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Long Time Estimates for the Heat Kernel Associated with a Uniformly Subellipic Symmetric Second Order Operator

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0. Introduction:

Second order subelliptic operators have been the subject of a considerable amount of research in recent years. Starting with the paper [R-S] by L. Rothschild and E. Stein, in which the sharp form of Hormander's famous subellipticity theorem is proved, and continuing through the work of C. Fefferman and D. Phong [F] and A. Sanchez-Calle [S], it has become increasingly clear that precise regularity estimates for these operators depend intimately on the geometry associated with the operator under consideration. For example, if the operator L is written as the sum of squares of vector fields $V_1,\dots,V_d \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and one defines $d(x,y)$ to be the $\{V_1,\dots,V_d\}$-control distance between x and y (cf. section 1)), then, under a suitably uniform version of Hormander's condition (cf. (3.14) in section 3)), one can show that the fundamental solution $p(t,x,y)$ to the Cauchy initial value problem for $\partial_t u = Lu$ satisfies an estimate of the form:

$$(0.1)\quad \frac{1}{M|B_d(x,t^{1/2})|}\exp[-Md(x,y)^2/t] \le p(t,x,y) \le \frac{M}{|B_d(x,t^{1/2})|}\exp[-d(x,y)^2/Mt]$$

for all $(t,x,y) \in (0,1]\times\mathbb{R}^N\times\mathbb{R}^N$, where $B_d(x,r) \equiv \{y \in \mathbb{R}^N: d(x,y) < r\}$. (This estimate was first derived by Sanchez [S] for $t \in (0,1]$ and x and y satisfying $d(x,y) \le t^{1/2}$. More recently, it was extended to $(t,x,y) \in (0,1]\times\mathbb{R}^N\times\mathbb{R}^N$ with $d(x,y) \le 1$ by D. Jerison and Sanchez [J-S]; and, at about the same time, it was proved for general x and y by the present authors [K-S,III].)

What (0.1) makes clear is that the local regularity (which is determined by the way in which $p(t,x,y)$ tends to δ_{x-y} as $t\downarrow 0$) of solutions to equations involving L is inextricably tied to the "differential geometry" for which $d(x,y)$ is the "geodesic distance." In particular, as is shown in [K-S,III], (0.1) leads very quickly to a quantitative Harnack's principle, in terms of the balls $B_d(x,r)$, for non-negative solutions to $\partial_t u + Lu = 0$. (At least for non-negative solutions to $Lu = 0$, the same Harnack's principle was derived at the same time by D. Jerison [J]. His proof is based on a Poincare inequality, which can also be derived as a consequence of (0.1).) In a related direction, Fefferman and Phong [F] have further strengthened the connection between local regulartity and intrinsic geometry by showing that, even when L cannot be written as the sum of squares of vector fields, precise subellipticity results are tied to the size relationship between the balls $B_d(x,r)$ and Euclidean balls.

As much as the results cited above say about the local regularity theory of equations involving the operator L, they say very little about global behavior. Based on probabilistic intuition, coming from the central limit theorem, one suspects that, at least when the operator L is symmetric, the detailed geometry should get blurred as time evolves, with the result that $p(t,x,y)$ should look increasing like a standard heat (i.e. Weirstrass) kernel for large time. This suspicion is further confirmed if one believes that (0.1) persists even when $t \in [1,\infty)$, since $d(x,y)$ is commensurate with the Euclidean distance for x and y which are far away from one another. However, the techniques used in the papers cited above give no hint how one might go about checking the validity of this suspicion.

The main purpose of the present article is to obtain bounds, from above and below, on $p(t,x,y)$, $t \in [1,\infty)$, in terms of standard heat kernels (cf. Theorem (3.9) and Corollary (3.13) below). (In other words, (0.1) does continue to hold for $t \in [1,\infty)$.) These estimates are based on comparison principles and are therefore much less delicate than the short time results like (0.1). For instance, they are proved under much less stringent smoothness requirements on the coefficients. In this sense they are reminiscent of the classical results proved by D. Aronson [A] in the uniformly elliptic setting; and, in fact, our methodology here is derived from the approach used in [F-S,2] to get Aronson's estimates.

Once we have the estimates mentioned above, we apply them in the concluding section, to prove a "large scale" Harnack's principle for non-negative solutions to $Lu = 0$. Again the mehtodology is similar to that developed in earlier articles, in particular [F-S,1] and [F-S,2].

1. Preliminary Results:

Let $a \in C_b^2(\mathbb{R}^N;\mathbb{R}^N \otimes \mathbb{R}^N)$ a symmetric, non-negative definite matrix-valued function. Denote by $\mathcal{L}$ the divergence form operator $\nabla \cdot a \nabla$ (i.e. $\mathcal{L}u$ is defined for $u \in C_0^2(\mathbb{R}^N)$ by $\mathcal{L}u(x) = \sum_{i,j=1}^{N} [\partial_{x_i}(a^{ij}\partial_{x_j}u)](x)$). Then it is an easy consequence of standard diffusion theory that there is a unique transition probability function $P(t,x,\cdot)$ on $\mathbb{R}^N$ such that the associated Markov semigroup $\{P_t: t > 0\}$ satisfies $P_t\varphi(x) - \varphi(x) = \int_0^t [P_s\mathcal{L}\varphi](x)ds$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. In addition, one can check that $\{P_t: t > 0\}$ is symmetric in $L^2(\mathbb{R}^N)$ in the sense that $(\varphi,P_t\psi) = (\psi,P_t\varphi)$ (when there is no danger of confusion, we will use $(\cdot,*)$ to denote the $L^2(\mathbb{R}^N)$-inner product) for all $\varphi,\psi \in C_0(\mathbb{R}^N)$. In particular, Lebesgue measure on $\mathbb{R}^N$ is $\{P_t: t > 0\}$-invariant and so $\|P_t\|_{q\to q} \le 1$ (i.e. $\|P_t\varphi\|_q \le \|\varphi\|_q$, $\varphi \in C_0^\infty(\mathbb{R}^N)$ where $\|\cdot\|_q$ denotes the $L^q(\mathbb{R}^N)$-norm) for each $q \in [1,\infty)$. Moreover, it is clear that each P_t admits a unique extention $\overline{P}_t$ as a self-adjoint, non-negativity preserving contraction on $L^2(\mathbb{R}^N)$ and that $\{\overline{P}_t: t > 0\}$ is a strongly continuous semigroup on $L^2(\mathbb{R}^N)$. Finally, let $\{E_\lambda: \lambda \in [0,\infty)\}$ denote the resolution of the identity determined by $\{\overline{P}_t: t > 0\}$ (i.e. $\overline{P}_t = \int_{[0,\infty)} e^{-\lambda t}dE_\lambda$, $t > 0$) and set $A = \int_{[0,\infty)} \lambda dE_\lambda$. Clearly $-A$ is the generator of $\{\overline{P}_t: t > 0\}$, and it is not hard to check that $-A$ is the Friedrich's extention of $\mathcal{L}$.

When discussing the semigroup $\{\overline{P}_t: t > 0\}$, an important role is played by the <u>Dirichlet form</u> $\mathcal{E}$ given by $\mathcal{E}(f,f) = \int_{[0,\infty)} \lambda d(E_\lambda f,f) \in [0,\infty]$ for $f \in L^2(\mathbb{R}^N)$. Clearly, $\mathcal{E}(\varphi,\varphi) = \int \nabla\varphi\cdot a\nabla\varphi dx$ for $\varphi \in C_o^1(\mathbb{R}^N)$, and it is not hard to see that $\mathcal{E}$ is just the closure of its restriction to $C_o^\infty(\mathbb{R}^N)$. In order to exploit the special properties of $\mathcal{E}$ resulting from its connection with a Markov transition probability function, we note first that $t \longrightarrow (f - \overline{P}_t f,f)$ is a non-dereasing function of $t > 0$ and that $\mathcal{E}(f,f) = \lim_{t\downarrow 0}(f - \overline{P}_t f,f)$ and conclude from this that

(1.1) $$\mathcal{E}(f,f) = \lim_{t\downarrow 0} \frac{1}{2t}\int_{\mathbb{R}^N\times\mathbb{R}^N} (f(y) - f(x))^2 m_t(dx\times dy),$$

where <u>m_t</u> is the measure on $\mathbb{R}^N\times\mathbb{R}^N$ given by $m_t(dx\times dy) = P(t,x,dy)m(dy)$. In particular, (1.1) brings out the basic property of Dirichlet forms, namely: $\mathcal{E}(|f|,|f|) \le \mathcal{E}(f,f)$.

Set <u>$\Gamma(\psi)$</u>$^2 = \| \nabla\psi\cdot a\nabla\psi \|_\infty$ for $\psi \in C^1(\mathbb{R}^N)$; and, for $x,y \in \mathbb{R}^N$, define <u>$D(x,y)$</u> $= \sup\{|\psi(y) - \psi(x)| : \Gamma(\psi) \le 1\}$. The following result contains special cases of Theorem (3.25) and Corollary (3.28) in [C-K-S] (cf. also section 5) of that article).

(<u>1.2</u>) <u>Theorem</u>: Assume that there exist $A \in (0,\infty)$, $\nu \in (0,\infty)$, and $\delta \in (0,\infty)$ such that:

(1.3) $$\|f\|_2^{2+4/\nu} \le A(\mathcal{E}(f,f) + \delta\|f\|_2^2)\|f\|_1^{4/\nu}, \quad f \in L^2(\mathbb{R}^N);$$

or, equivalently (cf. Theorem (2.1) in [C-K-S]), that there is a $B \in (0,\infty)$ such that

(1.4) $$\|P_t\|_{1\longrightarrow\infty} \le Be^{\delta t}/t^{\nu/2}, \quad t > 0.$$

Then, $P(t,x,dy) = p(t,x,y)dy$ and there is a $C \in (0,\infty)$, depending only on ν, such that for each $\rho \in (0,1]$ and all $(t,x) \in (0,\infty)\times\mathbb{R}^N$:

(1.5) $p(t,x,\cdot) \le C(A/\rho t)^{\upsilon/2} e^{\rho\delta t} \exp[-D(x,\cdot)^2/(1+\rho)t]$ a.e.

