Long–time asymptotics of kinetic models of granular flows

Giuseppe Toscani
Dipartimento di Matematica, Universitá di Pavia, 27100 Pavia, Italy,
e-mail: toscani@dimat.unipv.it

Hailiang Li
Institute of Mathematics, University of Vienna, A-1090 Vienna, Austria.
e-mail: hailiang.li@univie.ac.at

Abstract

We analyze the long-time asymptotics of certain one-dimensional kinetic models of granular flows, which have been recently introduced in [22] in connection with the quasi elastic limit of a model Boltzmann equation with dissipative collisions and variable coefficient of restitution. These nonlinear equations, classified as nonlinear friction equations, split naturally into two classes, depending whether their similarity solutions (homogeneous cooling state) extinguish or not in finite time. For both classes, we show uniqueness of the solution by proving decay to zero in the Vasershtein metric of any two solutions with the same mass and mean velocity. Furthermore, if the similarity solution extinguishes in finite time, we prove that any other solution with initially bounded support extinguishes in finite time, by computing explicitly upper bounds for the life-time of the solution in terms of the length of the support.

Key words. Granular gases, nonlinear friction equations, long-time behavior of solutions.

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1 Introduction

This paper is devoted to large-time behavior of solutions of the equation

$$\frac{\partial f(v,t)}{\partial t} = \lambda \frac{\partial}{\partial v}\left[f(v,t) \int_{\mathbb{R}} |v-w|^{\gamma}(v-w) f(w,t)\, dw \right], \tag{1}$$

where the unknown $f(\cdot, t)$ is a time–dependent probability density on $\mathbb{R}$, and $\gamma > -1$. This equation, from now on called nonlinear friction equation, arises in the study of granular flows, and has been introduced in [22], in connection with the quasi-elastic limit of a model Boltzmann equation for rigid spheres with dissipative collisions and variable coefficient of restitution.

Granular flows describe the evolution of materials composed of many small discrete grains, which are inherently inelastic. Once initialized with a certain velocity distribution, granular gases cool down due to inelastic collisions of their particles. Similar as molecular gases, granular gases can be described at a mesoscopic level within the concepts of classical statistical mechanics, by means of methods borrowed from the kinetic theory of rarefied gases. Many recent papers (see [3, 18, 19, 6] and the references therein), consider in fact Boltzmann-like equations for partially inelastic rigid spheres. This choice relies in the physical hypothesis that the grains must be cohesionless, which implies the hard-sphere interaction only, and no long-range forces of any kind. Collapse and clustering are collective phenomena which are peculiar of the dissipative nature of granular flows [14, 17]. These phenomena are difficult to observe in realistic models with few particles [13]. On the other hand, kinetic collisional models of Boltzmann type are extremely complicated to study.

In reason of this, the study of the cooling of a granular gas has recently been attacked by introducing simplified models, typically in one dimension of the velocity variable. The idea of considering a simplification of the Boltzmann equation for rigid spheres goes back to McNamara and Young [18], who derived equation (1), in their paper called the test-particle equation, with $\gamma = 1$. The nonlinear friction equation with $\gamma = 1$ was derived independently some year later in [3] in a suitable scaling limit from a one-dimensional system of N particles colliding inelastically. We remark here that $\gamma = 1$ corresponds to set the coefficient of restitution, which characterizes the loss of energy of two colliding grains, equal to a material constant. This assumption, however, does not only contradict experiments, but it contradicts even some basic mechanical laws [21]. In one dimension of the velocity space, a Boltzmann equation for dissipative collisions and a almost general variable coefficient of restitution has been recently considered in [22]. When collisions are close to be elastic [22], the Boltzmann collision operator simplifies, and it becomes convenient to use the nonlinear friction operator on the right-hand side of (1). This choice leads to various advantages. In fact, despite their relatively simple (with respect to the Boltzmann equation) structure, the nonlinear friction equations exhibit the main properties of any kinetic model with dissipative collisions, like conservation of mass and mean velocity and decay of the temperature. Likewise, the equilibrium state is given by a Dirac mass located at the mean velocity of particles. In addition, on the contrary to the Boltzmann models for granular flows (see the discussion in [5]), these equations exhibit similarity solutions, which are in general of noticeable importance to understand the cooling process of the granular flow, and to construct reasonable macroscopic equations [18].

The study of these similarity solutions shows that the speed of relaxation of the temperature (cooling process) is heavily dependent of the value of the dissipation parameter γ in (1). For positive γ the decay is shown to be proportional to $t^{-2/\gamma}$, which implies a decay as t^{-2} when $\gamma = 1$, in accord with the decay of the full Boltzmann equation with a constant coefficient of restitution found in [11]. The analysis of the cooling of the similarity solutions implies in addition that this decay, at least for $\gamma > 1$, can be very slow. This characteristic is peculiar to dissipative equations, in contrast to what happens for the elastic Boltzmann equation and related conservative models, where convergence to equilibrium is very fast [23], when not exponential in time. Thus, when the cooling process is very slow, the problem of understanding the role played by the similarity solution is of paramount importance.

A first attempt to give an answer to the aforementioned problem has been recently done in [7]. The main goal of this paper, which is restricted to the case $\gamma = 1$, is the proof of a condition from which one can argue that the similarity solution does not represent in a strong sense the intermediate asymptotics of any other solution with the same mass and momentum. Thus one arrives to the conclusion that, at least for $\gamma = 1$, the homogeneous cooling state is not a good candidate for playing the fundamental role attributed to locally Maxwellian states in the classical, elastic kinetic theory.

In this paper we push further the previous studies in two directions, which hopefully clarify the meaning of the result of [7]. First, we obtain for all acceptable values of the dissipation parameter γ an exact equation for the time-evolution of the Vasershtein metric, from which one can easily reckon explicit rates of its time-decay. When $\gamma > 0$ one concludes that there is convergence towards a similarity solution at least at the same rate at which the similarity solution approaches the equilibrium. In this sense, at least when the relaxation process is very slow, the similarity solution remains a candidate to approximate the cooling process of any other solution with the same mass and momentum. Second, we prove finite time extinction of the solution whenever $-1 < \gamma < 0$, by computing explicitly upper bounds for the life-time of the solution in terms of the length of the support. In our opinion in this case the cooling process is so rapid that it is difficult to justify a role for the similarity solution.

