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Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions

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A modulated Fourier expansion in time is used to show long-time near-conservation of the harmonic actions associated with spatial Fourier modes along the solutions of nonlinear wave equations with small initial data. The result implies the long-time near-preservation of the Sobolev-type norm that specifies the smallness condition on the initial data.

Key words. semilinear wave equation, nonlinear Klein–Gordon equation, action preservation, long-time regularity, modulated Fourier expansion

1. Introduction

We consider the one-dimensional semilinear wave equation (nonlinear Klein-Gordon equation)

$$u_{tt} - u_{xx} + \rho u + g(u) = 0 \tag{1}$$

for $t > 0$ and $-\pi \le x \le \pi$ subject to periodic boundary conditions. We assume $\rho > 0$ and a nonlinearity g that is a smooth real function with $g(0) = g'(0) = 0$. We take small initial data: in appropriate Sobolev norms, the initial data $u(\cdot, 0)$ and $\partial_t u(\cdot, 0)$ is bounded by a small parameter ε, but is not restricted otherwise. We are interested in studying the behaviour of the solutions over long times $t \le \varepsilon^{-N}$, with fixed, but arbitrary positive integer N. Under a non-resonance condition that restricts the possible values of ρ to a set of full measure, we show that for each Fourier mode, the harmonic action remains nearly constant over such long times, as does the Sobolev-type norm specifying the smallness of the initial data. The result slightly refines previous results by Bambusi [1] and Bourgain [5], using entirely different techniques.

The main novelty in the present paper lies in the technique of proof via a *modulated Fourier expansion* in time. This is a multiscale expansion that represents the solution as an asymptotic series of products of exponentials $e^{i\omega_j t}$ (with ω_j the frequencies of the linear equation) multiplied with coefficient functions that vary slowly in time. This approach was first used for showing long-time almost-conservation properties in [11], in that case of numerical methods for highly oscillatory Hamiltonian ordinary differential equations; also see [12, Ch. XIII] and further references therein. A modulated Fourier expansion appears similarly, and independently, in the work by Guzzo and Benettin [10] on the spectral formulation of the Nekhoroshev theorem for quasi-integrable Hamiltonian systems. In the context of wave equations, the expansion constructed here can be viewed as an extension to higher approximation order of a nonlinear geometric optics expansion given by Joly, Métivier, and Rauch [13]. Multiscale expansions and modulation equations have certainly been used in various types and for various purposes in many places, also with nonlinear wave equations; see, e.g., Whitham [16], Kalyakin [14], Kirrmann, Schneider, and Mielke [15], Craig and Wayne [8]. Unlike all these works, we here construct a two-scale expansion to arbitrary order in ε and use the Lagrangian/Hamiltonian structure of the modulation equations to infer long-time near-conservation and regularity properties over times ε^{-N}, beyond the time of validity ε^{-1} of the expansion.

In Section 2 we describe the technical framework and state the result on the long-time near-conservation of harmonic actions (Theorem 1). Section 3 gives the construction of the modulated Fourier expansion and proves the necessary bounds of its coefficients and of the remainder term, which are collected in Theorem 2. The expansion works with all frequencies in the system, without a cutoff of high frequencies. While Section 3 is the technical core of this paper, its conceptual heart is in Section 4. There, it is shown that the system determining the modulation functions has a Hamiltonian structure and a remarkable invariance property, which yields the existence of almost-invariants close to the harmonic actions (Theorems 3 and 4). Though the modulated Fourier expansion is constructed only as a short-time expansion (over time scale ε^{-1}), its almost-invariants can be patched together over very many short time intervals, which finally gives the long-time near-conservation of actions over times ε^{-N} with $N > 1$ as stated in Section 2.

The approach to the long-time analysis of (1) via modulated Fourier expansions does *not* use nonlinear coordinate transforms to a normal form, as is done in Bambusi [1] (see also [2–4, 6] and references therein) and as is typical in Hamiltonian perturbation theory. While normal form theory uses coordinate transforms to take the equation to a simpler form from which essential properties can be read off, the present approach can be viewed as instead embedding the original equation in a large system of modulation equations from which the desired properties can be read off.

We consider equation (1) only with periodic boundary conditions, but it appears that the problem with Dirichlet boundary conditions, as studied in [1,5], can be treated in the same way. Less general than [1], we do not treat problems with an additional dependence on x in ρ and g, though such an extension could be done without pain. As in these previous works, an extension of the results to problems in more than one space dimension over time scales ε^{-N} with $N > 1$ does not seem possible with the present techniques, mainly due to problems with small denominators. See, however, Delort and Szeftel [9] for existence results over time ε^{-2} for nonlinear wave equations with periodic boundary conditions in higher dimension. Moreover, Bambusi, Delort, Grébert and Szeftel [4] deal with the same equation on manifolds (e.g. spheres) of arbitrary dimension.

Corresponding to the authors' research background, the present work was originally motivated by numerical analysis, with the aim of understanding the long-time behaviour of discretization schemes for the nonlinear wave equation (1). Since the approach via modulated Fourier expansions does not use nonlinear coordinate transforms, it turns out to be applicable also to numerical discretizations of (1), as is shown in a companion paper to the present article [7].

2. Preparation and statement of result

In this section we describe basic concepts, introduce notation, formulate assumptions, and state the result on the long-time near-conservation of actions.

2.1. Modulated Fourier expansion

The spatially 2π-periodic solutions to the linear wave equation $\partial_t^2 u - \partial_x^2 u + \rho u = 0$ are superpositions of plane waves $\mathrm{e}^{\pm \mathrm{i}\omega_j t \pm \mathrm{i} jx}$, where j is an arbitrary integer and

$$\omega_j = \sqrt{\rho + j^2}$$

are the frequencies of the equation. If the nonlinearity g is evaluated at a superposition of plane waves, its Taylor expansion involves mixed products of such waves. This can be taken as a motivation to look for an approximation to the solution $u(x,t)$ of the nonlinear problem in the form of a *modulated Fourier expansion*, that is, a linear combination of products of plane waves with coefficient functions that change slowly in time, or more precisely, their derivatives with respect to the slow time $\tau = \varepsilon t$ are bounded independently of ε:

$$u(x,t) \approx \widetilde{u}(x,t) = \sum_{\|\mathbf{k}\| \le K} z^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} = \sum_{\|\mathbf{k}\| \le K} \sum_{j=-\infty}^{\infty} z_j^{\mathbf{k}}(\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t + \mathrm{i}jx}. \tag{2}$$

Here, the sum is over all

$$\mathbf{k} = (k_\ell)_{\ell \ge 0} \quad \text{with integers } k_\ell \text{ and } \ \|\mathbf{k}\| := \sum_{\ell \ge 0} |k_\ell| \le K$$

(at most K of the k_ℓ are non-zero) and we write

$$\mathbf{k}\cdot\boldsymbol{\omega} = \sum_{\ell\ge 0} k_\ell\,\omega_\ell\,.$$

For $K = 2N$, we will obtain an expansion (2) with an approximation error of size $\mathcal{O}(\varepsilon^{N+1})$ in the same norm in which the initial data is assumed to be bounded by ε, uniformly over times $\mathcal{O}(\varepsilon^{-1})$.

In the construction, a special role is played by the modulation functions $z_j^{\mathbf{k}}$ for $\mathbf{k} = \pm\langle j\rangle$, with the notation (Kronecker delta)

$$\langle j\rangle := (\delta_{|j|,\ell})_{\ell\ge 0}\,.$$

The function $z_j^{\pm\langle j\rangle}$ is multiplied with $\mathrm{e}^{\pm\mathrm{i}\omega_j t+\mathrm{i}jx}$ in (2). The $z_j^{\pm\langle j\rangle}$ will be determined from first-order differential equations, whereas the other $z_j^{\mathbf{k}}$ are obtained from equations of the form $\big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big)z_j^{\mathbf{k}} = \ldots$, where we need to divide by $\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|$. If this denominator is too small in absolute value (less than $\varepsilon^{1/2}$, say), then this corresponds to a situation where we cannot safely distinguish the exponentials $\mathrm{e}^{\pm\mathrm{i}\omega_j t}$ and $\mathrm{e}^{\pm\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$ and we just set $z_j^{\mathbf{k}} = 0$.

2.2. Non-resonance condition

The effect of ignoring contributions from near-resonant indices $(j,\mathbf{k})$ for which $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$ (+ or −), should not spoil the $\mathcal{O}(\varepsilon^{N+1})$ remainder term in the modulated Fourier expansion. This requirement is fulfilled under a *non-resonance condition*. With the abbreviations

$$|\mathbf{k}| = (|k_\ell|)_{\ell\ge 0} \quad\text{and}\quad \boldsymbol{\omega}^{\sigma|\mathbf{k}|} = \prod_{\ell\ge 0}\omega_\ell^{\sigma|k_\ell|} \tag{3}$$

and the set of near-resonant indices

$$\mathcal{R}_\varepsilon = \{(j,\mathbf{k}) : j\in\mathbb{Z} \text{ and } \mathbf{k}\ne\pm\langle j\rangle,\ \|\mathbf{k}\|\le 2N \text{ with } |\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}\}\,, \tag{4}$$

the non-resonance condition can be formulated as follows: there are $\sigma > 0$ and a constant C_0 such that

$$\sup_{(j,\mathbf{k})\in\mathcal{R}_\varepsilon} \frac{\omega_j^\sigma}{\boldsymbol{\omega}^{\sigma|\mathbf{k}|}}\,\varepsilon^{\|\mathbf{k}\|/2} \le C_0\,\varepsilon^N\,. \tag{5}$$

For $N = 1$, this condition is always satisfied for arbitrary $\sigma \ge 0$ and ρ in (1). For $N > 1$, it imposes a restriction on the choice of ρ, and the possible values of σ depend on N. The condition requires that a near-resonance can only appear with at least two large frequencies among the ω_ℓ with $k_\ell \ne 0$ (counted with their multiplicity $|k_\ell|$).

As we show next, condition (5) is implied, for sufficiently large σ, by the non-resonance condition of Bambusi [1], which reads as follows: for every positive integer r, there exist $\alpha = \alpha(r) > 0$ and $c > 0$ such that for all combinations of signs,

$$|\omega_j \pm \omega_k \pm \omega_{\ell_1} \pm \ldots \pm \omega_{\ell_r}| \geq c\, L^{-\alpha} \quad \text{for} \quad j \geq k \geq L = \ell_1 \geq \ldots \geq \ell_r \geq 0, \tag{6}$$

provided that the sum does not vanish because the terms cancel pairwise. In [1] it is shown that for almost all (w.r.t. Lebesgue measure) ρ in a fixed interval of positive numbers there is a $c > 0$ such that condition (6) holds with $\alpha = 16\, r^5$. It is also noted in [1] that an analogous condition is typically not satisfied for wave equations in spatial dimension greater than 1.

Lemma 1. *Under Bambusi's non-resonance condition (6), the bound (5) holds with* $\sigma = \max_{r+1<2N} (2N - r - 1)\, \alpha(r)$.

Proof. Consider $(j, \mathbf{k}) \in \mathcal{R}_\varepsilon$, so that $|\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|| < \varepsilon^{1/2}$. For $\mathbf{k}$ with $\|\mathbf{k}\| = r+1$, we write $\mathbf{k}\cdot\boldsymbol{\omega} = \pm\omega_m \pm \omega_{\ell_1} \pm \ldots \pm \omega_{\ell_r}$ with $m \geq L = \ell_1 \geq \ldots \geq \ell_r \geq 0$. We then have

$$\frac{\omega_j}{\boldsymbol{\omega}^{|\mathbf{k}|}} \leq \frac{|\mathbf{k}\cdot\boldsymbol{\omega}| + \varepsilon^{1/2}}{\boldsymbol{\omega}^{|\mathbf{k}|}} \leq \frac{\omega_m + \omega_{\ell_1} + \cdots + \omega_{\ell_r} + \varepsilon^{1/2}}{\omega_m \omega_{\ell_1} \ldots \omega_{\ell_r}} \leq \frac{C}{L}$$

where the constant C depends on a lower bound of ρ.

Now, under condition (6), a near-resonance $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$ can only appear with $cL^{-\alpha} < \varepsilon^{1/2}$, i.e., $L^{-1} < c^{-1/\alpha}\varepsilon^{1/(2\alpha)}$. We then have

$$\frac{\omega_j^\sigma}{\boldsymbol{\omega}^{\sigma|\mathbf{k}|}} \leq \frac{C^\sigma}{L^\sigma} \leq C_0\, \varepsilon^{\sigma/(2\alpha)}$$

with $C_0 = (C/c^{1/\alpha})^\sigma$. If σ is chosen so large that $\sigma/(2\alpha) \geq N - \frac{1}{2}(r+1)$, i.e., $\sigma \geq (2N - r - 1)\alpha$, then we obtain the bound (5). $\square$

With Bambusi's value $\alpha(r) = 16\, r^5$, the lemma yields $\sigma = 2^9$ already for $N = 2$. (The corresponding quantity in [1] is $s_* = 4M\, \alpha(2M)$ for $M = N+2$, which for $N = 2$ results in $s_* = 2^{19}$.) However, it should be noted that condition (5) may actually be satisfied with a much smaller exponent σ. This is suggested by testing (5) numerically for various values of ε, ρ, and N.

