Long-Wave Instabilities and Saturation in Thin Film Equations

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Abstract

Hocherman and Rosenau conjectured that long-wave unstable Cahn-Hilliard-type interface models develop finite-time singularities when the nonlinearity in the destabilizing term grows faster at large amplitudes than the nonlinearity in the stabilizing term (Phys. D 67, 1993, pp. 113–125). We consider this conjecture for a class of equations, often used to model thin films in a lubrication context, by showing that if the solutions are uniformly bounded above or below (e.g., are nonnegative), then the destabilizing term can be stronger than previously conjectured yet the solution still remains globally bounded.

For example, they conjecture that for the long-wave unstable equation

$$h_t = -(h^n h_{xxx})_x - (h^m h_x)_x \,,$$

$m > n$ leads to blowup. Using a conservation-of-volume constraint for the case $m > n > 0$, we conjecture a different critical exponent for possible singularities of nonnegative solutions. We prove that nonlinearities with exponents below the conjectured critical exponent yield globally bounded solutions. Specifically, for the above equation, solutions are bounded if $m < n+2$. The bound is proved using energy methods and is then used to prove the existence of nonnegative weak solutions in the sense of distributions. We present preliminary numerical evidence suggesting that $m \geq n+2$ can allow blowup. © 1998 John Wiley & Sons, Inc.

1 Introduction

Long-wave unstable equations are ubiquitous in the modeling of pattern formation in physical systems that involve interfaces. A now-classical example is the periodic Kuramoto-Sivashinsky equation that arises in modeling combustion [28, 29] and solidification [26, 27]

$$h_t = -h_{xxxx} - h_{xx} + h_x^2 \,, \qquad h(x+L) = h(x) \,. \tag{1.1}$$

The graph of h represents the position of the interface between the solid and liquid phases or the burnt and unburnt material. The equation arises from a series of approximations including both a "sharp interface" assumption and an assumption that the solution has a long-wave character.

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The KS equation (1.1) is long-wave unstable in that a small perturbation of a flat interface yields, to linear order, the solution $h = h_0 + \varepsilon g(t)\cos(2\pi kx/L)$ where $g(t) \sim e^{\sigma t}$ with linear growth rate

$$\sigma(k) = -k^2\left(\frac{2\pi}{L}\right)^2\left(k^2\left(\frac{2\pi}{L}\right)^2 - 1\right).$$

Only long-wave perturbations grow: $\sigma(k) > 0 \iff |k| < L/2\pi$. The growth of the linearized solution implies that the nonlinear terms must enter into the dynamics. In fact, the nonlinear term h_x^2 causes the solution to saturate—h remains bounded and smooth despite the solutions of the linearized equation growing exponentially in time [17, 35, 43]. The nonlinear term transports energy from longer (growing) wavelengths to shorter wavelengths, which then dissipate the energy.

The nonlinearity in the KS equation is advective and affects the dynamics differently than do other types of nonlinearities. For example, if the nonlinearity is destabilizing, it can cause finite-time blowup. The semilinear heat equation is an example of a second-order equation with such a nonlinear destabilizing term:

$$h_t = h_{xx} + h^p\,.$$

For $p > 1$ certain initial data can yield a finite-time blowup: $h(x^*, t) \uparrow \infty$ at some point x^* as $t \uparrow T^* < \infty$. Extensive rigorous work on this equation shows the existence and self-similarity of the blowup singularity [2, 32, 49].[1]

The *Childress-Spiegel* equation is a fourth-order equation with a nonlinear destabilizing term

$$v_t = -\frac{\partial^2}{\partial x^2}(v_{xx} + v + v^2)\,. \tag{1.2}$$

The equation arises as an interface model in biofluids [15], solar convection [20], and binary alloys [48]. It, too, can have a finite-time blowup: $v(x^*, t) \uparrow \infty$ at a point x^* as $t \uparrow T^* < \infty$. One way in which this equation differs from the KS equation is that if the period $L \approx 2\pi$, then the nontrivial steady states near the $k = 1$ mode are subcritical rather than supercritical. However, subcriticality of nontrivial states is not the driving force for blowup, as a recently studied generalization illustrates. The modified Kuramoto-Sivashinsky equation[2]

$$h_t = -h_{xxxx} - h_{xx} + (1-\lambda)h_x^2 + \lambda h_{xx}^2 \tag{1.3}$$

[1] This partial list of references is given simply as sources for further information.

[2] Note that the Childress-Spiegel equation is a special case of the modified Kuramoto-Sivashinsky equation with $v = h_{xx}$ and $\lambda = 1$.

arises as a model for the dynamics of a hypercooled melt [47]. Solutions of this equation can exhibit finite-time singularities in which $\int_0^{T^*} |h_{xx}|^2\,dt \uparrow \infty$. Numerics confirm the existence of a self-similar blowup profile in which $|h|_{l^\infty}$ grows like $-\log(T^* - t)$ [7].

A natural question, addressed by Hocherman and Rosenau, is under what conditions do such destabilizing nonlinearities allow finite-time blowup. For a generic Cahn-Hilliard model, they conjectured:

CONJECTURE 1 ([38]) *Consider the evolution equation*

$$u_t = -\frac{\partial}{\partial x}\left(M(u)\frac{\partial}{\partial x}[-Q(u) + R(u)u_{xx}]\right) \tag{1.4}$$

with periodic boundary conditions $u(x+L) = u(x)$. If $M(u), R(u) \geq 0$ and $Q'(u) < 0$, then the equation is long-wave unstable. In such a case, the behavior of $Q(s)/(sR(s))$ determines the presence or absence of a finite-time blowup.

Specifically,

$$\lim_{s\to\infty} \frac{Q(s)}{sR(s)} = \begin{cases} \infty & u \to \infty \textit{ in finite time}, \\ \textit{finite} & \textit{marginal case}, \\ 0 & \textit{globally stable solutions}. \end{cases} \tag{1.5}$$

This conjecture is consistent with the fact that the linear growth rate associated with linearized perturbations of a flat state u_0 is

$$\sigma = M(u_0)\left[Q'(u_0)k^2 - R(u_0)k^4\right],$$

and the band of unstable modes becomes infinite (vanishes) as $u_0 \uparrow \infty$ if $Q'/R \to \infty$ ($\to 0$).

In this article, we prove that while Hocherman and Rosenau's conjecture may stand for equations that have nondegenerate coefficients of diffusion, $M(u) \geq \alpha > 0$ for all u, it must be modified for degenerate diffusion coefficients. We propose an alternate conjecture for such cases.

All previously studied examples of (1.4) seemed to confirm Conjecture 1. However, these examples all had $M(u)$ constant. In this paper, we use the fact that if $M(u)$ is degenerate, then the solutions can have special behavior near points x_0 where $M(u(x_0)) = 0$. The vanishing of $M(u)$ at u_0 can stop the solution from crossing the line $u = u_0$. Such degeneracy of M can ensure that solutions are uniformly bounded above ($u \leq u_0$) or below ($u \geq u_0$). This "weak maximum principle" is not true for general fourth-order equations and requires a certain degree of degeneracy in the fourth-order term. Specifically, for certain equations with $M(0) = 0$, nonnegative initial data can be proven to yield nonnegative solutions.

Here we show that nonnegativity of solutions coupled with the conservation of volume, $\int u = \text{const}$, can lead to different behavior than that predicted by Conjecture 1. The conjecture and proven results generalize immediately to solutions that are uniformly bounded above or below.

1.1 Long-Wave Unstable Lubrication Models

Equations of the form (1.4) arise in modeling the dynamics of thin liquid films. In some physical situations, a destabilizing force causes the liquid film to bead up into isolated droplets. Such an instigator can be either an external force, such as gravity in the case of a thin liquid film hanging from the bottom of a horizontal surface [23], or intrinsic to the system, such as repulsive, long-range van der Waals forces that enter the evolution equation in the form of a disjoining pressure [19, 42, 50]. In such situations, a lubrication approximation reduces the evolution equation to one of the form

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x \,. \tag{1.6}$$

For simplicity we consider periodic boundary conditions.

For thin liquid films, the fourth-order term of (1.6) comes from surface tension between the liquid and air and also incorporates any slippage at the liquid/solid interface [36]. The general form for $f(h)$ is

$$f(h) = h^3 + \lambda h^p \tag{1.7}$$

where $0 < p < 3$ and $\lambda > 0$ determines a slip length [22, 37, 39, 40, 41]. There is a long-wave instability if the second-order term of (1.6) has $g(h) \geq 0$. In the gravity-destabilized thin film problem, $g(h) \sim h^3$ [23]. For the thin film problem with repulsive van der Waals forces [19, 42, 50],

$$g(h) = \begin{cases} \frac{A}{h}\,, & \text{3D film,} \\ \frac{A}{h^2}\,, & \text{2D film,} \end{cases} \tag{1.8}$$

for nonretarded interactions, and

$$g(h) = \begin{cases} \frac{B}{h^2}\,, & \text{3D film,} \\ \frac{B}{h^3}\,, & \text{2D film,} \end{cases} \tag{1.9}$$

for retarded interactions.