Some problem linked to dissipative equations containing the nonlinear friction operator in (1) has been addressed before. Indeed, the long-time behavior of these and more complex equations has been deeply investigated in [9]. In this paper, by means of suitable generalization of logarithmic Sobolev inequalities and mass transportation inequalities, the long-time asymptotics of certain nonlocal, diffusive equations with a gradient flow structure has been analyzed. In particular, the results of [9] cover the asymptotic behavior of the equation

$$\frac{\partial f}{\partial t} = \sigma \frac{\partial^2 f}{\partial v^2} + \mu \frac{\partial}{\partial v}(vf) + \lambda \frac{\partial}{\partial v}\left[f(v,t) \int_{\mathbb{R}} |v-w|^{\gamma}(v-w) f(w,t)\, dw \right]. \tag{2}$$

where $\gamma > 0$. The interest in (2) follows from the fact that a few years ago, Benedetto, Caglioti, Carrillo and Pulvirenti [4] studied the asymptotic behavior of (2) when $\gamma = 1$, via the study of the free energy, proving convergence to equilibrium in large time, without obtaining any rate. The proof in [9] follows along the lines of the recent paper by Otto [20], where the Vasershtein metric first entered as a main ingredient into the study of the long–time behavior of nonlinear diffusion equations. In particular, in [20] it has been shown that displacement convexity of the energy functional implies contraction in Vasershtein metric for gradient flow, which was the basis for the Carrillo–McCann–Villani paper. The case when there is no diffusion is included in the analysis of [9], but for interaction potentials which require $\gamma > 0$.

Since the main part of our analysis does not depend on the value of the constant γ in (1), we will give a unified treatment of all cases for a large part of the paper, working separately of the cases $\gamma > 0$ and $\gamma < 0$ only when they lead to different behaviors.

2 The extremal distributions and the Vasershtein distance

Having in mind that the equilibrium solution to equation (1) is a Dirac mass, any convergence result holds in weak*-measure sense. Denote by $\mathcal{M}_0$ the space of all probability measures in $\mathbb{R}$ and by

$$\mathcal{M}_p = \left\{ F \in \mathcal{M}_0 : \int_{\mathbb{R}} |v|^p dF(v) < +\infty, p \geq 0 \right\}, \tag{3}$$

the space of all Borel probability measures of finite momentum of order p, equipped with the topology of the weak convergence of the measures. Several types of metrics on $\mathcal{M}_p$ can be considered (see [28]). For the purposes of this paper we will consider a class of the so-called minimal metrics.

Let $\Pi = \Pi(F, G)$ be the set of all cumulative probability distribution functions H on $\mathbb{R}^2$ having F and G as marginals, where F and G have finite positive variances. Within Π there are cumulative probability distribution functions H^* and H_* discovered by Hoeffding [15] and Fréchet [12] which have maximum and minimum correlation. Let $x^+ = \max\{0, x\}$ and $x \wedge y = \min\{x, y\}$. Then, in $\Pi(F, G)$ for all $(x, y) \in \mathbb{R}^2$,

$$H^*(v, w) = F(v) \wedge G(w) \quad \text{and} \quad H_*(v, w) = [F(v) + G(w) - 1]^+.$$

The extremal distributions can also be characterized in another way, based on certain familiar properties of uniform distributions. Let $F^{-1}(w) = \inf\{v : F(v) > w\}$ denote the pseudo inverse function of the distribution function $F(v)$. If X is a real–valued random variable with distribution function F, and U is a random variable uniformly distributed on $[0, 1]$, it follows that $F^{-1}(U)$ has distribution function F, and, for any F, G with finite positive variances the pair $[F^{-1}(U), G^{-1}(U)]$ has cumulative distribution function H^* [27]. Let

$$T_p(F, G) = \inf_{H \in \Pi(F,G)} \int |v - w|^p \, dH(v, w). \tag{4}$$

Then $T_p^{1/p}$ metrizes the weak-* topology TW_* on $\mathcal{M}_p$. For a detailed discussion, and application of these distances to statistics and information theory, see Vajda [25]. We remark that $T_2^{1/2}$ is known as the Kantorovich-Vasershtein distance of F and G [16, 26]. In this case

$$d(F, G)^2 = T_2(F, G) = \inf_{H \in \Pi(F,G)} \int |v - w|^2 \, dH(v, w) = \int |v - w|^2 \, dH^*(v, w). \tag{5}$$

In fact, if the random vector (X, Y) has cumulative distribution function H with marginals F and G, thanks to a result by Hoeffding [15]

$$E(XY) - E(X)E(Y) = \int [H(v,w) - F(v)G(w)]dvdw \leq$$

$$\leq \int [H^*(v,w) - F(v)G(w)]dvdw,$$

and this implies (5). Recalling now that $[F^{-1}(U), G^{-1}(U)]$ has cumulative distribution function H^* [27], we conclude that the Vasershtein distance between F and G can be rewritten as the L^2-distance of the pseudo inverse functions

$$d(F,G) = \left(\int_0^1 [F^{-1}(\rho) - G^{-1}(\rho)]^2 \, d\rho \right)^{1/2}. \tag{6}$$

3 The nonlinear friction equation and the evolution of Vasershtein distance

In this section we shall consider the time evolution of the Vasershtein distance (6) along solutions of the nonlinear friction equation (1). For the sake of simplicity, we will assume from now on $\lambda = 1$. The general case will follow easily through a time scaling. Thus, we will study equation

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v} \left[f(v,t) \int_{\mathbb{R}} |v-w|^\gamma (v-w) f(w,t) \, dw \right]. \tag{7}$$

As briefly discussed in the introduction, equation (7) arises in the study of granular flows, and has been introduced in [22], in connection with the quasi-elastic limit of a model Boltzmann equation for rigid spheres with dissipative collisions and variable coefficient of restitution. The value of the dissipation parameter γ is linked to the coefficient of restitution in the binary collision. A value of $\gamma > 1$ corresponds to grains that are close to be elastic for small relative velocity. Of course, $\gamma < 1$ gives the opposite phenomenon, namely the grains are close to be elastic for large relative velocities. We will refer to this case as the case of "anomalous" granular materials. The separating case $\gamma = 1$ refers to a constant coefficient of restitution, namely to a binary collision in which the degree of inelasticity does not depend on the relative velocity. Since the dissipative Boltzmann collision operator in [22] can be defined only when $-1 < \gamma \leq 2$, connections of the present results with the Boltzmann equation will be possible only in this range of the parameter.

For each $t \in \mathbb{R}^+$, let $F^{-1}(\rho, t) = \inf\{v : F(v,t) > \rho\}$ denote the pseudo inverse function of the distribution function $F(v,t)$. Then, a direct computation shows that, if the probability density $f(v,t)$ satisfies (7), $F^{-1}(\rho, t)$ solves

$$\frac{\partial F^{-1}(\rho,t)}{\partial t} = -\frac{1}{f(v,t)} \frac{\partial F(v,t)}{\partial t} \Big|_{v=F^{-1}(\rho,t)}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}} |F^{-1}(\rho,t)-w|^{\gamma}(F^{-1}(\rho,t)-w)f(w,t)\,dw \\
&= -\int_0^1 |F^{-1}(\rho,t)-F^{-1}(p,t)|^{\gamma}(F^{-1}(\rho,t)-F^{-1}(p,t))\,dp\,. \qquad (8)
\end{aligned}
$$

By a weak solution of the initial value problem for equation (7), corresponding to the initial distribution $F_0(v) \in \mathcal{M}_2$ we shall mean any distribution function $F \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ satisfying