2.3. Functional-analytic setting: Sobolev algebras

For a 2π-periodic function $v \in L^2(\mathbb{T})$ (with the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), we denote by $(v_j)_{j\in\mathbb{Z}}$ the sequence of Fourier coefficients of $v(x) = \sum_{-\infty}^{\infty} v_j\, \mathrm{e}^{\mathrm{i}jx}$. We will work with functions (or coefficient sequences) for which the weighted ℓ^2 norm

$$\|v\|_s = \Big(\sum_{j=-\infty}^{\infty} \omega_j^{2s}\, |v_j|^2 \Big)^{1/2}$$

is finite. We denote, for $s \geq 0$, the Sobolev space

$$H^s = \{ v \in L^2(\mathbb{T}) : \|v\|_s < \infty \} = \{ v :\ (-\partial_x^2 + \rho)^{s/2} v \in L^2 \}.$$

For $s > \frac{1}{2}$, we have $H^s \subset C(\mathbb{T})$ and H^s is a normed algebra:

$$\|vw\|_s \le C_s \, \|v\|_s \, \|w\|_s \, . \tag{7}$$

It is convenient to rescale the norm such that $C_s = 1$.

2.4. Condition of small initial data

We assume that the initial position and velocity have small norms in H^{s+1} and H^s, resp., for an $s \ge \sigma + 1$ with σ of the non-resonance condition (5):

$$\Big(\|u(\cdot,0)\|_{s+1}^2 + \|\partial_t u(\cdot,0)\|_s^2 \Big)^{1/2} \le \varepsilon. \tag{8}$$

This is equivalent to requiring

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s+1} \Big(\frac{\omega_j}{2} \, |u_j(0)|^2 + \frac{1}{2\omega_j} \, |\partial_t u_j(0)|^2 \Big) \le \frac{1}{2} \, \varepsilon^2. \tag{9}$$

2.5. Long-time near-conservation of harmonic actions

Along every real solution $u(x,t) = \sum_{j=-\infty}^{\infty} u_j(t) \, \mathrm{e}^{\mathrm{i}jx}$ to the linear wave equation $\partial_t^2 u - \partial_x^2 u + \rho u = 0$, the *actions* (energy divided by frequency)

$$I_j(t) = \frac{\omega_j}{2} \, |u_j(t)|^2 + \frac{1}{2\omega_j} \, |\partial_t u_j(t)|^2$$

remain constant in time. For real solutions as considered here, we have $u_{-j} = \overline{u_j}$ and hence $I_{-j} = I_j$. For the *nonlinear* equation (1) with a smooth real nonlinearity satisfying $g(0) = g'(0) = 0$, and under conditions (5) and (8), we will show that the actions I_j and in fact also their weighted sums

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s+1} I_j(t) = \frac{1}{2} \|u(\cdot,t)\|_{s+1}^2 + \frac{1}{2} \|\partial_t u(\cdot,t)\|_s^2 \, ,$$

remain constant up to small deviations over long times.

Theorem 1. *Under the non-resonance condition (5) and assumption (8) on the initial data with $s \ge \sigma + 1$, the estimate*

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \, \frac{|I_\ell(t) - I_\ell(0)|}{\varepsilon^2} \le C\varepsilon \qquad \textit{for} \quad 0 \le t \le \varepsilon^{-N+1}$$

holds with a constant C which depends on s, N, and C_0, but not on ε and t.

This result is closely related to results by Bambusi [1] and Bourgain [5]. In particular, Bambusi shows that, under the non-resonance condition (6) and with the same assumption on the initial data, there is the estimate $|I_\ell(t) - I_\ell(0)|/\varepsilon^2 \le C\varepsilon\, \omega_\ell^{-2(s+1)}$, which is close to the above bound. Theorem 1 implies, in particular, that the same norm that specifies the smallness condition on the initial data, remains nearly constant along the solution over long times: for $t \le \varepsilon^{-N+1}$,

$$\|u(\cdot,t)\|_{s+1}^2 + \|\partial_t u(\cdot,t)\|_s^2 = \|u(\cdot,0)\|_{s+1}^2 + \|\partial_t u(\cdot,0)\|_s^2 + \mathcal{O}(\varepsilon^3)\,.$$

This could also be obtained as an immediate consequence of the theory of [1]. Theorem 1 can be further interpreted as saying that the solution $(u(t), \partial_t u(t))$ stays close to an infinite-dimensional torus in the $H^{s+1/2}$ norm for long times. This improves slightly on [1], where such an estimate is obtained only in weaker norms.

We remark that for *complex* solutions of (1) with a complex differentiable nonlinearity g, the statement of Theorem 1 remains valid with

$$I_\ell(t) = \frac{\omega_\ell}{2} u_{-\ell}(t)\, u_\ell(t) + \frac{1}{2\omega_\ell}\, \partial_t u_{-\ell}(t)\, \partial_t u_\ell(t),$$

with the same proof.

We emphasize that the main novelty of the present work is not in the result, but in the technique of proof via invariance properties of the system of equations that determine the coefficient functions in the modulated Fourier expansion (2). This approach is completely different from the techniques in [1,5] and turns out to be applicable also to numerical discretizations of (1), since it involves no transformations of coordinates.

2.6. Illustration of the near-conservation of actions

In this subsection we give numerical results that show long-time near-conservation of actions even in situations that are not covered by Theorem 1: for initial data that are not very smooth and, more remarkably, near-conservation of the actions corresponding to high frequencies even for initial data that are not small. At present we have no rigorous explanation of these phenomena.

We consider the nonlinear wave equation (1) with $\rho = 1$ and nonlinearity $g(u) = u^2$, subject to periodic boundary conditions. As initial data we choose

$$u(x,0) = \varepsilon\Big(1 - \frac{x^2}{\pi^2}\Big)^2, \qquad \partial_t u(x,0) = 0 \qquad \text{for} \quad -\pi \le x \le \pi. \tag{10}$$

The 2π-periodic extension of $u(x,0)$ has a jump in the third derivative, so that its Fourier coefficients $u_j(0)$ decay like j^{-4}. This function therefore lies in H^s with $s \le 3$.

Figure 1 shows the first 32 functions $I_\ell(t)$ on the time interval $[0, 1000]$ (they are computed numerically with high precision). We have chosen a

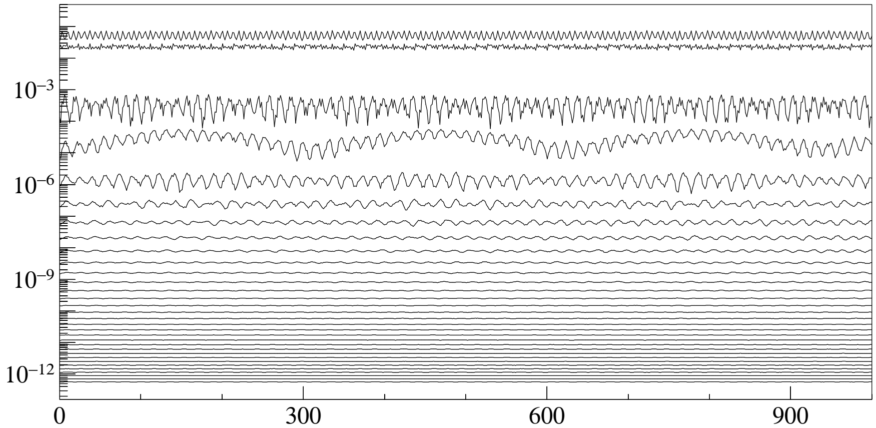

Fig. 1. Near-conservation of actions; the first 32 actions $I_\ell(t)$ are plotted as functions of time.

large $\varepsilon = 0.5$, so that we are able to see oscillations at least in the low frequency modes. The higher the frequency, the better the relative error of the corresponding action is conserved. For ε smaller than 0.1 only horizontal straight lines could be observed. Further experiments with this example have shown that the qualitative behaviour of Fig. 1 is insensitive with respect to the value of ρ, as long as it is not too small, and the good conservation holds on much longer time intervals.

3. The modulated Fourier expansion

Our principal tool for the long-time analysis of the nonlinearly perturbed wave equation is a short-time expansion constructed in this section.

3.1. Statement of result

We will prove the following result, where we use the abbreviation (3) and, for $\mathbf{k} = (k_\ell)_{\ell \geq 0}$ with integers k_ℓ and $\|\mathbf{k}\| = \sum_\ell |k_\ell|$, we set

$$[\![\mathbf{k}]\!] = \begin{cases} \frac{1}{2}(\|\mathbf{k}\| + 1), & \mathbf{k} \neq 0 \\ \frac{3}{2}, & \mathbf{k} = 0. \end{cases} \tag{11}$$

Theorem 2. *Consider the nonlinear wave equation (1) with frequencies ω_j satisfying the non-resonance condition (5), and with small initial data bounded by (8) with $s \geq \sigma + 1$. Then, the solution u admits an expansion (2),*

$$u(x,t) = \sum_{\|\mathbf{k}\| \leq 2N} z^{\mathbf{k}}(x, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} + r(x,t), \tag{12}$$

where the remainder is bounded by

$$\|r(\cdot,t)\|_{s+1} + \|\partial_t r(\cdot,t)\|_s \le C_1\,\varepsilon^N \qquad \textit{for} \quad 0 \le t \le \varepsilon^{-1}. \tag{13}$$

On this time interval, the modulation functions $z^{\mathbf{k}}$ *are bounded by*

$$\sum_{\|\mathbf{k}\|\le 2N} \left(\frac{\boldsymbol{\omega}^{|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}} \|z^{\mathbf{k}}(\cdot,\varepsilon t)\|_s \right)^2 \le C_2\,. \tag{14}$$

Bounds of the same type hold for any fixed number of derivatives of $z^{\mathbf{k}}$ *with respect to the slow time* $\tau = \varepsilon t$. *Moreover, the modulation functions satisfy* $z_{-j}^{-\mathbf{k}} = \overline{z_j^{\mathbf{k}}}$. *The constants* C_1 *and* C_2 *are independent of* ε, *but depend on* N, s, *on* C_0 *of (5), and on bounds of derivatives of the nonlinearity* g.

Apart from the relation $z_{-j}^{-\mathbf{k}} = \overline{z_j^{\mathbf{k}}}$, the theorem and its proof remain unchanged for *complex* solutions of (1) with a complex differentiable nonlinearity.

3.2. Formal modulation equations

Formally inserting the ansatz (2) into (1), equating terms with the same exponential $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t+\mathrm{i}jx}$ and Taylor expansion of g lead to the condition

$$\begin{aligned}\big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big) z_j^{\mathbf{k}} + 2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot z_j^{\mathbf{k}} \;+\; \varepsilon^2 \ddot z_j^{\mathbf{k}} \\ + \mathcal{F}_j \sum_m \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m} = 0\,.\end{aligned} \tag{15}$$

Here, $\mathcal{F}_j v = v_j$ denotes the jth Fourier coefficient of a function $v \in L^2(\mathbb{T})$, and the dots $(\cdot)$ on $z_j^{\mathbf{k}}(\tau)$ symbolize derivatives with respect to $\tau = \varepsilon t$. From this formal consideration, it becomes obvious that there will be three groups of modulation functions $z_j^{\mathbf{k}}$: for $\mathbf{k} = \pm\langle j\rangle$, the first term vanishes and the second term with the time derivative $\dot z_j^{\mathbf{k}}$ can be viewed as the dominant term. For $\mathbf{k} \ne \pm\langle j\rangle$, the first term is dominant if $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \ge \varepsilon^{1/2}$. Else, we simply set $z_j^{\mathbf{k}} \equiv 0$ and we will use the non-resonance condition (5) to ensure that the defect in (15) is only of size $\mathcal{O}(\varepsilon^{N+1})$ in an appropriate Sobolev-type norm.

In addition, the initial conditions $\widetilde u(\cdot,0) = u(\cdot,0)$ and $\partial_t \widetilde u(\cdot,0) = \partial_t u(\cdot,0)$ need to be taken care of. They will yield the initial conditions for the functions $z_j^{\pm\langle j\rangle}$:

$$\sum_{\mathbf{k}} z_j^{\mathbf{k}}(0) = u_j(0)\,, \qquad \sum_{\mathbf{k}} \Big(\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}}(0) + \varepsilon \dot z_j^{\mathbf{k}}(0) \Big) = \partial_t u_j(0). \tag{16}$$

3.3. Reverse Picard iteration

We now turn to an iterative construction of the functions $z_j^{\mathbf{k}}$ such that after $4N$ iteration steps, the defect in equations (15) and (16) is of size $\mathcal{O}(\varepsilon^{N+1})$ in the H^s norm. The iteration procedure we employ can be viewed as a reverse Picard iteration on (15) and (16): indicating by $[\cdot]^n$ that the nth iterate of all appearing variables $z_j^{\mathbf{k}}$ is taken within the bracket, we set for $\mathbf{k} = \pm\langle j\rangle$

$$\pm 2\mathrm{i}\varepsilon\omega_j \Big[\dot z_j^{\pm\langle j\rangle}\Big]^{n+1} = -\Big[\varepsilon^2 \ddot z_j^{\pm\langle j\rangle} + \mathcal{F}_j \sum_{m=2}^{N} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\pm\langle j\rangle} \frac{g^{(m)}(0)}{m!}\, z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m}\Big]^n$$

and for $\mathbf{k} \neq \pm\langle j\rangle$ and j with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \geq \varepsilon^{1/2}$ we set

$$\begin{aligned}\big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big)\Big[z_j^{\mathbf{k}}\Big]^{n+1} = -\ &\Big[2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot z_j^{\mathbf{k}} + \varepsilon^2 \ddot z_j^{\mathbf{k}} \\ &+\ \mathcal{F}_j \sum_{m=2}^{N} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{1}{m!}\, g^{(m)}(0)\, z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m}\Big]^n,\end{aligned}$$

whereas we let $z_j^{\mathbf{k}} = 0$ for $\mathbf{k} \neq \pm\langle j\rangle$ with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$.