In a recent work [13], we considered the case of attractive van der Waals forces ($g \leq 0$) and discussed the mathematical need for a cutoff of van der Waals interactions on a microscopic-length scale. We considered a cutoff such that $g(h) \sim h^m$ as $h \downarrow 0$ where $m > 0$. We call this a "porous media" cutoff because it can introduce behavior similar to that of the subdiffusive porous media equation [45] near the contact line. In this case, the choice of cutoff depends on the nature of the slip model used at the liquid/solid interface. With such a cutoff of the attractive van der Waal forces, we prove existence and long-time behavior results for nonnegative solutions. It is natural to consider a porous-media-type cutoff for repulsive van der Waals interactions ($g \geq 0$).

Another context in which equation (1.6) arises is the gravity-driven Hele-Shaw cell, for which $f(h) = g(h) = h$ [33, 34]. The fourth-order term comes from surface tension between the two liquids. The second-order term comes from the destabilization due to a density mismatch between the liquids. In [34], Goldstein et al. show that the initial disturbance leads to a finite-time pinching of the fluid neck ($h \downarrow 0$) and is due to a long-wave instability that persists up to times close to the singularity time. They present a scenario in which the higher modes of the system are slaved to a low mode. However, their slaving mechanism does not establish whether it is possible for the model to form finite-time "spikes" in which $h_{\text{max}} \to \infty$. In this paper we prove that their equation always leads to saturation in which the solution, while unstable to finite-wavelength perturbations of a flat state, does not grow without bound. Such saturation was observed in their numerical simulations.

In recent papers, authors studied the problem (1.6) for $g = 0$ [3, 6, 9] and $g \leq 0$ [13]. If $g \leq 0$, the second-order term is stabilizing. In both situations, solutions are uniformly bounded for all time, so that the only unresolved issue regarding singularity formation is whether $h \downarrow 0$ in finite time. However, in the $g \geq 0$ case considered here, the second-order term is destabilizing and two new concerns arise: The problem may be ill-posed near the contact line and the solution may blow up in finite time. Of course, $h \uparrow \infty$ is a clear violation of the assumptions made by the lubrication approximation, and the modeling equation has broken down.

To prove the problem is well-posed, one must prove that the solutions not only exist and depend smoothly on the initial data but are unique. While uniqueness of weak solutions is not known for this class of problems, we conjecture, based on linear stability theory, that ill-posedness is avoided if $f(h)$ dominates $g(h)$ in the $h \downarrow 0$ limit. Indeed, this condition proves to be sufficient to derive an existence theory for the problem. The question of blowup versus uniform boundedness presents an interesting case study for Conjecture 1. Writing equation (1.6) in the form of equation (1.4) with $R(u) = 1$, $M(u) = f(u)$, and

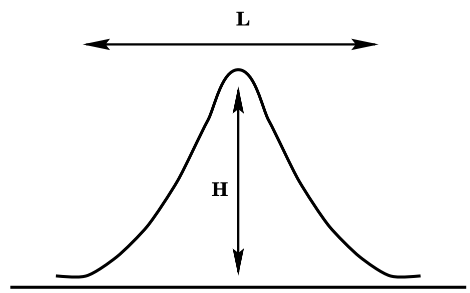

Figure 1.1. Length and height scales of a bump.

$Q'(u) = -g(u)/f(u)$, Conjecture 1 becomes "solutions can only blow up if $f(s)/g(s) < \infty$ as $s \to \infty$." Given an equation of the type (1.6) in which $f(h)$ and $g(h)$ are positive for $h > 0$, vanishing or diverging as $h \downarrow 0$ and $h \to \infty$, is Conjecture 1 true?

In this paper, we show that, for a class of nonnegative weak solutions of equation (1.6), Conjecture 1 must be revised. This is because equation (1.6) conserves volume ($\int h$) *and* has nonnegative solutions when $f(y)$ and $g(y)$ are sufficiently degenerate at $y = 0$. These two properties are sufficient to control the maximum of the solution for a range of cases where Conjecture 1 suggests blowup occurs.

A heuristic argument based on volume conservation suggests a different scaling than that of Conjecture 1. Consider a local maximum of the solution of height H (see Figure 1.1). Denote the characteristic-length scale of this bump by L. Conservation of volume requires that $HL \leq V$, where V is the total fluid volume. However, if the bump is growing without bound, the dynamics should have a balance between the nonlinear terms in equation (1.6)

$$\frac{f(H)H}{L^4} \sim \frac{g(H)H}{L^2} \Rightarrow \frac{f(H)}{g(H)} \sim L^2 \,.$$

This gives the constraint

$$\frac{H^2 f(H)}{g(H)} < V^2 \,,$$

suggesting that the solution can grow without bound only if

$$\lim_{s \to \infty} \frac{s^2 f(s)}{g(s)} < \infty \,.$$

We analytically show that such scaling arguments are valid by proving that $s^2 f(s)/g(s) \to \infty$ as $s \to \infty$ implies uniform boundedness for positive classical solutions (in Section 2) and for nonnegative weak solutions (in Section 3).

Including the h_t term from (1.6) in the scaling analysis, we get

$$\dot{H} \le \frac{g(H)H}{L^2} \sim \frac{g(H)^2}{f(H)} H\,.$$

This bound on $\dot{H}$ suggests that any blowup must take infinite time whenever

$$\lim_{s\to\infty} \frac{g(s)^2}{f(s)} = A < \infty$$

since the solution would be dominated by e^{At}. We prove this for positive classical solutions (in Section 2) and for nonnegative weak solutions (in Section 3). We conclude that finite-time singularities are only possible for equations of the type (1.6) in which

$$\lim_{s\to\infty} \frac{s^2 f(s)}{g(s)} < \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)^2}{f(s)} = \infty\,.$$

1.2 The Need for Nonnegative Solutions

The modified Kuramoto-Sivashinsky equation (1.3) provides a case study for the difference between the behavior predicted in Conjecture 1 and the results we prove here.

Computations of solutions [7] of the modified KS equation

$$h_t = -h_{xxxx} - h_{xx} + (1-\lambda)h_x^2 + \lambda h_{xx}^2$$

show that, as the solution becomes singular, the driving equation is

$$h_t = -h_{xxxx} + \lambda h_{xx}^2\,.$$

Rewriting this equation with $v = h_{xx}$, it is of the form we consider (1.6)

$$v_t = -v_{xxxx} + 2\lambda(v v_x)_x\,. \tag{1.10}$$

With $f(v) = 1$, $g(v) = 2\lambda v$, and $\lim s^2 f(s)/g(s) = \infty$, Conjecture 1 states that this equation produces a blowup in which $v \to \infty$. The preceding argument for sign-preserving solutions suggests that if the solution to (1.10) has a fixed sign, then it cannot blow up because $\lim_{s\to\infty} s^2 f(s)/g(s) = \infty$. On the other hand, solutions of (1.10) do have finite-time singularities with self-similar structure in which $\max\{v(x,t)\} \to \infty$ and $\min\{v(x,t)\} \to -\infty$ as

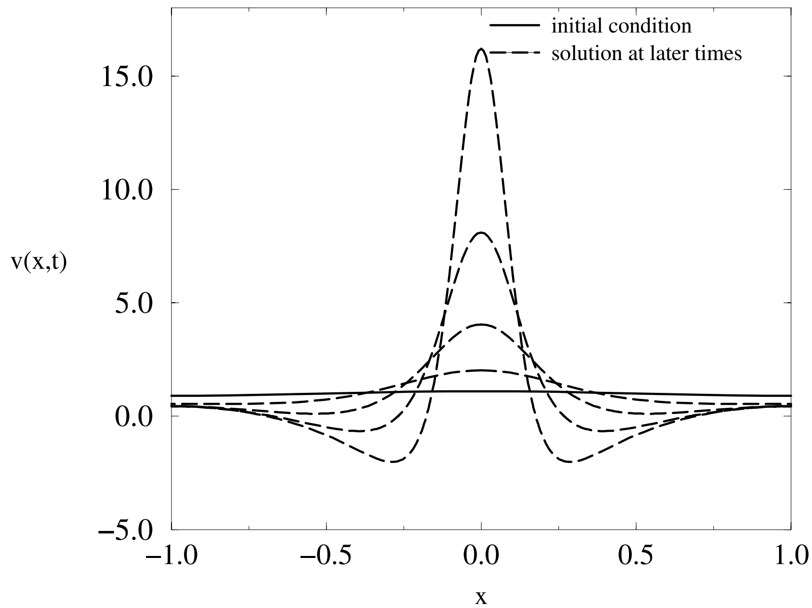

Figure 1.2. The beginning of a blowup of the solution to (1.3) with initial data $h_0(x) = 1 + 0.1\cos(\pi x)$ and $2(1-\lambda) = 40$. Note that although the initial data is positive, the solution changes sign. The blowup occurs as in [7] with the solution having a self-similar structure in which $\max\{h_{xx}(x,t)\} \to \infty$ and $\min\{h_{xx}(x,t)\} \to -\infty$ as $t \to T^*$.

$t \to T^*$ while $\int v = 0$ [7]. Figure 1.2 presents the early evolution of a solution that ultimately blows up in finite time. Note that although the initial data is positive, the solution changes sign. In Section 2.3 we prove that all finite-time singularities of the modified Kuramoto-Sivashinsky equation must be the type where the solution changes sign and its second derivatives blow up and down to $\pm\infty$.

1.3 The Revised Conjecture

The scaling arguments of Section 1.1 lead us to make the following revised conjecture in order to consider degenerate fourth-order equations.

