varrho) \quad \text{for all} \quad t \in \mathbb{R}$$

by (2.5). In particular Theorem 3.1 applies to any $u_+ \in Y(\varrho)$ as well as to any $U(s)u_+$ for any $s \in \mathbb{R}$ provided ϱ satisfies the smallness condition of that theorem. Let $W(t)$ be the full non-linear evolution group, namely $W(t): u(0) \to u(t)$, where u is the solution of the Cauchy problem for the equation (1.1), (1.2) with prescribed $u(0)$. We can now state the intertwining property as follows.

Proposition 3.2. *Let $n = 2$ or 3 and $p - 1 = 2/n$. Let ϱ satisfy the smallness condition of Theorem 3.1. Then for any $u_+ \in Y(\varrho)$ and any $s \in \mathbb{R}$,*

$$W(s)\Omega_+ u_+ = \Omega_+ U(s) u_+. \tag{3.67}$$

Proof. Let $v(t, u_+)$ be any of the asymptotic dynamics considered in Theorem 3.1 and let $u(t, u_+)$ be the solution of the equation (1.1), (1.2) constructed in that theorem for $u_+ \in Y(\varrho)$. That solution is characterized by the fact that

$$u(\cdot, u_+) - v(\cdot, u_+) \in X_{\theta,r}([T,\infty)) \quad \text{for some} \quad T > 0 \tag{3.68}$$

for some (equivalently for all) admissible θ and r. Furthermore

$$\Omega_+ u_+ = u(0, u_+)$$

and

$$W(s)\Omega_+ u_+ = u(s, u_+). \tag{3.69}$$

Similarly

$$\Omega_+ U(s) u_+ = u(0, U(s)u_+), \tag{3.70}$$

where the solution $u(0, U(s)u_+)$ is characterized by the fact that

$$u(\cdot, U(s)u_+) - v(\cdot, U(s)u_+) \in X_{\theta,r}([T, \infty)) \quad \text{for some} \quad T > 0 \tag{3.71}$$

for admissible θ, r. The intertwining property (3.67) therefore reduces to

$$u(s, u_+) = u(0, U(s)u_+)$$

or equivalent, by the uniqueness for the Cauchy problem

$$u(s + \cdot, u_+) = u(\cdot, U(s)u_+) . \tag{3.72}$$

It follows from Theorem 3.1, in particular from (3.68) and (3.71), and from (3.72) that the intertwining property holds provided

$$v(s + \cdot, u_+) - v(\cdot, U(s)u_+) \in X_{\theta,r}([T, \infty)) \quad \text{for some} \quad T > 0 \tag{3.73}$$

for some admissible θ, r. That property in turns follows from estimates very similar to those of Lemma 3.4. We choose $v = v_1$ and both for $n = 2$ and $n = 3$,

$$S(t, \xi) = (\ln t) g(|\hat{u}_+|^2) = \lambda (\ln t) \, |\hat{u}_+(\xi)|^{2/n} . \tag{3.74}$$

Then

$$v_1(t + s, u_+) - v_1(t, U(s)u_+) = U(t + s) \, (\exp[-iS(t + s)] - \exp[-iS(t)]) u_+ . \tag{3.75}$$

By the same estimate as in Lemma 3.4, we obtain [cf. (3.51)]

$$\begin{aligned} \|v_1(t + s, u_+) - v_1(t, U(s)u_+)\|_r &\leq C_r |t + s|^{-\delta} \\ &\times \| \, |x|^\delta (\exp[-iS(t + s)] - \exp[-iS(t)]) u_+ \|_2 \end{aligned} \tag{3.76}$$

for $2 \leq r < \infty$, $\delta = \delta(r)$. In particular for $r = 2$,

$$\begin{aligned} \|v_1(t + s, u_+) - v_1(t, U(s)u_+)\|_2 &\leq \|(S(t + s) - S(t)) u_+\|_2 \\ &= |\ln(1 + s/t)| \, \|g(|\hat{u}_+)|^2) \hat{u}_+\|_2 \qquad (3.77) \\ &\leq |\lambda \ln(1 + s/t)| \varrho^{2/n} \|u_+\|_2 \equiv m . \qquad (3.78) \end{aligned}$$