On the initial conditions we iterate by

$$\begin{aligned}\Big[z_j^{\langle j\rangle}(0) + z_j^{-\langle j\rangle}(0)\Big]^{n+1} &= \Big[u_j(0) - \sum_{\mathbf{k}\neq\pm\langle j\rangle} z_j^{\mathbf{k}}(0)\Big]^n \\ \mathrm{i}\omega_j\Big[z_j^{\langle j\rangle}(0) - z_j^{-\langle j\rangle}(0)\Big]^{n+1} &= \Big[\partial_t u_j(0) - \sum_{\mathbf{k}\neq\pm\langle j\rangle} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}}(0) - \varepsilon \sum_{\|\mathbf{k}\|\leq K} \dot z_j^{\mathbf{k}}(0)\Big]^n.\end{aligned}$$

In all the above formulas, we tacitly assume $\|\mathbf{k}\| \leq K = 2N$ and $\|\mathbf{k}^i\| \leq K$. In each iteration step, we thus have an initial value problem of first-order differential equations for $z_j^{\pm\langle j\rangle}$ ($j \in \mathbb{Z}$) and algebraic equations for $z_j^{\mathbf{k}}$ with $\mathbf{k} \neq \pm\langle j\rangle$.

The starting iterates ($n = 0$) are chosen as $z_j^{\mathbf{k}} = 0$ for $\mathbf{k} \neq \pm\langle j\rangle$, and $z_j^{\pm\langle j\rangle}(\tau) = z_j^{\pm\langle j\rangle}(0)$ with $z_j^{\pm\langle j\rangle}(0)$ determined from the above formula with right-hand sides $u(0)$ and $\partial_t u(0)$.

For real initial data we have $u_{-j}(0) = \overline{u_j(0)}$ and $\partial_t u_{-j}(0) = \overline{\partial_t u_j(0)}$, and we observe that the above iteration yields $\big[z_{-j}^{-\mathbf{k}}\big]^n = \overline{\big[z_j^{\mathbf{k}}\big]^n}$ for all iterates n and all $j, \mathbf{k}$ and hence gives real approximations (2).

3.4. Inequalities for the frequencies

We collect a few inequalities involving the frequencies ω_ℓ, which are needed later on. These inequalities only rely on the growth property $\omega_\ell \sim \ell$ for large ℓ, but do not depend on any diophantine relations between the frequencies.

Lemma 2. *For $s > \frac{1}{2}$,*

$$\sum_{\|\mathbf{k}\| \le K} \boldsymbol{\omega}^{-2s|\mathbf{k}|} \le C_{K,s} < \infty \,, \tag{17}$$

where we have used the short-hand notation (3)*. For $s > \frac{1}{2}$ and $m \ge 2$, we have*

$$\sup_{\|\mathbf{k}\| \le K} \sum_{\mathbf{k}^1 + \ldots + \mathbf{k}^m = \mathbf{k}} \frac{\boldsymbol{\omega}^{-2s(|\mathbf{k}^1| + \cdots + |\mathbf{k}^m|)}}{\boldsymbol{\omega}^{-2s|\mathbf{k}|}} \le C_{m,K,s} < \infty \,, \tag{18}$$

where the sum is taken over $(\mathbf{k}^1, \ldots, \mathbf{k}^m)$ satisfying $\|\mathbf{k}^i\| \le K$. For $s \ge 1$, we further have

$$\sup_{\|\mathbf{k}\| \le K} \frac{\sum_{\ell \ge 0} |k_\ell| \, \omega_\ell^{2s+1}}{\boldsymbol{\omega}^{2s|\mathbf{k}|} \, (1 + |\mathbf{k} \cdot \boldsymbol{\omega}|)} \le C_{K,s} < \infty \,. \tag{19}$$

Proof. We notice that

$$\sum_{0 < \|\mathbf{k}\| \le K} \boldsymbol{\omega}^{-2s|\mathbf{k}|} \le 2 \sum_{q=1}^{K} \Bigl(\sum_{\ell=0}^{\infty} \omega_\ell^{-2s} \Bigr)^q .$$

The term $\omega_{\ell_1}^{-2sq_1} \cdots \omega_{\ell_m}^{-2sq_m}$ with $0 \le \ell_1 < \ldots < \ell_m$ and $q_1 + \ldots + q_m = q$ ($q_j > 0$) appears exactly 2^m times in the left-hand expression and $\binom{q}{q_1, \ldots, q_m}$ times in $\bigl(\sum_{\ell=0}^{\infty} \omega_\ell^{-2s}\bigr)^q$ (multinomial theorem). The estimate thus follows from the bound

$$2^{m-1} \le \binom{q}{q_1, \ldots, q_m}$$

which is obtained by induction on m. The statement of the first inequality (17) is thus a consequence of the facts that $\omega_\ell \sim \ell$ and $\sum_{\ell \ge 1} \ell^{-2s} < \infty$.

The second inequality (18) is proved as follows: whenever $\mathbf{k}^1 + \ldots + \mathbf{k}^m = \mathbf{k}$ and $\|\mathbf{k}^i\| \le K$, there exist q (with $0 \le q \le mK$) integers $\ell_1, \ldots, \ell_q \ge 0$ such that

$$|\mathbf{k}^1| + \cdots + |\mathbf{k}^m| = |\mathbf{k}| + \langle \ell_1 \rangle + \cdots + \langle \ell_q \rangle \,.$$

Conversely, for any choice of non-negative integers $\ell_1, \ldots, \ell_q$ with $q \le mK$, the number of $(\mathbf{k}^1, \ldots, \mathbf{k}^m)$ satisfying $\mathbf{k}^1 + \ldots + \mathbf{k}^m = \mathbf{k}$ and the above equation, is bounded by a constant $M_{m,K}$. Therefore,

$$\begin{aligned} \sum_{\mathbf{k}^1 + \ldots + \mathbf{k}^m = \mathbf{k}} \frac{\boldsymbol{\omega}^{-2s(|\mathbf{k}^1| + \cdots + |\mathbf{k}^m|)}}{\boldsymbol{\omega}^{-2s|\mathbf{k}|}} &\le M_{m,K} \sum_{q=0}^{mK} \sum_{\ell_1, \ldots, \ell_q \ge 0} \boldsymbol{\omega}^{-2s(\langle \ell_1 \rangle + \cdots + \langle \ell_q \rangle)} \\ &= M_{m,K} \sum_{q=0}^{mK} \sum_{\ell_1=0}^{\infty} \omega_{\ell_1}^{-2s} \cdots \sum_{\ell_q=0}^{\infty} \omega_{\ell_q}^{-2s} \le C_{m,K,s} \,, \end{aligned}$$

which proves (18).

For the proof of (19) we split the set of $\mathbf{k}$ with $\|\mathbf{k}\| \le K$ into two sets: for those $\mathbf{k}$ with $|k_L| = 1$ and $k_\ell = 0$ for all $\ell \ne L$ with $\omega_\ell \ge \omega_L^{1/2}$, we have $\sum_{\ell\ge 0} |k_\ell|\,\omega_\ell^{2s+1} \le \omega_L^{2s+1} + K\omega_L^{s+1/2}$ but $\boldsymbol{\omega}^{2s|\mathbf{k}|} \ge c_K\omega_L^{2s}$ with $c_K = \min(1,\rho^{2sK})$ and $|\mathbf{k}\cdot\boldsymbol{\omega}| \ge \omega_L - K\omega_L^{1/2}$, and hence the quotient of (19) is uniformly bounded on this subset of $\mathbf{k}$. On the complementary subset, we have $\sum_{\ell\ge 0}|k_\ell|\,\omega_\ell^{2s+1} \le K\omega_L^{2s+1}$ for the largest integer L for which $k_L \ne 0$, but here the product in the denominator is bounded from below as $\boldsymbol{\omega}^{2s|\mathbf{k}|} = \prod_{\ell\ge 0}\omega_\ell^{2s|k_\ell|} \ge c_K\big(\omega_L^{1/2}\big)^{2s}\cdot\omega_L^{2s}$, and hence the quotient is uniformly bounded on this subset for $s\ge 1$. This proves (19). $\square$

3.5. Rescaling and estimation of the nonlinear terms

Since we aim for (14), for the following analysis it is convenient to work with rescaled functions

$$c_j^{\mathbf{k}} = \frac{\boldsymbol{\omega}^{|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}}\, z_j^{\mathbf{k}}\,, \qquad c^{\mathbf{k}}(x) = \sum_{j=-\infty}^{\infty} c_j^{\mathbf{k}}\,\mathrm{e}^{\mathrm{i}jx} = \frac{\boldsymbol{\omega}^{|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}}\, z^{\mathbf{k}}(x) \tag{20}$$

where we use the notation (11) and (3). The superscripts $\mathbf{k}$ are in

$$\mathcal{K} = \{\mathbf{k} = (k_\ell)_{\ell\ge 0} \text{ with integers } k_\ell \,:\, \|\mathbf{k}\| \le K = 2N\},$$

and we will work in the Hilbert space

$$\mathbf{H}^s := (H^s)^{\mathcal{K}} = \{\mathbf{c} = (c^{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} : c^{\mathbf{k}} \in H^s\}$$

$$\text{with norm} \quad |\!|\!|\mathbf{c}|\!|\!|_s^2 = \sum_{\mathbf{k}\in\mathcal{K}} \|c^{\mathbf{k}}\|_s^2 = \sum_{\mathbf{k}\in\mathcal{K}}\sum_{j=-\infty}^{\infty} \omega_j^{2s}|c_j^{\mathbf{k}}|^2.$$

We now express the nonlinearity in (15),

$$v^{\mathbf{k}}(\mathbf{z}) = \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m}\,,$$

with $\|\mathbf{k}^i\| \le K$ in the sum, in rescaled variables as the map $\mathbf{f} = (f^{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} : \mathbf{H}^s \to \mathbf{H}^s$ given by

$$f^{\mathbf{k}}(\mathbf{c}) = \frac{\boldsymbol{\omega}^{|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}} \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \frac{\varepsilon^{[\![\mathbf{k}^1]\!]+\cdots+[\![\mathbf{k}^m]\!]}}{\boldsymbol{\omega}^{|\mathbf{k}^1|+\cdots+|\mathbf{k}^m|}}\, c^{\mathbf{k}^1}\ldots c^{\mathbf{k}^m}.$$

Using the triangle inequality, the inequality $(\sum_{m=1}^N a_m)^2 \le N\sum_{m=1}^N a_m^2$, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\!|\!|\mathbf{f}(\mathbf{c})|\!|\!|_s^2 &= \sum_{\|\mathbf{k}\|\le K} \|f^{\mathbf{k}}(\mathbf{c})\|_s^2 \\ &\le \sum_{\|\mathbf{k}\|\le K} N \sum_{m=2}^{N}\Big(\frac{g^{(m)}(0)}{m!}\Big)^2 \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \bigg(\frac{\varepsilon^{[\![\mathbf{k}^1]\!]+\cdots+[\![\mathbf{k}^m]\!]}}{\varepsilon^{[\![\mathbf{k}]\!]}}\,\frac{\boldsymbol{\omega}^{-(|\mathbf{k}^1|+\cdots+|\mathbf{k}^m|)}}{\boldsymbol{\omega}^{-|\mathbf{k}|}}\bigg)^2 \\ &\qquad\qquad \times \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \|c^{\mathbf{k}^1}\ldots c^{\mathbf{k}^m}\|_s^2\,. \end{aligned}$$

Since H^s is a normed algebra, and since we have the bound (18) (with 1 in place of s there) and the obvious lower estimate $[\![\mathbf{k}^1]\!] + \cdots + [\![\mathbf{k}^m]\!] \geq \frac{m-1}{2} + [\![\mathbf{k}]\!]$, this is further estimated as

$$
\begin{aligned}
&\sum_{\|\mathbf{k}\| \leq K} \|f^{\mathbf{k}}(\mathbf{c})\|_s^2 \\
&\quad \leq N \sum_{m=2}^{N} \Big(\frac{g^{(m)}(0)}{m!} \Big)^2 \varepsilon^{m-1} C_{m,K,1} \sum_{\|\mathbf{k}\| \leq K} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} \|c^{\mathbf{k}^1}\|_s^2 \ldots \|c^{\mathbf{k}^m}\|_s^2 \\
&\quad \leq N \sum_{m=2}^{N} \Big(\frac{g^{(m)}(0)}{m!} \Big)^2 \varepsilon^{m-1} C_{m,K,1} \Big(\sum_{\|\mathbf{k}\| \leq K} \|c^{\mathbf{k}}\|_s^2 \Big)^m = \varepsilon\, P(|\!|\!|\mathbf{c}|\!|\!|_s^2) \qquad (21)
\end{aligned}
$$

where the polynomial $P(\mu) = N \sum_{m=2}^{N} \Big(\frac{g^{(m)}(0)}{m!} \Big)^2 C_{m,K,1}\, \varepsilon^{m-2} \mu^m$ has coefficients bounded independently of ε.