CONJECTURE 2 *Consider the evolution equation*

$$u_t = -\frac{\partial}{\partial x}\left(f(u)\frac{\partial}{\partial x}\left[-\tilde{G}'(u) + u_{xx}\right]\right) \tag{1.11}$$

with periodic boundary conditions $u(x+L) = u(x)$*. If* $f(u) \geq 0$ *and* $\tilde{G}''(u) \leq 0$*, then the equation is long-wave unstable. Suppose* $\tilde{G}''(u)$ *is bounded as* $u \downarrow 0$ *and* f *is degenerate,* $f(u) \downarrow 0$ *as* $u \downarrow 0$*. In such a case, the behavior of* $\tilde{G}''(u)/u^2$ *as* $u \to \infty$ *determines the presence or absence of blowup for nonnegative solutions.*

Specifically,

$$\lim_{s\to\infty} \frac{\tilde{G}''(s)}{s^2} = \begin{cases} \infty & \textit{blowup}, \\ \textit{finite} & \textit{marginal case}, \\ 0 & \textit{globally bounded solutions}. \end{cases} \tag{1.12}$$

Defining

$$A = \lim_{s\to\infty} \sqrt{f(s)}G''(s)\,,$$

if $A = \infty$*, then it is possible that the blowup will occur in finite time.*

The conjecture is written for equations where the degeneracy in $f(y)$ occurs at $y = 0$; hence it considers nonnegative solutions. The conjecture transforms in a natural manner to consider equations with degeneracy $f(u_0) = 0$ and solutions $u \leq u_0$ or $u \geq u_0$.

We prove the global boundedness part of this conjecture for equations in which $f(u)$ is sufficiently degenerate at $u = 0$. In Section 4 we present a preliminary numerical computation suggesting that Conjecture 2 is sharp.

The paper is organized as follows. In Section 2.1 we introduce the Lyapunov function upon which the work depends. In Section 2 we use this Lyapunov function to prove the global boundedness part of Conjecture 2 for positive classical solutions of equations of the form (1.11). In Section 2.3 we discuss the modified Kuramoto-Sivashinsky equation (1.3) and, by using energy estimates, prove that when a solution to (1.3) blows up, the second derivative h_{xx} must simultaneously blow up to $+\infty$ and blow down to $-\infty$ (behavior observed in numerical simulations [7]). In Section 3 we use the global boundedness results for positive classical solutions to prove global existence of nonnegative weak solutions of a class of equations of type (1.6). The proof follows arguments from previous papers [3, 9] with $G = 0$. In Section 4 we present preliminary numerical simulations that confirm the blowup part of Conjecture 2. We will study the detailed structure of this blowup in a separate paper. Finally, in Section 5 we review the results of this paper and consider the case of higher-space dimensions.

2 H^1 Bounds for Long-Wave Unstable Equations

2.1 The Lyapunov Function

We start by considering steady periodic solutions of the evolution equation (1.6). These satisfy

$$0 = (f(h)h_{xxx})_x + (g(h)h_x)_x \,.$$

By integrating, we get

$$C = f(h)h_{xxx} + g(h)h_x \,.$$

Assuming that $f(y)$ and $g(y)$ vanish only at $y = 0$, we show that $C = 0$. If there is a point at which h vanishes, then $C = 0$. If there is no such point, then $f(h) > 0$ and

$$\frac{C}{f(h)} = h_{xxx} + \frac{g(h)}{f(h)}h_x = h_{xxx} + F(h)_x \,.$$

Integration yields

$$C\int_{\mathbb{S}^1} \frac{1}{f(h)}\,dx = 0 \iff C = 0\,.$$

Hence if $h \geq 0$ then $C = 0$, and integrating gives

$$D = h_{xx} + F(h) \quad \text{where} \quad F'(y) = \frac{g(y)}{f(y)}\,.$$

The constant D is determined by the steady state, $D = \int F(h)$. Steady states are extrema of the Lyapunov function

$$\mathcal{E}(h) = \int_{\mathbb{S}^1} \left(\frac{1}{2}h_x^2 - \tilde{G}(h) + Dh\right) dx \quad \text{where} \quad \tilde{G}''(y) = \frac{g(y)}{f(y)}\,.$$

This Lyapunov function is crucial in proving the uniform boundedness of positive and nonnegative solutions.

2.2 A Global Bound for Positive Solutions

We now consider positive smooth solutions of

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x \tag{2.1}$$

with periodic boundary conditions

$$h(0,t) = h(1,t)$$

and initial condition $h(x,t)|_{t=0} = h_0(x)$. In Section 3, we prove that given certain assumptions,[3] such equations yield positive solutions from positive initial data. Slightly weaker assumptions on f yield nonnegative solutions from nonnegative initial data.

In this section, we prove uniform boundedness of positive smooth solutions of equations of the form (2.1) that satisfy the further constraint that

$$\frac{g(y)}{y^2 f(y)} \to 0 \quad \text{as } y \to \infty\,.$$

(This is the condition of Conjecture 2.) The methods of this section require only that the solution be nonnegative and smooth. As strictly positive solutions are guaranteed to be smooth, we consider this case.

Equation (2.1) is a conservation law; therefore a smooth solution conserves mass:

$$\frac{d}{dt}\int_{\mathbb{S}^1} h\,dx = 0 \Longrightarrow \overline{h} = \int_{\mathbb{S}^1} h\,dx = \int_{\mathbb{S}^1} h_0\,dx\,.$$

Moreover, smooth solutions satisfy

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{S}^1} h_x^2\,dx = -\int_{\mathbb{S}^1} f(h)h_{xxx}^2\,dx - \int_{\mathbb{S}^1} g(h)h_x h_{xxx}\,dx\,.$$

Finally, for any $G(h)$, a smooth solution satisfies

$$\frac{d}{dt}\int_{\mathbb{S}^1} G(h)\,dx = \int_{\mathbb{S}^1} G''(h)f(h)h_x h_{xxx}\,dx + \int_{\mathbb{S}^1} G''(h)g(h)h_x^2\,dx\,.$$

Choosing $\tilde{G}(y)$ so that $\tilde{G}''(y) = g(y)/f(y)$ yields the Lyapunov function

$$\begin{aligned}(2.2)\qquad &\frac{d}{dt}\int_{\mathbb{S}^1}\left(\frac{1}{2}h_x^2 - \tilde{G}(h) + Dh\right)dx\\ &\quad= -\int_{\mathbb{S}^1} f(h)\left[h_{xxx} + \frac{g(h)}{f(h)}h_x\right]^2 dx \le 0\,.\end{aligned}$$

If in (2.2) the minus sign in front of $\tilde{G}(h)$ were a plus sign, the Lyapunov function would be a sum of positive quantities and its dissipation would immediately guarantee that the solution remains bounded in $H^1(\mathbb{S}^1)$ for all time. This is the case when the second-order term is stabilizing ($g \le 0$) [13]. In the following, we show that for some $\tilde{G}$, the Lyapunov function can be used to control $|h|_{H^1}$ despite its mixed sign.

[3] Both $f(y)$ and $g(y)$ positive for $y > 0$, and as $y \downarrow 0$, $g(y) \downarrow 0$, $g(y)/f(y) < M$, and $f(y) \downarrow 0$ "sufficiently strongly."

We assume for simplicity that both f and g behave as power laws in the large and small y limits: $f(y) \sim y^{n_1}$, $g(y) \sim y^{m_1}$ for $y \gg 1$, and $f(y) \sim y^{n_2}$, $g(y) \sim y^{m_2}$ for $y \ll 1$. The further assumption[4] $m_2 > n_2 - 2$ gives the crude bound

$$\tilde{G}(y) \leq \begin{cases} ay^p & \text{for } y \geq 1\,, \\ C & \text{for } y \leq 1\,, \end{cases} \tag{2.3}$$

where $p = \max\{2, m_1 - n_1 + 2\}$. The following lemma states that given bound (2.3) with $p < 4$, the H^1-norm of a positive function can be bounded by its mean and the Lyapunov function $\mathcal{E}(h)$:

LEMMA 2.1 *Let $\tilde{G}$ be a function on $[0, \infty)$ such that the bound* (2.3) *holds for some exponent $p < 4$. Define a functional on H^1 by*

$$\mathcal{E}(h) = \int_{\mathbb{S}^1} \left(\frac{1}{2} h_x^2 - \tilde{G}(h) + Dh \right) dx < \infty$$

and define

$$q = \begin{cases} \max\left\{2, \frac{2+p}{4-p}\right\} & \text{if } \overline{h} > 1\,, \\ 2 & \text{if } \overline{h} \leq 1\,. \end{cases} \tag{2.4}$$

Then there exist positive constants c_1 and c_2 such that for all nonnegative $h \in H^1(\mathbb{S}^1)$,

$$\frac{1}{4} |h|_{H^1}^2 < \mathcal{E}(h) + c_2 \overline{h}^q + c_1 + \frac{1}{4} \overline{h}^2\,. \tag{2.5}$$

The critical case $p = 4$ is discussed in a remark following the proof of this lemma. In one space dimension, the H^1-norm bounds the L^∞-norm, providing a uniform upper bound for the solution. The bound (2.5) depends only on $\mathcal{E}(h)$, $\overline{h} = \int h$, and the quantities used to bound $\tilde{G}(h)$.

Lemma 2.1 immediately yields the following uniform boundedness result for positive smooth solutions of equation (2.1):

PROPOSITION 2.2 *Let $h(x,t)$ be a smooth positive solution on $[0, T]$ to* (2.1)*. Let $\tilde{G}''(y) = g(y)/f(y)$ be such that $\tilde{G}$ satisfies the conditions of Lemma* 2.1*.*

[4] For the existence of nonnegative weak solutions with nonnegative initial data, the stronger requirement of $m_2 \geq n_2$ is needed.