We next obtain by interpolation

$$\| \cdot \|_r \leq C_r |t + s|^{-\delta} m^{1-\delta} \|\nabla (\exp[-iS(t + s)] - \exp[-iS(t)]) \hat{u}_+\|_2^\delta \tag{3.79}$$

for $0 \leq \delta \leq 1$, $\delta < 1$ if $n = 2$. Now

$$\begin{aligned} &\nabla (\exp[-iS(t + s)] - \exp[-iS(t)]) \hat{u}_+ \\ &\quad = -i \exp[-iS(t + s)] \, (\nabla S(t + s) - \nabla S(t)) \hat{u}_+ \\ &\qquad + (\exp[-iS(t + s)] - \exp[-iS(t)]) \, (\nabla \hat{u}_+ - i \hat{u}_+ \nabla S(t)) \end{aligned}$$

so that by (3.74)

$$|\cdot| \leq |\ln(1 + s/t)| \, \{ |\nabla g(|\hat{u}_+|^2) \hat{u}_+| + g(|\hat{u}_+|^2) \, (|\nabla \hat{u}_+| + (\ln t) \, |\hat{u}_+ \nabla g(|\hat{u}_+|^2)|) \} , \tag{3.80}$$

and therefore

$$\begin{aligned} |_2 &\leq C |\ln(1 + s/t)| \, \{ \| \, |\hat{u}_+|^{2/n} \nabla \hat{u}_+ \|_2 + (\ln t) \, \| \, |\hat{u}_+|^{4/n} \nabla \hat{u}_+ \|_2 \} \\ &\leq C |\ln(1 + s/t)| \, (\varrho^{2/n} + (\ln t) \varrho^{4/n}) \, \|x u_+\|_2 . \end{aligned} \tag{3.81}$$

Substituting (3.81) into (3.79) yields

$$\begin{aligned}\|v_1(t+s,u_+) - v_1(t,U(s)u_+)\|_r &\le C|t+s|^{-\delta}|\ln(1+s/t)| \\ &\times \varrho^{2/n}\|u_+\|_2^{1-\delta}(1+\varrho^{2/n}\ln t)^{\delta}\|xu_+\|_2^{\delta}\end{aligned} \tag{3.82}$$

and (3.73) for $v = v_1$ follows immediately from (3.78) and (3.82) for all $\theta < 1$ and $0 < \delta < 1$. QED

4. The Hartree Equation

In this section we prove the existence of modified wave operators for the Hartree equation (1.1) with non-linearity

$$f(u) = (V * |u|^2)u = \lambda(|x|^{-1} * |u|^2)u \tag{1.3}$$

in space dimension $n \ge 2$. As for the case of the NLS equation, the main step of the argument consists in solving the equation (2.36) for u in a neighborhood of infinity in time by a contraction method. We use the same Banach spaces $X_{\theta,r}(T)$ defined by (3.1) as for the NLS equation. The basic existence and uniqueness result is the following analogue of Proposition 3.1.

Proposition 4.1. *Let $n \ge 2$. Let $k_\pm$ satisfy $\delta(k_\pm) = (1 \pm \eta)/2$ with $0 < \eta \le 1$ so that $k_- < 2n/(n-1) < k_+$. Let $T_0 > 0$ and let $v \in \mathscr{C}([T_0,\infty),L^2) \cap L^\infty([T_0,\infty),L^{k_+})$ satisfy*

$$\|v(t)\|_{k_\pm} \le c_\infty t^{-\delta(k_\pm)} \quad \text{for} \quad t \ge T_0\,, \tag{4.1}$$

$$\|\tilde{f}(t)\|_2 \le Ct^{-(1+\theta_0)} \quad \text{for} \quad t \ge T_0\,, \tag{4.2}$$

for some $\theta_0 > 1/4$, where

$$\tilde{f} = (i\partial_t + (1/2)\Delta)v - f(v)\,. \tag{4.3}$$

Then for c_∞ sufficiently small (depending only on λ and n) for $0 < 2/q = \delta(r) < 1$ and $1/4 < \theta < \theta_0$, the equation (1.1), (1.3) *has a unique solution*

$$u \in \mathscr{C}(\mathbb{R},L^2) \cap L^q_{\text{loc}}(\mathbb{R},L^r)$$

such that $u - v \equiv w \in X_{\theta,r}(T_0)$.