Moreover, if, in addition to (1.3) or (1.4), one has for some $\mu \in (0,\upsilon]$, either that

(1.6) $\|f\|_2^{2+4/\mu} \le A\mathcal{E}(f,f)\|f\|_1^{4/\mu}$ for $f \in L^2(\mathbb{R}^N)$ with $\mathcal{E}(f,f) \le \|f\|_1^2$

or equivalently (cf. Theorem (2.9) in [C-K-S]) that

(1.7) $\|P_t\|_{1\longrightarrow\infty} \le B/t^{\mu/2}$ for $t \in [1,\infty)$

for some $B \in (0,\infty)$, then, for each $\rho \in (0,1]$:

(1.8) $$p(t,x,\cdot) \le \begin{matrix} C(\rho t)^{-\upsilon/2}\exp[-D(x,\cdot)^2/(1+\rho)t], & t \in (0,1] \\ C(\rho t)^{-\mu/2}\exp[-D(x,\cdot)^2/(1+\rho)t], & t \in [1,\infty), \end{matrix}$$

a.e., where $C \in (0,\infty)$ depends only on A or B, μ and υ.

(<u>1.9</u>) <u>Remark</u>: It should be obvious that (1.4) is equivalent to both

(1.4') $$\|P_t\|_{1\longrightarrow\infty} \le B'/(t\wedge 1)^{\upsilon/2}, \quad t > 0,$$

and

(1.4'') $$\|P_t\|_{1\longrightarrow\infty} \le B'/t^{\upsilon/2}, \quad t \in (0,1]$$

where $B' = Be^{\delta}$. Also, if any one of (1.3) or the various forms of (1.4) holds and if $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y)$ is continuous, then it follows from (1.5) that:

(1.10) $$\overline{\lim_{t\downarrow 0}}\ t\log(p(t,x,y)) \le -D(x,y)^2/4, \quad x,y \in \mathbb{R}^N.$$

In addition to the preceding, we will also need the following variant of Corollary (4.9) in [C-K-S].

<u>(1.11)</u> <u>Theorem</u>: Assume that $P(t,x,dy) = p(t,x,y)dy$, where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y) \in [0,\infty)$ is continuous. Further, assume that there exist $\epsilon > 0$, $r > 0$, $B \in (0,\infty)$, and $T \in (0,1]$ such that $\epsilon \le p(T,\cdot,*) \le B$ on $\{(x,y) \in \mathbb{R}^N\times\mathbb{R}^N: |x-y| \le r\}$. Then there is a $C \in (0,\infty)$, depending only on N, B, ϵ, and r, such that

(1.12) $$p(t,x,y) \le C/t^{N/2}, \quad (t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N.$$

In particular, if, in addition, either $\mathcal{E}$ satisfies (1.3) or $\{P_t: t > 0\}$ satisfies (1.4) for some $\nu \in [N,\infty)$, then there is a C, depending only on A or B, N, ν, ϵ, and r, such that

(1.13) $$p(t,x,y) \le \begin{cases} C(\rho t)^{-\nu/2}\exp[-D(x,y)^2/(1+\rho)t], & t \in (0,1] \\ C(\rho t)^{-N/2}\exp[-D(x,y)^2/(1+\rho)t], & t \in [1,\infty), \end{cases}$$

for each $\rho \in (0,1]$.

Proof: Clearly the second assertion follows immediately from the first when combined with the second part of Theorem (1.2).

To prove the first part, choose $\rho \in C_0^\infty(B(0,r))^+$ so that $\rho = \epsilon$ on $B(0,r/2)$. Then, $p(T,x,y) \ge \rho(x - y)$ for all $x,y \in \mathbb{R}^N$; and there is an $\epsilon' > 0$ (depending only on N, r, and ϵ) such that $\int(1 - \cos(\xi\cdot y))\rho(y)dy \ge \epsilon'|\xi|^2$ for $\xi \in \mathbb{R}^N$ with $|\xi| \le 1$. Now taking $\pi(x,y) = p(T,x,y)$ in Corollary (4.9) of [C-K-S], we conclude that $p(nT,x,y) \le C'/n^{N/2}$, for some $C' \in (0,\infty)$, depending only on N, B, r, and ϵ, and all $n \ge 1$. Hence, if $nT \le t \le (n+1)T$, then

$$p(t,x,y) = \int p(nT,x,\xi)P(t-nT,y,d\xi) \le C'/n^{N/2} \le C/t^{N/2}$$

for some $C \in (0,\infty)$ having the required dependence. Q.E.D.

We next turn to a primative version of the large deviation theory for the short time behavior of diffusions. Throughout this discussion, the function $a:\mathbb{R}^N \longrightarrow \mathbb{R}^N\otimes\mathbb{R}^N$ will be as above, $b:\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a bounded uniformly Lipschitz continuous function, and $\underline{L}$ is the operator $\sum_{i,j=1}^{N} a^{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{N} b^i(x)\partial_{x_i}$. Then there is a unique transition probability function $Q(t,x,\cdot)$ on $\mathbb{R}^N$ such that the associated semigroup $\{Q_t: t > 0\}$ satisfies

$$Q_t\varphi(x) = \varphi(x) + \int_0^t [Q_s L\varphi](x)ds, \ (t,x) \in (0,\infty)\times\mathbb{R}^N,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. In order to study $Q(t,x,\cdot)$, we introduce the Itô stochastic integral equations

$$X^{\epsilon,h}(t,x) = x + \epsilon\int_0^t \sigma(X^{\epsilon,h}(s,x))d\beta(s) + \int_0^t [\epsilon^2 b(X^{\epsilon,h}(s,x)) + \sigma(X^{\epsilon,h}(s,x))\dot{h}(s)]ds, \ t \geq 0,$$

where $\epsilon \in (0,1]$, $\sigma:\mathbb{R}^N \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^d$ is a uniformly Lipschitz continuous function satisfying $2a^{ij} = \sum_{k=1}^{d} \sigma_k^i \sigma_k^j$ for some $d \in Z^+$ (i.e. $2a = \sigma\sigma^\dagger$), $h \in H \equiv \{h \in C([0,\infty);\mathbb{R}^d): h(0) = 0 \text{ and } \dot{h} \in L^2([0,\infty);\mathbb{R}^d)\}$, and $\beta(\cdot)$ is a $\mathbb{R}^d$-valued Brownian motion on some filtered probability space $(\Omega,\mathcal{F}_t,P)$. If $X^\epsilon(\cdot,x) \equiv X^{\epsilon,0}(\cdot,x)$, then $Q(t,x,\cdot) = P\circ(X^1(t,x))^{-1}$, $P\circ(X^\epsilon(\cdot,x))^{-1} = P\circ(X^1(\epsilon^2\cdot,x))^{-1}$, and

$$\frac{dP\circ(X^{\epsilon,h}(1,x))^{-1}}{dP\circ(X^\epsilon(1,x))^{-1}} = R^{\epsilon,h}$$

$$\equiv \exp\left[\frac{1}{\epsilon}\int_0^1 \dot{h}(s)\cdot d\beta(s) - \frac{1}{2\epsilon^2}\int_0^1 |\dot{h}(s)|^2 ds\right].$$

In particular, for all $\Gamma \in \mathcal{B}$ (the Borel field over $\mathbb{R}^N$) and any $q \in (1,\infty)$:

$$P(X^{\epsilon,h}(1,x) \in \Gamma) = E^P\left[R^{\epsilon,h},\ X^1(\epsilon^2,x) \in \Gamma\right]$$

$$\leq \exp[(q-1)\|h\|_H^2/2\epsilon^2]Q(\epsilon^2,x,\Gamma)^{1/q'}$$

where $\|h\|_H^2 \equiv \|\dot{h}\|_{L^2([0,\infty);\mathbb{R}^d)}$ and q' is the Holder conjugate of q. Hence, for all $q \in (1,\infty)$ and $h \in H$:

$$Q(\epsilon^2,x,\Gamma) \geq \exp[-q\|h\|_H^2/2\epsilon^2]P(X^{\epsilon,h}(1,x) \in \Gamma)^{q'}. \tag{1.14}$$

Next, given $h \in H$, define $Y^h(\cdot,x)$ by

$$Y^h(t,x) = x + \int_0^t \sigma(Y^h(s,x))\dot{h}(s)ds, \quad t \geq 0,$$

and set $\Delta^{\epsilon,h}(\cdot,x) = X^{\epsilon,h}(\cdot,x) - Y^h(\cdot,x)$. Then

$$\Delta^{\epsilon,h}(t,x) = \epsilon\int_0^t \sigma(X^{\epsilon,h}(s,x))d\beta(s) + \epsilon^2\int_0^t b(X^{\epsilon,h}(s,x))ds$$
$$+ \int_0^t [\sigma(X^{\epsilon,h}(s,x)) - \sigma(Y^h(s,x))]\dot{h}(s)ds.$$

In particular, there is a $K \in (0,\infty)$, depending only on the upper bounds on a and b and the Lipschitz constant for σ, such that $E^P\left[|\Delta^{\epsilon,h}(1,x)|^2\right] \leq K\epsilon^2\exp[K\|h\|_H^2]$; and this, together with (1.14), yields

$$\begin{aligned} &Q(t,x,B(Y(1,h),r)) \\ &\geq \left[1 - (Kt\exp[K\|h\|_H^2]/r^2)\wedge 1\right]^{q'} \exp[-q\|h\|_H^2/2t] \end{aligned} \tag{1.15}$$

for all $q \in (1,\infty)$, $r \in (0,1]$, and $t \in (0,1]$.

Finally, we define <u>d(x,y)</u> for $x,y \in \mathbb{R}^N$ as $\inf\{2^{1/2}\|h\|_H : h \in H$ and $Y^h(1,x) = y\}$ ($\equiv \infty$ if no such h exists).

(<u>1.16</u>) <u>Remark</u>: It is easy to check that the value of d(x,y) does not depend on the particular choice of Lipschitz continuous σ satisfying $2a = \sigma\sigma^\dagger$. In particular, we can take $\sigma = (2a)^{1/2}$, in which case the Lipschitz constant of σ can be bounded in terms of the C_b^2-norm of a. In addition, it is obvious that $D(x,y) \leq d(x,y)$. What is less trivial, but is nonetheless not very difficult, is the fact that

(1.17) $$d(x,y) = D(x,y)$$

if $d(x,\cdot)$ is continuous at y (cf. Lemma (5.43) in [C-K-S]).

The following result is an essentially immediate consequence of the preceding discussion.

(1.18) Lemma: For each $R \in (0,\infty)$ there is a $\gamma \in (0,1)$, depending only on R, the upper bounds on a and b, and the Lipschitz constant for σ, such that

(1.19) $$Q(t,x,B(y,r)) \geq 2^{-q'}\exp[-qd(x,y)^2/4t]$$

for all $q \in (1,\infty)$, $r \in (0,1]$, and $(t,x,y) \in (0,\gamma r^2]\times\mathbb{R}^N\times\mathbb{R}^N$ with $|x - y| \leq R$.