If the initial data $h_0 \in H^1$, then $|h(\cdot,t)|_{H^1}$ is uniformly bounded by the initial data

$$\begin{aligned}\frac{1}{4}|h|^2_{H^1} &< \mathcal{E}(h) + c_2\overline{h}^q + c_1 + \frac{1}{4}\overline{h}^2 \\ &< \mathcal{E}(h_0) + c_2\overline{h}^q + c_1 + \frac{1}{4}\overline{h}^2 < \infty\,.\end{aligned} \tag{2.6}$$

The constants c_1 and c_2 and the exponent q are as in Lemma 2.1.

The proof follows directly from Lemma 2.1 and the fact that for a smooth solution h of (2.1), $\mathcal{E}(h,t) \le \mathcal{E}(h_0)$ and $\int h = \int h_0$.

In the proof of Lemma 2.1, we use the interpolation inequality (see, e.g., [31, theorem 10.1, p. 27]):

Interpolation Lemma. *Let $p > 1$. Then there exists a constant C_1 depending only on p so that for all $u \in H^1(\mathbb{S}^1)$*

$$|u|_{L^p} \le C_1|u|^{2(p-1)/3p}_{H^1}\,|u|^{(2+p)/3p}_{L^1}. \tag{2.7}$$

PROOF OF LEMMA 2.1: Bound (2.3) on $\tilde{G}(h)$ implies

$$\begin{aligned}\int_{\mathbb{S}^1}\tilde{G}(h)\,dx &\le a\int_{\{h\ge1\}} h^p\,dx + \int_{\{h<1\}} C\,dx \\ &\le a\int_{\mathbb{S}^1} h^p\,dx + C\,.\end{aligned}$$

For $p > 0$,

$$\begin{aligned}\mathcal{E}(h) + c_2\overline{h}^q + \frac{1}{4}\,\overline{h}^2 &= \frac{1}{2}\int_{\mathbb{S}^1} h_x^2\,dx + \frac{1}{4}\,\overline{h}^2 - \int_{\mathbb{S}^1}\tilde{G}(h)\,dx + D\overline{h} + c_2\overline{h}^q \\ &\ge \frac{1}{4}|h|^2_{H^1} - C - a\int_{\mathbb{S}^1} h^p\,dx + D\overline{h} + c_2\overline{h}^q \\ &\ge \frac{1}{4}|h|^2_{H^1} - C - aC_1^p\,|h|^{2(p-1)/3}_{H^1}\,\overline{h}^{(p+2)/3} + D\overline{h} + c_2\overline{h}^q \qquad (2.8)\\ &\ge \frac{1}{8}|h|^2_{H^1} - C + D\overline{h} + \frac{c_2}{2}\overline{h}^q \ge \frac{1}{8}|h|^2_{H^1} - C\,. \qquad (2.9)\end{aligned}$$

In (2.8) we use the interpolation inequality (2.7) coupled with the key observation that for $h \ge 0$, $|h|_1 = \int h = \overline{h}$. Step (2.9) uses the fact that (2.8) is of the form

$$\frac{1}{4}A - C - \beta A^{\frac{p-1}{3}}B^{\frac{p+2}{3}} + DB + c_2B^q$$

where $\beta = aC_1^p$ and the following elementary lemma:[5]

[5] This lemma is proved by considering two cases: $A \ge \varepsilon B^q$ and $A \le \varepsilon B^q$ where $\varepsilon = (8\beta)^{3/(4-p)}$ and $c_2 \ge 2\beta\varepsilon^{(p-1)/3}$.

Lemma. Given $0 < p < 4$ and $\beta \in \mathbb{R}$, there exists a constant c_2 such that for all $A, B \geq 0$,

$$\frac{1}{4}A - \beta A^{(p-1)/3} B^{(p+2)/3} + c_2 B^{(2+p)/(4-p)} \geq \frac{1}{8}A + \frac{c_2}{2} B^{(2+p)/(4-p)}\,.$$

∎

Remark Regarding the Critical Case $p = 4$. In this case, $p = 4$, (2.8) implies that there exists a constant c_{crit}, depending on a, the asymptotic prefactor for $\tilde{G}$ in (2.3), such that if $\bar{h} < c_{\text{crit}}$ then Lemma 2.1 holds. Thus an a priori upper bound also occurs for the critical case when the initial data has sufficiently small mean.

As the scaling argument in Section 1.1 suggests, if $p \geq 4$ in the bound (2.3), smooth solutions have controlled growth if $m_1 \leq n_1/2$. To see this, we assume that $m_2 \geq n_2/2$ and find that the solution satisfies

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\mathbb{S}^1} h_x^2 &= -\int_{\mathbb{S}^1} f(h) h_{xxx}^2 - \int_{\mathbb{S}^1} g(h) h_x h_{xxx} \\ &= -\int_{\mathbb{S}^1} f(h)\left(h_{xxx} + \frac{1}{2}\frac{g(h)}{f(h)} h_x\right)^2 + \frac{1}{4}\int_{\mathbb{S}^1} \frac{g(h)^2}{f(h)} h_x^2 \\ &\leq \frac{1}{4}\left|\frac{g(h)^2}{f(h)}\right|_{L^\infty} \int_{\mathbb{S}^1} h_x^2 \leq C \int_{\mathbb{S}^1} h_x^2\,. \end{aligned} \tag{2.10}$$

It follows immediately that if $m_1 \leq n_1/2$, then the H^1-norm of h grows at most exponentially in time.

In the $m_1 > n_1/2$ case, g^2/f is not in L^∞, and the final step (2.10) is not valid. Instead, one finds

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{S}^1} h_x^2 \leq C\left(\int_{\mathbb{S}^1} h_x^2 + \overline{h}^2\right)^{m_1 - n_1/2 + 1}.$$

This does not preclude a finite-time blowup but does ensure that the H^1-norm of h is bounded for $t < (\int_{\mathbb{S}^1} h_{0x}^2 + \bar{h}^2)^{n_1/2 - m_1}/(C(m_1 - n_1/2))$:

PROPOSITION 2.3 *Let* $h(x,t)$ *be a smooth positive solution of* (2.1) *with initial data* $h_0 \in H^1(\mathbb{S}^1)$. *Assume that* $m_2 \geq n_2/2$.

If $m_1 \leq n_1/2$, *then* $|h(\cdot,t)|_{H^1}$ *grows at most exponentially on any time interval* $[0,T]$.

If $m_1 > n_1/2$, *then* $|h(\cdot,t)|_{H^1}$ *has controlled growth on a finite-time interval* $[0, T_0)$ *with* $T_0 = C(\int h_{0x}^2 + \overline{h}^2)^{n_1/2 - m_1}$.

As uniform boundedness is known whenever $m_1 < n_1 + 2$, the exponential bound is only relevant for the case $n_1 + 2 \leq m_1 \leq n_1/2$. This implies both m_1 and n_1 are negative. The algebraic bound is for the case $n_1 + 2 \leq m_1$ and $m_1 > n_2/2$. These conditions, suggestive of a possible finite-time blowup, can be satisfied by both positive and negative exponents. We present such an equation in the conclusions, Section 5.

Proposition 2.2 can be generalized to solutions that are no longer restricted to be nonnegative if the solution h is smooth enough to admit integration by parts and if for some $1 < p < 4$ one has

$$\int_{\mathbb{S}^1} \tilde{G}(h)\, dx \leq a \int_{\mathbb{S}^1} |h|^p\, dx + C\,.$$

In such a case, if the solution h has either an upper or lower pointwise bound, then the Lyapunov function $\mathcal{E}(h)$ bounds the H^1-norm of h. The proof is a minor modification of the proof of Proposition 2.2. Proposition 2.3 can be analogously generalized.

In the next subsection, we expand upon this observation to show that the finite-time singularity of the MKS equation (1.3) must be of the form where $h_{xx} \uparrow \infty$ and $h_{xx} \downarrow -\infty$ simultaneously as $t \uparrow T^*$. Furthermore, $|h_{xx}|_{L^1} \to \infty$. The rest of the article is independent of the next subsection. Those readers who wish to read immediately about the existence of nonnegative weak solutions of equation (2.1) should skip to Section 3.

2.3 Classifying the Finite-Time Singularity of the Modified Kuramoto-Sivashinsky Equation

The methods used to prove Proposition 2.3 can be used to prove sharper results concerning blowup of the modified Kuramoto-Sivashinsky equation. In [7], Bernoff and Bertozzi consider periodic solutions of the modified Kuramoto-Sivashinsky equation

$$h_t = -h_{xxxx} - h_{xx} + (1-\lambda)h_x^2 + \lambda h_{xx}^2\,. \tag{2.11}$$

The $\lambda = 0$ case is the Kuramoto-Sivashinksy equation that is known to have globally bounded smooth solutions.