Proof. The main step consists again in solving the equation (2.36) or equivalently (3.5) for w by contraction in $X_{\theta,r}(T)$ for $T \ge T_0$, T sufficiently large, where f is now defined by (1.3). We estimate $w^{(0)}$ as before by (3.6), (3.7). We next estimate the integrand in $F(w)$ for $w \in X(T)$. Now

$$\begin{aligned}f(v+w) - f(v) &= (V * |v|^2)w + (V * 2\,\mathrm{Re}\,\bar{v}w)v \\ &\quad + (V * 2\,\mathrm{Re}\,\bar{v}w)w + (V * |w|^2)v + (V * |w|^2)w \\ &\equiv L_1 + L_2 + Q_1 + Q_2 + f(w)\,,\end{aligned} \tag{4.4}$$

where L_i and Q_i, $i = 1, 2$ are the terms linear and quadratic in w. By Lemma 2.2 part (2), we estimate

$$\begin{aligned}&\|F(w); L^\infty([t,\infty),L^2)\| + \|F(w); L^q([t,\infty),L^r)\| \\ &\le C\{\|L_1 + L_2; L^1([t,\infty),L^2)\| + \|f(w); L^{\bar{q}_1'}([t,\infty),L^{\bar{r}_1'})\| \\ &\quad + \|Q_1 + Q_2; L^{\bar{q}_2'}([t,\infty),L^{\bar{r}_2'})\|\}\end{aligned} \tag{4.5}$$

for $0 \le 2/q'_i = \delta(r'_i) \equiv \delta'_i < 1$ $(i = 1, 2)$, r'_i otherwise arbitrary. We consider the various terms in the right-hand side of (4.5) successively. We note for future reference that by the Hölder and Hardy-Littlewood-Sobolev (HLS) inequalities,

$$\|(V * (v_1 v_2))v_3\|_{\bar{r}'} \le \|V * (v_1 v_2)\|_m \|v_3\|_{s_3} \le \|v_1\|_{s_1} \|v_2\|_{s_2} \|v_3\|_{s_3} \, , \tag{4.6}$$

provided

$$n/\bar{r}' - n/s_3 \equiv \delta' + \delta(s_3) = n/m = n/s_1 + n/s_2 - n + 1 \equiv 1 - \delta(s_1) - \delta(s_2)$$

and $n < m < \infty$, or equivalently

$$0 < \delta' + \delta(s_3) = 1 - \delta(s_1) - \delta(s_2) < 1 \, . \tag{4.7}$$

We first estimate the linear terms in (4.5). Now

$$\|(V * |v|^2)w\|_2 \le \|V * |v|^2\|_\infty \|w\|_2 \, . \tag{4.8}$$

We estimate

$$\begin{aligned} |V * |v|^2| &= |\lambda| \left(|x|^{-1}\chi(|x| \le a) * |v|^2 + |x|^{-1}\chi(|x| \ge a) * |v|^2 \right) \\ &\le C(a^\eta \|v\|_{k_+}^2 + a^{-\eta} \|v\|_{k_-}^2) \end{aligned}$$

by the Hölder inequality,

$$\ldots \le C \|v\|_{k_+} \|v\|_{k_-} \tag{4.9}$$

by optimizing with respect to a. From (4.8), (4.9) and the assumption (4.1), we obtain for all $t \ge T \ge T_0$,

$$\|(V * |v|^2)w; L^1([t, \infty), L^2)\| \le C c_\infty^2 \|w; X(T)\| t^{-\theta} \, . \tag{4.10}$$

The second linear term is estimated by (4.6) with $r' = 2$, $v_1 = v_3 = v$, $s_1 = s_3 = k$, $v_2 = w$, $s_2 = 2$ as

$$\|(V * 2\,\mathrm{Re}\, \bar{v} w)v\|_2 \le C \|v\|_k^2 \|w\|_2$$

and (4.7) is obviously fulfilled with $\delta(k) = 1/2$. Taking the L^1 norm in time and using (4.1) and the fact that $k_- < k < k_+$, we obtain for all $t \ge T \ge T_0$,

$$\|(V * 2\,\mathrm{Re}\, \bar{v} w)v; L^1([t, \infty), L^2)\| \le C c_\infty^2 \|w; X(T)\| t^{-\theta} \, . \tag{4.11}$$

We postpone the study of the quadratic terms in (4.5) and consider next the cubic term. Dropping the subscript 1 for brevity, we estimate by (4.6) with $v_1 = v_2 = v_3 = w$ and $s_1 = s_2 = s_3 = s$,

$$\|(V * |w|^2)w\|_{\bar{r}'} \le C \|w\|_s^3 \, , \tag{4.12}$$

provided $\delta' + 3\delta(s) = 1$ and $0 < 2\delta(s) < 1$, the latter being automatically satisfied for $0 \le \delta' < 1$. We next estimate for $0 \le \delta(s) \le \delta$,

$$\|w\|_s \le \|w\|_2^{1-\delta(s)/\delta} \|w\|_r^{\delta(s)/\delta} \, . \tag{4.13}$$