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \varphi(v)dF(v,t) &= -\int_{\mathbb{R}} |v-w|^{\gamma}(v-w)\varphi'(v)dF(v,t)dF(w,t) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |v-w|^{\gamma}(v-w)\left[\varphi'(w)-\varphi'(v)\right]dF(v,t)dF(w,t) \qquad (9)
\end{aligned}
$$

for $t > 0$ and all $\varphi \in C^1(\mathbb{R})$, and such that for all $\varphi \in C^1(\mathbb{R})$

$$
\lim_{t\to 0}\int_{\mathbb{R}} \varphi(v)dF(v,t) = \int_{\mathbb{R}} \varphi(v)dF_0(v). \qquad (10)
$$

Alternatively, we can define the same concept of weak solution, in terms of the pseudo inverse function $F^{-1}(\rho,t)$. Under the same hypotheses on φ and on the initial distribution $F_0(v)$, we say that $F(v,t)$ is a weak solution to (7) if $F^{-1}(\rho,t)$ satisfies the following equation

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \varphi(F^{-1}(\rho,t))d\rho \\
&= -\int_0^1\int_0^1 |F^{-1}(\rho,t)-F^{-1}(p,t)|^{\gamma}(F^{-1}(\rho,t)-F^{-1}(\rho,t))\varphi'(F^{-1}(p,t))\,d\rho\,dp \\
&= -\frac{1}{2}\int_0^1\int_0^1 |H(\rho,p)|^{\gamma}H(\rho,p)\left[\varphi'(F^{-1}(\rho,t))-\varphi'(F^{-1}(p,t))\right]d\rho dp \qquad (11)
\end{aligned}
$$

where, to simplify notations, here and throughout this paper we set

$$
H(\rho,p) =: F^{-1}(\rho) - F^{-1}(p). \qquad (12)
$$

Let the random variable X have distribution function $F(\cdot,t)$, and let U be a random variable uniformly distributed on $[0,1]$. The equivalence between (9) and (11) can be easily verified also recalling that, since $F^{-1}(U)$ has distribution function F,

$$
\int_{\mathbb{R}} \varphi(v)dF(v,t) = E(\varphi(X)) = \int_0^1 \varphi(F^{-1}(\rho))d\rho. \qquad (13)
$$

Choosing $\varphi(v) = v$ into (9) shows that the total momentum is conserved in time. For this reason, and without loss of generality, we will consider in the rest of the paper as initial values only probability measures with expectation equal to zero. In view of (8), the time evolution of the square of the Vasershtein metric (6) is easily found to satisfy

$$
\frac{d}{dt}\int_0^1 [F^{-1}(\rho)-G^{-1}(\rho)]^2\,d\rho
$$

$$=2\int_0^1 d\rho[F^{-1}(\rho)-G^{-1}(\rho)]\left(\frac{\partial}{\partial t}F^{-1}(\rho)-\frac{\partial}{\partial t}G^{-1}(\rho)\right)$$
$$=-2\int_0^1 dp\int_0^1 d\rho[F^{-1}(\rho)-G^{-1}(\rho)]\left(|H(\rho,p)|^\gamma H(\rho,p)-|K(\rho,p)|^\gamma K(\rho,p)\right), \quad (14)$$

with $H(\rho,q)$ defined by (12) and

$$K(\rho,p)=: G^{-1}(\rho)-G^{-1}(p). \quad (15)$$

Since the equation (7) conserves mass and mean value, whenever X and Y are random variables distributed according to $F(v,t)$ and $G(v,t)$ respectively, and $E(X-Y)=0$, by definition, for all $F(v,w)\in\Pi(F,G)$ it holds

$$\int_{\mathbb{R}^2}(v-w)dF(v,w)=0. \quad (16)$$

Now, $F^{-1}(U)$ and $G^{-1}(U)$ have joint-distribution in the same class, which implies

$$\int_0^1[F^{-1}(\rho)-G^{-1}(\rho)]\,d\rho=\int_{\mathbb{R}^2}(v-w)dF^*(v,w)=0. \quad (17)$$

Hence, we have the equality

$$F^{-1}(\rho)-G^{-1}(\rho)=\int_0^1 dq[F^{-1}(\rho)-F^{-1}(q)-\big(G^{-1}(\rho)-G^{-1}(q)\big)], \quad (18)$$

and the equation (14) becomes

$$\frac{d}{dt}\int_0^1[F^{-1}(\rho)-G^{-1}(\rho)]^2\,d\rho=-Q(H,K)$$
$$=-2\int_0^1 d\rho\int_0^1 dp\int_0^1 dq[H(\rho,q)-K(\rho,q)]\cdot$$
$$\cdot[|H(\rho,p)|^\gamma H(\rho,p)-|K(\rho,p)|^\gamma K(\rho,p)]. \quad (19)$$

Let us use the variable transformation $\rho\to p$ and $p\to\rho$ into Q. We obtain

$$Q(H,K)=-2\int_0^1 d\rho\int_0^1 dp\int_0^1 dq[H(p,q)-K(p,q)]\cdot$$
$$\cdot[|H(\rho,p)|^\gamma H(\rho,p)-|K(\rho,p)|^\gamma K(\rho,p)]$$
$$=\int_0^1 d\rho\int_0^1 dp\int_0^1 dq[H(\rho,q)-H(p,q)-(K(\rho,q)-K(p,q))]$$
$$\cdot[|H(p,\rho)|^\gamma H(\rho,p)-|K(p,\rho)|^\gamma K(\rho,p)]$$
$$=\int_0^1 d\rho\int_0^1 dp\,[H(\rho,p)-K(\rho,p)][|H(\rho,p)|^\gamma H(\rho,p)-|K(\rho,p)|^\gamma K(\rho,p)]. \quad (20)$$

Finally, from (17), it follows that

$$\int_0^1 d\rho\int_0^1 dq[F^{-1}(\rho)-F^{-1}(q)-\big(G^{-1}(\rho)-G^{-1}(q)\big)]^2$$

$$
\begin{aligned}
&=\int_0^1 d\rho \int_0^1 dq[F^{-1}(\rho) - G^{-1}(\rho) - \big(F^{-1}(p) - G^{-1}(q)\big)]^2 \\
&=2[d(F,G)]^2 - 2\int_0^1 d\rho \int_0^1 dq[F^{-1}(\rho) - G^{-1}(\rho)]\big(F^{-1}(q) - G^{-1}(q)\big) \\
&=2[d(F,G)]^2,
\end{aligned} \tag{21}
$$

which implies

$$
[d(F,G)]^2 = \frac{1}{2}\int_0^1 d\rho \int_0^1 dp\, [H(\rho,p) - K(\rho,p)]^2. \tag{22}
$$

To this point, let us remark that both $F^{-1}(\rho)$ and $G^{-1}(\rho)$ are nondecreasing for $0 \le \rho \le 1$. Thus it follows that both $H(\rho,p)$ and $K(\rho,p)$ have the same sign; they are nonnegative if $\rho \ge p$, while they are nonpositive if $\rho \le p$. Thanks to the previous remark, and grouping together (19) and (22) we finally obtain the evolution equation for the square of the Vasershtein Metric