For all values of $\lambda \neq 0$, Bernoff and Bertozzi prove that there exist periodic initial conditions that lead to finite-time singularities in which $|h_{xx}|_{L^\infty} \to \infty$ as $t \uparrow T^*$. Moreover, their computations combined with asymptotic methods suggest that the finite-time singularities are of a self-similar form in which $\max\{h_{xx}\} \uparrow \infty$ and $\min\{h_{xx}\} \downarrow -\infty$ simultaneously as $t \uparrow T^*$. We prove here that $|h_{xx}|_{L^1}$ must become singular when a blowup occurs. This then implies that both $\max\{h_{xx}\} \uparrow \infty$ and $\min\{h_{xx}\} \downarrow -\infty$ simultaneously, since for periodic solutions

$$|h_{xx}|_{L^1} = \int h_{xx}^+ - \int h_{xx}^- = 2\int h_{xx}^+ = -2\int h_{xx}^- \,.$$

Here h_{xx}^+ and h_{xx}^- denote the positive and negative parts of h_{xx}.

THEOREM 2.4 *Consider the modified Kuramoto-Sivashinsky equation*

$$h_t = -h_{xxxx} - h_{xx} + (1-\lambda)h_x^2 + \lambda h_{xx}^2\,, \quad x \in \mathbb{S}^1\,. \tag{2.12}$$

For $\lambda \neq 0$ *there exist smooth periodic initial data that yield a finite-time singularity in which*

$$|h_{xx}|_{L^1} \to \infty \quad \text{as } t \uparrow T^*\,. \tag{2.13}$$

Moreover, all finite-time singularities from smooth periodic solutions to (2.12) *must satisfy* (2.13).

PROOF: We first note that Bernoff and Bertozzi proved a continuation lemma for the problem—all finite-time singularities must be accompanied by a blowup of $|h_{xx}|_{L^\infty}$. For this reason, we can assume that control of $|h_{xx}|_{L^\infty}$ implies there is no finite-time singularity.

Our proof relies on the construction of a kind of Lyapunov function for the variable $v = h_{xx}$. The proof is by contradiction: We prove that if v has bounded L^1-norm, then its L^∞-norm is controlled.

The equation for $v = h_{xx}$ is

$$v_t = -v_{xxxx} - (g(v)v_x)_x + (1-\lambda)(h_x^2)_{xx} \tag{2.14}$$

where $g(v) = 1 - 2\lambda v$. Taking $G''(v) = g(v)$,

$$\mathcal{E}(v) = \int_{\mathbb{S}^1} \left(\frac{1}{2}v_x^2 - G(v)\right) dx$$

satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(v) = &- \int_{\mathbb{S}^1} [v_{xxx} + g(v)v_x]^2 \, dx \\ &+ 2(1-\lambda)\int_{\mathbb{S}^1} [v_{xxx} + g(v)v_x]\, h_x v \, dx\,.\end{aligned} \tag{2.15}$$

Equation (2.15) implies

$$\frac{d}{dt}\mathcal{E}(v) \le (1-\lambda)^2 |h_x|^2_{L^\infty} \int_{\mathbb{S}^1} v^2 \,.$$

Since $|h_x|_{L^\infty} \le |v|_{L^1}$,

$$\frac{d}{dt}\mathcal{E}(v) \le (1-\lambda)^2 |v|^2_{L^1} |v|^2_{L^2} \,. \tag{2.16}$$

By an argument similar to the proof of Lemma 2.1, there exist constants c_1 and C so that for all $v \in H^1$,

$$\frac{1}{8}|v|^2_{H^1} \le \mathcal{E}(v) + C|v|^5_{L^1} + c_1 \,.$$

The integrated form of (2.16) then implies

$$\frac{1}{8}|v|^2_{H^1} \le \mathcal{E}(v_0) + C|v|^5_{L^1} + c_1 + \int_0^t (1-\lambda)^2 |v|^2_{L^1} |v|^2_{H^1} \,.$$

We assume

$$|v|_{L^1} < M \,. \tag{2.17}$$

This assumption, combined with Grönwall's lemma, yields

$$|v|^2_{H^1} \le 8(\mathcal{E}(v_0) + CM^5 + c_1) e^{8(1-\lambda)^2 M^2 t} \,.$$

In short, on any finite-time interval, $|v|_{H^1}$ and hence $|v|_\infty$ is bounded by a function of the initial data and the maximum of its L^1-norm. This is a contradiction if the initial data is in the class for which Bernoff and Bertozzi proved $|h_{xx}|_\infty = |v|_\infty \to \infty$ in finite time. Therefore for such initial data, assumption (2.17) must be false, finishing the proof. ∎

As $\int h_{xx} = 0$ is conserved by the evolution, a pointwise upper or lower bound for h_{xx} implies a bound on the L^1-norm of h_{xx}. This observation yields a corollary predicting the simultaneous blowup of h_{xx} found in numerical simulations:

COROLLARY 2.5 *For* $\lambda \ne 0$ *there exist smooth periodic initial data for* (2.12) *that yield a finite-time singularity in which*

$$\min\{h_{xx}\} \downarrow -\infty \quad \textit{and} \quad \max\{h_{xx}\} \uparrow \infty \tag{2.18}$$

simultaneously as the blowup occurs. Moreover, all finite-time singularities from smooth periodic solutions to (2.12) *must satisfy* (2.18).

3 Weak Solutions: Existence, Positivity, and Behavior near the Edge of Support

Section 2 addressed the question of global boundedness of positive smooth solutions of the long-wave unstable diffusion equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x \,. \tag{3.1}$$

This equation has a Lyapunov function of the form

$$\mathcal{E}(h) = \int_{\mathbb{S}^1} \left(\frac{1}{2}h_x^2 - \tilde{G}(h) + Dh \right) dx$$

where $\tilde{G}''(h) = g(h)/f(h)$. As before, we make the assumption that both f and g behave as power laws in the large and small y limits:

$$f(y) \sim \begin{cases} y^{n_1} & \text{for } y \gg 1\,, \\ y^{n_2} & \text{for } y \ll 1\,, \end{cases} \tag{3.2}$$

$$g(y) \sim \begin{cases} y^{m_1} & \text{for } y \gg 1\,, \\ y^{m_2} & \text{for } y \ll 1\,, \end{cases} \tag{3.3}$$

and thus for $y \gg 1$

$$\tilde{G}(y) \sim \begin{cases} y^{m_1 - n_1 + 2} & \text{if } m_1 - n_1 \neq -2, -1, \\ \log(y) & \text{if } m_1 - n_1 = -2, \\ y\log(y) - y & \text{if } m_1 - n_1 = -1. \end{cases}$$

In Section 2, we proved that if $m_1 - n_1 + 2 < 4$ (i.e., $g(h)/(h^2 f(h)) \to 0$ as $h \to \infty$), then positive smooth solutions are uniformly bounded in H^1 and thus in L^∞.

For thin films, both positive and nonnegative solutions are of interest. In particular, nonnegative solutions can be used to describe coating flows with moving contact lines. In this section, we derive a global existence theory, similar to that derived for equation (3.1) without the long-wave instability [3, 6, 9, 13]. Furthermore, we discuss cases for which the contact line of a nonnegative solution can be shown to have finite speed of propagation. Such finite speed of propagation is relevant as it proves the solutions to be physically reasonable. Some of the techniques used here are very similar to those in previous work [3, 4, 5, 9, 13]. For this reason, we present sketches of the proofs and refer the reader to previous papers whenever possible.

Positive solutions are smooth since given a priori upper and lower bounds, the equation is uniform parabolic [25, 30]. On the other hand, nonnegative solutions can be positive to one side of a point and zero to the other side. Such a point is denoted a "contact line" and an asymptotic expansion near this point suggests that the solution may be C^1 but not C^2.[6]

For this reason, any formulation of an existence theory for nonnegative solutions requires weak solutions rather than classical solutions. This is typically done by integrating the evolution equation against a test function φ and then integrating by parts. Taking $\varphi \in C_0^\infty(0,T;\mathbb{S}^1)$, we have

$$\iint_{Q_T} h\varphi_t \, dx\, dt = -\iint_{Q_T} f(h)h_{xxx}\varphi_x \, dx\, dt - \iint_{Q_T} g(h)h_x\varphi_x \, dx\, dt \,. \tag{3.4}$$

Q_T is the parabolic cylinder $[0,T] \times \mathbb{S}^1$. The above formulation is a weak solution in the sense of distributions and requires control of h_{xxx} at the contact line. Bernis and Friedman [6] introduced a weaker form of (3.4) to define a weak solution for (3.1) with $g = 0$ that does not consider h_{xxx} at the contact line:

DEFINITION 3.1 A *BF weak solution*[7] of equation (3.1) is a function h satisfying the conditions

$$\begin{gathered} h \in C^{\frac{1}{2},\frac{1}{8}}([0,T] \times \mathbb{S}^1) \cap L^\infty(0,T;H^1(\mathbb{S}^1)) \,, \\ h \in C^{4,1}(\mathcal{P}) \,, \quad \text{and} \quad \sqrt{f(h)}h_{xxx} \in L^2(\mathcal{P}) \,, \end{gathered}$$

where $\mathcal{P}$ is the set $[0,T] \times \mathbb{S}^1/(h = 0 \cup t = 0)$ and h satisfies (3.1) in the sense of

$$\iint_{Q_T} h\varphi_t \, dx\, dt + \iint_{\mathcal{P}} f(h)h_{xxx}\varphi_x + \iint_{Q_T} g(h)h_x\varphi_x \, dx\, dt = 0 \,.$$

For the $g = 0$ case, Bernis [4] denotes a "strong solution" to be one that satisfies Definition 3.1 and in addition has $h(\cdot,t) \in C^1(\mathbb{S}^1)$ for almost all $t > 0$. This definition is motivated by the results in [3, 9, 13] showing that for $f(h) = h^n$, $0 < n < 3$, there exists a weak solution in the sense of Definition 3.1 that is $C^1(\mathbb{S}^1)$ for almost all $t > 0$.