Taking the appropriate norm in the time variable and eliminating $\delta(s)$, we estimate for $t \geq T > 0$,

$$\begin{aligned}\|f(w); L^{\bar{q}'}([t,\infty), L^{\bar{r}'})\| &\leq C\|w; X(T)\|^{3-(1-\delta')/\delta} \\ &\quad\times \left\{ \int_t^\infty d\tau\, \tau^{-(3-(1-\delta')/\delta)\theta\bar{q}'} \|w(\tau)\|_r^{\bar{q}'(1-\delta')/\delta} \right\}^{1/\bar{q}'} \\ &\leq C\|w; X(T)\|^{3-(1-\delta')/\delta} \|w; L^q([t,\infty), L^r)\|^{(1-\delta')/\delta} \\ &\quad\times t^{-(3-(1-\delta')/\delta)\theta+1/\bar{q}'-(1-\delta')/\delta q} \end{aligned} \tag{4.14}$$

by the Hölder inequality in time, provided $q\delta \geq \bar{q}'(1-\delta')$, which is equivalent to $1-\delta' \leq 2\bar{q}' \equiv 2-\delta'$ and is therefore automatically satisfied, and provided the final time integral converges, or equivalently

$$\theta(3-(1-\delta')/\delta) > 1/\bar{q}' - (1-\delta')/\delta q \equiv 1/2\,. \tag{4.15}$$

From (4.14), from the definition of $X(T)$ and the last equality in (4.15), it follows that for all $t \geq T > 0$,

$$\|f(w); L^{\bar{q}'}([t,\infty). L^{\bar{r}'})\| \leq C\|w; X(T)\|^3 t^{-(\theta+\varepsilon)} \tag{4.16}$$

with

$$\varepsilon = 2(\theta - 1/4) > 0\,. \tag{4.17}$$

The condition (4.15) is automatically satisfied if $\varepsilon > 0$ and $\delta \geq 1-\delta'$. In particular all the conditions required on the various parameters are satisfied by imposing (4.17) and

$$0 < 1-\delta' = 3\delta(s) = \delta < 1\,. \tag{4.18}$$

We next come back to the quadratic terms in (4.5), omitting the subscript 2 for brevity. Using (4.6) with $v_1 = w$, $s_1 = s$, with (v_2, v_3) being $(v. w)$ and (s_2, s_3) being (k, s) in the two possible ways, we estimate

$$\|Q_1 + Q_2\|_{\bar{r}'} \leq C\|v\|_k \|w\|_s^2 \tag{4.19}$$

with $\delta' + \delta(k) + 2\delta(s) = 1$ and provided $0 < 2\delta(s) < 1$ and $0 < \delta(s) + \delta(k) < 1$. We next use (4.13) and take the appropriate norm in the time variable to obtain for $t \geq T \geq T_0$,

$$\begin{aligned}\|Q_1 + Q_2; L^{\bar{q}'}([t,\infty), L^{\bar{r}'})\| &\leq Cc_\infty \|w; X(T)\|^{2(1-\delta(s)/\delta)} \\ &\quad\times \left\{ \int_t^\infty d\tau\, \tau^{-(\delta(k)+2\theta(1-\delta(s)/\delta))\bar{q}'} \|w(\tau)\|_r^{2\bar{q}'\delta(s)/\delta} \right\}^{1/\bar{q}'} \\ &\leq Cc_\infty \|w; X(T)\|^{2(1-\delta(s)/\delta)} \|w; L^q([t,\infty), L^r)\|^{2\delta(s)/\delta} \\ &\quad\times t^{-(\delta(k)+2\theta(1-\delta(s)/\delta))+1/\bar{q}'-2\delta(s)/\delta q} \end{aligned} \tag{4.20}$$

by the Hölder inequality in time, provided $k_- \leq k \leq k_+$, provided $q\delta \geq 2\bar{q}'\delta(s)$ which is equivalent to $\delta' + 2\delta(s) \leq 2$ and is always satisfied for $\delta(k) \geq 0$, and provided the final time integral converges, or equivalently

$$2\theta(1-\delta(s)/\delta) > 1/\bar{q}' - \delta(s) - \delta(k) \equiv (1-\delta(k))/2\,. \tag{4.21}$$