(1.20) Remark: Although it is not in the direction in which we are headed, we note the following complement to the remark (1.9). Namely, suppose that $Q(t,x,dy) = q(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow q(t,x,y) \in [0,\infty)$ is continuous. Further. assume that

(1.21) $$\lim_{t\downarrow 0} t\log\left[\inf\{q(t,x,y):\ |y - x| \leq Kt^{1/2}\}\right] = 0$$

for each $K \in (0,\infty)$. Then the preceding line of reasoning leads quickly to

(1.22) $$\underline{\lim}_{t\downarrow 0}\ t\log(q(t,x,y)) \geq -d(x,y)^2/4,\quad x,y \in \mathbb{R}^N.$$

Indeed, given $x,y \in \mathbb{R}^N$ with $d(x,y) < \infty$, choose γ and T from $(0,1)$ so that $Q(t,x,B(y,(t/\gamma)^{1/2}) \geq 2^{-q'}\exp[-qd(x,y)^2/4t]$ for all $q \in (1,\infty)$ and $t \in (0,T]$. Then, for any $\rho \in (0,1)$,

$$q(t,x,y) \geq \int_{B(y,(t/\gamma)^{1/2})} q(\rho t,\xi,y)Q((1-\rho)t,x,d\xi);$$

and so, by (1.22),

$$\underline{\lim_{t\downarrow 0}}\ t\log(q(t,x,y)) \geq -qd(x,y)^2/4(1-\rho)$$

for all $q \in (1,\infty)$ and $\rho \in (0,1)$. In particular, in the case when $L = \mathcal{L}$ (and therefore $q(t,x,y) = p(t,x,y)$) and remark (1.9) applies, we have

$$-d(x,y)^2/4 \leq \underline{\lim_{t\downarrow 0}}\ t\log(p(t,x,y)) \tag{1.23}$$
$$\leq \overline{\lim_{t\downarrow 0}}\ t\log(p(t,x,y)) \leq -D(x,y)^2/4.$$

Thus, when, in addition, $d(x,\cdot)$ is continuous at y:

$$\lim_{t\downarrow 0} t\log(p(t,x,y)) = -d(x,y)^2/4. \tag{1.24}$$

Since the uniform Hormander condition in (3.14) below implies both (0.1) as well as (3.23), it follows immediately that (1.24) holds whenever (3.14) is satisfied. This observation is the subject of articles by R. Leandre announced in [L]

(1.25) Theorem: Assume that there is an $R \in (0,\infty)$ such that $d(x,y) \leq R$ whenever $|y - x| \leq 1$. Then, for each $r \in (0,1]$ there exists an $\alpha = \alpha(r) \in (0,1)$, depending only on R, the upper bounds on a and b, and the Lipschitz constant for σ, such that

$$Q(t,x,B(y,r)) \geq \alpha\exp[-d(x,y)^2/\alpha t],\ (t,x,y) \in (0,2]\times\mathbb{R}^N\times\mathbb{R}^N. \tag{1.26}$$

In particular, if, in addition, $Q(t,x,dy) = q(t,x,y)dy$ where $(t,x,y)\longrightarrow q(t,x,y)$ is continous, and if there is an $\epsilon > 0$ with the property that $q(1/2,x,y) \geq \epsilon$ whenever $|y - x| \leq \epsilon$, then there is a $\gamma \in (0,1)$, depending only on ϵ and $\alpha(\epsilon)$, such that

$$q(t,x,y) \geq \gamma\exp[-|y - x|^2/\gamma t],\ (t,x,y) \in [1,2]\times\mathbb{R}^N\times\mathbb{R}^N. \tag{1.27}$$

Proof: Let $r \in (0,1/4)$ be given. Then, by (1.19) with

$q = 2$, we know that $T \in (0,1]$ can be chosen so that $Q(t,x,B(y,r/2)) \geq \exp[-d(x,y)^2/2t]/4$ for all $t \in (0,T]$ and $|y - x| \leq 1$. Hence, if $|y - x| < r/2$, then $Q(t,x,B(y,r)) \geq Q(t,x,B(x,r/2)) \geq 1/4$ for all $t \in (0,T]$. On the other hand, if $t \in (0,T]$ and $r/2 \leq |y - x| \leq 1$, then $Q(t,x,B(y,r)) \geq \exp[-d(x,y)^2/2t]/4 \geq \exp[-2R^2|y - x|^2/r^2t]/4$. Finally, if $|y - x| > 1$, let n be the smallest integer exceeding $4|y - x|$ and set $x_m = \frac{n-m}{n}x + \frac{m}{n}y$ and $B_m = B(x_m,r)$ for $0 \leq m \leq n$; and, given $t \in (0,T]$, set $\tau = t/n$. Then

$$Q(t,x,B(y,r)) \geq \int_{B_1\times\cdots\times B_{n-1}} Q(\tau,x,\xi_1)Q(\tau,\xi_1,d\xi_2)\cdots Q(\tau,\xi_{n-1},B(y,r))$$

Since $|\xi_{m+1} - \xi_m| \leq 1$ for all $0 \leq m \leq n$, it follows from this that $Q(t,x,B(y,r)) \geq \left[\exp[-nR^2/t]/4\right]^n = \exp[-n^2R^2/t]/16^n$. Thus, we have now proved that (1.26) holds for all $t \in (0,T]$. To extend the estimate to all $t \in (0,2]$, suppose that $t \in (T,2]$ and let n be the smallest integer for which $t/n \in (0,T]$. Then, by (1.26) for τ's in $(0,T]$,

$$\begin{aligned} Q(t,x,B(y,r)) &\geq \int_{B(x,r)^{n-1}} Q(\tau,x,d\xi_1)Q(\tau,\xi_1,d\xi_2)\cdots Q(\tau,\xi_{n-1},B(y,r)) \\ &\geq \left[\alpha\exp[-nr^2/\alpha t]\right]^{n-1}\alpha\exp[-(r + |y - x|)^2/\alpha t]. \end{aligned}$$

Hence, since $n \leq 2/T + 1$, we can now adjust α so that (1.26) holds for all $t \in (0,2]$.

Finally, to prove (1.27), set $\alpha = \alpha(\epsilon)$. Then, by (1.26),

$$q(t,x,y) \geq \int_{B(y,\epsilon)} q(t/2,\xi,y)Q(t/2,x,d\xi) \geq \epsilon\alpha\exp[-2|y - x|/\alpha t]$$

for all $(t,x,y) \in [1,2]\times\mathbb{R}^N\times\mathbb{R}^N$. Q.E.D.

2. A Spectral Gap Estimate:

Let a and $\mathcal{L}$ be as in section 1), and define $P(t,x,\cdot)$, $\{P_t: t > 0\}$, etc. accordingly. Set $\omega(x) = \exp[-2(1 + |x|^2)^{1/2}]$ and use ω to also denote the measure $\omega(dx) = \omega(x)dx$. In this section we will be studying the Dirichlet forms $\hat{\mathcal{E}}_\lambda$, $\lambda \in [1,\infty)$, obtained by closing $\varphi \in C_o^\infty(\mathbb{R}^N) \longrightarrow \int \nabla\varphi \cdot a_\lambda \nabla\varphi d\omega$ in $L^2(\omega)$ (the L^2-space of functions on $\mathbb{R}^N$ with respect to the weight ω) where $a_\lambda(\cdot) \equiv a(\lambda\cdot)$. In fact, what we want to do is find conditions which guarantee that there exists a $K \in (0,\infty)$ with the property that

$$(2.1) \qquad \|f - \bar{f}\|^2_{L^2(\omega)} \le K\hat{\mathcal{E}}_\lambda(f,f), \ f \in L^2(\omega) \text{ and } \lambda \in [1,\infty),$$

where $\bar{f} \equiv \int f d\omega/\omega(\mathbb{R}^N)$. We begin by showing that such a K exists when $a \equiv I$.

Note: In order to distinguish the case $a \equiv I$, we will use a superscript "o" on quantities associated with it.

(2.2) Lemma: There is a $K^o \in (0,\infty)$ such that (2.1) holds for $\hat{\mathcal{E}}^o_\lambda$.

Proof: Obviously, what we have to do is show that if $\hat{\mathcal{L}}^o\varphi = [\nabla\cdot(\omega\nabla\varphi)]/\omega$ for $\varphi \in C_o^\infty(\mathbb{R}^N)$ and if $\hat{A}^o$ denotes the Friedrich's extention of $-\hat{\mathcal{L}}^o$ in $L^2(\omega)$, then 0 is a simple and isolated eigenvalue of $\hat{A}^o$. To this end, it is convenient to use the unitary map $U: L^2(\mathbb{R}^N) \longrightarrow L^2(\omega)$ given by $Uf \equiv f/\omega^{1/2}$. Indeed, since $\int_{\mathbb{R}^N} |\nabla(U\varphi)|^2 d\omega = \int_{\mathbb{R}^N} (|\nabla\varphi|^2 + V\varphi^2)dx$, where $V \equiv \Delta(\log\omega^{1/2})$, we see that $\hat{A}^o$ is unitarily equivalent to the Schrodinger operator $-\Delta + V$ on $L^2(\mathbb{R}^N)$. Hence, the problem becomes that of showing that 0 is a simple and isolated eigenvalue of $-\Delta + V$.

First note that $\text{spec}(-\Delta + V) = \text{spec}(\hat{A}^o) \subseteq [0,\infty)$ and that $\omega^{1/2}$ is an eigenfunction for $-\Delta + V$ with eigenvalue 0. Hence, by familiar reasoning, the fact that $0 = \inf(\text{spec}(-\Delta + V))$ guarantees that it must be a simple eigenvalue. In order to prove that 0 is an isolated eigenvalue, note that $V \in C_b(\mathbb{R}^N)$ and that $V - 1$ tends to 0 at ∞. Hence, $-\Delta + V$ is obtained from $-\Delta + 1$ by a relatively compact perturbation; and so $\text{spec}(-\Delta + V)$ can differ from $\text{spec}(-\Delta + 1) = [1,\infty)$ only by the addition of isolated eigenvalues. In particular, this shows that 0 must be isolated. Q.E.D.