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 d\rho \int_0^1 dp\, (|H(\rho,p)| - |K(\rho,p)|)^2 \\
=&-2\int_0^1 d\rho \int_0^1 dp\, (|H(\rho,p)| - |K(\rho,p)|)(|H(\rho,p)|^{1+\gamma} - |K(\rho,p)|^{1+\gamma}).
\end{aligned} \tag{23}
$$

Theorem 3.1 *Let $\gamma > -1, \gamma \neq 0$, and let $F(v,t), G(v,t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ be two solutions to the initial value problem for equation (7) corresponding to the initial distributions $F_0(v), G_0(v) \in \mathcal{M}_2$, respectively. Then, for all time $t \ge 0$, (23) holds, which implies*

$$
d(F(t), G(t)) \le d(F(0), G(0)).
$$

Equation (23) shows that the Vasershtein metric is non expanding with time along trajectories of the nonlinear friction equation. As an immediate corollary, we obtain that the solution to the Cauchy problem for equation (9) is unique.

Corollary 3.2 *Let $F_0(v) \in \mathcal{M}_2$ be a nonnegative measure with finite variance. Then, there exists a unique weak solution $F(t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ of equation (7), such that $F(0) = F_0$.*

4 Similarity solutions

In this section, we will show that the use of the pseudo inverse functions permit a simple analysis of both the finding of the similarity solutions, and their importance as intermediate asymptotics. The study of self-similar solutions to (7) has been performed in [22], as a first step in the direction of understanding the cooling process of the granular gas. Classical methods developed in [2] allow to conclude that these self–similar solutions are obtained, though a suitable scaling in time, from the stationary solutions of the equation

$$
\frac{\partial}{\partial t} g(v,t) = \frac{\partial}{\partial v}\left[g(v,t)\int_{\mathbb{R}} |v-w|^\gamma (v-w) g(w,t)\, dw - v g(v,t)\right], \tag{24}
$$

More precisely, if $g_\infty(v)$ is a stationary solution to (24), $g_s(v,t) = \alpha(t)^{-1} g_\infty(\alpha(t)^{-1}v)$ is a self–similar solution to (7). These solutions are also called homogeneous cooling states of the equation (7). The function $\alpha(t)$ is given by

$$\alpha(t) = \left[\frac{1}{\xi^\gamma + \gamma t}\right]_+^{\frac{1}{\gamma}}, \quad \gamma \ge -1, \; \gamma \ne 0, \tag{25}$$

In (25) ξ is a positive constant. Moreover, we denoted by f_+ the positive part of the function f. Looking at the form of the function $\alpha(t)$, one concludes that, while for $\gamma > 0$ there is no cooling in finite time, when $\gamma < 0$ one has cooling in finite time, and the solution concentrates at time $t_c = \xi^\gamma/|\gamma|$. This fact led to the conjecture that any solution to the nonlinear friction equation corresponding to $\gamma < 0$ ceases to exist after a finite time [22].

Writing equation (24) in terms of the pseudo inverse function, as we did in (8), we can recover these stationary solutions in a simpler and direct way, which in addition gives their uniqueness. To this aim, consider that equation (24) is equivalent to

$$\begin{aligned}\frac{\partial F^{-1}(\rho,t)}{\partial t} &= -\int_0^1 |F^{-1}(\rho,t) - F^{-1}(p,t)|^\gamma (F^{-1}(\rho,t) - F^{-1}(p,t))\, dp + F^{-1}(\rho,t) \\ &= -\int_0^1 \left(|F^{-1}(\rho,t) - F^{-1}(p,t)|^\gamma - 1\right)(F^{-1}(\rho,t) - F^{-1}(p,t))\, dp\,. \end{aligned} \tag{26}$$

In (26) we used the equality

$$F^{-1}(\rho,t) = \int_0^1 (F^{-1}(\rho,t) - F^{-1}(p,t))\, dp, \tag{27}$$

which holds for any distribution function with mean equal to zero. Hence, the stationary solutions to (26) solve the equation

$$\int_0^1 \left(|F^{-1}(\rho) - F^{-1}(p)|^\gamma - 1\right)(F^{-1}(\rho) - F^{-1}(p))\, dp\, = 0. \tag{28}$$

It is immediate to conclude that the solutions to (28) are such that, for almost all $\rho \ne p$, $|F^{-1}(\rho) - F^{-1}(p)| = 1$, or $F^{-1}(\rho) = F^{-1}(p)$. Since $F^{-1}(\rho)$ is nondecreasing, and

$$\int_0^1 F^{-1}(p)\, dp = 0 \tag{29}$$

in addition to the trivial solution $F^{-1}(\rho) = 0 \quad a.e.$ we have only the possibility

$$|F^{-1}(\rho)| = 1 \quad a.e.,$$

which implies

$$F_\infty^{-1}(\rho) = -\frac{1}{2} \quad 0 \le \rho < \frac{1}{2}; \qquad F_\infty^{-1}(\rho) = \frac{1}{2} \quad \frac{1}{2} \le \rho < 1. \tag{30}$$

The solution (30) in nothing but the sum of two Dirac masses located symmetrically which respect to the origin,

$$g_\infty(v) = \frac{1}{2}\delta\left(v + \frac{1}{2}\right) + \frac{1}{2}\delta\left(v - \frac{1}{2}\right). \tag{31}$$

For the similarity solution generated by (31), the constant ξ in (25) is easily recognized to be related to the initial energy. One has

$$\xi = \frac{1}{2}\left(\int_{\mathbb{R}} v^2 g_s(v,0)\,dv\right)^{-1/2}. \tag{32}$$

The previous construction enlights remarkable similarities between equation (24) and the Fokker–Planck like equations studied in [8]. In both cases the non trivial stationary solution, for any given mass, is unique. In the Fokker–Planck equation this stationary solution corresponds to a minimum of a convex functional (the free–energy or entropy), which is used to get explicit rate of convergence towards the steady state of the solution.