From (4.20), from the last equality in (4.21) and from the definition of $X(T)$, it follows that for all $t \geq T \geq T_0$,

$$\|Q_1 + Q_2; L^{\bar{q}'}([t,\infty), L^{\bar{r}'})\| \leq C c_\infty \|w; X(T)\|^2 t^{-(\theta+\varepsilon_2)} \tag{4.22}$$

with

$$\varepsilon_2 = \theta - (1 - \delta(k))/2 \,. \tag{4.23}$$

It is natural at this point to choose $\delta(k) = 1/2$, so that in particular $k_- < k < k_+$ and (4.1) can be used. With that choice, $\delta' + 2\delta(s) = 1/2$, $\varepsilon_2 = \theta - 1/4 = \varepsilon/2$ and all the conditions required on the various parameters are easily seen to be satisfied provided $\delta' \geq 0$ and

$$0 < 1/2 - \delta' = 2\delta(s) \leq \delta < 1 \,. \tag{4.24}$$

Finally, under the conditions $0 < \delta < 1$ and (4.17), we obtain from (4.10), (4.11), (4.22), and (4.16),

$$\begin{aligned} \|F(w); X(T)\| \leq C\{ & c_\infty^2 \|w; X(T)\| + c_\infty T^{-\varepsilon/2} \|w; X(T)\|^2 \\ & + T^{-\varepsilon} \|w; X(T)\|^3\} \end{aligned} \tag{4.25}$$

for all $T \geq T_0$. In exactly the same way, since in the present case $f(v+w)$ is a polynomial in w and $\bar{w}$, we obtain for $w_1, w_2 \in X(T)$,

$$\begin{aligned} \|F(w_1) - F(w_2); X(T)\| \leq C \Bigg\{ & c_\infty^2 + c_\infty T^{-\varepsilon/2} \sum_{i=1,2} \|w_i; X(T)\| \\ & + T^{-\varepsilon} \sum_{i=1,2} \|w_i; X(T)\|^2 \Bigg\} \|w_1 - w_2; X(T)\| \,. \end{aligned} \tag{4.26}$$

With the estimates (4.25) and (4.26) available, the end of the proof is identical with that of Proposition 3.1. QED

The previous result calls for the following remarks.

Remark. As in the case of Proposition 3.1, existence and uniqueness for one special choice of r with $0 < \delta < 1$ implies uniqueness for any other such choice and existence of the solution in the intersection over all such r of the spaces $X_{\theta,r}$ (see Remark 3.1). The smallness condition on c_∞ has therefore to be imposed for one single value of δ. It can furthermore be taken to be valid for any admissible θ. Proposition 4.1 then yields uniqueness in $X_{\theta,r}$ for any admissible θ, and the solution belongs to the intersection over all admissible θ of such spaces (see Remark 3.2).

Remark 4.2. The Hartree equation is more regular than the NLS equation. This appears in the result at two places. First, the (optimal) decay condition on v has to be imposed only in $L^{k_\pm}$ instead of L^∞, where $k_- < 2n/(n-1) < k_+$ [compare (4.1) with (3.2)]. Second, the lower bound on θ is $\theta < 1/4$ instead of $\theta > n/4$, thereby allowing for the result to hold in arbitrary dimension $n \geq 2$.

Remark 4.3. As in the case of the NLS equation, the smallness condition on v could be dropped if the potential V were short range, namely $|V(x)| \leq C|x|^{-\gamma}$ for some $\gamma > 1$. Since however that case has been treated in [16] by more direct methods, we shall not elaborate on this possibility.

We now come back to the main problem. We shall apply Proposition 4.1 by taking for v the function v_2 defined by (2.28) and we shall need to ensure that v_2 satisfies the conditions (4.1) and (4.2). We first consider (4.1). If follows from (2.22) that v_2 satisfies (4.1) with

$$c_\infty = \operatorname*{Max}_{\pm} \|\hat{u}_+\|_{k\pm}, \tag{4.27}$$

provided the right-hand side is finite, which is always the case for $u_+ \in H^{0,2}$. We next consider the condition (4.2). The function $\tilde{f}$ is defined by (1.3), (4.3) and we take for S the obvious choice (3.23) which in the present case reduces to

$$S(t,\xi) = (\ln t)\,(V * |\hat{u}_+|^2)\,(\xi) \tag{4.28}$$

by an elementary computation. The relevant estimate is given by the following lemma.

Lemma 4.1. *Let $n \ge 2$. Let $u_+ \in H^{0,2}$ and let S be defined by (4.28). Then the following estimate holds for any $t > 0$:*

$$\begin{aligned}\|\Delta \exp[-iS]\hat{u}_+\|_2 \le \|x^2 u_+\|_2 &+ C|\ln t|\, \|u_+\|_2 \|xu_+\|_2 \|x^2 u_+\|_2 \\ &+ C(\ln t)^2 \|u_+\|_2 \|xu_+\|_2^4 .\end{aligned} \tag{4.29}$$

In particular $\tilde{f}$ defined by (4.3) with $v = v_2$ defined by (2.28) and S defined by (4.28) satisfies the condition (4.2) for any $T_0 > 0$ and any $\theta_0 < 1$.