In considering more general a's, it is useful to observe that (2.1) is equivalent to

(2.3) $$\|f - \overline{f}^\lambda\|^2_{L^2(\omega_\lambda)} \le K\lambda^2\tilde{\mathcal{E}}_\lambda(f,f),\ \lambda \in [1,\infty) \text{ and } f \in L^2(\omega_\lambda),$$

where $\underline{\omega}_\lambda(\cdot) = \omega(\lambda\cdot)$, $\overline{f}^\lambda = \int f d\omega_\lambda/\omega_\lambda(\mathbb{R}^N)$, and $\underline{\tilde{\mathcal{E}}}_\lambda$ is the Dirichlet form obtained by closing $\varphi \in C_0^\infty(\mathbb{R}^N) \longrightarrow \int \nabla\varphi\cdot a\nabla\varphi d\omega_\lambda$ in $L^2(\omega_\lambda)$.

(<u>2.4</u>) <u>Lemma</u>: The transition probability function $\tilde{P}_\lambda(t,x,\cdot)$ associated with $\tilde{\mathcal{E}}_\lambda$ satisfies

(2.5) $$\exp[-M(t + |y - x|)]P(t,x,\cdot) \le \tilde{P}_\lambda(t,x,\cdot) \le \exp[-M(t + |y - x|)]P(t,x,\cdot),$$

where M depends only on the C_b^1- norm of a but not on either $\lambda \in [1,\infty)$ or $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$.

<u>Proof</u>: Define $\tilde{\mathcal{L}}_\lambda\varphi = [\nabla\cdot(\omega_\lambda a\nabla\varphi)]/\omega_\lambda = \mathcal{L}\varphi + \nabla\omega_\lambda\cdot a\nabla\varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^N)$, and note that $\tilde{\mathcal{E}}_\lambda(f,f) = (f,\tilde{A}_\lambda f)_{L^2(\omega_\lambda)}$, $f \in \mathcal{D}(\tilde{A}_\lambda)$, where $\tilde{A}_\lambda$

is the Friedrich's extention of $-\tilde{\mathcal{L}}_\lambda$ in $L^2(\omega_\lambda)$. Next, set $V_\lambda = \mathcal{L}\omega_\lambda/\omega_\lambda$, and note that $\tilde{\mathcal{L}}_\lambda\varphi = [(\mathcal{L} - V_\lambda)(\omega_\lambda\varphi)]/\omega_\lambda$. Hence, if $\{R_t^\lambda: t > 0\}$ is the semigroup determined by

$$R_t^\lambda\varphi = P_t\varphi - \int_0^t P_{t-s}(V_\lambda R_s^\lambda\varphi)ds, \quad t > 0 \text{ and } \varphi \in C_b(\mathbb{R}^N),$$

and $\tilde{P}_t^\lambda\varphi = [R_t^\lambda(\omega_\lambda\varphi)]/\omega_\lambda$, $t > 0$ and $\varphi \in C_b(\mathbb{R}^N)$, then $\{\tilde{P}_t^\lambda: t > 0\}$ is the unique Markov semigroup satisfying

$$\tilde{P}_t^\lambda\varphi = \varphi + \int_0^t \tilde{P}_s^\lambda(\tilde{\mathcal{L}}_\lambda\varphi)ds, \quad t > 0 \text{ and } \varphi \in C_0^\infty(\mathbb{R}^N);$$

and as such, $\{\tilde{P}_t^\lambda: t > 0\}$ is the Markov semigroup associated with $\tilde{\mathcal{E}}_\lambda$. Finally, note that $R_t^\lambda\varphi = \int\varphi(y)R_\lambda(t,\cdot,dy)$ where $\exp[t(\inf(V_\lambda))]P(t,x,\cdot) \le R_\lambda(t,x,\cdot) \le \exp[t(\sup(V_\lambda))]P(t,x,\cdot)$. Hence, if $\tilde{P}_\lambda(t,x,dy) \equiv [\omega_\lambda(y)R_\lambda(t,x,dy)]/\omega_\lambda(x)$, then $\tilde{P}_t^\lambda\varphi = \int\varphi(y)\tilde{P}_\lambda(t,\cdot,dy)$ and so $\tilde{P}_\lambda(t,x,\cdot)$ is the transition probability function associated with $\tilde{\mathcal{E}}_\lambda$. In addition, it is clear from the preceding representation of $\tilde{P}_\lambda(t,x,\cdot)$ that (2.5) holds with an M having the required dependence.

Q.E.D.

(<u>2.9</u>) <u>Theorem</u>: Assume that there exists an $R > 0$ such that $d(x,y) \le R$ whenever $|x - y| \le 1$. Also, assume that $P(t,x,dy) = p(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y)$ is continuous and $p(1/2,x,y) \ge \epsilon$ for some $\epsilon > 0$ and all $x,y \in \mathbb{R}^N$ with $|x - y| \le \epsilon$. Then there exists a $K \in (0,\infty)$, depending only on R, ϵ, and the C_b^2- norm of a, such that (2.1) holds.

<u>Proof</u>: We need only show that (2.3) holds for an appropriate K. To this end, note that, by Lemma (2.2), (2.3) holds with $K = K^o$ for $\tilde{\mathcal{E}}_\lambda^o$. Hence, using the spectral representation for the

$L^2(\omega_\lambda)$-semigroup determined by $\tilde{P}^o(t,x,\cdot)$, one sees that

$$(\lambda^2/2t)\int_{\mathbb{R}^N\times\mathbb{R}^N}(f(y) - f(x))^2\ \tilde{P}^o_\lambda(t,x,dy)\omega_\lambda(dx)$$
$$\geq \lambda^2(1 - \exp[-t/(K^o\lambda^2)])\|f - \bar{f}^\lambda\|^2_{L^2(\omega_\lambda)}$$

for all $\lambda \in [1,\infty)$ and $t > 0$. At the same time,

$$\tilde{\mathcal{E}}^\lambda(f,f) \geq 1/2\int_{\mathbb{R}^N\times\mathbb{R}^N}(f(y) - f(x))^2\ \tilde{P}_\lambda(1,x,dy)\omega_\lambda(dx).$$

Hence we will be done once we show that $\tilde{P}_\lambda(1,x,\cdot) \geq \tau\tilde{P}^o_\lambda(t,x,\cdot)$ for some choice of $t,\tau \in (0,1)$ depending only on R, ϵ, and the C^2_b-norm of a. But, since $P^o(t,x,dy) = (4\pi t)^{-N/2}\exp[-|y - x|^2/4t]dy$, the existence of such t and τ is easily deduced from Lemma (2.4) combined with Theorem (1.25).

Q.E.D.

3. Long Time Estimates on the Fundamental Solution:

Our first goal in this section is to prove the following result. Our proof is patterned on the method used in [F-S,2] which, in turn, uses ideas introduced by J. Nash in his famous paper [N].

(3.1) Theorem: Assume that there exist r, B, and K from $(0,\infty)$ such that

(3.2) $$P(t,x,B(x,rt^{1/2})) \geq 1/2, \quad (t,x) \in [1/4,\infty)\times\mathbb{R}^N,$$

(3.3) $$\|P_{1/4}\|_{1\longrightarrow\infty} \leq B,$$

and (2.1) holds. Then there is an $\alpha \in (0,1]$, depending only on r, B, K, and the upper bound on a, such that

(3.4) $$P_t\varphi(0) \geq \frac{\alpha}{t^{N/2}}\int\varphi(y)dy, \quad t \in [1,\infty) \text{ and } \varphi \in C_o(B(0,rt^{1/2}))^+.$$

As a first step, we observe that (3.4) is equivalent to

(3.4') $$P_1^\lambda\varphi(0) \geq \alpha\int\varphi(y)dy, \quad \lambda \in [1,\infty) \text{ and } \varphi \in C_o(B(0,r))^+,$$

where $\{P_t^\lambda: t > 0\}$ is the semigroup associated with the transition probability function $P_\lambda(t,x,\cdot)$ given by $P_\lambda(t,x,\Gamma) = P(\lambda^2 t,\lambda x,\lambda\Gamma)$ for $(t,x,\Gamma) \in (0,\infty)\times\mathbb{R}^N\times\mathcal{B}$. We next set $\mathcal{L}_\lambda = \nabla\cdot(a_\lambda\nabla)$ (recall that $a_\lambda(\cdot) = a(\lambda\cdot))$ and remark that $\{P_t^\lambda: t > 0\}$ is the only Markov semigroup which satisfies $P_t^\lambda\varphi = \varphi + \int_0^t P_s^\lambda\mathcal{L}_\lambda\varphi ds$, $t \in (0,\infty)$, for all $\varphi \in C_o^\infty(\mathbb{R}^N)$. In particular, $(t,x)\in [0,T]\times\mathbb{R}^N\longrightarrow P_t^\lambda\varphi(x)$ is an element of $C_b^{1,2}([0,T]\times\mathbb{R}^N)$ for each $T > 0$ and

(3.5) $$\partial_t P_t^\lambda\varphi(x) = [\mathcal{L}_\lambda P_t^\lambda\varphi](x), \quad (t,x) \in (0,\infty)\times\mathbb{R}^N,$$

for all $\varphi \in C_o^\infty(\mathbb{R}^N)$ (cf. Theorem 3.2.4 in [S-V]).