It is tempting to apply the same physical idea of looking for the decay of a free-energy. Unlikely, we can only obtain weaker conclusions. As remarked in the paper [4], and subsequently used in [9], we can associate to equation (24) a free–energy like functional, we denote by $H(F)$

$$H(F) = \frac{1}{2+\gamma}\int_{\mathbb{R}^2} |v-w|^{2+\gamma} dF(v)dF(w) - \frac{1}{2}\int_{\mathbb{R}^2} |v-w|^2 dF(v)dF(w). \tag{33}$$

$H(F)$ can be rewritten in terms of the pseudo inverse functions, to give the expression

$$H(F) = \frac{1}{2+\gamma}\int_0^1\int_0^1 |F^{-1}(\rho) - F^{-1}(p)|^{2+\gamma}\,dpd\rho - \frac{1}{2}\int_0^1\int_0^1 |F^{-1}(\rho) - F^{-1}(p)|^2\,dpd\rho. \tag{34}$$

The time evolution of the free-energy is shown to satisfy

$$\frac{dH(F(t))}{dt} = -I(F(t)), \tag{35}$$

where

$$I(F) = \int_0^1 \left[\int_0^1 \left(|F^{-1}(\rho) - F^{-1}(p)|^\gamma - 1\right)\left(F^{-1}(\rho) - F^{-1}(p)\right)\,dp\right]^2 d\rho \tag{36}$$

is the entropy production. It is immediate to recognize that the entropy production is equal to zero if and only if $F^{-1}(\rho)$ is a stationary solution. Moreover, if $\gamma > 0$, the free–energy, among all pseudoinverse functions satisfying (29) attains the minimum in correspondance to the stationary solution (30). Hence, if we start with a distribution different from a stationary one, by means of (35) we can conclude that the free energy will converge towards its minimum. We conjecture that, when $\gamma > 0$, the free-energy satisfies a Csiszar-Kullback type inequality with respect to the Vasershtein metric, namely, for some universal constant c, and for any distribution function F with zero mean,

$$d(F, F_\infty)^2 \le c[H(F) - H(F_\infty)]. \tag{37}$$

The proof of this conjecture, coupled with a detailed analysis of the relationship between free-energy and entropy production, will produce, as for general Fokker-Planck like equations, an explicit rate of decay of the solution to (26) towards the stationary solution.

The case $\gamma < 0$ is completely different. In this range of the parameter $H(F)$ is not bounded from below, and we can repeat the previous analysis to conclude that among all the distributions with zero mean, the free–energy attains the maximum in correspondance to the stationary solution F_∞. Hence, since the free energy decays, the solution to equation (24) will never converge to the stationary solution F_∞.

The previour analysis helps to recognize that, at least for $\gamma < 0$, the stationary solutions to equation (24) do not play a general role of attraction, like it happens for a large class of nonlinear Fokker-Planck like equations studied in [1, 8]. Reverting to the old variables, the same conclusion can be drawn for the nonlinear friction equation. For this dissipative model, the self–similar solutions do not play the same role of attracting any other solution like it happens in linear and nonlinear diffusion equations.

5 The decay of Vasershtein metric for $\gamma > 0$

We will start the analysis of the time–decay of the Vasershtein metric with the relatively simpler case $\gamma > 0$. We prove

Theorem 5.1 *Let $F(v,t), G(v,t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ be two solutions to the initial value problem for equation (7) corresponding to the initial distributions $F_0(v), G_0(v) \in \mathcal{M}_2$, respectively. Then, if $\gamma > 0$, the Vasershtein distance of $F(v,t)$ and $G(v,t)$ is monotonically decreasing with time, and the following decay holds*

$$d(F(t), G(t)) \le d(F_0, G_0)\left[1 + 2^{\gamma/2}\gamma d(F_0, G_0)^\gamma t\right]^{-\frac{1}{\gamma}}. \tag{38}$$

`Proof` We make use of the differential equation (23) to prove Theorem 5.1. For simplicity, we often use H, K instead of $H(\rho, p)$ and $K(\rho, p)$. Without loss of generality, we choose $H > 0$ and $K > 0$, $H \ge K$. Then, $H/K \ge 1$ and it holds

$$\begin{aligned}(H-K)(H^{1+\gamma} - K^{1+\gamma}) =& K^{\gamma+2}\left(\frac{H}{K} - 1\right)\left(\left(\frac{H}{K}\right)^{\gamma+1} - 1\right)\\ \ge& K^{\gamma+2}\left(\frac{H}{K} - 1\right)\left(\frac{H}{K} - 1\right)^{\gamma+1} = (H-K)^{\gamma+2}\end{aligned} \tag{39}$$

because the function

$$W(a) = a^{1+\gamma} - (a-1)^{1+\gamma} - 1, \quad \gamma > 0,$$

increases strictly for $a > 1$.

Therefore, by (23), and the fact

$$\int_0^1 d\rho \int_0^1 dp\,[H-K]^2 \le \left\{ \int_0^1 d\rho \int_0^1 dp\,|H-K|^{\gamma+2} \right\}^{\frac{2}{\gamma+2}},$$

we obtain the differential inequality

$$\frac{d}{dt}\int_0^1 d\rho \int_0^1 dp\,[H-K]^2 \le -2\left\{ \int_0^1 d\rho \int_0^1 dp\,[H-K]^2 \right\}^{\frac{\gamma+2}{2}}. \tag{40}$$

The Gronwall's Lemma together with (22) yields (38). □

Remark. Theorem 5.1 enlights the importance of the dissipation parameter γ on the asymptotic decay of the solution. In [7], the main objection about the role of the similarity solution as intermediate asymptotics of any other solution comes out from Corollary 1. This Corollary shows that, if $\gamma = 1$, given a probability measure $F_0(v) \in \mathcal{M}_2$, which is not a symmetric convex combination of two delta masses, the corresponding solution $F(v,t)$ is such that there exists some constant $K > 0$, depending on F_0, such that, as $T \to \infty$,

$$\int_0^T d(F(t), G_s(t))\Delta t \ge K \log\log T.$$

¿From this inequality they conclude that extremely little is gained by replacing the equilibrium solution by the similarity solution, in that the gain is at best logarithmic in time [7]. Theorem (5.1) shows that the case studied in [7] is critical. Indeed, if $\gamma > 1$ we can draw the same conclusions of Corollary 1 in [7]. On the other hands, if $0 < \gamma < 1$, the bound (38) implies that

$$\int_0^\infty d(F(t), G_s(t))\Delta t = C < \infty.$$

Hence, at least in this range of the parameter, the objection of [7] does not hold, and some other argument has to be used to obtain similar conclusions.

We now proceed to study the time evolution of the support of the solution to equation (7). This study can be easily done by passing to equation (8). In fact, denoting with $S(dF(t))$ the measure of the support of the solution to equation (7), we have the identity

$$S(dF(t)) = \|F^{-1}(.,t)\|_{L^\infty} \tag{41}$$

We prove the following

Theorem 5.2 *Let the initial distribution $dF_0(v)$ have bounded support, $S(dF_0) = L < \infty$. Then, if $\gamma > 0$, the support of the solution to (7) decays to zero, and the following bound holds*

$$S(dF(t)) \le S(dF_0)\,[1 + \gamma S(F_0)^\gamma t]^{-\frac{1}{\gamma}}. \tag{42}$$

`Proof` To avoid inessential heavy notations, in the remaining of this section let us simply set $F^{-1}(\rho,t) = h(\rho,t)$. Then, Eq.(8) reads

$$\frac{dh}{dt} = -\int_0^1 |h(\rho,t) - h(p,t)|^\gamma (h(\rho,t) - h(p,t))dp. \tag{43}$$