Proof. With S defined by (4.28) and omitting the subscript + for brevity, we obtain

$$\nabla S = (\ln t)\,(V * 2\,\mathrm{Re}\,\bar{\hat{u}}\nabla\hat{u}), \tag{4.30}$$

$$\Delta S = (\ln t)\,(V * 2(\mathrm{Re}\,\bar{\hat{u}}\Delta\hat{u} + |\nabla\hat{u}|^2)). \tag{4.31}$$

We substitute (4.30) and (4.31) into the right-hand side of (3.28) and estimate the various terms in L^2. By the Hölder and HLS inequalities, we estimate [see especially (4.6), (4.7)]

$$\begin{aligned}\|\nabla\hat{u}\cdot\nabla S\|_2 &\le \|\nabla\hat{u}\|_k \|\nabla S\|_m \\ &\le C|\ln t|\, \|\nabla\hat{u}\|_k \|\nabla\hat{u}\|_2 \|\hat{u}\|_s ,\end{aligned}$$

provided

$$0 < \delta(k) = n/m = 1 - \delta(s) < 1 ,$$

$$\ldots \le C|\ln t|\, \|\Delta\hat{u}\|_2^\eta \|\nabla\hat{u}\|_2^{3-2\eta} \|\hat{u}\|_2^\eta$$

with $\eta = \delta(k)$, by Sobolev inequalities,

$$\ldots \le C|\ln t|\, \|\Delta\hat{u}\|_2 \|\nabla\hat{u}\|_2 \|\hat{u}\|_2 \tag{4.32}$$

by interpolation. We estimate the next term in (3.28) as

$$\|\hat{u}\cdot\Delta S\|_2 \le 2|\ln t|\, \|\hat{u}(V * (|\hat{u}\Delta\hat{u}| + |\nabla\hat{u}|^2))\|_2 . \tag{4.33}$$

Now

$$\begin{aligned}\|\hat{u}(V * |\hat{u}\Delta\hat{u}|)\|_2 &\le \|\hat{u}\|_s \|V * |\hat{u}\Delta\hat{u}|\|_m \\ &\le C\|\hat{u}\|_s^2 \|\Delta\hat{u}\|_2 ,\end{aligned}$$

provided

$$0 < \delta(s) = n/m = 1 - \delta(s) < 1 ,$$

which is satisfied by $\delta(s) = 1/2$ and $m = 2n$,

$$\ldots \le C\|\hat{u}\|_2 \|\nabla\hat{u}\|_2 \|\Delta\hat{u}\|_2 . \tag{4.34}$$

Finally

$$\|\hat{u}(\nabla S)^2\|_2 \le 4(\ln t)^2 \|\hat{u}\|_s \|V * |\hat{u}\nabla\hat{u}|\,\|_m^2$$
$$\le C(\ln t)^2 \|\hat{u}\|_s^3 \|\nabla\hat{u}\|_2^2 ,$$

provided

$$0 < \delta(s)/2 = n/m = 1 - \delta(s) < 1 ,$$

which is satisfied by $\delta(s) = 2/3$ and $m = 3n$,

$$\ldots \le C(\ln t)^2 \|\hat{u}\|_2 \|\nabla\hat{u}\|_2^4 . \tag{4.35}$$

Collecting (4.32), (4.33), (4.34), and (4.35) yields (4.29) immediately. The property stated for $\tilde{f}$ follows from (4.29) and from the fact that the function $\tilde{f}$ satisfies (3.27). QED

Remark 4.4. Some parts of the estimate (4.29) can be somewhat sharpened. For instance, from the fact that

$$\Delta S = 4\pi(\lambda \ln t)\,|\hat{u}_+|^2 \qquad \text{for } n = 3 ,$$
$$\Delta S = -(n-3)\,(\lambda \ln t)\,(|\xi|^{-3} * |\hat{u}_+|^2) \quad \text{for } n \ge 4 ,$$

one obtains for any $n \ge 3$,

$$\|\hat{u}_+ \Delta S\| \le C|\ln t|\,\|\hat{u}_+\|_{2^*}^3 \le C|\ln t|\,\|xu_+\|_2^3 . \tag{4.36}$$

Since one must assume that $u_+ \in H^{0,2}$ anyway, this does not result in any significant improvement of Lemma 4.1.

As in the case of the NLS equation, the previous lemma will enable us to apply Proposition 4.1 by choosing for v the asymptotic dynamics v_2 defined by (2.28). In order to extend the results to the asymptotic dynamics v_1 and v_3 defined by (2.27) and (2.34) we shall as before rely on Lemma 3.3, which does not make any reference to the specific choice of the interaction f in (1.1).