(3.6) Lemma: There is a $C \in (0,\infty)$, depending only on r, N, B, K, and the upper bound on a, such that

(3.7) $$\int \log\left[P^{\lambda}_{1/2}\varphi\right] d\omega \geq -C\int\varphi(y)dy, \ \lambda \in [1,\infty) \text{ and } \varphi \in C_o(B(0,r)).$$

Proof: Given $\lambda \in [1,\infty)$, $\varphi \in C_o^{\infty}(B(0,r))^{+}$ with $\int\varphi(y)dy = 1$, and $\delta > 0$, set $u(t,x) = P^{\lambda}_t\varphi(x) + \delta$, $v = \log(u)$, and $G(t) = \int v(t,y)\omega(dy)/\omega(\mathbb{R}^N)$. Then, by (3.5), integration by parts, and (2.1):

$$\omega(\mathbb{R}^N)G'(t) = \int\frac{\omega(y)}{u(t,y)}[\mathcal{L}_{\lambda}u(t,\cdot)](y)dy$$

$$= -\int \nabla(\log(\omega))\cdot a_{\lambda}\nabla(\log(u(t,\cdot)))d\omega + \int\nabla(v(t,\cdot))\cdot a_{\lambda}\nabla(v(t,\cdot))d\omega$$

$$\geq -1/2\int\nabla(\log(\omega))\cdot a_{\lambda}\nabla(\log(\omega))d\omega + 1/2\hat{\mathcal{E}}_{\lambda}(v,v)$$

$$\geq -A + (1/2K)\int(v(t,\cdot) - G(t))^2 d\omega,$$

where $A \in (0,\infty)$ depends only on the upper bound on a. Next, note that the function $\xi \in [e^{2+G(t)},\infty) \longrightarrow (\log(\xi) - G(t))^2/\xi$ is non-increasing and that $u(t,\cdot) \leq B$ for $t \in [1/4,1/2]$. Thus, if $\Gamma_t \equiv \{y \in \mathbb{R}^N: u(t,y) \geq e^{2+G(t)}\}$, then

$$\omega(\mathbb{R}^N)G'(t) \geq -A + \frac{(\log(B) - G(t))^2}{2K\log(B)}\int_{\Gamma_t} u(t,y)\omega(dy)$$

for all $t \in [1/4,1/2]$. At the same time,

$$\frac{1}{\omega(\mathbb{R}^N)}\int_{\Gamma_t} u(t,y)\omega(dy) \geq \frac{1}{\omega(\mathbb{R}^N)}\int u(t,y)\omega(dy) - e^{2+G(t)};$$

and, by (3.2),

$$\int u(t,y)\omega(dy) \geq \int_{B(0,2r)} P^{\lambda}_t\varphi(y)\omega(dy) \geq e^{-2(1+4r^2)^{1/2}}\int_{B(0,2r)} P^{\lambda}_t\varphi(y)dy$$

$$= e^{-2(1+4r^2)^{1/2}}(\chi_{B(0,2r)}, P^\lambda_t\varphi)_{L^2(\mathbb{R}^N)}$$
$$= e^{-(1+4r^2)^{1/2}}\int \varphi(x)P(\lambda^2 t,\lambda x,B(x,\lambda r))dx \geq \frac{1}{2}e^{-2(1+4r^2)^{1/2}}.$$

From this and the preceding, it is easy to see that there exist $\gamma \in (0,1]$ and $M \in (0,\infty)$, depending only on r, B , K, and A, such that

$$G'(t) \geq \gamma G(t)^2, \quad t \in [1/4,1/2],$$

so long as $G(t) \leq -M$ for $t \in [1/4,1/2]$. Since, in any case, $G'(t) \geq -A/2\omega(\mathbb{R}^N)$, we therefore conclude that $G(1/2) \geq -4/\gamma$ if $G(1/2) \leq -M - A/\omega(\mathbb{R}^N)$. In other words, $G(1/2) \geq -[(M + A/2\omega(\mathbb{R}^N))\vee(4/\gamma)]$.

Q.E.D.

<u>Proof of (3.1)</u>: As we have said, it suffices to check (3.4') with an α having the required dependence. To this end, let $\varphi \in C_0^\infty(B(0,r))^+$ with $\int \varphi(y)dy = 1$ be given, and suppose that ψ is a second such function. Then, by (3.7) and Jensen's inequality:

$$\log\left[\omega(\mathbb{R}^N)^{-1}(\psi,P^\lambda_1\varphi)_{L^2(\mathbb{R}^N)}\right] = \log\left[\omega(\mathbb{R}^N)^{-1}(P^\lambda_{1/2}\psi,P^\lambda_{1/2}\varphi)_{L^2(\mathbb{R}^N)}\right]$$

$$\geq \log\left[\omega(\mathbb{R}^N)^{-1}\int(P^\lambda_{1/2}\psi)(P^\lambda_{1/2}\varphi)d\omega\right]$$

$$\geq \omega(\mathbb{R}^N)^{-1}\left[\int \log(P^\lambda_{1/2}\psi)d\omega + \int \log(P^\lambda_{1/2}\varphi)d\omega\right] \geq -2C/\omega(\mathbb{R}^N)$$

for all $\lambda \in [1,\infty)$. Hence, if $\alpha = \omega(\mathbb{R}^N)\exp[-2C/\omega(\mathbb{R}^N)]$, then $(\psi,P^\lambda_1\varphi)_{L^2(\mathbb{R}^N)} \geq \alpha$. Finally, replace ψ by $\psi_\epsilon \equiv \epsilon^{-N/2}\psi(\cdot/\epsilon)$ and let $\epsilon \downarrow 0$.

Q.E.D.

Before drawing conclusions from Theorem (3.1) it is useful to have the following simple observation.

(3.8) Lemma: Suppose that $P(t,x,dy) = p(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y)$ is continuous. If there exist $\alpha, r \in (0,\infty)$ such that $p(t,x,y) \geq \alpha/t^{N/2}$ for all $(t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$ with $|y - x| \leq rt^{1/2}$, then there is a $\beta \in (0,\infty)$, depending only on N, α, r, such that

(3.9) $$p(t,x,y) \geq (\beta/t^{N/2})\exp[-|y - x|^2/\beta t]$$

for all $(t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$ with $|y - x| \leq rt/4$. If, in addition, there is a $T \in (0,1]$ such that $P(t,x,B(y,r)) \geq \alpha\exp[-|y - x|^2/\alpha t]$ for all $(t,x,y) \in (0,T]\times\mathbb{R}^N\times\mathbb{R}^N$, then $\beta \in (0,\infty)$, depending only on N, α, r, and T, can be chosen so that (3.9) holds for all $(t,x,y) \in [2,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$.

Proof: First suppose that $t \in [1,\infty)$ and $rt^{1/2} \leq |y - x| \leq rt/4$, and let n be the smallest integer which exceeds $9|y - x|^2/r^2t$. Clearly $9|y - x|^2/r^2t \leq n \leq 10|y - x|^2r^2t$ and $3|y - x|/n \leq r(t/n)^{1/2}$. Thus, if $\delta \equiv |y - x|/n$ and $\tau \equiv t/n$, then $3\delta \leq r\tau^{1/2}$ and $\tau \geq (rt)^2/10|y - x|^2 > 1$. Now set $x_m = \frac{n-m}{n}x + \frac{m}{n}y$ and note that $|\xi_{m+1} - \xi_m| \leq r\tau^{1/2}$ for $\xi_\mu \in B(x_\mu,\delta)$, $1 \leq \mu \leq n$. Hence, if $B_\mu = B(x_\mu,\delta)$, then

$$p(t,x,y) \geq \int_{B_1\times\cdots\times B_{n-1}} p(\tau,x,\xi_1)p(\tau,\xi_1,\xi_2)\cdots p(\tau,\xi_{n-1},y)d\xi_1\cdots d\xi_{n-1}$$
$$\geq (\alpha/\tau^{N/2})(\Omega_N\delta^N)^{n-1} \geq (\alpha/t^{N/2})(\alpha\Omega_N r/10^{1/2})^{n-1};$$

and clearly the first part follows from this.

To prove the second part, suppose that $t \in [2,\infty)$ and $|y - x| \geq rt/4$ are given. Then with n the smallest integer exceeding

$(t-1)/T$, $x_m = \frac{n-m}{n}x + \frac{m}{n}y$, and $B_\mu = B(x_\mu, r)$

$$P(t-1,x,B(y,r)) \geq \int_{B_1\times\cdots\times B_{n-1}} P(\tfrac{t-1}{n},x,d\xi_1)P(\tfrac{t-1}{n},\xi_1,d\xi_2)\cdots P(\tfrac{t-1}{n},\xi_{n-1},d\xi_n)$$

$$\geq \alpha^n \exp[-8(|y - x|^2 + r^2n^2)Vn^2/\alpha t].$$

Since $n \leq t/T \leq (|y - x|/rT)\wedge(|y - x|^2/r^2t)$ and $p(t,x,y) \geq \int_{B(y,r)} p(1,\xi,y)P(t-1,x,d\xi)$, the second part follows. Q.E.D.

(<u>3.9</u>) <u>Theorem</u>: Assume that $P(t,x,dy) = p(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y)$ is continuous and satisfies $p(1/2,x,y) \geq \epsilon$ when $|y - x| \leq r$ and $p(1/4,\cdot,*) \leq B$ for some ϵ, r, and B from $(0,\infty)$. Further, assume that there is an $R \in (0,\infty)$ such that $d(x,y) \leq R$ whenever $|y - x| \leq 1$. Then there is a $\beta \in (0,1]$, depending only on N, ϵ, r, B, R, and $\|a\|_{C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)}$, such that

(3.10) $$p(t,x,y) \geq \beta\exp[-|y - x|^2/\beta t]$$

for all $(t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$.

<u>Proof</u>: In view of Lemma (3.8) and Theorem (1.25), all that we have to do is check that there are r and α from $(0,1]$ such that $p(t,x,y) \geq \alpha/t^{N/2}$ for all $(t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$ with $|y - x| \leq rt^{1/2}$. Moreover, since our assumptions are translation invariant, it suffices for us to check that $p(t,0,y) \geq \alpha/t^{N/2}$ for all $(t,y) \in [1,\infty)\times\mathbb{R}^N$ with $|y| \leq rt^{1/2}$; and, by Theorem (3.1), this reduces to showing that $P(t,x,B(x,rt^{1/2})) \geq 1/2$ for some appropriately chosen $r \in (0,\infty)$. But, by standard estimates (cf. Theorem (4.2.1) in [S-V]), $P(t,x,B(x,rt^{1/2})^c) \leq 2N\exp[-(r - M)^2/4AN^{1/2}]$ for $r > M$,

where $M^2 = \sup\{\sum_{i=1}^{N}\left[\sum_{j=1}^{N}\partial_{x_j}a^{ij}(x)\right]^2 : x \in \mathbb{R}^N\}$ and $A = \sup\{(\eta,a(x)\eta)_{\mathbb{R}^N} : x \in \mathbb{R}^N \text{ and } \eta \in S^{N-1}\}$. Hence, it is clear how to choose r.

Q.E.D.

(<u>3.11</u>) <u>Corollary</u>: Assume that either (1.3) or (1.4) holds for some $\upsilon \in [N,\infty)$, $\delta \in (0,1]$, and A or B from $(0,\infty)$ and also that there is an $R \in (0,\infty)$ for which $d(x,y) \le R$ whenever $|y - x| \le 1$. In addition, assume that $P(t,x,dy) = p(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y)$ is continuous and satisfies $p(1/2,x,y) \ge \epsilon$ for all $|y - x| \le r$ and some positive r and ϵ. Then there exists an $M \in [1,\infty)$, depending only on N, υ, R, r, ϵ, A or B, and $\|a\|_{C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)}$, such that

$$\frac{1}{Mt^{N/2}}\exp[-M|y-x|^2/t] \le p(t,x,y) \le \frac{M}{t^{N/2}}\exp[-|y-x|^2/Mt] \tag{3.12}$$

for all $(t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$.