For $\mathbf{k} = \pm\langle j \rangle$ we note that if $m \geq 2$ and $\mathbf{k}^1 + \cdots + \mathbf{k}^m = \pm\langle j \rangle$, then necessarily $[\![\mathbf{k}^1]\!] + \cdots + [\![\mathbf{k}^m]\!] \geq 5/2$. Hence, for the restriction to this case the bound improves to a factor ε^3 instead of ε:

$$
\sum_{j=-\infty}^{\infty} \|f^{\pm\langle j \rangle}(\mathbf{c})\|_s^2 \leq \varepsilon^3\, P_1(|\!|\!|\mathbf{c}|\!|\!|_s^2), \qquad (22)
$$

where P_1 is another polynomial with coefficients bounded independently of ε.

Since H^s is a normed algebra and the map $\mathbf{f}$ is an absolutely convergent sum of polynomials in the functions $c^{\mathbf{k}}$, we also obtain that $\mathbf{f}$ is arbitrarily differentiable with correspondingly bounded derivatives on bounded subsets of $\mathbf{H}^s$.

Instead of (20), we could also have worked with a different rescaling:

$$
\widehat{c}_j^{\mathbf{k}} = \frac{\boldsymbol{\omega}^{s|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}}\, z_j^{\mathbf{k}}, \qquad \widehat{c}^{\mathbf{k}}(x) = \sum_{j=-\infty}^{\infty} \widehat{c}_j^{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}jx} = \frac{\boldsymbol{\omega}^{s|\mathbf{k}|}}{\varepsilon^{[\![\mathbf{k}]\!]}}\, z^{\mathbf{k}}(x), \qquad (23)
$$

considered in the space $\mathbf{H}^1 = (H^1)^{\mathcal{K}}$ with norm $|\!|\!|\widehat{\mathbf{c}}|\!|\!|_1^2 = \sum_{\|\mathbf{k}\| \leq K} \|\widehat{c}^{\mathbf{k}}\|_1^2$. For $\widehat{f}^{\mathbf{k}}$ defined in the same way as $f^{\mathbf{k}}$ above, but with $\boldsymbol{\omega}^{s|\mathbf{k}|}$ in place of $\boldsymbol{\omega}^{|\mathbf{k}|}$, we then have the bounds

$$
\begin{aligned}
\sum_{\|\mathbf{k}\| \leq K} \|\widehat{f}^{\mathbf{k}}(\widehat{\mathbf{c}})\|_1^2 &\leq \varepsilon P(|\!|\!|\widehat{\mathbf{c}}|\!|\!|_1^2) \\
\sum_{j=-\infty}^{\infty} \|\widehat{f}^{\pm\langle j \rangle}(\widehat{\mathbf{c}})\|_1^2 &\leq \varepsilon^3\, P_1(|\!|\!|\widehat{\mathbf{c}}|\!|\!|_1^2)\,.
\end{aligned} \qquad (24)
$$

3.6. Abstract reformulation of the iteration

For $\mathbf{c} = (c_j^{\mathbf{k}}) \in \mathbf{H}^s$ with $c_j^{\mathbf{k}} = 0$ for all $\mathbf{k} \neq \pm\langle j\rangle$ with $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| < \varepsilon^{1/2}$, we split the components of $\mathbf{c}$ corresponding to $\mathbf{k} = \pm\langle j\rangle$ and $\mathbf{k} \neq \pm\langle j\rangle$ and collect them in $\mathbf{a} = (a_j^{\mathbf{k}}) \in \mathbf{H}^s$ and $\mathbf{b} = (b_j^{\mathbf{k}}) \in \mathbf{H}^s$, respectively:

$$\begin{aligned} a_j^{\mathbf{k}} &= c_j^{\mathbf{k}} \quad \text{ if } \mathbf{k} = \pm\langle j\rangle, \quad \text{and 0 else} \\ b_j^{\mathbf{k}} &= c_j^{\mathbf{k}} \quad \text{ if } |\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \geq \varepsilon^{1/2}, \quad \text{and 0 else.} \end{aligned} \tag{25}$$

We then have $\mathbf{a}+\mathbf{b} = \mathbf{c}$ and $|||\mathbf{a}|||_s^2 + |||\mathbf{b}|||_s^2 = |||\mathbf{c}|||_s^2$. We define the multiplication operator on $\mathbf{H}^s$,

$$(\boldsymbol{\Omega}^{-1}\mathbf{c})_j^{\mathbf{k}} = \frac{1}{\omega_j + |\mathbf{k}\cdot\boldsymbol{\omega}|}\, c_j^{\mathbf{k}} \quad \text{ for } \quad \mathbf{c} \in \mathbf{H}^s,$$

and note in particular that $(\boldsymbol{\Omega}^{-1}\mathbf{c})_j^{\pm\langle j\rangle} = \frac{1}{2\omega_j} c_j^{\pm\langle j\rangle}$. In terms of $\mathbf{a}$ and $\mathbf{b}$, the iteration of Subsection 3.3 written in the scaled variables (20) then becomes of the form

$$\begin{aligned} \dot{\mathbf{a}}^{(n+1)} &= A\mathbf{a}^{(n)} + \boldsymbol{\Omega}^{-1}\mathbf{F}(\mathbf{a}^{(n)}, \mathbf{b}^{(n)}) \\ \mathbf{b}^{(n+1)} &= B\mathbf{b}^{(n)} + \boldsymbol{\Omega}^{-1}\mathbf{G}(\mathbf{a}^{(n)}, \mathbf{b}^{(n)}), \end{aligned} \tag{26}$$

with the linear differential operators A and B given by

$$(A\mathbf{a})_j^{\pm\langle j\rangle} = \pm\frac{\mathrm{i}\varepsilon}{2\omega_j}\ddot{a}_j^{\pm\langle j\rangle}\,, \qquad (B\mathbf{b})_j^{\mathbf{k}} = \frac{-2\mathrm{i}\varepsilon(\mathbf{k}\cdot\boldsymbol{\omega})}{\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2}\,\dot{b}_j^{\mathbf{k}} - \frac{\varepsilon^2}{\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2}\,\ddot{b}_j^{\mathbf{k}}\,,$$

for $\big|\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|\big| \geq \varepsilon^{1/2}$, and nonlinear maps $\mathbf{F}$ and $\mathbf{G}$ given by

$$\begin{aligned} \big(\mathbf{F}(\mathbf{a},\mathbf{b})\big)_j^{\pm\langle j\rangle} &= \pm\mathrm{i}\varepsilon^{-1} f_j^{\pm\langle j\rangle}(\mathbf{a}+\mathbf{b}) \\ \big(\mathbf{G}(\mathbf{a},\mathbf{b})\big)_j^{\mathbf{k}} &= -\frac{\varepsilon^{1/2}}{\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|}\,\varepsilon^{-1/2} f_j^{\mathbf{k}}(\mathbf{a}+\mathbf{b})\,. \end{aligned}$$

In view of (21) – (22), $\mathbf{F}$ and $\mathbf{G}$ are arbitrarily differentiable maps, bounded in $\mathbf{H}^s$ by $\mathcal{O}(\varepsilon^{1/2})$ and $\mathcal{O}(1)$, respectively, with all derivatives bounded in the same way on bounded subsets of $\mathbf{H}^s$. The loss of a factor $\varepsilon^{1/2}$ in the bound for $\mathbf{G}$ results from the condition $|\omega_j \pm \mathbf{k}\cdot\boldsymbol{\omega}| \geq \varepsilon^{1/2}$ in (25). We further note the bounds

$$\begin{aligned} &|||(A\mathbf{a})(\tau)|||_s \leq C\varepsilon\, |||\ddot{\mathbf{a}}(\tau)|||_s\,, \\ &|||(B\mathbf{b})(\tau)|||_s \leq C\varepsilon^{1/2} |||\dot{\mathbf{b}}(\tau)|||_s + C\varepsilon^{3/2} |||\ddot{\mathbf{b}}(\tau)|||_s\,. \end{aligned} \tag{27}$$

The initial value for $\mathbf{a}^{(n+1)}$ is determined by an equation of the form

$$\mathbf{a}^{(n+1)}(0) = \mathbf{v} + P\mathbf{b}^{(n)}(0) + Q\big(\dot{\mathbf{a}}^{(n)}(0) + \dot{\mathbf{b}}^{(n)}(0)\big) \tag{28}$$

where the nonzero components $v_j^{\pm\langle j\rangle}$ of $\mathbf{v}$ are given by

$$v_j^{\pm\langle j\rangle} = \frac{\omega_j}{\varepsilon}\Big(\frac{1}{2}u_j(0) \pm \frac{1}{2}(\mathrm{i}\omega_j)^{-1}\partial_t u_j(0)\Big)$$

so that $\mathbf{v}$ is bounded in $\mathbf{H}^s$ by the assumption on the initial values, and with the operators P and Q given by

$$(P\mathbf{b})_j^{\pm\langle j\rangle} = -\frac{1}{2\varepsilon}\sum_{\mathbf{k}\neq\pm\langle j\rangle}\frac{\varepsilon^{[\![\mathbf{k}]\!]}}{\boldsymbol{\omega}^{|\mathbf{k}|}}(\omega_j\pm\mathbf{k}\cdot\boldsymbol{\omega})b_j^{\mathbf{k}}$$
$$(Q\mathbf{c})_j^{\pm\langle j\rangle} = \pm\frac{\mathrm{i}}{2}\sum_{\|\mathbf{k}\|\leq K}\frac{\varepsilon^{[\![\mathbf{k}]\!]}}{\boldsymbol{\omega}^{|\mathbf{k}|}}c_j^{\mathbf{k}}\,,$$

for which we have the bounds, using (17) with 1 in the role of s there,

$$|||P\mathbf{b}|||_s \leq C\,|||\boldsymbol{\Omega}\mathbf{b}|||_s\,, \qquad |||Q\mathbf{c}|||_s \leq C\varepsilon\,|||\mathbf{c}|||_s\,.$$

The starting iterate is $\mathbf{a}^{(0)}=\mathbf{v}$ and $\mathbf{b}^{(0)}=0$.

3.7. Bounds of the modulation functions

The iterates $\mathbf{a}^{(n)}$ and $\mathbf{b}^{(n)}$ and, by differentiation of the iteration equations (26), also their derivatives with respect to the slow time $\tau=\varepsilon t$ are thus bounded in $\mathbf{H}^s$ for $0\leq\tau\leq 1$ and $n\leq 4N$: more precisely, the $4N$-th iterates satisfy, with constants depending on N,

$$|||\mathbf{a}^{(4N)}(0)|||_s \leq C\,, \qquad |||\boldsymbol{\Omega}\dot{\mathbf{a}}^{(4N)}|||_s \leq C\varepsilon^{1/2}\,, \qquad |||\boldsymbol{\Omega}\mathbf{b}^{(4N)}|||_s \leq C\,. \quad (29)$$

We also obtain the bound $|||\boldsymbol{\Omega}\dot{\mathbf{b}}^{(4N)}|||_s \leq C$ and similarly for higher derivatives with respect to $\tau=\varepsilon t$. For $z_j^{\mathbf{k}}=\varepsilon^{[\![\mathbf{k}]\!]}\boldsymbol{\omega}^{-|\mathbf{k}|}\,c_j^{\mathbf{k}}$ with $(c_j^{\mathbf{k}})=\mathbf{c}=\mathbf{c}^{(4N)}=\mathbf{a}^{(4N)}+\mathbf{b}^{(4N)}$, the bounds for $\mathbf{a}$ and $\mathbf{b}$ together yield the bound (14).