We are now in a position to state the main result of this section.

Theorem 4.1. *Let $n \ge 2$. Let $k_\pm$ satisfy $\delta(k_\pm) = (1 \pm \eta)/2$ with $0 < \eta \le 1$. Let $u_+ \in H^{0,2}$ with c_∞ as defined by* (4.27) *sufficiently small. Let S be defined by* (4.28) *and let v_i ($i = 1,2,3$) be defined by* (2.27), (2.28), (2.34). *Then the equation* (1.1), (1.3) *has a unique solution $u \in \mathscr{C}(\mathbb{R}, L^2)$ such that for any r with $0 < 2/q = \delta(r) < 1$, $u \in L^q_{\mathrm{loc}}(\mathbb{R}, L^r)$ and for any such r and any θ with $1/4 < \theta < 1$, any $i = 1, 2, 3$ and any $t > 0$,*

$$\|u(t) - v_i(t)\|_2 \le Ct^{-\theta} , \tag{4.37}$$
$$\|u - v_i; L^q([t,\infty), L^r)\| \le Ct^{-\theta} . \tag{4.38}$$

A similar result holds for negative times.

Proof. The proof closely follows that of Theorem 3.1. We first prove the result for $i = 2$. It follows from Lemma 4.1 that $\tilde{f}$ defined by (4.3) with $v = v_2$ satisfies the assumption (4.2) for any $T_0 > 0$ and any $\theta_0 < 1$ while v_2 satisfies (4.1) with c_∞ given by (4.27). The result for v_2 is then an immediate application of Proposition 4.1.

The result for v_1 and v_3 follows from the fact that $v_1 - v_2$ and $v_3 - v_2$ belong to $X_{\theta,r}(T)$ for all relevant values of r and θ by Lemma 3.3 and Lemma 4.1. QED

Remark 4.5. In the same way as for the NLS case, uniqueness of u holds for any special choice of θ, r in the allowed range, and the decay estimates (4.37), (4.38) hold for all θ, r in that range (see Remark 3.5 and 4.1).

Remark 4.6. As in the NLS case, one can derive additional decay estimates of u from Theorem 4.1. For instance it follows from the fact that v_2 satisfies (3.60) also in the present case that u also satisfies (3.62) for all $t > 0$ and for $0 \le 2/q = \delta(r) < 1$. Similarly, the free dynamics $U(t)u_+$ yields correctly the asymptotic behaviour of $|u(t)|$ and Remark 3.7 applies verbatim in the Hartree case.

We conclude this section by proving that the wave operator $\Omega_+ : u_+ \to u(0)$ defined through Theorem 4.1 satisfies the intertwining property on a suitable set of asymptotic states. We now define

$$Y(\varrho) = \{u_+ \in H^{2,0} \cap H^{0,2} : \operatorname*{Max}_{\pm} \|\hat{u}_+\|_{k_\pm} < \varrho\} . \tag{4.39}$$

Clearly $Y(\varrho)$ is invariant under the free evolution for any $\varrho > 0$. We still denote by $W(t)$ the non-linear evolution group $W(t): u(0) \to u(t)$, where u is the solution of the Cauchy problem for the equation (1.1), (1.3) with prescribed $u(0)$. We can state the intertwining property as follows.

Proposition 4.2. *Let $n \ge 2$ and $\delta(k_\pm) = (1 \pm \eta)/2$ with $0 < \eta \le 1$. Let ϱ satisfy the smallness condition of Theorem 4.1. Then for any $u_+ \in Y(\varrho)$ and any $s \in \mathbb{R}$,*

$$W(s)\Omega_+ u_+ = \Omega_+ U(s) u_+ . \tag{4.40}$$

Proof. The proof follows closely that of Proposition 3.2. As in the latter it is sufficient to show that

$$v_1(s + \cdot, u_+) - v_1(\cdot, U(s)u_+) \in X_{\theta,r}([T,\infty)) \quad \text{for some} \quad T > 0 . \tag{4.41}$$

We estimate that difference in L^2 by (3.77) followed by (4.8), (4.9) and we obtain

$$\begin{aligned} \|\cdot\|_2 &\le C|\ln(1 + s/t)|\, \|\hat{u}_+\|_{k_+} \|\hat{u}_+\|_{k_-} \|u_+\|_2 \\ &\le C|\ln(1 + s/t)|\varrho^2 \|u_+\|_2 \equiv m . \end{aligned} \tag{4.42}$$