<u>Proof</u>: The right hand side of (3.12) comes from Theorem (1.11) and the assumption that $d(x,y) \le R$ for $|y - x| \le 1$. The left hand side of (3.12) is an simple application of Theorem (3.9) once one notices that, again by (1.11), the required upper bound on $p(1/2,x,y)$ is a consequence of either (1.3) or (1.4).

Q.E.D.

(<u>3.13</u>) <u>Corollary</u>: Let $P(t,x,\cdot)$ corresponding to a be as in Corollary (3.11) above. Suppose that $\hat{a}: \mathbb{R}^N \longrightarrow \mathbb{R}^N\otimes\mathbb{R}^N$ is a second symmetric matrix valued function in $C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)$ and let $\hat{P}(t,x,\cdot)$ be the transition probability function determined by the operator

$\hat{\mathcal{L}} = \nabla\cdot(\hat{a}\nabla)$. If $\hat{a}(\cdot) \geq a(\cdot)$, then $\hat{P}(t,x,dy) = \hat{p}(t,x,y)dy$ where $(t,x,y,) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \hat{p}(t,x,y)$ is measurable and

(3.14) $$\frac{1}{\hat{M}t^{N/2}}\exp[-\hat{M}|y-x|^2/t] \leq \hat{p}(t,x,y) \leq \frac{\hat{M}}{t^{N/2}}\exp[-|y-x|^2/\hat{M}t]$$

for all $(t,x) \in [1,\infty)\times\mathbb{R}^N$ and almost every $y \in \mathbb{R}^N$, where $\hat{M} \in [1,\infty)$ depends only on N and $\|\hat{a}\|_{C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)}$ as well as the quantities υ, R, ϵ, A or B, and $\|a\|_{C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)}$ from Corollary (3.11).

<u>Proof</u>: Let $\hat{\mathcal{E}}$ denote the Dirichlet form determined by $\hat{a}$ and note that $\hat{\mathcal{E}} \geq \mathcal{E}$. Thus, with the same A, υ, and δ as for $\mathcal{E}$,

(3.15) $$\|f\|_2^{2+4/\upsilon} \leq A(\hat{\mathcal{E}}(f,f) + \delta\|f\|_2^2)\|f\|_1^{4/\upsilon}, \quad f \in L^2(\mathbb{R}^N).$$

In addition, since $\|P_t\|_{1\longrightarrow\infty} \leq M/t^{N/2}$, $t \in [1,\infty)$, Theorem (2.9) in [C-K-S] says that $\|f\|_2^{2+4/\upsilon} \leq \hat{B}\mathcal{E}(f,f)\|f\|_1^{4/\upsilon}$ for all $f \in L^2(\mathbb{R}^N)$ satisfying $\mathcal{E}(f,f) \leq \|f\|_1^2$, where $\hat{B} \in (0,\infty)$ depends only on M and N. Hence, we also have

(3.16) $$\|f\|_2^{2+4/\upsilon} \leq \hat{B}\hat{\mathcal{E}}(f,f)\|f\|_1^{4/\upsilon} \text{ if } f \in L^2(\mathbb{R}^N) \text{ with } \hat{\mathcal{E}}(f,f) \leq \|f\|_1^2.$$

Combining (3.15), (3.16), and Theorem (1.2), we conclude that there is a $\hat{C} \in (0,\infty)$, depending only on N, M, $\hat{B}$, υ, and R, such that

(3.17) $$\hat{p}(t,x,y) \leq (\hat{C}/t^{N/2})\exp[-|y - x|^2/\hat{C}t]$$

for all $(t,x) \in [1/4,\infty)\times\mathbb{R}^N$ and a.e. $y \in \mathbb{R}^N$. (We have used here the fact that $\hat{D}(x,y) \leq d(x,y) \leq 2R|y - x|$ for $|y - x| \geq 1$.) In particular, this completes the proof of the right hand side of (3.14).

To prove the left hand side of (3.14), assume, for the moment, that a continous version of $\hat{p}$ exists. Next, note that, by (3.17), both (3.2) and (3.3) hold with P replaced by $\hat{P}$ and constants depending only on N and $\hat{C}$. Also, since our assumptions are translation invariant and because we already know that (2.1) holds for all translates of a with a K having the required dependence, we can proceed in precisely the same way as we did in the proof of Theorem (3.9) to get the left hand side of (3.14). Finally, in order to remove the assumption that $\hat{p}$ is contiuous, proceed as follows. Given $\epsilon > 0$, set $\hat{a}_\epsilon = \hat{a} + \epsilon I$. Then, for each $\epsilon > 0$, the corresponding $\hat{p}_\epsilon$ will be continuous. In addition, (3.14) will be satisfied for $\hat{p}_\epsilon$ with an $\hat{M}$ which can be taken independent of $\epsilon \in (0,1]$. Hence, since $\hat{P}_\epsilon(t,x,\cdot)$ tends weakly to $\hat{P}(t,x,\cdot)$ as $\epsilon \downarrow 0$, it is easy to see that (3.14) will hold for each $(t,x) \in [1,\infty)\times\mathbb{R}^N$ and almost every $y \in \mathbb{R}^N$. Q.E.D.

(<u>3.18</u>) <u>Remark</u>: It should be clear that the right hand side of (3.14) holds with an $\hat{M}$ whose only dependence on $\hat{a}$ is in terms of the upper bound $\hat{A}$ of $\hat{a}$. Also (cf. Lemma (3.8)), so long as one restricts ones attention to a region $\{(t,x,y) \in [1,\infty)\times\mathbb{R}^N\times\mathbb{R}^N: |y - x| \le \rho t\}$ for some $\rho \in (0,\infty)$, the $\hat{M}$ on the left hand side can be chosen to depend on $\hat{a}$ only through $\hat{A}$. Thus, it is only to get the left hand side of (3.14) for all $x,y \in \mathbb{R}^N$ that we need to allow $\hat{M}$ to depend on $\|\hat{a}\|_{C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)}$. It is not clear to us whether this dependence is real or simply an flaw in our method.

This problem does not arrise in the uniformly elliptic case (treated in [F-S,2]) because, in that case, one has that $p(t,x,y) \geq \alpha/t^{N/2}$ for some $\alpha \in (0,1]$ and all $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$ with $|y - x| \leq \alpha t^{1/2}$ (not just for $t \geq 1$); and therefore one can extend the argument used to prove the first part of Lemma (3.8) to cover the whole of $\mathbb{R}^N\times\mathbb{R}^N$.

We are now ready to prove the main results of this article. Namely, we are going to describe a class of non-elliptic a's to which the above apply. To this end, assume that $2a = \sigma\sigma^\dagger$, where $\sigma \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^d)$; define $d(x,y)$ accordingly, as in section 1); and, for $1 \leq k \leq d$, set $V_k = \sum_{j=1}^N \sigma_k^j\partial_{x_j}$. For $\alpha \in \bigcup_{\ell=1}^\infty (\{1,\ldots,d\})^\ell$, set $|\alpha| = \ell$ if $\alpha \in (\{1\ldots,d\})^\ell$, $\ell \in Z^+$, and define $V_\alpha = V_k$ if $\alpha = (k)$ and $V_\alpha = [V_k,V_{(\alpha_1,\ldots\alpha_{\ell-1})}]$ if $\ell \geq 2$, $1 \leq k \leq d$, and $\alpha = (\alpha_1,\ldots,\alpha_{\ell-1},k)$. (We use $[V,W]$ to denote the commutator, or Lie product, of vector fields V and W.) Identifying $T_x(\mathbb{R}^N)$ with $\mathbb{R}^N$, we define

$$\overline{A}_\ell(x) = \sum_{1\leq|\alpha|\leq\ell} V_\alpha(x)\otimes V_\alpha(x) \tag{3.19}$$

for $\ell \in Z^+$. The following theorem summarizes a few results which, in one form or another, have been derived by various authors (cf., for example, Corollary (3.25) in [K-S,II] and Lemma (3.17) in [K-S,III]).

(<u>3.20</u>) <u>Theorem</u>: Referring to the preceding, assume that

$$\overline{A}_\ell(x) \geq \epsilon I,\ x \in \mathbb{R}^N. \tag{3.21}$$

for some $\ell \in Z^+$ and $\epsilon > 0$. Then $P(t,x,dy) = p(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow p(t,x,y)$ is smooth. Moreover, for each $n \geq 0$, there exist C_n, μ_n, and υ_n from $(0,\infty)$ such that

$$(3.22) \qquad |\partial_t^m \partial_x^\beta \partial_y^\gamma p(t,x,y)| \leq (C_n/t^{\upsilon_n/2})\exp[-\mu_n|y - x|^2/t]$$

for all $(m,\beta,\gamma) \in Z^+\times\mathcal{N}^d\times\mathcal{N}^d$ satisfying $m + |\beta| + |\gamma| \leq n$ and $(t,x,y) \in (0,1]\times\mathbb{R}^N\times\mathbb{R}^N$. Finally, there is a $R \in [1,\infty)$ such that

$$(3.23) \qquad (1/R)|y - x| \leq d(x,y) \leq R|y - x|^{1/\ell}$$

for all $x,y \in \mathbb{R}^N$ with $|y - x| \leq 1$.

Plugging these results about the "short time" properties of $p(t,x,y)$ into the machinery which we have been developing in the present article, we obtain the following "long time" estimates.

(<u>3.2</u>4) <u>Theorem</u>: Let a be as in the preceding and assume that (3.21) holds for some $\ell \in Z^+$ and $\epsilon > 0$. Suppose that $\hat{a} \in C_b^2(\mathbb{R}^N;\mathbb{R}^N\otimes\mathbb{R}^N)$ is a second non-negative, symmetric matrix-valued function, and define $\hat{P}(t,x,\cdot)$ accordingly. If $\hat{a}(\cdot) \geq a(\cdot)$, then $\hat{P}(t,x,dy) = \hat{p}(t,x,y)dy$ where $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \hat{p}(t,x,y)$ is measurable and satisfies (3.14) for some $\hat{M} \in (0,\infty)$. Moreover, $\hat{M}$ can be chosen so that its only direct dependence on $\hat{a}$ is in terms of $\|\hat{a}\|_{C_b^2(\mathbb{R}^N;\mathbb{R}^N\times\mathbb{R}^N)}$.