For any convex function $\phi(r)$, $r \geq 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 \phi(h)d\rho &= -\int_0^1 \phi'(h(\rho,t))\int_0^1 |h(\rho,t) - h(p,t)|^\gamma (h(\rho,t) - h(p,t))dpd\rho \\ &= \int_0^1 \phi'(h(p,t))\int_0^1 |h(p,t) - h(\rho,t)|^\gamma (h(p,t) - h(\rho,t))dpd\rho.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 \phi(h)d\rho = -\frac{1}{2}\int_0^1 dp \int_0^1 d\rho |h(p,t) - h(\rho,t)|^\gamma \cdot \\ (h(p,t) - h(\rho,t))(\phi'(h(p,t)) - \phi'(h(p,t)).\end{aligned} \tag{44}$$

Inequality (44) implies that equation (7) dissipates along any convex functional of the solution, that is

$$\frac{d}{dt}\int_0^1 \phi(F^{-1}(\rho,t))d\rho \leq 0, \quad t \geq 0.$$

In particular, the L^p-norm of $F^{-1}(\rho,t)$ is non-increasing with respect to time for $p \in (1,\infty)$. Since $\|F^{-1}(\rho,t)\|_{L^\infty}$ is the measure of the support of $dF(v,t)$, we conclude that the support of $f(\rho,t)$ is non-increasing with respect to time.

Let us go further to analyze the evolution of the L^{2n}-norm. Equation (44) implies the formula

$$\begin{aligned}\frac{d}{dt}\|h(.,t)\|_{L^{2n}} &= \frac{d}{dt}\left(\left[\int_0^1 |h(\rho,t)|^{2n}d\rho\right]^{1/2n}\right) \\ = -&\frac{\int_0^1 dp \int_0^1 d\rho |h(\rho) - h(p)|^\gamma (h(\rho) - h(p))(h(\rho)^{2n-1} - h(p)^{2n-1})}{2\int_0^1 h(\rho)^{2n}d\rho}\|h(.,t)\|_{L^{2n}}.\end{aligned} \tag{46}$$

Here and after we just use $h(\rho)$ instead of $h(\rho,t)$ for simplicity. Since the total momentum is conserved in time and is set to be zero,

$$\int_0^1 xdF(\rho,t) = \int_0^1 F^{-1}(\rho,t)d\rho = 0, \quad t \geq 0,$$

we obtain

$$\begin{aligned}&\int_0^1 dp \int_0^1 d\rho \,[h(\rho) - h(p)]^2\, h(\rho)^{2n-2} \\ =&\int_0^1 dp \int_0^1 d\rho \left[h(\rho)^{2n} - 2h(p)h(\rho)^{2n-1} + h(p)^2 h(\rho)^{2n-2}\right]\end{aligned}$$

$$= \int_0^1 d\rho h(\rho)^{2n} + \int_0^1 dp h(p)^2 \int_0^1 d\rho h(\rho)^{2n-2}. \tag{48}$$

We finally obtain the inequality

$$\int_0^1 d\rho h(\rho)^{2n} \leq \int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p)\right]^2 h(\rho)^{2n-2} \tag{49}$$

Now, consider that, by Hölder inequality, for all $\gamma > 0$

$$\begin{aligned}
&\int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p)\right]^2 h(\rho)^{2n-2} \leq \\
&\left(\int_0^1 d\rho h(\rho)^{2n-2}\right)^{\gamma/(\gamma+2)} \left(\int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p)\right]^{2+\gamma} h(\rho)^{2n-2}\right)^{2/(\gamma+2)} \leq \\
&\left[\left(\int_0^1 d\rho h(\rho)^{2n}\right)^{\frac{\gamma}{\gamma+2}}\right]^{\frac{2n-2}{2n}} \left(\int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p)\right]^{2+\gamma} h(\rho)^{2n-2}\right)^{2/(\gamma+2)}
\end{aligned} \tag{50}$$

Hence, from inequality (49) we obtain

$$\begin{aligned}
\left(\int_0^1 d\rho h(\rho)^{2n}\right)^{1+\gamma/2n} &\leq \int_0^1 dp \int_0^1 d\rho \left|h(\rho) - h(p)\right|^{2+\gamma} h(\rho)^{2n-2} \\
&\leq \frac{1}{2} \int_0^1 dp \int_0^1 d\rho \left|h(\rho) - h(p)\right|^{2+\gamma} \left[h(\rho)^{2n-2} + h(p)^{2n-2}\right] \\
\leq \frac{1}{2} \int_0^1 dp \int_0^1 d\rho & \left|h(\rho) - h(p)\right| \left(h(\rho) - h(p)\right) \left[h(\rho)^{2n-1} - h(p)^{2n-1}\right].
\end{aligned} \tag{51}$$

Substituting this inequality into (46) gives a differential inequality for the L^{2n}-norm.

$$\frac{d}{dt} \|h(.,t)\|_{L^{2n}} \leq \|h(.,t)\|_{L^{2n}}^{1+\gamma}. \tag{52}$$

This inequality can be easily solved to give, reverting to old notations

$$\|F^{-1}(.,t)\|_{L^{2n}} \leq \|F_0^{-1}\|_{L^{2n}} \left[1 + \gamma \|F_0^{-1}\|_{L^{2n}}^{\gamma} t\right]^{-\frac{1}{\gamma}}. \tag{53}$$

Letting $n \to \infty$, we prove the result. □

6 Finite time extinction

Let us now study the asymptotic behavior of the solution to (7) for $-1 < \gamma < 0$. Looking at the finite-time extinction of the similarity solution, it was conjectured in [22] that any solution with initially bounded support extinguishes in finite time. A partial answer to this conjecture has been given in [24], where the finite extinction of the support was proven, without any estimate of the time-decay of the support itself.

In what follows we improve this result, obtaining un upper bound of the life-time of the solution in terms of the initial support. To study the time evolution of the support of the solution to equation (7) for $\gamma > 0$, we proceed as in the previous section. We prove the following

Theorem 6.1 *Let the initial distribution $dF_0(v)$ have bounded support, $S(dF_0) = L < \infty$. Then, if $-1 < \gamma < 0$, the support of the solution to (7) decays to zero in finite time, and the following bound holds*

$$S(dF(t)) \leq \left[S(dF_0)^{|\gamma|} - \frac{|\gamma|}{2^{|\gamma|}} t \right]_+^{\frac{1}{|\gamma|}}. \tag{54}$$

`Proof` We proceed as in the proof of Theorem 5.2. The starting point will be the formulas (46) and (48). Setting as before $F^{-1}(\rho, t) = h(\rho, t)$, and using once more the conservation of momentum, from (48) we finally obtain the inequality

$$\begin{aligned} \int_0^1 d\rho h(\rho)^{2n} \leq & \int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p) \right]^2 h(\rho)^{2n-2} \\ \leq & \frac{1}{2} \int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p) \right]^2 \left[h(\rho)^{2n-2} + h(p)^{2n-2} \right] \\ \leq & \frac{1}{2} \int_0^1 dp \int_0^1 d\rho \left[h(\rho) - h(p) \right] \left[h(\rho)^{2n-1} - h(p)^{2n-1} \right]. \end{aligned} \tag{55}$$