[6] See, e.g., [14] for traveling wave solutions to the equation with only the fourth-order term or [13] for traveling wave solutions to the problem with a stabilizing second-order term.

[7] We introduce the name "BF weak solution" to differentiate this solution from a weak solution in the sense of distributions.

In [9, 13] (for $g = 0$ and $g \leq 0$) we showed that, for $1 < n_2 < 3$, a nonnegative weak solution constructed to satisfy Definition 3.1 also satisfies the equation in the sense of distributions in one of two ways:

$$\begin{aligned} &\iint_{Q_T} h\varphi_t \, dx\, dt \\ &\quad = \iint_{Q_T} f(h) h_{xx} \varphi_{xx} \, dx\, dt + \iint_{Q_T} f'(h) h_x h_{xx} \varphi_x \, dx\, dt \\ &\quad\quad - \iint_{Q_T} g(h) h_x \varphi_x \, dx\, dt \end{aligned} \tag{3.5}$$

or

$$\begin{aligned} &- \iint_{Q_T} h\varphi_t \, dx\, dt \\ &\quad = \iint_{Q_T} f(h) h_x \varphi_{xxx} \, dx\, dt + \frac{3}{2} \iint_{Q_T} f'(h) h_x^2 \varphi_{xx} \, dx\, dt \\ &\quad\quad + \frac{1}{2} \iint_{Q_T} f''(h) h_x^3 \varphi_x \, dx\, dt + \iint_{Q_T} g(h) h_x \varphi_x \, dx\, dt \, . \end{aligned} \tag{3.6}$$

Being a weak solution in the sense of (3.5) is stronger than being a weak solution in the sense of (3.6) in that (3.5) requires some control of h_{xx}.

Using the techniques developed in the above-mentioned papers, we outline the analogous existence results for various weak solutions of (3.1) with $g \geq 0$.

3.1 Initial Data and Problems of Interest

The weak solution theory considers two types of initial conditions, $h_0 > 0$ and $h_0 \geq 0$.

First, we consider positive initial data $h_0 > 0$. In this case, we prove existence of nonnegative weak solutions for all $n_2 > 0$. If $n_2 > 3.5$, we show that the solution remains smooth and positive for all time, while if $n_2 \leq 3.5$ finite-time singularities might occur in which a contact line spontaneously forms ($\min\{h\} \downarrow 0$). A finite-time pinching singularity has been observed numerically for the $f(h) = g(h) = h$ case, although the authors did not consider the evolution of the resultant weak solution [34]. The formation and evolution of a contact line was studied numerically for the equation with $f(h) = h^{1/2}$ and $g = 0$ [10]. The contact line was observed to move with finite speed as was later proven in [4].

Second, we consider nonnegative initial data $h_0 \geq 0$. For this case we prove global existence theorems when $0 < n_2 < 3$, $n_2 \leq m_2$, and f and g satisfy the conditions of Proposition 2.2. We show that if there is a contact line, then it moves with finite speed and its position is a Hölder continuous function of time if $\frac{1}{2} < n_2 < 3$. The finite speed of propagation results follow immediately from methods similar to those derived in [4, 5].

3.2 Approximate Problem

To prove the existence theory, we construct a family of smooth approximate solutions. For the porous medium equation, a degenerate second-order equation, $h_t = (h^m h_x)_x$, $m > 0$, a natural approximate equation is the uniformly parabolic equation $h_{\varepsilon t} = ((h_\varepsilon{}^m + \varepsilon) h_{\varepsilon x})_x$. Because the approximate equation is second order, one can apply the maximum principle to find that positive initial data yields positive solutions. A subsequence of these positive approximate solutions will have a nonnegative $\varepsilon \to 0$ limit, which can be proven to be a weak solution of the porous medium equation. The lubrication equation, $h_t = -(h^n h_{xxx})_x$, $n > 0$, is a degenerate fourth-order equation. However, the maximum principle does not apply so that the analogous approximate equation, $h_{\varepsilon t} = -((h_\varepsilon{}^n + \varepsilon) h_{\varepsilon xxx})_x$, can take positive initial data to a solution that may be negative in regions.

To find strictly positive approximate solutions, we consider the problem with "lifted" initial data

$$h_{\varepsilon 0}(x) = h_0(x) + \delta(\varepsilon) > 0 \tag{3.7}$$

and approximate equation

$$h_{\varepsilon t} = -(f_\varepsilon(h_\varepsilon) h_{\varepsilon xxx})_x - (g_\varepsilon(h_\varepsilon) h_{\varepsilon x})_x\,, \tag{3.8}$$

where

$$f_\varepsilon(y) = \begin{cases} \frac{f(y)}{y^{n_2}} \frac{y^{n_2+4}}{\varepsilon y^{n_2} + y^4} & \text{if } n_2 < 4, \\ f(y) & \text{if } n_2 \geq 4, \end{cases} \tag{3.9}$$

and

$$g_\varepsilon(y) = \begin{cases} \frac{g(y)}{y^{m_2}} \frac{y^{m_2+4}}{\varepsilon y^{m_2} + y^4} & \text{if } n_2 < 4 \text{ and } m_2 < 4, \\ g(y) & \text{otherwise.} \end{cases} \tag{3.10}$$

This is a slight modification of a regularization first suggested by Bernis and Friedman [6] and later used in [3, 9] for equation (3.1) with $g = 0$. The regularizations (3.9–3.10) preserve the large y behavior of $f(y)$ and $g(y)$ while modifying the small y behavior of $f(y)$ and $g(y)$.

Like the regularizations used for the $g = 0$ case [3, 6, 9], $f_\varepsilon(y) \sim y_2^n$ for $y \ll 1$ with $n \geq 4$. This ensures that the approximate solutions are strictly positive, smooth, and unique. The existence theory [13] for the lubrication equation with a porous medium term, $h_t = -(h^n h_{xxx})_x + (h^m h_x)_x$, uses the approximation

$$h_{\varepsilon 0}(x) = h_0(x) + \delta(\varepsilon) > 0\,, \quad h_{\varepsilon t} = -(f_\varepsilon(h_\varepsilon) h_{\varepsilon xxx})_x + ({h_\varepsilon}^m h_{\varepsilon x})_x\,.$$

The porous medium term does not need to be regularized to ensure that the approximate solutions be positive, smooth, and unique. However, as we demonstrate below, if the second-order term is destabilizing, then positivity, smoothness, and uniqueness of the approximate solutions are only guaranteed if g is regularized whenever f is regularized.

PROPOSITION 3.2 (Global Existence and Positivity of Approximate Solutions) *Given a time T and positive initial data $h_{\varepsilon 0} = h_0 + \delta(\varepsilon) \in H^1(\mathbb{S}^1)$ with $h_0 \geq 0$, with $\delta(\varepsilon) = \varepsilon^\theta$, $\varepsilon < 1$, and $\theta < \frac{2}{5}$, the approximate equation* (3.8) *with $m_1 < n_1 + 2$ and $n_2 \leq m_2$ has a unique positive smooth solution for all time. The approximate solution h_ε has a pointwise lower bound M_ε that depends on ε but not on T*:

$$0 < M_\varepsilon \leq h_\varepsilon(x,t)\,, \quad t \in [0,T]\,.$$

Moreover, if $0 < n_2 < 3$ and $h_0 > 0$, then for all $-\frac{1}{2} < s < 1$, there exists a constant C independent of both T and ε such that the following uniform-in-ε bounds are satisfied for all approximate solutions:

$$|h_\varepsilon(\cdot,t)|_{H^1}^2 \leq C\,, \tag{3.11}$$

$$\iint_{Q_T} \left({h_\varepsilon}^{s/2+1}\right)_{xx}^2 \,dx\,dt \leq CT\,, \tag{3.12}$$

$$\iint_{Q_T} \left({h_\varepsilon}^{s/4+1/2}\right)_{x}^4 \,dx\,dt \leq CT\,. \tag{3.13}$$

If $h_0 \geq 0$, then the a priori bounds (3.12) *and* (3.13) *hold for all s with* $\max\{-\frac{1}{2}, n_2 - 2\} < s < 1$.

SKETCH OF PROOF: The proof essentially follows from the uniform boundedness of positive smooth solutions shown in Section 2 and methods from previous papers. First, short-time existence is proved following the same steps Bernis and Friedman used for the $g = 0$ case. To prove that the solution can be continued in time and will remain positive, we note that from Proposition 2.2 there is a global bound on the H^1-norm (3.11). As in Bernis and Friedman, this implies that the solutions are uniformly in $C^{1/2,1/8}(\Omega_T)$. A pointwise lower bound then suffices to continue this positive solution indefinitely in time.