<u>Proof</u>: In view of Corollary (3.13), we need only check the case when $\hat{a} = a$; and, because of Corollary (3.11), this reduces to showing that $p(1/2,x,y) \geq \epsilon$ for some $\epsilon > 0$ and all $x,y \in \mathbb{R}^N$ with

$|y - x| \le \epsilon$. But, as we noted in the proof of Theorem (3.9), $P(1/4,x,B(x,r)) \ge 1/2$, $x \in \mathbb{R}^N$, for some $r \in (0,\infty)$. Hence, since $p(1/4,\cdot,*)$ is symmetric,

$$p(1/2,x,x) \ge \int_{B(x,r)} p(1/4,x,\xi)^2 d\xi \ge \frac{1}{|B(x,r)|}\left[\int_{B(x,r)} p(1/4,x,\xi)d\xi\right]^2$$

$$= (1/\Omega_N r^N)P(1/4,x,B(x,r))^2 \ge (1/\Omega_N r^N)/4.$$

At the same time, by (3.22), we see that there is a $\delta > 0$ such that $|p(1/2,x,y) - p(1/2,x,x)| \le (1/8\Omega_N r^N)$ for all $x,y \in \mathbb{R}^N$ with $|y - x| \le \delta$. Hence, we can take $\epsilon = \delta\wedge(1/8\Omega_N r^N)$. Q.E.D.

(3.25) Corollary: Let $\hat{a} \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N\times\mathbb{R}^N)$ be a non-negative definite, symmetric matrix-valued function. Given $1 \le k \le N$, set $\hat{V}_k = \sum_{j=1}^N \hat{a}^{ik}\partial_{x_i}$, and define $\hat{V}_\alpha$ ($\alpha \in (\{1,...N\})^\ell$ and $\ell \in Z^+$) in terms of $\{\hat{V}_1,...,\hat{V}_N\}$ accordingly. If there is an $\epsilon > 0$ and an $\ell \in Z^+$ such that

$$\sum_{1\le|\alpha|\le\ell} (\hat{V}_\alpha(x),\eta)^2_{\mathbb{R}^N} \ge \epsilon/2,\ x \in \mathbb{R}^N \text{and } \eta \in S^{N-1}, \tag{3.26}$$

then $\hat{P}(t,x,dy) = \hat{p}(t,x,y)dy$ where $\hat{p}$ is measurable and satisfies (3.14) for some $\hat{M} \in (0,\infty)$.

Proof: Without loss in generality, we assume that $\hat{a}(\cdot) \le I$ and therefore that $a(\cdot) \equiv (\hat{a}(\cdot))^2 \le \hat{a}(\cdot)$. If we now take $\sigma = 2^{1/2}\hat{a}$, then (3.26) implies (3.21) for the $\overline{A}_\ell(x)$ defined relative to this σ. Hence our result follows from Theorem (3.25) applied to the pair $\hat{a}$ and a. Q.E.D.

(<u>3.27</u>) <u>Remark</u>: By combining the results in [F-P] with ideas from [O-R], C. Fefferman and A. Sanchez-Calle remark in [F-S] that the condition on $\hat{a}$ in Corollary (3.26) is necessary and sufficient for the corresponding operator $\hat{\mathcal{L}}$ to be sub-elliptic. In particular, one can use this observation to conclude that the $\hat{p}$ in (3.26) is smooth.

(<u>3.28</u>) <u>Remark</u>: The reader who remembers (0.1) in the introduction may well be wondering why we have bothered to state Theorem (3.20) or to derive the lower bound in the proof of Theorem (3.24). Our reason is that the results in (3.20) are considerably easier to prove than is (0.1) and that they suffice for our present purposes.

4. Applications to a Large Scale Harnack's Inequality:

In [F-S,1], [K-S,III], and [F-S,2], various estimates on fundamental solutions are shown to lead to Harnack's inequality. In this section we will use similar techniques to derive a "large scale" Harnack's inequality from the "long time" estimate obtained in the previous section.

Throughout this section we will assume that the $P(t,x,\cdot)$ associated with $\mathcal{L} = \nabla\cdot(a\nabla)$ admits a smooth density $p(t,x,y)$ for which there exist an $M \in [1,\infty)$ and a $\upsilon \in [N,\infty)$ such that

(4.1) $$p(t,x,y) \leq (M/t^{\upsilon/2})\exp[-|y - x|^2/Mt], \quad t \in (0,1],$$

and

(4.2) $$\frac{1}{Mt^{N/2}}\exp[-M|y - x|^2/t] \leq p(t,x,y) \leq \frac{M}{t^{N/2}}\exp[-|y - x|^2/Mt], \quad t \in [1,\infty),$$

for all $(t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$.

(4.3) Remark: Note that if a is $\hat{a}$ in either Theorem (3.24) or Corollary (3.25), then such M and υ exist. Indeed, the existence of M is the content of those results, whereas the existence of υ comes from the comparison of Dirichlet forms and an application of the first part of Theorem (1.2).

Let $(\beta(t),\mathcal{F}_t,P)$ be a Brownian motion on $\mathbb{R}^N$; and define $X(\cdot,x)$, $x \in \mathbb{R}^N$, by the Itô stochastic integral equation

(4.4) $$X(t,x) = x + \int_0^t a^{1/2}(X(s,x))d\beta(s) + \int_0^t b(X(s,x))ds, \quad t \geq 0,$$

where $b^i = \sum_{i=1}^{N} \partial_{x_j} a^{ij}$, $1 \le i \le N$. Given $x^o \in \mathbb{R}^N$ and $r \in (0,\infty)$, define

$P_{x^o,r}(t,x,\Gamma) = P(X(t,x) \in \Gamma$ and $X(s,x) \in B(x^o,r)$ for $s \in [0,t])$.

In the terminology of analysis, the density $p_{x^o,r}(t,x,y)$ of $P_{x^o,r}(t,x,\cdot)$ is the fundamental solution for $\mathcal{L}$ in $B(x^o,r)$ with boundary condition 0 (i.e. Dirichlet boundary conditions). The key to much of our analysis is contained in the following.

(4.5) Lemma: There exist an $\epsilon \in (0,1]$ and $R \in [1/\epsilon,\infty)$, depending only on N, M and ν, such that, for each $x^o \in \mathbb{R}^N$ and $r \in [R,\infty)$,

$$p_{x^o,r}((\epsilon r)^2,x,y) \ge \epsilon/r^N \tag{4.6}$$

for all $x,y \in B(x^o,r/2)$.

Proof: Without loss in generality, we assume that $x^o = 0$, and we will use $p_r(t,x,y)$ to denote $p_{0,r}(t,x,y)$.

Denote by $\zeta_r(x)$ the first time when $X(\cdot,x)$ exits from $B(0,r)$. Then, for $\epsilon \in (0,1]$, $r \ge 1/\epsilon$, and $x,y \in B(0,r/2)$:

$$\begin{aligned} p_r((\epsilon r)^2,x,y) &= p((\epsilon r)^2,x,y) \\ &\quad - E^P\Big[p((\epsilon r)^2 - \zeta_r(x),X(\zeta_r(x),x),y),\ \zeta_r(x) < (\epsilon r)^2\Big] \\ &\ge \frac{1}{M(\epsilon r)^N}\exp[-M/\epsilon^2] - M \sup_{s\le(\epsilon r)^2}\Big[\exp[-r^2/4Ms]/\rho(s)\Big] \\ &= \frac{\exp[-M/\epsilon^2]}{M(\epsilon r)^N}\Big[1 - M^2(\epsilon r)^2 \sup_{s\le(\epsilon r)^2}\Big[\exp[M/\epsilon^2 - r^2/4Ms]/\rho(s)\Big]\Big]. \end{aligned}$$

where $\rho(s) \equiv (s^{\nu} \vee s^{N})^{1/2}$. It is not hard to deduce from this that the required inequality holds as soon as ϵ is sufficiently small and r is sufficiently large, depending only on N, M, and ν.

Q.E.D.

(4.7) Theorem: Let ϵ and R be as in Lemma (4.6). Then, for every $x^o \in \mathbb{R}^N$, $r \in [R,\infty)$, and $u \in C^2(B(x^o,r))^+$ satisfying $\mathcal{L}u \leq 0$ in $B(x^o,r)$,

$$(4.8) \qquad u(x) \geq (\epsilon/r^N)\int_{B(x^o,r/2)} u(y)\,dy \;,\; x \in B(x^o,r/2).$$

In particular, there exists a $\rho \in (0,1)$, depending only on N and ϵ, such that for any $x^o \in \mathbb{R}^N$, $r \in [R,\infty)$, and $u \in C^2(B(x^o,r))\cap C_b(B(x^o,r))$ satisfying $\mathcal{L}u = 0$ in $B(x^o,r)$:

$$(4.9) \qquad \max_{x,y\in B(x^o,r/2)} [u(y) - u(x)] \leq \rho\left[\max_{x,y\in B(x^o,r)} [u(y) - u(x)]\right].$$

Thus, if $u \in C^2(\mathbb{R}^N)\cap C_b(\mathbb{R}^N)$ and $\mathcal{L}u = 0$ in $\mathbb{R}^N$, then u is constant.

Proof: Again we assume that $x^o = 0$. Let $u \in C^2(B(0,r))^+$ satisfying $\mathcal{L}u \leq 0$ be given. By a standard application of Itô's formula

$$\begin{aligned} u(x) &\geq E^P[u(X(\zeta_r(x)\wedge(\epsilon r)^2,x))] \\ &\geq E^P[u(X((\epsilon r)^2,x)),\ \zeta_r(x) > (\epsilon r)^2] - \int_{B(0,r/2)} u(y)p_r((\epsilon r)^2,x,y)dy, \end{aligned}$$

where the notation is the same as that in the proof of Lemma (4.6). Hence, by that lemma, (4.8) follows.

To prove (4.9), let σ and Σ denote, respectively, the infemum and supremum of u in B(0,r), and set $\Gamma = \{x \in B(0,r/2): u(x) \geq \frac{\Sigma + \sigma}{2}\}$. Assuming that $|\Gamma| \geq \frac{1}{2}|B(0,r/2)|$ and applying (4.8) to

$u - \sigma$, we have, from (4.8), that $u(x) - \sigma \geq (\epsilon\Omega_N/2^{N+1})\frac{\Sigma - \sigma}{2}$ for all $x \in B(0,r/2)$. Hence, if σ' and Σ' are the infemum and supremum of u in $B(0,r/2)$, then $\sigma' - \sigma \geq (\epsilon\Omega_N/2^{N+1})\frac{\Sigma - \sigma}{2}$; and so $\Sigma' - \sigma' \leq \rho(\Sigma - \sigma)$, where $\rho \equiv (1 - (\epsilon\Omega_N/2^{N+1}))/2$. If, on the other hand, $|\Gamma| \leq \frac{1}{2}|B(0,r/2)|$, then we repeat the preceding with $\Sigma - u$ replacing $u - \sigma$. Thus, in either case, (4.9) holds.