Assume now that the support of $dF(v,t)$ is bounded initially, i.e., $\|F^{-1}(., t = 0)\|_{L^\infty} = L$. Using inequality (55) into (46), we obtain

$$\begin{aligned} & \left| \frac{d}{dt} \left(\left[\int_0^1 |h(\rho, t)|^{2n} d\rho \right]^{1/2n} \right) \right| \\ \leq & \left[\int_0^1 |h(\rho, t)|^{2n} d\rho \right]^{\frac{1-2n}{2n}} \cdot \int_0^1 dp \int_0^1 d\rho |h(\rho) - h(p)|^{1-|\gamma|} h(\rho)^{2n-1} \\ \leq & (2L)^{1-|\gamma|} \cdot \left[\int_0^1 |h(\rho, t)|^{2n} d\rho \right]^{\frac{1-2n}{2n}} \cdot \left[\int_0^1 |h(\rho, t)|^{2n} d\rho \right]^{\frac{2n-1}{2n}} \\ \leq & (2L)^{1-|\gamma|}. \end{aligned} \tag{56}$$

This implies the bound

$$\int_0^T \left| \frac{d}{dt} \left(\left[\int_0^1 |h(\rho, t)|^{2n} d\rho \right]^{1/2n} \right) \right|^2 dt \leq T(2L)^{2-2|\gamma|}.$$

Moreover, since

$$\left[\int_0^1 |F^{-1}(\rho, t)|^{2n} d\rho \right]^{1/2n} \leq \|F^{-1}(., t)\|_{L^\infty} \leq L,$$

we have the bound

$$\int_0^T \left(\left[\int_0^1 |F^{-1}(\rho,t)|^{2n} d\rho \right]^{1/2n} \right)^2 dt \le L^2 T.$$

Let us set

$$f_n(t) = \left[\int_0^1 |F^{-1}(\rho,t)|^{2n} d\rho \right]^{1/2n}, \quad n \ge 1,$$

We just proved that $f_n \in H^1(0,T)$. Since $f_n(t)$ converges pointwise to $\|F^{-1}(.,t)\|_{L^\infty}$, we conclude easily that $f_n(t)$ also converges uniformly to $\|F^{-1}(.,t)\|_{L^\infty}$ on $[0,T]$. In fact, let us suppose that the sequence $f_n(t)$ does not converges uniformly to $f_\infty(t) =: \|F^{-1}(.,t)\|_{L^\infty}$ on $C(0,T)$. Then there are $\epsilon > 0$ and a sequence $\{n_k\}_{k=1}^\infty$ satisfying $n_k \to \infty$ as $k \to \infty$ such that

$$\|f_{n_k} - f_\infty\|_{C(0,T)} > \epsilon.$$

On the other hand, since $\{f_{n_k}\}_{k=1}^\infty$ is uniformly bounded in $H^1(0,T)$ and hence is relatively compact on $C(0,T)$, there is a subsequence (still denoted by $\{f_{n_k}\}_{k=1}^\infty$) converging strongly in $C(0,T)$ towards a function $f \in C(0,T)$, which is different from f_∞. This leads to a contradiction.

¿From the previous discussion we deduce that, for any given $\epsilon > 0$, we can find $\bar{n} = n(\epsilon)$ such that for all $t \in [0,T]$ and $n \ge \bar{n}$ it holds

$$f_n^{|\gamma|}(t) + \epsilon \ge f_\infty^{|\gamma|}(t), \quad t \ge 0. \tag{57}$$

Let t be such that $f_\infty^{|\gamma|}(t) \ge \delta > 0$. Then by (55) we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 dp \int_0^1 d\rho \, [h(\rho) - h(p)]^\gamma \, (h(\rho) - h(p)) \left[h(\rho)^{2n-1} - h(p)^{2n-1} \right] \\ \ge& \frac{1}{2^{1+|\gamma|} \|F^{-1}(.,t)\|^{|\gamma|}} \int_0^1 dp \int_0^1 d\rho \, [h(\rho) - h(p)]^2 \left[h(\rho)^{2n-1} - h(p)^{2n-1} \right] \\ \ge& \frac{1}{2^{|\gamma|} f_\infty^{|\gamma|}(t)} \int_0^1 d\rho h(\rho)^{2n}. \end{aligned} \tag{58}$$

Let us choose $\epsilon > 0$ and $\bar{n} = n(\epsilon)$ such that (57) holds. Then, (46) implies

$$\frac{d}{dt} f_n(t) \le -\frac{f_n(t)}{2^{|\gamma|}(f_n^{|\gamma|}(t) + \epsilon)} = -\frac{f_n(t)}{2^{|\gamma|} f_n^{|\gamma|}(t) + 2^{|\gamma|}\epsilon}.$$

We solve the above inequality and we obtain

$$\frac{2^{|\gamma|}}{|\gamma|} \left[f_n^{|\gamma|}(t) - f_n^{|\gamma|}(0) \right] + 2^{|\gamma|} \epsilon \log \frac{f_n(t)}{f_n(0)} \le -t. \tag{60}$$

If $f_\infty(t) \ge \delta$ at time $t \ge 0$, we get

$$\log \frac{f_n(0)}{f_n(t)} = \log \frac{f_n(0)}{f_\infty(0)} + \log \frac{f_\infty(0)}{f_\infty(t)} + \log \frac{f_\infty(t)}{f_n(t)}$$

$$\leq -\log \frac{\delta}{L} + \log \frac{f_\infty(t)}{f_n(t)}. \tag{61}$$

Let us choose $\epsilon \leq \delta^{|\gamma|}/2$. Then, by (57)

$$f_n^{|\gamma|}(t') \geq f_\infty^{|\gamma|}(t') - \epsilon \geq \delta^{|\gamma|} - \frac{1}{2}\delta^{|\gamma|} = \frac{1}{2}\delta^{|\gamma|} \;\Rightarrow\; f_n(t') \geq 2^{-|\gamma|}\delta, \quad t' \in [0,t].$$

Thus, as $\epsilon \leq \delta^{|\gamma|}/2$, it follows from (61) that

$$\log \frac{f_n(0)}{f_n(t)} \leq -\log \frac{\delta}{L} + \log \frac{2^{|\gamma|}L}{\delta} \leq \left|\log 2^{|\gamma|}L^2\delta^{-2}\right|,$$

and then from (60), for all $n \geq \bar{n}$, that

$$\frac{2^{|\gamma|}}{|\gamma|}\left[f_n^{|\gamma|}(t) - f_n^{|\gamma|}(0)\right] \leq -t + 2^{|\gamma|}\left|\log 2^{|\gamma|}L^2\delta^{-2}\right|\epsilon.$$

By letting $n \to \infty$ in the above inequality, we obtain

$$\frac{2^{|\gamma|}}{|\gamma|}\left[f_\infty^{|\gamma|}(t) - f_\infty^{|\gamma|}(0)\right] \leq -t + 2^{|\gamma|}\left|\log 2^{|\gamma|}L^2\delta^{-2}\right|\epsilon,$$

which implies, by letting $\epsilon \to 0$, that

$$f_\infty^{|\gamma|}(t) \leq f_\infty^{|\gamma|}(0) - \frac{|\gamma|}{2^{|\gamma|}}t. \tag{66}$$

Reverting to the old notations, we obtain the inequality

$$\|F^{-1}(.,t)\|_{L^\infty} \leq \left[\|F^{-1}(.,0)\|_{L^\infty}^{|\gamma|} - \frac{|\gamma|}{2^{|\gamma|}}t\right]_+^{\frac{1}{|\gamma|}}, \tag{67}$$

which yields the finite time extinction of the support, as well as un upper bound on the life-time of the support. □

Remark. The result of Lemma 6.1 is optimal, in that the decay in time to zero of the support is of the same order of the corresponding similarity solution.