To obtain the pointwise lower bound, we follow the argument of theorem 3.1 in [9]. Defining $G_\varepsilon(y)$ to satisfy $G_\varepsilon''(y) = 1/f_\varepsilon(y)$,[8]

$$\frac{d}{dt}\int_{\mathbb{S}^1} G_\varepsilon(h_\varepsilon) = -\int_{\mathbb{S}^1} {h_\varepsilon}_{xx}^2 - \int_{\mathbb{S}^1} \frac{g_\varepsilon(h_\varepsilon)}{f_\varepsilon(h_\varepsilon)} {h_\varepsilon}_x^2 . \tag{3.14}$$

The assumption that $g(y)/f(y)$ is bounded for all $M > y > 0$ and the a priori bound (3.11) yield

$$\int_{\mathbb{S}^1} G_\varepsilon(h_\varepsilon) \leq CT . \tag{3.15}$$

Equation (3.14) shows why g must be regularized whenever f is regularized: to have $|g_\varepsilon(h_\varepsilon)/f_\varepsilon(h_\varepsilon)|$ pointwise bounded independently of ε. The pointwise lower bound follows from the bound (3.15) and the Hölder continuity of the approximate solution h_ε. Specifically, if $\delta(t)$ is the minimum value of $h_\varepsilon(\cdot,t)$ occurring at the point $x_0(t)$, then $h_\varepsilon(x,t) \leq \delta(t) + C|x - x_0(t)|^{1/2}$. As f_ε and g_ε have been regularized to behave like y^4 for $y \ll 1$, (3.15) implies $\int G_\varepsilon(h_\varepsilon) \sim \int \varepsilon/{h_\varepsilon}^2 \leq CT$. This and the Hölder continuity yield

$$0 < M_\varepsilon = e^{-C_2/\varepsilon} \leq \delta(t) .$$

Finally, we define $G_\varepsilon^s(y)$ to satisfy $G_\varepsilon^{s\prime\prime}(y) = y^s/f_\varepsilon(y)$ for the bounds (3.12) and (3.13). In the $g \leq 0$ case this determines a family of entropies that are dissipated as the solution evolves (if $\max\{-\frac{1}{2}, n-2\} < s < 1$). Here we show that although these entropies are not necessarily dissipated in time, they continue to provide a framework in which to derive the bounds (3.12) and (3.13). Specifically, their growth can be controlled for all time:

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}^1} G_\varepsilon^s(h_\varepsilon)\,dx &= \int_{\mathbb{S}^1} {h_\varepsilon}^s {h_\varepsilon}_x {h_\varepsilon}_{xxx}\,dx + \int_{\mathbb{S}^1} {h_\varepsilon}^s \frac{g_\varepsilon(h_\varepsilon)}{f_\varepsilon(h_\varepsilon)} {h_\varepsilon}_x^2\,dx \\ &= -\int_{\mathbb{S}^1} {h_\varepsilon}^s {h_\varepsilon}_{xx}^2 dx + \frac{s(s-1)}{3}\int_{\mathbb{S}^1} {h_\varepsilon}^{s-2} {h_\varepsilon}_x^4 \qquad (3.16) \\ &\quad + \int_{\mathbb{S}^1} {h_\varepsilon}^s \frac{g_\varepsilon(h_\varepsilon)}{f_\varepsilon(h_\varepsilon)} {h_\varepsilon}_x^2\,dx \\ &\leq -\int_{\mathbb{S}^1} ({h_\varepsilon}^{s/2+1})_{xx}^2 + C\int_{\mathbb{S}^1} ({h_\varepsilon}^{s/2+1})_x^2 \qquad (3.17) \\ &\leq -\frac{1}{2}\int_{\mathbb{S}^1} ({h_\varepsilon}^{s/2+1})_{xx}^2 + C_1 . \end{aligned}$$

[8] G_ε is different from $\tilde{G}$ in the Lyapunov function of the previous section.

The condition $-\frac{1}{2} < s < 1$ is used in step (3.17), and the condition $s > n_2 - 2$ is needed for $\int_{\mathbb{S}^1} G^s_\varepsilon(h_{\varepsilon 0})$ to be bounded independently of ε. Following the arguments in [3, 9], the above yields the bounds (3.12) and (3.13). ∎

If the initial data is strictly positive, $h_0 > 0$, then the condition $0 < n_2 < 3$ can be broadened to include all $0 < n_2$ and the condition on s to include all $-\frac{1}{2} < s < 1$.

3.3 The $\varepsilon \to 0$ Limit

We use the a priori bounds of Proposition 3.2 to prove the existence of nonnegative weak solutions. In the following, we present the proofs for the $n_2 > 0$ case with positive initial data $h_0 > 0$ and for the $0 < n_2 < 3$ case with nonnegative initial data $h_0 \geq 0$.

THEOREM 3.3 (Weak Solution from Positive Initial Data) *Given $T < \infty$, $0 < n_2$, $n_2 \leq m_2$, $m_1 < n_1 + 2$, initial data $h_0 \in H^1(\mathbb{S}^1)$ and $h_0 > 0$, and h_ε the approximate solution of Proposition* 3.2 *on time interval* $[0, T]$, *there exists a subsequence of $\{h_\varepsilon\}$ that converges pointwise uniformly and weakly in*

$$L^2(0, T; H^2(\mathbb{S}^1)) \cap L^\infty(0, T; H^1(\mathbb{S}^1))$$

as $\varepsilon \to 0$ to a nonnegative BF-weak solution h. Furthermore, for $1 < n_2$ the solution h also satisfies the equation in the sense of distributions (3.5) *and inherits the a priori bounds* (3.12) *and* (3.13) *of Proposition* 3.2 *for all $-\frac{1}{2} < s < 1$. Finally, if $n_2 \geq 3.5$, then the weak solution is a positive classical solution.*

SKETCH OF PROOF: Given the a priori bounds of Proposition 3.2, the proof follows identically those in section 4 of [13] and [9]. We refer the reader to these papers. The result on global positivity when $n_2 \geq 3.5$ follows the analogous proof for the $g = 0$ case [11]. In particular, one uses the a priori bound on $G^s(h)$ for s arbitrarily close to $-\frac{1}{2}$ combined with the a priori H^1 (and hence $C^{1/2}$) bound. ∎

As in [13], [9], and other references, the condition $h_0 > 0$ can be weakened to include nonnegative data for which the entropies $\int_{\mathbb{S}^1} G^s_0(h_0)$ are bounded.

THEOREM 3.4 (Weak Solutions from Nonnegative Initial Data) *Given $T < \infty$, $0 < n_2 < 3$, $n_2 \leq m_2$, $m_1 < n_1 + 2$, and initial data $h_0 \in H^1(\mathbb{S}^1)$ and $h_0 \geq 0$, let h_ε be the approximate solution of Proposition* 3.2 *on time interval* $[0, T]$.

For $0 < n_2 < 2$*, there exists a subsequence of* $\{h_\varepsilon\}$ *that converges pointwise uniformly and weakly in*

$$L^2(0, T; H^2(\mathbb{S}^1)) \cap L^\infty(0, T; H^1(\mathbb{S}^1))$$

to a nonnegative BF-weak solution h*. Furthermore, for* $1 < n_2 < 1$*,* h *also satisfies the equation in the sense of distributions* (3.5) *and inherits the a priori bounds* (3.12) *and* (3.13) *of Proposition* 3.2 *for all* $\max(-\frac{1}{2}, n_2 - 2) < s < 1$.

For $2 < n_2 < 3$*, there exists a subsequence of* $\{h_\varepsilon\}$ *that converges pointwise uniformly and weakly in*

$$L^\infty(0, T; H^1(\mathbb{S}^1))$$

to a nonnegative BF-weak solution h*. Furthermore,* h *also satisfies the equation in the sense of distributions* (3.6) *and inherits the a priori bounds* (3.12) *and* (3.13) *of Proposition* 3.2 *for all* s *such that* $n_2 - 2 < s < 1$.

SKETCH OF PROOF: Again, the a priori bounds of Proposition 3.2 yield the results. The proof follows those in section 4 of [13] and [9]. ∎

In fact, the nonnegative weak solutions of Theorem 3.4 are weak solutions in the sense of distributions for $\frac{3}{8} < n_2 \leq 1$ and $n_2 = 2$. The weak solution formulation and analogous proofs can be found in [13] and [9].

We proved in Section 2 that the H^1-norm of positive smooth solutions can grow in a controlled manner if $m_1 \geq n_1 + 2$ and $m_1 \leq n_1/2$. This H^1 control then implies a priori bounds, propositions, and theorems analogous to Proposition 3.2 and Theorems 3.3 and 3.4.

If $m_1 \geq n_1 + 2$ and $m_1 > n_1/2$, then Conjecture 2 suggests a finite-time blowup is possible. However, Proposition 2.3 does provide short-time control of the H^1-norm of approximate solutions. Short-time equivalents of Proposition 3.2 and Theorems 3.3 and 3.4 then follow.

3.4 The Contact Line

In this section we establish two results. First, following the work of [3, 9, 13], we consider the asymptotic behavior of the solution near the edge of support. Second, following the work of [4, 5], we establish that the support of a weak solution has finite speed of propagation.

For almost all $t \in [0, T]$, the weak solution satisfies the bounds (3.12) and (3.13). Suppose that the contact line is at the point $a(t)$ with $h(x, t) > 0$ for $x < a(t)$ and $h(x, t) = 0$ for $x \geq a(t)$. If the leading-order asymptotics of the solution can be described by a power law behavior, $h(x,t) \sim C(t)(a(t)-x)^{\beta}$, the power law must satisfy the restrictions:

$$\begin{aligned} &\beta \geq 2\,, \quad 0 < n_2 < \frac{3}{2}\,, \\ &\beta \geq \frac{3}{n}\,, \quad \frac{3}{2} < n_2 < 3\,. \end{aligned} \tag{3.18}$$

The constraints in (3.18) are identical to those computed for the $g = 0$ case in [9]. The condition of $g(y)/f(y)$ remaining bounded as $y \downarrow 0$ is simply that the second-order term cannot dominate the solution near the contact line; hence the constraints in (3.18) depend only on n_2. As in [9], we believe that the a priori bounds (3.12) and (3.13) are sharp since the exponents $\beta = 2$ for $0 < n_2 < \frac{3}{2}$ and $\beta = 3/n$ for $\frac{3}{2} < n_2 < 3$ are exactly those of the $g = 0$ case. The $m_2 \geq n_2$ condition needed for existence of solutions implies that near the contact line, the evolution equation is like the $g = 0$ case, suggesting that the exponents would be sharp for this case as well.