Refined estimates are obtained for components corresponding to the non-resonant set $\mathcal{N}=\{(j,\mathbf{k})\,:\,\big|\omega_j^2-(\mathbf{k}\cdot\boldsymbol{\omega})^2\big|\geq c\}$, where $c>0$ is independent of ε. For indices in this set we gain the factor $\varepsilon^{1/2}$ in the estimate of $\boldsymbol{\Omega}^{-1}\mathbf{G}$, so that from the iteration (26) we obtain, with $\mathbf{b}=\mathbf{b}^{(4N)}$,

$$\Big(\sum_{(j,\mathbf{k})\in\mathcal{N}}\omega_j^{2s}\big|b_j^{\mathbf{k}}\big|^2\Big)^{1/2} \leq C\varepsilon^{1/2}\,. \qquad (30)$$

In particular $\mathcal{N}$ contains all $(j,\mathbf{k})$ with $\mathbf{k}=\mathbf{0}$, and those with $\mathbf{k}=\pm\langle j_1\rangle\pm\langle j_2\rangle$ and $j=j_1+j_2$ with all combinations of signs. Using (22), an even better bound is obtained for $\|\mathbf{k}\|=1$:

$$\Big(\sum_{\|\mathbf{k}\|=1}\|\boldsymbol{\Omega}b^{\mathbf{k}}\|_s^2\Big)^{1/2} \leq C\varepsilon^{3/2}\,. \qquad (31)$$

The bounds (29) imply $|||\mathbf{c}(\tau)-\mathbf{a}(0)|||_{s+1} \le C$ for $\mathbf{c}=\mathbf{c}^{(4N)}$ and $\mathbf{a}=\mathbf{a}^{(4N)}$, and hence give a bound of the expansion (2) in the H^{s+1} norm:

$$
\begin{aligned}
\|\widetilde{u}(\cdot,t)\|_{s+1}^2 &= \sum_{j=-\infty}^{\infty} \omega_j^{2(s+1)} \Big| \sum_{\|\mathbf{k}\|\le K} z_j^{\mathbf{k}}(\varepsilon t) \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big|^2 \\
&\le \sum_{j=-\infty}^{\infty} \omega_j^{2(s+1)} \Big(\frac{\varepsilon}{\omega_j} \big(|a_j^{\langle j\rangle}(0)| + |a_j^{-\langle j\rangle}(0)| \big) \\
&\qquad\qquad + \sum_{\|\mathbf{k}\|\le K} \frac{\varepsilon^{[\![\mathbf{k}]\!]}}{\boldsymbol{\omega}^{|\mathbf{k}|}} \, |c_j^{\mathbf{k}}(\varepsilon t) - a_j^{\mathbf{k}}(0)| \Big)^2 \\
&\le 4\varepsilon^2 |||\mathbf{a}(0)|||_s^2 + C_{K,1}\,\varepsilon^2 \sum_{j=-\infty}^{\infty} \omega_j^{2(s+1)} \sum_{\|\mathbf{k}\|\le K} |c_j^{\mathbf{k}}(\varepsilon t) - a_j^{\mathbf{k}}(0)|^2 \\
&= 4\varepsilon^2 |||\mathbf{a}(0)|||_s^2 + C_{K,1}\,\varepsilon^2 |||\mathbf{c}(\varepsilon t) - \mathbf{a}(0)|||_{s+1}^2 ,
\end{aligned}
$$

where we noted $a_j^{\mathbf{k}}=0$ for $\mathbf{k}\ne\pm\langle j\rangle$ and where we used the Cauchy-Schwarz inequality and (17) in the last inequality. So we have

$$
\|\widetilde{u}(\cdot,t)\|_{s+1} \le C\varepsilon \quad \text{for} \quad t\le\varepsilon^{-1}. \tag{32}
$$

With the alternative scaling (23) we obtain, again for $\tau=\varepsilon t\le 1$,

$$
|||\widehat{\mathbf{a}}^{(4N)}(0)|||_1 \le C\,, \qquad |||\boldsymbol{\Omega}\dot{\widehat{\mathbf{a}}}^{(4N)}|||_1 \le C\varepsilon^{1/2}\,, \qquad |||\boldsymbol{\Omega}\widehat{\mathbf{b}}^{(4N)}|||_1 \le C\,. \tag{33}
$$

The bounds for $\widehat{\mathbf{a}}$ follow trivially from (29) and $|||\widehat{\mathbf{a}}|||_1 = |||\mathbf{a}|||_s$, those for $\widehat{\mathbf{b}}$ are obtained from the rescaled iteration (26) for $\widehat{\mathbf{b}}^{(n)}$ and the bounds (24), without consideration of the starting values for $\widehat{\mathbf{a}}^{(n)}$. We also obtain an analogous improvement to (30) and

$$
\Big(\sum_{\|\mathbf{k}\|=1} \|\boldsymbol{\Omega}\widehat{b}^{\mathbf{k}}\|_1^2 \Big)^{1/2} \le C\varepsilon^{3/2}\,. \tag{34}
$$

In addition to these bounds, we also obtain that the map

$$
B_\varepsilon \subset H^{s+1}\times H^s \to \mathbf{H}^1 :\ (u(\cdot,0),\partial_t u(\cdot,0)) \mapsto \widehat{\mathbf{c}}(0)
$$

(with B_ε the ball of radius ε centered at 0) is Lipschitz continuous with a Lipschitz constant proportional to ε^{-1}: at $t=0$,

$$
|||\widehat{\mathbf{a}}_2 - \widehat{\mathbf{a}}_1|||_1^2 + |||\boldsymbol{\Omega}(\widehat{\mathbf{b}}_2 - \widehat{\mathbf{b}}_1)|||_1^2 \le \frac{C}{\varepsilon^2}\Big(\|u_2-u_1\|_{s+1}^2 + \|\partial_t u_2 - \partial_t u_1\|_s^2 \Big). \tag{35}
$$

3.8. Defects

For the functions $z_j^{\mathbf{k}}$ obtained as the $4N$-th iterate of the reverse Picard iteration of Section 3.3, we consider the defect $\mathbf{d} = (d_j^{\mathbf{k}})$ in (15),

$$d_j^{\mathbf{k}} = \big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big) z_j^{\mathbf{k}} + 2\mathrm{i}\varepsilon(\mathbf{k}\cdot\omega)\dot z_j^{\mathbf{k}} \;+\; \varepsilon^2 \ddot z_j^{\mathbf{k}} \\ + \mathcal{F}_j \sum_{m=2}^{N} \frac{1}{m!}\, g^{(m)}(0) \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m}\,. \tag{36}$$

This is to be considered for $\|\mathbf{k}\| \le NK$, where we set $z_j^{\mathbf{k}} = 0$ for $\|\mathbf{k}\| > K = 2N$. The approximation $\widetilde u$ given by (2) inserted into the wave equation (1) yields the defect

$$\delta = \partial_t^2 \widetilde u - \partial_x^2 \widetilde u + \rho \widetilde u + g(\widetilde u) \tag{37}$$

with

$$\delta(x,t) = \sum_{\|\mathbf{k}\|\le NK} d^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} + R_{N+1}(\widetilde u(x,t)), \tag{38}$$

where R_{N+1} is the remainder term of the Taylor expansion of g after N terms. By (32), we have $\|R_{N+1}(\widetilde u)\|_{s+1} \le C\varepsilon^{N+1}$. We need to bound

$$\begin{aligned} \Big\| \sum_{\|\mathbf{k}\|\le NK} d^{\mathbf{k}}(\cdot,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big\|_s^2 &= \sum_{j=-\infty}^{\infty} \omega_j^{2s} \Big| \sum_{\|\mathbf{k}\|\le NK} d_j^{\mathbf{k}}(\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big|^2 \\ &\le C_{NK,1} \sum_{j=-\infty}^{\infty} \sum_{\|\mathbf{k}\|\le NK} \omega_j^{2s} \Big| \boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}(\varepsilon t) \Big|^2 \\ &= C_{NK,1} \sum_{\|\mathbf{k}\|\le NK} \Big\| \boldsymbol{\omega}^{|\mathbf{k}|}\, d^{\mathbf{k}}(\cdot,\varepsilon t) \Big\|_s^2 . \end{aligned} \tag{39}$$

For the inequality we have used (17) with 1 in place of s and the Cauchy-Schwarz inequality. In the next three subsections we estimate the right-hand side of (39) by $C\varepsilon^{2(N+1)}$, separately for truncated modes $\|\mathbf{k}\| > K$ and near-resonant modes $(j,\mathbf{k}) \in \mathcal{R}_\varepsilon$, where $z_j^{\mathbf{k}} = 0$ in both cases, and for non-resonant modes with $z_j^{\mathbf{k}}$ constructed above.

3.9. Defect in the truncated modes

For $\|\mathbf{k}\| > K$ we have $z_j^{\mathbf{k}} = 0$, and the defect reads

$$d_j^{\mathbf{k}} = \mathcal{F}_j \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} z^{\mathbf{k}^1}\ldots z^{\mathbf{k}^m} = \varepsilon^{[\![\mathbf{k}]\!]}\, \boldsymbol{\omega}^{-|\mathbf{k}|}\, f_j^{\mathbf{k}}$$

with $|\!|\!|\mathbf{f}|\!|\!|_s^2 \le C_s\,\varepsilon$ by (29) and (21), used with NK in place of K. We then have

$$\sum_{\|\mathbf{k}\|>K} \sum_{j=-\infty}^{\infty} \omega_j^{2s} \big| \boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}} \big|^2 = \sum_{\|\mathbf{k}\|>K} \sum_{j=-\infty}^{\infty} \omega_j^{2s} \big| f_j^{\mathbf{k}} \big|^2\, \varepsilon^{2[\![\mathbf{k}]\!]}$$

and hence, since $2[\![\mathbf{k}]\!] = \|\mathbf{k}\| + 1 \geq K + 2 = 2(N+1)$,

$$\sum_{\|\mathbf{k}\|>K} \sum_{j=-\infty}^{\infty} \omega_j^{2s} \big|\boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}\big|^2 \leq C_s \varepsilon^{2(N+1)}. \tag{40}$$

3.10. Defect in the near-resonant modes

For $(j,\mathbf{k})$ in the set $\mathcal{R}_\varepsilon$ of near-resonances defined by (4) we have set $z_j^{\mathbf{k}} = 0$. The defect corresponding to the near-resonant modes is thus

$$d_j^{\mathbf{k}} = \mathcal{F}_j \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} z^{\mathbf{k}^1} \ldots z^{\mathbf{k}^m} = \varepsilon^{[\![\mathbf{k}]\!]}\, \boldsymbol{\omega}^{-s|\mathbf{k}|}\, \widehat{f}_j^{\mathbf{k}}$$

with $|\!|\!|\widehat{\mathbf{f}}|\!|\!|_1^2 \leq C_1\varepsilon$ by (33) and (24). We then have

$$\begin{aligned}\sum_{(j,\mathbf{k})\in\mathcal{R}_\varepsilon} \omega_j^{2s} \big|\boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}\big|^2 &= \sum_{(j,\mathbf{k})\in\mathcal{R}_\varepsilon} \frac{\omega_j^{2(s-1)}}{\boldsymbol{\omega}^{2(s-1)|\mathbf{k}|}}\, \varepsilon^{2[\![\mathbf{k}]\!]}\, \omega_j^2 |\widehat{f}_j^{\mathbf{k}}|^2 \\ &\leq C_1 \sup_{(j,\mathbf{k})\in\mathcal{R}_\varepsilon} \frac{\omega_j^{2(s-1)}\, \varepsilon^{2[\![\mathbf{k}]\!]+1}}{\boldsymbol{\omega}^{2(s-1)|\mathbf{k}|}}\,.\end{aligned}$$

Condition (5) is formulated such that the supremum is bounded by $C_0^2\, \varepsilon^{2(N+1)}$, and hence

$$\sum_{(j,\mathbf{k})\in\mathcal{R}_\varepsilon} \omega_j^{2s} \big|\boldsymbol{\omega}^{|\mathbf{k}|}\, d_j^{\mathbf{k}}\big|^2 \leq C\varepsilon^{2(N+1)}. \tag{41}$$

3.11. Defect in the non-resonant modes

The scaled defect (36), as it appears in (39), reads as follows in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b} = \mathbf{a}^{(4N)} + \mathbf{b}^{(4N)}$ defined in the iteration (26), which corresponds to the rescaling (20):

$$\boldsymbol{\omega}^{|\mathbf{k}|} d_j^{\mathbf{k}} = \big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big)\varepsilon^{[\![\mathbf{k}]\!]}\, c_j^{\mathbf{k}} + 2\mathrm{i}(\mathbf{k}\cdot\omega)\varepsilon^{1+[\![\mathbf{k}]\!]}\, \dot{c}_j^{\mathbf{k}} \;+\; \varepsilon^{2+[\![\mathbf{k}]\!]}\, \ddot{c}_j^{\mathbf{k}} + \varepsilon^{[\![\mathbf{k}]\!]}\, f_j^{\mathbf{k}}(\mathbf{c}). \tag{42}$$

Expressing, for the cases $\mathbf{k} = \pm\langle j\rangle$ and $\big|\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|\big| > \varepsilon^{1/2}$, the nonlinearity in terms of the functions $\mathbf{F}$ and $\mathbf{G}$ of the iteration (26), we find

$$\omega_j d_j^{\pm\langle j\rangle} = \pm 2\mathrm{i}\omega_j\, \varepsilon^2 \Big(\big[\dot{a}_j^{\pm\langle j\rangle}\big]^{(4N)} - \big[\dot{a}_j^{\pm\langle j\rangle}\big]^{(4N+1)}\Big) \tag{43}$$

$$\boldsymbol{\omega}^{|\mathbf{k}|} d_j^{\mathbf{k}} = \big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big)\, \varepsilon^{[\![\mathbf{k}]\!]} \Big(\big[b_j^{\mathbf{k}}\big]^{(4N)} - \big[b_j^{\mathbf{k}}\big]^{(4N+1)}\Big) \tag{44}$$

with the second formula again valid for $\big|\omega_j - |\mathbf{k}\cdot\boldsymbol{\omega}|\big| > \varepsilon^{1/2}$. This suggests to reconsider the iteration (26) in the transformed variables $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ given as

$$\begin{aligned}\widetilde{a}_j^{\pm\langle j\rangle} &= (\boldsymbol{\alpha}\mathbf{a})_j^{\pm\langle j\rangle} := \pm\mathrm{i}\varepsilon^2\, a_j^{\pm\langle j\rangle} \\ \widetilde{b}_j^{\mathbf{k}} &= (\boldsymbol{\beta}\mathbf{b})_j^{\mathbf{k}} := \big(\omega_j^2 - (\mathbf{k}\cdot\boldsymbol{\omega})^2\big)\, \varepsilon^{[\![\mathbf{k}]\!]}\, b_j^{\mathbf{k}}\,.\end{aligned}$$