A direct consequence of the previous lemma is the finite time decay to zero of the Vasershtein metric.

Corollary 6.2 *Let $F(v,t), G(v,t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_{2n})$, $n \in I\!N^+$, be two solutions to the initial value problem for equation (7) corresponding to the initial distributions $F_0(v)$, $G_0(v)$ with bounded support. Then, if $-1 < \gamma < 0$, the Vasershtein distance of $F(v,t)$ and $G(v,t)$ decays to zero at finite time, and the following time-decay holds*

$$d(F(t),G(t)) \leq d(F_0,G_0)\left[1 - \frac{|\gamma|}{(2L)^{|\gamma|}}t\right]_+^{\frac{1}{2}\left(\frac{1}{|\gamma|}-1\right)}, \tag{68}$$

where L denotes the maximum of the supports.

`Proof` Let $-1<\gamma<0$. We claim that the following inequality holds

$$(H-K)^2 \le \frac{1}{1+\gamma}(H-K)(|H|^\gamma H - |K|^\gamma K)(|H|+|K|)^{|\gamma|}. \tag{69}$$

Let $H\cdot K>0$. It is enough to prove (69) for $H\ge K>0$. This follows from the fact that, if $0\le\beta=|\gamma|<1$ the function

$$W(a)=(a+1)^\beta(a^{1-\beta}-1)-(1-\beta)(a-1) \tag{70}$$

(which is equal to zero for $a=1$) increases strictly for $a=H/K>1$. When $H\cdot K<0$, it is obvious that

$$(H-K)=(|H|+|K|)\le(|H|^{1+\gamma}+|K|^{1+\gamma})(|H|+|K|)^{|\gamma|}. \tag{71}$$

Assume the initial distributions F_0 and G_0 have compact supports, so that $dF_0(v)=0$ and $dG_0(v)=0$ for $v\ge L$ for some $L>0$. Once we have a formula for the time-decay of the support, we can proceed to prove the decay of Vasershtein metric simply making use of inequality (54) into (69). We have

$$(1+\gamma)\frac{(H-K)^2}{(|H|+|K|)^{|\gamma|}} \le (H-K)(|H|^\gamma H-|K|^\gamma K). \tag{72}$$

Now,

$$\begin{aligned}|H| &= |F^{-1}(\rho,t)-F^{-1}(p,t)| \le 2\|F^{-1}(.,t)\|_{L^\infty}\\ |K| &= |G^{-1}(\rho,t)-G^{-1}(p,t)| \le 2\|G^{-1}(.,t)\|_{L^\infty},\end{aligned}$$

and

$$\begin{aligned}&(|H|+|K|)^{|\gamma|}\\ &\le\left(2\left[\|F^{-1}(.,0)\|_{L^\infty}^{|\gamma|}-|\gamma|2^{-|\gamma|}t\right]_+^{\frac{1}{|\gamma|}}+2\left[\|G^{-1}(.,0)\|_{L^\infty}^{|\gamma|}-|\gamma|2^{-|\gamma|}t\right]_+^{\frac{1}{|\gamma|}}\right)^{|\gamma|}\\ &\le 2^{|\gamma|}\left[\|F^{-1}(.,0)\|_{L^\infty}^{|\gamma|}-\frac{|\gamma|}{2^{|\gamma|}}t\right]_+ + 2^{|\gamma|}\left[\|G^{-1}(.,0)\|_{L^\infty}^{|\gamma|}-\frac{|\gamma|}{2^{|\gamma|}}t\right]_+\\ &=: A(t).\end{aligned} \tag{73}$$

Assume $\bar t$ is the maximal time before extinction

$$\bar t=\max\left\{\frac{2^{|\gamma|}\|F^{-1}(0)\|_{L^\infty}^{|\gamma|}}{|\gamma|},\quad \frac{2^{|\gamma|}\|G^{-1}(0)\|_{L^\infty}^{|\gamma|}}{|\gamma|}\right\}, \tag{74}$$

then, for $t\in[0,\bar t]$ it follows from (23), (72) and (73) that

$$d\left(\log\int_0^1 d\rho\int_0^1 dp(H-K)^2\right)\le -2(1-|\gamma|)\frac{dt}{A(t)}. \tag{75}$$

If $\|F^{-1}(0)\|_{L^\infty} = \|G^{-1}(0)\|_{L^\infty} = L$, the solution is trivial. In fact, in this case for $t \le \bar{t} =: (2L)^{|\gamma|}/|\gamma|$

$$A(t) = 2(2L)^{|\gamma|} - 2|\gamma|t. \tag{76}$$

Integrating (75) over $[0,t]$ with $t \le \bar{t}$, we have

$$\begin{aligned}
\log \frac{\int_0^1 d\rho \int_0^1 dp(H-K)^2(t)}{\int_0^1 d\rho \int_0^1 dp(H-K)^2(0)} &\le -\,2\,(1-|\gamma|) \int_0^t \frac{ds}{2(2L)^{|\gamma|} - 2|\gamma|s} \\
&= -\left(\frac{1}{|\gamma|} - 1\right) \int_0^t \frac{ds}{|\gamma|^{-1}(2L)^{|\gamma|} - s} \\
&= \left(\frac{1}{|\gamma|} - 1\right) \log \frac{(2L)^{|\gamma|} - |\gamma|t}{(2L)^{|\gamma|}} \\
&= \log \left[1 - \frac{|\gamma|}{(2L)^{|\gamma|}} t\right]_+^{\frac{1}{|\gamma|}-1}
\end{aligned}$$

Thus, if both functions have the same support initially, we obtain

$$d(F,G)(t) \le d(F(0),G(0)) \left[1 - \frac{|\gamma|}{(2L)^{|\gamma|}} t\right]_+^{\frac{1}{2}\left(\frac{1}{|\gamma|}-1\right)}.$$

The previous formula is applicable also to the general case, i.e. when the functions have different support initially. This can be done by applying a similar argument, and by choosing L the maximum of both supports. In this case the extinction time is given by (74). We omit the details here. □

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