Finally, the assertion that a global, bounded solution to $\mathcal{L}u = 0$ is constant follows easily since, by repeated application of (4.9), we have that $\max_{x,y\in B(0,r)}[u(y) - u(x)] \leq 2\rho^n\|u\|_{C_b(\mathbb{R}^N)}$ for all $r \geq R$ and $n \in Z^+$.

Q.E.D.

According to the scheme introduced by N. Trudinger [T], the inequality (4.8) is one half of Harnack's inequality. To prove the other half, we follow an argument similar to that given in [F-S,1] to show that there exists a $C \in (0,\infty)$, depending only on N, M, υ, and the upper bound A on a, such that for every $x^o \in \mathbb{R}^N$, $r \in [1,\infty)$, and $u \in C^2(B(x^o,r))^+$ which satisfies $\mathcal{L}u \geq 0$ in $B(x^o,r)$:

$$(4.10) \qquad u(x) \leq \frac{C}{r^N}\int_{B(x^o,r/2)} u(y)dy \ , \ x \in B(x^o,r/4).$$

Given $r \in [1,\infty)$, define $g_r(x,y) = \int_0^{r^2} p(t,x,y)dt$ for $x \neq y$. It is then an easy matter to check that

$$(4.11) \qquad [\mathcal{L}(g_r(x,\cdot))](y) = p(r^2,x,y) \geq 0, \ x \neq y.$$

Also, from the estimates (4.1) and (4.2), it is easy to check that there exist $C_1 \in (0,\infty)$, depending only on N, M, and ν, such that

$$\max_{x \in B(0,r\rho)} \left[\int_{\Gamma_r((2\rho+\sigma)/3,\sigma)} g_r(x,y)^2 dy \right]^{1/2} \le C_1 r^{2-N/2}/(\sigma - \rho)^{\nu} \tag{4.12}$$

for all $r \in [1,\infty)$ and $0 < \rho < \sigma \le 1$, where $\underline{\Gamma_r(\alpha,\beta)} \equiv \{x \in \mathbb{R}^N: r\alpha \le |x| \le r\beta\}$ for $\alpha < \beta$.

We next recall the standard Caccioppli inequality. Namely, given an open G in $\mathbb{R}^N$, $v \in C^2(G)^+$ satisfying $\mathcal{L}v \ge 0$ and a $\psi \in C_0^\infty(G)$

$$\left[\int_G \psi^2 (\nabla v \cdot a \nabla v) dy \right]^{1/2} \le 2A^{1/2} \|\nabla\psi\|_\infty \left[\int_{\mathrm{supp}(\psi)} v^2 \, dy \right]^{1/2}. \tag{4.13}$$

(This is an application of integration by parts followed by Schwartz's inequality.) We are now prepared to prove the following result, from which (4.10) will be an easy step.

(4.14) Lemma: There is a $C_2 \in (0,\infty)$, depending only on N, M, A, and ν, such that for all $x^o \in \mathbb{R}^N$, $r \in [1,\infty)$, and $u \in C^2(B(x^o,r))^+$ satisfying $\mathcal{L}u \ge 0$ in $B(x^o,r)$:

$$u(x) \le (C_2/(\sigma - \rho)^{\lambda}) \left[\frac{1}{r^N} \int_{B(x^o,r\sigma)} u(y)^2 dy \right]^{1/2}, \quad x \in B(x^o,r\rho), \tag{4.15}$$

for all $0 < \rho < \sigma \le 1$, where $\lambda = 2V\nu$.

Proof: As usual, we assume that $x^o = 0$. Choose smooth functions $\eta_{\rho,\sigma}$ and $\psi_{\rho,\sigma}$ for $0 < \rho < \sigma \le 1$ so that $0 \le \eta_{\rho,\sigma}, \psi_{\rho,\sigma} \le 1$, $\eta_{\rho,\sigma} = 1$ on $B(0,(\rho+\sigma)/2)$ and 0 off of $B(0,(\rho+2\sigma)/3)$, $\psi_{\rho,\sigma} = 0$ on $B(0,(\rho+2\sigma)/3) \cup B(0,\sigma)^c$ and 1 on $\Gamma_1((\rho+\sigma)/2,(\rho+2\sigma)/3)$, and $\|\nabla\eta_{\rho,\sigma}\|_\infty \vee \|\nabla\psi_{\rho,\sigma}\|_\infty \le C_3/(\sigma - \rho)$ for some $C_3 \in (0,\infty)$. For $r \in [1,\infty)$, set $\eta_{\rho,\sigma,r} = \eta_{\rho,\sigma}(\cdot/r)$ and $\psi_{\rho,\sigma,r} = \psi_{\rho,\sigma}(\cdot/r)$.

Now suppose that r, u, ρ, and σ are given, and let $x \in B(0,r\rho)$. Then, using η to denote $\eta_{\rho,\sigma,r}$, we have

$$u(x) = (\eta u)(x) = \int(\eta u)(y)p(r^2,x,y)dy - \int[\mathcal{L}(\eta u)](y)g_r(x,y)dy.$$

By (4.2),

$$\int(\eta u)(y)p(r^2,x,y)dy \le \frac{M}{r^N}\int_{B(0,r\sigma)} u(y)dy \le \Omega_N^{1/2}M\left[\frac{1}{r^N}\int_{B(0,r\sigma)} u(y)^2dy\right]^{1/2}.$$

At the same time, since $\mathcal{L}u \ge 0$:

$$-\int[\mathcal{L}u](y)g_r(x,y)dy \le -2\int(\nabla\eta\cdot a\nabla u)(y)g_r(x,y)dy - \int u(y)[\mathcal{L}u](y)g_r(x,y)dy$$

$$= -\int(\nabla\eta\cdot a\nabla u)(y)g_r(x,y)dy + \int u(y)(\nabla\eta\cdot a\nabla g_r(x,\cdot))(y)dy$$

$$\le \left[\int_{\mathrm{supp}(\nabla\eta)} g_r(x,y)^2dy\right]^{1/2}\left[\int(\nabla\eta\cdot a\nabla u)(y)^2dy\right]^{1/2} + \left[\int_{\mathrm{supp}(\nabla\eta)} u(y)^2dy\right]^{1/2}\left[\int(\nabla\eta)\cdot a\nabla g_r(x,\cdot))(y)^2dy\right]^{1/2}$$

$$\le \frac{A^{1/2}C_3}{r(\sigma-\rho)}\left[\left[\int_\Gamma g_r(x,y)^2dy\right]^{1/2}\left[\int_\Gamma(\nabla u\cdot a\nabla u)(y)dy\right]^{1/2} + \left[\int_\Gamma u(y)^2dy\right]^{1/2}\left[\int_\Gamma(\nabla g_r(x,\cdot)\cdot a\nabla g_r(x,\cdot))(y)dy\right]^{1/2}\right]$$

where $\Gamma \equiv \mathrm{supp}(\nabla\eta) \subseteq \Gamma_r((\rho+\sigma)/2,(\rho+2\sigma)/3)$. Note that by (4.13) with $\psi = \psi_{\rho,\sigma,r}$:

$$\left[\int_\Gamma(\nabla u\cdot a\nabla u)(y)dy\right]^{1/2} \le \frac{2A^{1/2}C_3}{r(\sigma-\rho)}\left[\int_{B(0,r\sigma)} u(y)^2dy\right]^{1/2}$$

and

$$\left[\int_{\Gamma}(\nabla g_r(x,\cdot)\cdot a\nabla g_r(x,\cdot))(y)dy\right]^{1/2} \le \frac{2A^{1/2}C_3}{r(\sigma - \rho)}\left[\int_{\Gamma_r((\rho+2\sigma)/3,\sigma)} g_r(x,y)^2\, dy\right]^{1/2}.$$

Combined with the preceding and (4.12), this now yields (4.15).
Q.E.D.

A particular case of (4.15) is the inequality

$$\text{(4.16)} \qquad u(x) \le C_4\left[\frac{1}{r^N}\int_{B(x^o,r/3)} u(y)^2\, dy\right]^{1/2},\ x \in B(x^o,r/4),$$

where $C_4 = 6^{\lambda}C_3$. Hence, we will have proved (4.10) once we show that the left hand side of (4.16) can be estimated in terms of $\frac{1}{r^N}\int_{B(x^o,r/2)} u(y)\, dy$. To this end, assume that $x^o = 0$ and set $v(x) = u(rx)$ for $x \in B(0,1)$. Then, (4.15) becomes the statement that $v(x) \le (C_2/(\sigma - \rho)^{\lambda})\left[\int_{B(0,\sigma)} v(y)^2 dy\right]^{1/2}$ for all $0 < \rho < \sigma \le 1$ and $x \in B(0,\rho)$. Hence, by an easy argument due to Dahlberg and Kenig (cf. the last part of the proof of Lemma (3.2) in [F-S,1]), there is a $K \in (0,\infty)$, depending only on C_2 and λ, such that $\left[\int_{B(0,1/3)} v(y)^2 dy\right]^{1/2} \le K\int_{B(0,1/2)} v(y)\, dy$; and clearly this transforms back into the required statement about u. In other words, we have now proved (4.10): which, in combination with Theorem (4.7) gives the following version of Harnack's inequality.

(<u>4.17</u>) <u>Theorem</u>: There exist R and K from $(0,\infty)$, depending only on N, M, A, and ν, such that for any $x^o \in \mathbb{R}^N$, $r \in [R,\infty)$, and $u \in C^2(B(x^o,r))^+$ satisfying $\mathcal{L}u = 0$ in $B(x^o,r)$, $u(y) \le Ku(x)$ for all

$x, y \in B(x^o, r/4)$. In particular, the only global, non-negative solutions to $\mathcal{L}u = 0$ are constant.

(4.18) Remark: It should be clear that our assumption that $(t,x,y) \longrightarrow p(t,x,y)$ is not essential and can be circumvented by a procedure like the one which we used to conclude the proof of Corollary (3.13). Also, we point out that had we worked a little harder we could have derived the preceding Harnack's inequality for non-negative solutions to the parabolic equation $\partial_t u - \mathcal{L}u = 0$ (cf. [F-S,2]).

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