(We do not include the factor $2\omega_j$ in $\widetilde{a}_j^{\mathbf{k}}$, because we can bound $\boldsymbol{\Omega}\dot{\mathbf{a}}$ and $\mathbf{a}$ in $\mathbf{H}^s$, but not $\boldsymbol{\Omega}\mathbf{a}$.) In these variables the iteration (26) becomes

$$\begin{aligned} \dot{\widetilde{\mathbf{a}}}^{(n+1)} &= A\widetilde{\mathbf{a}}^{(n)} + \boldsymbol{\Omega}^{-1}\widetilde{\mathbf{F}}(\widetilde{\mathbf{a}}^{(n)},\widetilde{\mathbf{b}}^{(n)}), \\ \widetilde{\mathbf{b}}^{(n+1)} &= B\widetilde{\mathbf{b}}^{(n)} + \widetilde{\mathbf{G}}(\widetilde{\mathbf{a}}^{(n)},\widetilde{\mathbf{b}}^{(n)}), \end{aligned} \tag{45}$$

with the transformed nonlinearities

$$\widetilde{\mathbf{F}}(\widetilde{\mathbf{a}},\widetilde{\mathbf{b}}) = \boldsymbol{\alpha}\mathbf{F}(\boldsymbol{\alpha}^{-1}\widetilde{\mathbf{a}},\boldsymbol{\beta}^{-1}\widetilde{\mathbf{b}}), \qquad \widetilde{\mathbf{G}}(\widetilde{\mathbf{a}},\widetilde{\mathbf{b}}) = \boldsymbol{\beta}\boldsymbol{\Omega}^{-1}\mathbf{G}(\boldsymbol{\alpha}^{-1}\widetilde{\mathbf{a}},\boldsymbol{\beta}^{-1}\widetilde{\mathbf{b}}).$$

(In the definition of $\widetilde{\mathbf{G}}$ we have now included the factor $\boldsymbol{\Omega}^{-1}$, which therefore in the iteration no longer appears in front of $\widetilde{\mathbf{G}}$.) We note that

$$\widetilde{\mathbf{F}}(\widetilde{\mathbf{a}},\widetilde{\mathbf{b}}) + \widetilde{\mathbf{G}}(\widetilde{\mathbf{a}},\widetilde{\mathbf{b}}) = E\mathbf{f}(\boldsymbol{\alpha}^{-1}\widetilde{\mathbf{a}} + \boldsymbol{\beta}^{-1}\widetilde{\mathbf{b}}) \qquad \text{with} \quad (E\mathbf{f})_j^{\mathbf{k}} = \varepsilon^{[\![\mathbf{k}]\!]} f_j^{\mathbf{k}}.$$

The iteration (28) for the initial values becomes

$$\widetilde{\mathbf{a}}^{(n+1)}(0) = \alpha\mathbf{v} + \widetilde{P}\widetilde{\mathbf{b}}^{(n)}(0) + Q\dot{\widetilde{\mathbf{a}}}^{(n)}(0) + \widetilde{Q}\dot{\widetilde{\mathbf{b}}}^{(n)}(0)$$

with $\widetilde{P} = \boldsymbol{\alpha}P\boldsymbol{\beta}^{-1}$, $\widetilde{Q} = \boldsymbol{\alpha}Q\boldsymbol{\beta}^{-1}$ bounded by

$$|\!|\!|\widetilde{P}\widetilde{\mathbf{b}}|\!|\!|_s \le C\varepsilon^{1/2}|\!|\!|\widetilde{\mathbf{b}}|\!|\!|_s, \qquad |\!|\!|\widetilde{Q}\widetilde{\mathbf{b}}|\!|\!|_s \le C\varepsilon^{3/2}|\!|\!|\widetilde{\mathbf{b}}|\!|\!|_s. \tag{46}$$

With the aim of estimating the differences $[\Delta\dot{\widetilde{\mathbf{a}}}]^{(4N)} := [\dot{\widetilde{\mathbf{a}}}]^{(4N+1)} - [\dot{\widetilde{\mathbf{a}}}]^{(4N)}$, $[\Delta\widetilde{\mathbf{b}}]^{(4N)} := [\widetilde{\mathbf{b}}]^{(4N+1)} - [\widetilde{\mathbf{b}}]^{(4N)}$, and $[\Delta\widetilde{\mathbf{a}}]^{(4N)}(0) := [\widetilde{\mathbf{a}}]^{(4N+1)}(0) - [\widetilde{\mathbf{a}}]^{(4N)}(0)$, we first have to determine suitable Lipschitz bounds for the functions $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{G}}$. By repeating the computation of Subsection 3.5 for the partial derivatives of $f^{\mathbf{k}}(\mathbf{c})$ we find that, in an $\mathbf{H}^s$-neighbourhood of 0 where the bounds (29) hold, the derivatives of $\widetilde{F}$ with respect to $\widetilde{\mathbf{a}},\widetilde{\mathbf{b}}$ and of $\widetilde{G}$ with respect to $\widetilde{\mathbf{b}}$ are bounded in $\mathbf{H}^s$ by $\mathcal{O}(\varepsilon^{1/2})$, whereas that of $\widetilde{G}$ with respect to $\widetilde{\mathbf{a}}$ is bounded only by $\mathcal{O}(1)$. We thus have from (45)

$$\begin{aligned} |\!|\!|\boldsymbol{\Omega}[\Delta\dot{\widetilde{\mathbf{a}}}]^{(n+1)}|\!|\!|_s &\le C\varepsilon^{1/2}|\!|\!|[\Delta\widetilde{\mathbf{a}}]^{(n)}|\!|\!|_s + C\varepsilon^{1/2}|\!|\!|[\Delta\widetilde{\mathbf{b}}]^{(n)}|\!|\!|_s \\ &\quad + C\varepsilon|\!|\!|\boldsymbol{\Omega}[\Delta\ddot{\widetilde{\mathbf{a}}}]^{(n)}|\!|\!|_s \\ |\!|\!|[\Delta\widetilde{\mathbf{b}}]^{(n+1)}|\!|\!|_s &\le C|\!|\!|[\Delta\widetilde{\mathbf{a}}]^{(n)}|\!|\!|_s + C\varepsilon^{1/2}|\!|\!|[\Delta\widetilde{\mathbf{b}}]^{(n)}|\!|\!|_s \\ &\quad + C\varepsilon^{1/2}|\!|\!|[\Delta\dot{\widetilde{\mathbf{b}}}]^{(n)}|\!|\!|_s + C\varepsilon^{3/2}|\!|\!|[\Delta\ddot{\widetilde{\mathbf{b}}}]^{(n)}|\!|\!|_s \\ |\!|\!|[\Delta\widetilde{\mathbf{a}}(0)]^{(n+1)}|\!|\!|_s &\le C\varepsilon^{1/2}|\!|\!|[\Delta\widetilde{\mathbf{b}}]^{(n)}(0)|\!|\!|_s + C\varepsilon|\!|\!|[\Delta\dot{\widetilde{\mathbf{a}}}]^{(n)}(0)|\!|\!|_s \\ &\quad + C\varepsilon^{3/2}|\!|\!|[\Delta\dot{\widetilde{\mathbf{b}}}]^{(n)}(0)|\!|\!|_s, \end{aligned}$$

where we have used the estimates (27) for the operators A and B, and (46) for $\widetilde{P}$ and $\widetilde{Q}$. The presence of first and second derivatives in the right-hand side prevent a direct treatment of these inequalities. However, differentiation of (45) with respect to τ leads to the same estimates, where for all appearing functions the derivative is raised by one. Using the estimates (29) for higher derivatives with respect to τ, this procedure can be repeated so that similar estimates for higher derivatives are obtained. Let now

$$\eta_n := \max_{\ell=0,\ldots,2(4N-n)} \sup_{0\le\tau\le1} |||\boldsymbol{\Omega}[\Delta\widetilde{\mathbf{a}}^{(\ell+1)}]^{(n)}(\tau)|||_s$$
$$\mu_n := \max_{\ell=0,\ldots,2(4N-n)} \sup_{0\le\tau\le1} |||[\Delta\widetilde{\mathbf{b}}^{(\ell)}]^{(n)}(\tau)|||_s$$
$$\nu_n := |||[\Delta\widetilde{\mathbf{a}}]^{(n)}(0)|||_s,$$

where $[\Delta\widetilde{\mathbf{a}}^{(\ell+1)}]^{(n)}$ denotes the $(\ell+1)$th derivative of the nth iterate. Noticing that $|||[\Delta\widetilde{\mathbf{a}}]^{(n)}(\tau)|||_s \le |||[\Delta\widetilde{\mathbf{a}}]^{(n)}(0)|||_s + \sup_{0<\sigma<\tau} |||[\Delta\dot{\widetilde{\mathbf{a}}}]^{(n)}(\sigma)|||_s$, we obtain

$$\begin{pmatrix} \nu_{n+1} \\ \eta_{n+1} \\ \mu_{n+1} \end{pmatrix} \le C \begin{pmatrix} 0 & \varepsilon & \varepsilon^{1/2} \\ \varepsilon^{1/2} & \varepsilon^{1/2} & \varepsilon^{1/2} \\ 1 & 1 & \varepsilon^{1/2} \end{pmatrix} \begin{pmatrix} \nu_n \\ \eta_n \\ \mu_n \end{pmatrix}.$$

In the scaled variables $(\varepsilon^{-1/4}\nu_n, \varepsilon^{-1/4}\eta_n, \mu_n)$, the iteration matrix has norm $\mathcal{O}(\varepsilon^{1/4})$ in the maximum norm, which implies that

$$\max\bigl(\varepsilon^{-1/4}\nu_{4N}, \varepsilon^{-1/4}\eta_{4N}, \mu_{4N}\bigr) \le C_N \varepsilon^N \max\bigl(\varepsilon^{-1/4}\nu_0, \varepsilon^{-1/4}\eta_0, \mu_0\bigr).$$

Recalling $[\widetilde{\mathbf{a}}]^{(0)}(\tau) = \alpha\mathbf{v}$ and $[\widetilde{\mathbf{b}}]^{(0)}(\tau) = \mathbf{0}$, we have for $n = 0$

$$[\dot{\widetilde{\mathbf{a}}}]^{(1)} - [\dot{\widetilde{\mathbf{a}}}]^{(0)} = [\boldsymbol{\Omega}^{-1}\widetilde{\mathbf{F}}(\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})]^{(0)}, \qquad [\widetilde{\mathbf{b}}]^{(1)} - [\widetilde{\mathbf{b}}]^{(0)} = [\widetilde{\mathbf{G}}(\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}})]^{(0)},$$

and $[\widetilde{\mathbf{a}}]^{(1)}(0) - [\widetilde{\mathbf{a}}]^{(0)}(0) = \mathbf{0}$. All derivatives of these differences vanish identically. Using the bounds $\widetilde{\mathbf{F}} = \mathcal{O}(\varepsilon^{5/2})$ and $\widetilde{\mathbf{G}} = \mathcal{O}(\varepsilon^2)$, we thus obtain $\eta_0 = \mathcal{O}(\varepsilon^{5/2})$, $\mu_0 = \mathcal{O}(\varepsilon^2)$, and $\nu_0 = 0$, so that η_{4N}, μ_{4N}, and ν_{4N} are all of size $\mathcal{O}(\varepsilon^{N+2})$.

With (43)-(44), these bounds yield the desired bound for the defect,

$$\Bigl(\sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{|\mathbf{k}|} d^{\mathbf{k}}(\cdot,\tau)\|_s^2 \Bigr)^{1/2} \le C\varepsilon^{N+1} \quad \text{for} \quad \tau \le 1, \tag{47}$$

where we recall that here the sum is over non-resonant modes $(j,\mathbf{k}) \notin \mathcal{R}_\varepsilon$.