We next estimate the same difference in L^r by (3.79), (3.80) followed by estimates of the type (4.6) and (4.9) and we obtain

$$\begin{aligned} \|\cdot\|_r &\le C|t + s)|^{-\delta} |\ln(1 + s/t)|\varrho^2 \|u_+\|_2^{1-\delta} \\ &\quad \times (1 + \varrho^2 \ln t)^\delta \|xu_+\|_2^\delta \end{aligned} \tag{4.43}$$

for $0 \le \delta \equiv \delta(r) \le 1$, $\delta < 1$ if $n = 2$. Then (4.41) follows from (4.42) and (4.43) for all $\theta < 1$ and $0 < \delta < 1$. QED

References

1. Barab, J.E.: Nonexistence of asymptotically free solutions for nonlinear Schrödinger equation. J. Math. Phys. **25**, 3270–3273 (1984)
2. Cazenave, T.: An introduction to nonlinear Schrödinger equations. Textos de métodos Mathemáticas vol. 22, Instituto de Matématica, Rio de Janeiro 1989
3. Cazenave, T., Weissler, F.: Rapidly decaying solutions of the non-linear Schrödinger equation, Commun. Math. Phys. **147**, 75–100 (1992)
4. Ginibre, J.: Unpublished (summarized in [10])
5. Ginibre, J., Velo, G.: On a class of non-linear Schrödinger equations. I. The Cauchy problem, general case; II Scattering theory, general case. J. Funct. Anal. **32**, 1–71 (1979)
6. Ginibre, J., Velo, G.: On a class of non-linear Schrödinger equations with non-local interactions, Math. Z. **170**, 109–136 (1980)
7. Ginibre, J., Velo, G.: Scattering theory in the energy space for a class of non-linear Schrödinger equations. J. Math. Pures Appl. **64**, 363–401 (1985)
8. Hayashi, N., Ozawa, T.: Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for some Schrödinger equations. Ann. Inst. Henri Poincaré, Physique théorique **48**, 17–37 (1988)
9. Hayashi, N., Tsutsumi, M.: $L^\infty(\mathbb{R}^n)$-decay of classical solutions for non-linear Schrödinger equations. Proc. Royal Soc. Edinburgh **104**, 309–327 (1986)
10. Hayashi, N., Tsutsumi, Y.: Scattering theory for Hartree type equations. Ann. Inst. Henri Poincaré, Physique théorique **46**, 187–213 (1987)
11. Hayashi, N., Tsutsumi, Y.: Remarks on the scattering problem for non-linear Schrödinger equations, Lect. Notes in Math., Vol. 1285, Berlin, Heidelberg, New York: Springer 1987, pp. 162–168
12. Hörmander, L.: The analysis of linear partial differential operators. IV. Chap. XXX, Berlin, Heidelberg, New York: Springer 1985
13. Kato, T.: Nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Physique théorique **46**, 113–129 (1987)
14. Kato, T.: Nonlinear Schrödinger equations. In: Schrödinger Operators. Holden, H., Jensen, A. (eds.). Lect. Notes in Phys., vol. 345. Berlin, Heidelberg, New York: Springer 1989
15. Kitada, H., Yajima, K.: A scattering theory for time dependent long range potentials. Duke Math. J. **49**, 341–376 (1982); **50**, 1005–1016 (1983)
16. Nawa, H., Ozawa, T.: Non-linear scattering with non-local interaction. Commun. Math. Phys. **146**, 259–275 (1992)
17. Ozawa, T.: Long range scattering for nonlinear Schrödinger equations in one space dimension. Commun. Math. Phys. **139**, 479–493 (1991)
18. Strauss, W.A.: Non-linear scattering theory. In: Scattering theory in mathematical physics. Lavita, J., Marchand, J.P. (eds.) Dordrecht: Reidel 1974
19. Strauss, W.A.: Non-linear scattering theory at low energy. J. Funct. Anal. **41**, 110–133 (1981)
20. Tsutsumi, Y.: Scattering problem for nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Physique théorique **43**, 321–347 (1985)
21. Tsutsumi, Y.: Global existence and asymptotic behavior of solutions for nonlinear Schrödinger equations. Doctoral Thesis, University of Tokyo (1985)
22. Tsutsumi, Y.: L^2 solutions for nonlinear Schrödinger equations and nonlinear groups. Funkcialaj Ekvacioj **30**, 115–125 (1987)
23. Tsutsumi, Y., Yajima, K.: The asymptotic behavior of nonlinear Schrödinger equations. Bull. Am. Math. Soc. **11**, 186–188 (1984)
24. Yajima, K.: Existence of solutions for Schrödinger evolution equations. Commun. Math. Phys. **110**, 415–426 (1987)

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