With the alternative scaling $\hat{c}_j^{\mathbf{k}} = \boldsymbol{\omega}^{s|\mathbf{k}|} z_j^{\mathbf{k}}$ we obtain in the same way

$$\Bigl(\sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{s|\mathbf{k}|} d^{\mathbf{k}}(\cdot,\tau)\|_1^2 \Bigr)^{1/2} \le C\varepsilon^{N+1} \quad \text{for} \quad \tau \le 1\,. \tag{48}$$

For the defect in the initial conditions (16) we obtain from $\nu_{4N} \le C\varepsilon^{N+2}$ that

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s} \Bigl| \omega_j \sum_{\|\mathbf{k}\|\le K} z_j^{\mathbf{k}}(0) - \omega_j\, u_j(0) \Bigr|^2 \le C\,\varepsilon^{2(N+1)} \tag{49}$$

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s} \Bigl| \sum_{\|\mathbf{k}\|\le K} \Bigl(\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega}) z_j^{\mathbf{k}}(0) + \varepsilon \dot{z}_j^{\mathbf{k}}(0) \Bigr) - \partial_t u_j(0) \Bigr|^2 \le C\,\varepsilon^{2(N+1)}\,. \tag{50}$$

3.12. Defect in the wave equation

We estimate the defect δ of (37). By (40), (41), and (47), we now have

$$\Big\| \sum_{\|\mathbf{k}\| \le NK} d^{\mathbf{k}}(\cdot, \varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \Big\|_s \le C\varepsilon^{N+1} \quad \text{for} \quad t \le \varepsilon^{-1},$$

so that indeed, by (38) and (39),

$$\|\delta(\cdot, t)\|_s \le C\varepsilon^{N+1} \quad \text{for} \quad t \le \varepsilon^{-1}. \tag{51}$$

We also note that, by (49)–(50), the deviations in the initial values are bounded by

$$\|\widetilde{u}(\cdot, 0) - u(\cdot, 0)\|_{s+1} + \|\partial_t \widetilde{u}(\cdot, 0) - \partial_t u(\cdot, 0)\|_s \le C\,\varepsilon^{N+1}. \tag{52}$$

3.13. Remainder term of the modulated Fourier expansion

Using the well-posedness of the nonlinear wave equation in $H^{s+1} \times H^s$, we now conclude from a small defect to a small error by a standard argument: we rewrite (1) and (37) in terms of the Fourier coefficients as

$$\begin{aligned} \partial_t^2 u_j + \omega_j^2\, u_j + \mathcal{F}_j g(u) &= 0 \\ \partial_t^2 \widetilde{u}_j + \omega_j^2\, \widetilde{u}_j + \mathcal{F}_j g(\widetilde{u}) &= \delta_j \end{aligned}$$

and subtract the equations. With the variation-of-constants formula, the error $r_j = u_j - \widetilde{u}_j$ satisfies

$$\begin{aligned} \begin{pmatrix} r_j(t) \\ \omega_j^{-1}\dot{r}_j(t) \end{pmatrix} &= \begin{pmatrix} \cos(\omega_j t) & \sin(\omega_j t) \\ -\sin(\omega_j t) & \cos(\omega_j t) \end{pmatrix} \begin{pmatrix} r_j(0) \\ \omega_j^{-1}\dot{r}_j(0) \end{pmatrix} \\ &\quad - \int_0^t \omega_j^{-1} \begin{pmatrix} \sin(\omega_j(t-\theta)) \\ \cos(\omega_j(t-\theta)) \end{pmatrix} \Big(\mathcal{F}_j g(u(\cdot,\theta)) - \mathcal{F}_j g(\widetilde{u}(\cdot,\theta)) + \delta_j(\cdot,\theta) \Big) d\theta\,. \end{aligned}$$

The Taylor expansion of the nonlinearity g at 0 and the fact that H^s is a normed algebra, yield the Lipschitz bound

$$\|g(v) - g(w)\|_s \le C\varepsilon\, \|v - w\|_s \quad \text{for } v, w \in H^s \text{ with } \|v\|_s \le M\varepsilon,\ \|w\|_s \le M\varepsilon.$$

Comparing the solution u with 0, this Lipschitz bound and the Gronwall inequality give $\|u(\cdot, t)\|_{s+1} \le M\varepsilon$ for $t \le \varepsilon^{-1}$. Comparing u and $\widetilde{u}$ gives, together with (51) and (52),

$$\|\widetilde{u}(\cdot, t) - u(\cdot, t)\|_{s+1} + \|\partial_t \widetilde{u}(\cdot, t) - \partial_t u(\cdot, t)\|_s \le C(1+t)\varepsilon^{N+1} \tag{53}$$

for $t \le \varepsilon^{-1}$. This completes the proof of Theorem 2.

3.14. Remark

The analysis of the modulated Fourier expansion could be done more neatly in weighted Wiener algebras $W^s = \{v \in C(\mathbb{T}) : \sum_{-\infty}^{\infty} \omega_j^s \, |v_j| < \infty\}$. Unfortunately, this ℓ^1 framework is not suited for the analysis of the almost-invariants studied in the next section, which are quadratic quantities and therefore require an ℓ^2-based framework.

4. Almost-invariants

We now show that the system of equations determining the modulation functions has almost-invariants close to the actions. The arguments are modelled after those of [12, Ch. XIII] for finite-dimensional oscillatory Hamiltonian systems.

4.1. The extended potential

Corresponding to the modulation functions $z^{\mathbf{k}}(x,\varepsilon t)$ we introduce

$$\mathbf{y} = (y^{\mathbf{k}})_{\|\mathbf{k}\|\le K} \qquad \text{with} \qquad y^{\mathbf{k}}(x,t) = z^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t} \tag{54}$$

and denote the Fourier coefficients of $y^{\mathbf{k}}(x,t)$ by $y_j^{\mathbf{k}}(t)$. By construction, the functions $y^{\mathbf{k}}$ satisfy

$$\partial_t^2 y^{\mathbf{k}} - \partial_x^2 y^{\mathbf{k}} + \rho y^{\mathbf{k}} + \sum_{m=2}^{N} \frac{g^{(m)}(0)}{m!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^m=\mathbf{k}} y^{\mathbf{k}^1} \ldots y^{\mathbf{k}^m} = e^{\mathbf{k}}\,, \tag{55}$$

where the defects $e^{\mathbf{k}}(x,t) = d^{\mathbf{k}}(x,\varepsilon t)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$ are bounded by $C\varepsilon^{N+1}$ in $\mathbf{H}^s$, see (41) and (47). In (1), the nonlinearity $g(u)$ is the gradient of the potential $U(u) = \int_0^u g(v)\, dv$. The sum in (55) is recognized as the functional gradient $\nabla^{-\mathbf{k}}\mathcal{U}(\mathbf{y})$ with respect to $y^{-\mathbf{k}}$ of the *extended potential* $\mathcal{U} : \mathbf{H}^1 \to \mathbb{R}$ defined, for $\mathbf{y} = (y^{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathbf{H}^1$, by

$$\mathcal{U}(\mathbf{y}) = \sum_{m=2}^{N} \frac{U^{(m+1)}(0)}{(m+1)!} \sum_{\mathbf{k}^1+\cdots+\mathbf{k}^{m+1}=\mathbf{0}} \frac{1}{2\pi}\int_{-\pi}^{\pi} y^{\mathbf{k}^1} \ldots y^{\mathbf{k}^{m+1}}\, \mathrm{d}x\,, \tag{56}$$

where we note that by Parseval's formula,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} y^{\mathbf{k}^1} \ldots y^{\mathbf{k}^{m+1}}\, \mathrm{d}x = \sum_{j_1+\cdots+j_{m+1}=0} y_{j_1}^{\mathbf{k}^1} \ldots y_{j_{m+1}}^{\mathbf{k}^{m+1}}\,.$$

Hence, the modulation system (55) can be rewritten as

$$\partial_t^2 y^{\mathbf{k}} - \partial_x^2 y^{\mathbf{k}} + \rho y^{\mathbf{k}} + \nabla^{-\mathbf{k}}\mathcal{U}(\mathbf{y}) = e^{\mathbf{k}}, \tag{57}$$

or equivalently in terms of the Fourier coefficients,

$$\partial_t^2 y_j^{\mathbf{k}} + \omega_j^2\, y_j^{\mathbf{k}} + \nabla_{-j}^{-\mathbf{k}}\mathcal{U}(\mathbf{y}) = e_j^{\mathbf{k}}\,,$$

where $\nabla_{-j}^{-\mathbf{k}}\mathcal{U}$ is the partial derivative of $\mathcal{U}$ with respect to $y_{-j}^{-\mathbf{k}}$.

4.2. Invariance under group actions

The key to the existence of almost-invariants for the system (57) is, in the spirit of Noether's theorem, the *invariance of the extended potential under continuous group actions*: for an arbitrary real sequence $\boldsymbol{\mu} = (\mu_\ell)_{\ell \geq 0}$ and for $\theta \in \mathbb{R}$, let

$$S_{\boldsymbol{\mu}}(\theta)\mathbf{y} = \left(\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu})\theta} y^{\mathbf{k}}\right)_{\|\mathbf{k}\| \leq K}. \tag{58}$$

Since the sum in the definition of $\mathcal{U}$ is over $\mathbf{k}^1 + \cdots + \mathbf{k}^{m+1} = \mathbf{0}$, we have

$$\mathcal{U}(S_{\boldsymbol{\mu}}(\theta)\mathbf{y}) = \mathcal{U}(\mathbf{y}) \quad \text{ for } \quad \theta \in \mathbb{R}.$$

Differentiating this relation with respect to θ yields

$$0 = \frac{d}{d\theta}\Big|_{\theta=0} \mathcal{U}(S_{\boldsymbol{\mu}}(\theta)\mathbf{y}) = \sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{\mathbf{k}} \, \nabla^{\mathbf{k}} \mathcal{U}(\mathbf{y}) \, \mathrm{d}x \,. \tag{59}$$

In fact, the full Lagrangian of the system (57) without the perturbations $e^{\mathbf{k}}$,

$$\mathcal{L}(\mathbf{y}, \partial_t \mathbf{y}) = \frac{1}{2} \sum_{\|\mathbf{k}\| \leq K} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\partial_t \mathbf{y}^{-\mathbf{k}} \partial_t \mathbf{y}^{\mathbf{k}} - \partial_x \mathbf{y}^{-\mathbf{k}} \partial_x \mathbf{y}^{\mathbf{k}} - \rho \mathbf{y}^{-\mathbf{k}} \mathbf{y}^{\mathbf{k}}\right) \mathrm{d}x - \mathcal{U}(\mathbf{y}),$$

is invariant under the action of the one-parameter groups $S_{\boldsymbol{\mu}}(\theta)$.

4.3. Almost-invariants of the modulation system

We now multiply (57) with $\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) y^{-\mathbf{k}}$, integrate over $[-\pi, \pi]$, and sum over $\mathbf{k}$ with $\|\mathbf{k}\| \leq K$. Thanks to (59) and a partial integration, we obtain

$$\sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(y^{-\mathbf{k}} \, \partial_t^2 y^{\mathbf{k}} + \partial_x y^{-\mathbf{k}} \, \partial_x y^{\mathbf{k}} + \rho y^{-\mathbf{k}} \, y^{\mathbf{k}}\right) \mathrm{d}x$$
$$= \sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, e^{\mathbf{k}} \, \mathrm{d}x \,.$$

Since the second and third terms under the left-hand integral cancel in the sum, the left-hand side simplifies to

$$\sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, \partial_t^2 y^{\mathbf{k}} \, \mathrm{d}x = -\frac{d}{dt} \mathcal{J}_{\boldsymbol{\mu}}(\mathbf{y}, \partial_t \mathbf{y})$$

with

$$\begin{aligned} \mathcal{J}_{\boldsymbol{\mu}}(\mathbf{y}, \partial_t \mathbf{y}) &= - \sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \frac{1}{2\pi} \int_{-\pi}^{\pi} y^{-\mathbf{k}} \, \partial_t y^{\mathbf{k}} \, \mathrm{d}x \\ &= - \sum_{\|\mathbf{k}\| \leq K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \sum_{j=-\infty}^{\infty} y_{-j}^{-\mathbf{k}} \partial_t y_j^{\mathbf{k}}, \end{aligned} \tag{60}$$

where the last equality holds by Parseval's formula. This yields

$$\frac{d}{dt}\mathcal{J}_{\boldsymbol{\mu}}(\mathbf{y},\partial_t\mathbf{y}) = -\sum_{\|\mathbf{k}\|\le K} \mathrm{i}(\mathbf{k}\cdot\boldsymbol{\mu}) \sum_{j=-\infty}^{\infty} y_{-j}^{-\mathbf{k}} e_j^{\mathbf{k}}. \tag{61}$$

Recalling the $\mathcal{O}(\varepsilon^{N+1})$-bound of $\mathbf{e} = (e^{\mathbf{k}})$ on the right-hand side, we see that $\mathcal{J}_{\boldsymbol{\mu}}$ is almost conserved.

In the following it will be more convenient to consider the almost-invariant $\mathcal{J}_{\boldsymbol{\mu}}$ for $\boldsymbol{\mu} = \langle \ell \rangle = (0,\dots,0,1,0,0,\dots)$ with the only entry at the ℓth position as a function of the modulation sequence $\mathbf{z}(\varepsilon t)$ rather than of $\mathbf{y}(t)$ defined by (54). We write

$$\mathcal{J}_\ell(\mathbf{z},\dot{\mathbf{z}}) = \mathcal{J}_{\langle\ell\rangle}(\mathbf{y},\partial_t\mathbf{y}).$$

By (61) we have

$$\varepsilon\,\frac{d}{d\tau}\mathcal{J}_\ell(\mathbf{z},\dot{\mathbf{z}}) = -\sum_{\|\mathbf{k}\|\le K} \mathrm{i}\,k_\ell \sum_{j=-\infty}^{\infty} z_{-j}^{-\mathbf{k}} d_j^{\mathbf{k}}. \tag{62}$$

Theorem 3. *Under the conditions of Theorem 2,*

$$\sum_{\ell\ge 0} \omega_\ell^{2s+1} \left| \frac{d}{d\tau}\mathcal{J}_\ell(\mathbf{z}(\tau),\dot{\mathbf{z}}(\tau))\right| \le C\,\varepsilon^{N+1} \qquad \textit{for} \quad \tau \le 1.$$

Proof. From the rescaling (23) we have

$$z_j^{\mathbf{k}} \;=\; \frac{\varepsilon^{[\![\mathbf{k}]\!]}}{\boldsymbol{\omega}^{s|\mathbf{k}|}}\,\widehat{c}_j^{\mathbf{k}} \;=\; \frac{\varepsilon}{\omega_j^s}\,\widehat{a}_j^{\mathbf{k}} \;+\; \frac{\varepsilon^{[\![\mathbf{k}]\!]}}{\boldsymbol{\omega}^{s|\mathbf{k}|}}\,\widehat{b}_j^{\mathbf{k}} \tag{63}$$

with the estimates $|\!|\!|\widehat{\mathbf{a}}|\!|\!|_1 \le C$ and $|\!|\!|\boldsymbol{\Omega}\widehat{\mathbf{b}}|\!|\!|_1 \le C$ by (33). For the defect, split as $\mathbf{d} = \mathbf{p} + \mathbf{q}$ into the diagonal and non-diagonal parts, we note that

$$|\!|\!|\mathbf{p}|\!|\!|_s^2 + \sum_{\|\mathbf{k}\|\le K} \|\omega^{s|\mathbf{k}|}\mathbf{q}^{\mathbf{k}}\|_0^2 = \sum_{\|\mathbf{k}\|\le K} \|\omega^{s|\mathbf{k}|}\mathbf{d}^{\mathbf{k}}\|_0^2,$$

which is bounded by $(C\varepsilon^{N+1})^2$ by (48). The result now follows from Lemma 3 below. Notice that resonant indices need not be considered in the sum (62), because $z_{-j}^{-\mathbf{k}} = 0$ for $(j,\mathbf{k}) \in \mathcal{R}_\varepsilon$ by definition. $\square$

Lemma 3. *For $\mathbf{c} = \mathbf{a} + \mathbf{b} \in \mathbf{H}^{s+1}$ and $\mathbf{r} = \mathbf{p} + \mathbf{q} \in \mathbf{H}^s$ split into diagonal and non-diagonal parts as in (25), we have the estimate*

$$\begin{aligned}\sum_{\ell\ge 0}\omega_\ell^{2s+1}\left| \sum_{\|\mathbf{k}\|\le K} k_\ell \sum_{j=-\infty}^{\infty} c_{-j}^{-\mathbf{k}} r_j^{\mathbf{k}}\right| \;&\le\; |\!|\!|\mathbf{a}|\!|\!|_{s+1}|\!|\!|\mathbf{p}|\!|\!|_s \\ &+ \Big(\sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{s|\mathbf{k}|}(1+|\mathbf{k}\cdot\boldsymbol{\omega}|)\mathbf{b}^{\mathbf{k}}\|_0^2\Big)^{1/2} \Big(\sum_{\|\mathbf{k}\|\le K} \|\boldsymbol{\omega}^{s|\mathbf{k}|}\mathbf{q}^{\mathbf{k}}\|_0^2\Big)^{1/2}.\end{aligned}$$

Proof. In the expression to be estimated we treat the terms with $\mathbf{k} = \pm\langle j\rangle$ separately (notice that for $\mathbf{k} = \pm\langle j\rangle$ we have $k_\ell = 0$ for $\ell \neq j$) and bound it by

$$\sum_{j=-\infty}^{\infty} \omega_j^{2s+1} \left| a_{-j}^{-\langle j\rangle} p_j^{\langle j\rangle} + a_{-j}^{\langle j\rangle} p_j^{-\langle j\rangle} \right|$$
$$+ \sum_{j=-\infty}^{\infty} \sum_{\mathbf{k}\neq\pm\langle j\rangle} \frac{\sum_{\ell\geq 0} |k_\ell|\, \omega_\ell^{2s+1}}{\boldsymbol{\omega}^{2s|\mathbf{k}|}(1+|\mathbf{k}\cdot\boldsymbol{\omega}|)}\, \boldsymbol{\omega}^{s|\mathbf{k}|}(1+|\mathbf{k}\cdot\boldsymbol{\omega}|)|b_{-j}^{-\mathbf{k}}|\, \boldsymbol{\omega}^{s|\mathbf{k}|}|q_j^{\mathbf{k}}|.$$

By (19) and the Cauchy–Schwarz inequality the stated estimate follows. □

4.4. Relationship of almost-invariants and actions

We now show that the almost-invariant $\mathcal{J}_\ell$ of the modulated Fourier expansion is close to the corresponding harmonic actions of the solution of the nonlinear wave equation,

$$J_\ell = I_\ell + I_{-\ell} = 2I_\ell \quad \text{for} \quad \ell \geq 1, \qquad J_0 = I_0$$

where for $u, v \in L^2(\mathbb{T})$ with Fourier coefficients u_j, v_j,

$$I_j(u,v) = \frac{\omega_j}{2}\,|u_j|^2 + \frac{1}{2\omega_j}\,|v_j|^2\,.$$

Theorem 4. *Under the conditions of Theorem 2, along the solution $u(t) = u(\cdot,t)$ of Eq. (1) and the associated modulation sequence $\mathbf{z}(\varepsilon t)$, it holds that*

$$\mathcal{J}_\ell\big(\mathbf{z}(\varepsilon t), \dot{\mathbf{z}}(\varepsilon t)\big) = J_\ell\big(u(t), \partial_t u(t)\big) + \gamma_\ell(t)\,\varepsilon^3$$

for $t \leq \varepsilon^{-1}$ and for all $\ell \geq 0$, with $\sum_{\ell\geq 0} \omega_\ell^{2s+1}\gamma_\ell(t) \leq C$.

Proof. Inserting in (60) the functions $y_j^{\mathbf{k}}(t) = z_j^{\mathbf{k}}(\varepsilon t)\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$, we have[1]

$$\begin{aligned} \mathcal{J}_\ell(\mathbf{z},\dot{\mathbf{z}}) &= -\sum_{\|\mathbf{k}\|\leq K} \mathrm{i}k_\ell \sum_{j=-\infty}^{\infty} z_{-j}^{-\mathbf{k}}\Big(\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})z_j^{\mathbf{k}} + \varepsilon\dot{z}_j^{\mathbf{k}}\Big) \\ &= \sum_{\|\mathbf{k}\|\leq K} k_\ell \sum_{j=-\infty}^{\infty} \Big((\mathbf{k}\cdot\boldsymbol{\omega})|z_j^{\mathbf{k}}|^2 - \mathrm{i}\varepsilon\, z_{-j}^{-\mathbf{k}}\,\dot{z}_j^{\mathbf{k}}\Big)\,. \end{aligned} \tag{64}$$

Using (63) and the bounds (33)-(34), an application of Lemma 3 shows that (64) is of the form

$$\mathcal{J}_\ell = \omega_\ell\Big(|z_\ell^{\langle\ell\rangle}|^2 + |z_\ell^{-\langle\ell\rangle}|^2\Big) + \omega_\ell\Big(|z_{-\ell}^{\langle\ell\rangle}|^2 + |z_{-\ell}^{-\langle\ell\rangle}|^2\Big) + \mathcal{O}_\ell(\varepsilon^3)$$

[1] The second equation is the only place in this paper where we use the relationship $z_{-j}^{-\mathbf{k}} = \overline{z_j^{\mathbf{k}}}$ that is valid only for real solutions of (1).

where $\mathcal{O}_\ell(\varepsilon^3)$ stands for a term $\alpha_\ell\,\varepsilon^3$ with $\sum_{\ell\ge 0}\omega_\ell^{2s+1}\alpha_\ell \le C$ (only one of the two terms is present for $\ell=0$). In terms of the Fourier coefficients of the modulated Fourier expansion $\widetilde u_j(t)=\sum_{\|\mathbf{k}\|\le K} z_j^{\mathbf{k}}(\varepsilon t)\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\omega})t}$,

$$
\begin{aligned}
\mathcal{J}_\ell &= \frac{\omega_\ell}{4}\Big(\big|\widetilde u_\ell+(\mathrm{i}\omega_\ell)^{-1}\partial_t\widetilde u_\ell\big|^2+\big|\widetilde u_\ell-(\mathrm{i}\omega_\ell)^{-1}\partial_t\widetilde u_\ell\big|^2\Big)\\
&\quad+\frac{\omega_\ell}{4}\Big(\big|\widetilde u_{-\ell}+(\mathrm{i}\omega_\ell)^{-1}\partial_t\widetilde u_{-\ell}\big|^2+\big|\widetilde u_{-\ell}-(\mathrm{i}\omega_\ell)^{-1}\partial_t\widetilde u_{-\ell}\big|^2\Big)+\mathcal{O}_\ell(\varepsilon^3)\\
&= J_\ell(\widetilde u,\partial_t\widetilde u)+\mathcal{O}_\ell(\varepsilon^3)\\
&= J_\ell(u,\partial_t u)+\mathcal{O}_\ell(\varepsilon^3),
\end{aligned}
$$

where we have used $\widetilde u_\ell(t)=z_\ell^{\langle\ell\rangle}(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega_\ell t}+z_\ell^{-\langle\ell\rangle}(\varepsilon t)\mathrm{e}^{-\mathrm{i}\omega_\ell t}+r_\ell$ with $\|r\|_{s+1}\le C\varepsilon^2$, which follows from the bounds (29)-(31). The last equality is a consequence of the remainder bound of Theorem 2. $\square$

4.5. From short to long time intervals

We apply Theorem 3 repeatedly on intervals of length ε^{-1}, for modulated Fourier expansions corresponding to different starting values $(u(t_n),\partial_t u(t_n))$ at

$$t_n = n\varepsilon^{-1}$$

along the solution $u(t)=u(\cdot,t)$ of (1). As long as u satisfies the smallness condition (8) (with 2ε in place of ε), Theorem 2 gives a modulated Fourier expansion $\widetilde u^n(t)$ that corresponds to starting values $(u(t_n),\partial_t u(t_n))$. We denote the sequence of modulation functions of this expansion by $\mathbf{z}_n(\varepsilon t)$. We now show that

$$\sum_{\ell=0}^{\infty}\omega_\ell^{2s+1}\Big|\mathcal{J}_\ell(\mathbf{z}_n(1),\dot{\mathbf{z}}_n(1))-\mathcal{J}_\ell(\mathbf{z}_{n+1}(0),\dot{\mathbf{z}}_{n+1}(0))\Big|\le C\varepsilon^{N+1}. \qquad (65)$$

This bound is obtained as follows: Theorem 2 shows that

$$\Big(\|\widetilde u^n(\varepsilon^{-1})-u(t_{n+1})\|_{s+1}^2+\|\partial_t\widetilde u^n(\varepsilon^{-1})-\partial_t u(t_{n+1})\|_s^2\Big)^{1/2}\le C\varepsilon^N.$$

By the Lipschitz continuity (35) of Section 3.7, by the decomposition (63), and by Lemma 3, this bound yields (65).

The bound (65) and Theorem 3 now yield

$$\sum_{\ell=0}^{\infty}\omega_\ell^{2s+1}\Big|\mathcal{J}_\ell(\mathbf{z}_{n+1}(0),\dot{\mathbf{z}}_{n+1}(0))-\mathcal{J}_\ell(\mathbf{z}_n(0),\dot{\mathbf{z}}_n(0))\Big|\le C\varepsilon^{N+1}$$

and hence, for $\tau\le 1$ and $n\ge 1$,

$$\sum_{\ell=0}^{\infty}\omega_\ell^{2s+1}\Big|\mathcal{J}_\ell(\mathbf{z}_n(\tau),\dot{\mathbf{z}}_n(\tau))-\mathcal{J}_\ell(\mathbf{z}_0(0),\dot{\mathbf{z}}_0(0))\Big|\le C\,n\,\varepsilon^{N+1},$$

which is smaller than $C\varepsilon^3$ for $n \le \varepsilon^{-N+2}$, i.e., for $t_n = n\varepsilon^{-1} \le \varepsilon^{-N+1}$. By Theorem 4 and Theorem 2, this implies

$$\sum_{\ell=0}^{\infty} \omega_\ell^{2s+1} \Big| J_\ell\big(u(t), \partial_t u(t)\big) - J_\ell\big(u(0), \partial_t u(0)\big) \Big| \le C\varepsilon^3 \quad \text{for} \quad t \le \varepsilon^{-N+1}.$$

This is the estimate of Theorem 1. It also shows that the smallness condition (8) remains indeed satisfied (with 2ε instead of ε, say) at $t_0, t_1, t_2, \dots$ up to times $t \le \varepsilon^{-N+1}$, so that the construction of the modulated Fourier expansions on each of the subintervals of length ε^{-1} is indeed feasible with bounds that hold uniformly in n. The proof of Theorem 1 is thus complete.

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