We now show that the support of the weak solutions of Theorems 3.3 and 3.4 have a property known as finite speed of propagation. This property is not enjoyed by solutions of uniformly parabolic equations; however, it is a well-known property of solutions of the "porous media" equation $h_t = (h^m h_x)_x$, $m > 0$. Recently it was shown that for the $g = 0$ case, the "strong"[9] solutions have support that propagates with finite speed [4, 5].

The key ideas of these papers are strong and local versions of the entropy equation (3.17) for the weak solutions constructed in Theorems 3.3 and 3.4. Without presenting all the details, we show how to extend these ideas to the problem considered here.

DEFINITION 3.5 A function $h(x,t) : \mathbb{S}^1 \times [0,\infty) \to \mathbb{R}$ is said to have *finite speed of propagation* if for all $t_0 > 0$, $x_0 \in \mathbb{S}^1$, and $r_0 > 0$, such that $B_{r_0}(x_0) \subset \mathbb{S}^1$ and $h(x,0) \equiv 0$ almost everywhere in $B_{r_0}(x_0)$, there exists a $T_* > 0$ and a continuous function $r : [t_0, t_0 + T_*) \to \mathbb{R}^+$ with $r(t_0) = r_0$ such that

$$h(x,t) = 0 \quad \text{a.e. for all } t \in [t_0, t_0 + T_*) \quad \text{and} \quad x \in B_{r(t)}(x_0)\,.$$

With this definition, we have the following theorem:

THEOREM 3.6 (Finite Speed of Propagation) *The weak solutions of Theorems* 3.3 *and* 3.4 *have finite speed of propagation.*

---

[9] As used by Bernis in [4] to denote weak solutions that are C^1 in space for a.e. time.

The proof follows by considering local versions of the estimates in (3.17) and using the Lyapunov function from Section 2. We sketch the proof below for the $0 < n_2 < 2$ case.

LEMMA 3.7 *Assume the hypotheses of either Theorem* 3.3 *or* 3.4 *with approximate solutions from Proposition* 3.2*. If* $0 < n_2 < 2$*, then for any cutoff function* $\xi(x)$ *satisfying* $\xi \in C^2(\mathbb{S}^1)$*,* $\xi \geq 0$*, there exists a positive constant* C *depending only on* s *and* n_2 *so that for all* $T > 0$ *and* $\varepsilon > 0$*, the approximate solutions satisfy*

$$\begin{aligned}
&\int_{\mathbb{S}^1} \xi(x)(G_\varepsilon^s(h_\varepsilon(x,T)) - G_\varepsilon^s(h_0(x)+\varepsilon^\theta)dx \\
&\quad + \iint_{Q_T} \xi \bar{h}_\varepsilon^s h_{\varepsilon xx}^{\;2} + \iint_{Q_T} {h_\varepsilon}^{s-2} h_{\varepsilon x}^{\;4} \\
&\quad \leq C\left(\iint_{Q_T} |\xi' {h_\varepsilon}^{s-1} h_{\varepsilon x}^{\;3}| + \iint_{Q_T} |\xi' {h_\varepsilon}^{s} h_{\varepsilon x} h_{\varepsilon xx}| \right. \\
&\qquad \left. + \iint_{Q_T} |\xi'' {h_\varepsilon}^{s+1} h_{\varepsilon xx}| \right) \\
&\qquad + \iint_{Q_T} \left| \frac{\xi {h_\varepsilon}^{s} f(h_\varepsilon)}{g_\varepsilon(h_\varepsilon) h_{\varepsilon x}^{\;2}} \right| + \iint_{Q_T} |\xi' G_\varepsilon^{s\prime}(h_\varepsilon) h_{\varepsilon x}| \,.
\end{aligned} \tag{3.19}$$

The proof of this lemma is identical to that of lemma 4.3 in [4]. The key aspect of (3.19) is that because $f_\varepsilon/g_\varepsilon \leq M$, the last two terms on the right-hand side, which arise from the long-wave unstable term in (3.1), can be absorbed into the relevant terms when taking $\varepsilon \to 0$.

Choosing a cutoff function φ_r of the form

$$\varphi_r(x) = r\varphi_1\left(\frac{x}{r}\right), \quad r > 0\,,\ \varphi_1 \geq 0\,,\ \varphi^4 \in C_0^2(\mathbb{R})\,, \tag{3.20}$$

we pass to the $\varepsilon = 0$ limit to obtain a bound analogous to that of lemma 4.5 in [4]:

LEMMA 3.8 *Given a cutoff function* $\xi = \varphi_r^4$ *where* φ_r *satisfies* (3.20)*, let* $0 < n_2 < 2$ *and* h *be the weak solution of either Theorem* 3.3 *or* 3.4*. Then there exists a positive constant* C_2 *depending only on* s*,* n_2*, and* φ_1 *such that for all* $T > 0$*,*

$$\begin{aligned}&\frac{1}{(1-n_2+s)(2-n_2+s)}\int_{\mathbb{S}^1}\varphi_r^4(x)h^{2-n_2+s}(x,T)dx\\&\quad-\frac{1}{(1-n_2+s)(2-n_2+s)}\int_{\mathbb{S}^1}\varphi_r^4(x)h^{2-n_2+s}(x,0)\\&\quad+\iint_{Q_T}\varphi_r^4(x)(h^{(s+2)/2})_{xx}^2\\&\qquad\le C_2\iint_{Q_T\cap\{\varphi_r>0\}}h^{s+2}\,.\end{aligned}$$

The constant C_2 is slightly different from the constant in lemma 4.5 in [4] because the lower-order destabilizing terms are also absorbed into the estimate.

Following the arguments in [4], Lemma 3.8 is sufficient to imply that the weak solution has finite speed of propagation.

The $2 < n_2 < 3$ case similarly follows the argument in [5] for the $g = 0$ case. In fact, since $g < Cf$ and $g_\varepsilon < Cf_\varepsilon$, all the relevant inequalities in [5] can be proved for the weak solutions constructed here. The details regarding the Hölder continuity in time of the contact line in time for $\frac{1}{2} < n_2 < 3$ follow the arguments from [4] and [5].

4 Computational Results

In this section we present numerical simulations of

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x\,, \tag{4.1}$$

which confirm the results of Sections 2 and 3 and support Conjecture 2.

We use a numerical method that is a modification of the finite difference scheme used for related problems [10, 11, 12, 13]. The previously used schemes were found to have numerical instabilities on nonuniform meshes near points at which $h \downarrow 0$ when used to study the long-wave unstable problem. We avoid this numerical instability by modifying the numerical scheme to use an "entropy dissipating" form [51] of the nonlinear term $f(h)$.

4.1 Hanging Drops

In [23], Ehrhard considered a viscous fluid hanging from the bottom of a smooth horizontal plate. In that work, the author did not compute solutions of the evolution equation. Instead, solutions of a quasi-static approximation were computed: a sequence of steady states satisfying time-dependent boundary conditions. Here we present solutions of the evolution equation.

In the isothermal case, the equation for the dynamic evolution of the film height is [24, eq. (4.8p)]

$$h_t = -\left(\left(\frac{h^3}{3} + \beta h^2\right)(h_{xxx} - Gh_x)\right)_x = -\left(\left(\frac{h^3}{3} + \beta h^2\right)\mathcal{L}(h)\right)_x . \tag{4.2}$$

$\beta = 0$ corresponds to a no-slip boundary condition at the liquid/solid interface—this precludes contact line motion. A slip length $\beta > 0$ is introduced to allow slippage near the contact line. For hanging drops, $G < 0$, there is a long-wave instability. This reflects the Rayleigh-Taylor instability arising when a heavier fluid is above a lighter one.

Figure 4.1 presents a numerical solution of equation (4.2) with $G = -80$, $\beta = 0$, and initial data

$$h_0(x) = 1 + 0.1\cos(\pi x) . \tag{4.3}$$

The initial data is positive and the computation shows that the solution remains positive (and hence smooth) for all time,[10] apparently approaching a nonnegative weak solution as $t \to \infty$. The black lines denote successive times of the height profile starting at $t = 0$ and ending at $t = 100$. The weak solution is a periodic array of separated droplets and is denoted with circles:

$$h_\infty(x) = \begin{cases} 1.66(1 + \cos(\sqrt{80}x)) , & |x| < \frac{\pi}{\sqrt{80}} , \\ 1.18(1 + \cos(\sqrt{80}(1 - |x|))) , & 1 - |x| < \frac{\pi}{\sqrt{80}} , \\ 0 , & \text{elsewhere.} \end{cases} \tag{4.4}$$

A steady weak solution of equation (4.2) must satisfy $\mathcal{L}(h_\infty) \equiv 0$ wherever $h_\infty \neq 0$: The solutions are shifted cosines of period $2\pi/\sqrt{80}$. The above steady state has zero contact angle and $\int h_\infty = \int h_0$.

4.2 Growth and Saturation in the Subcritical Case

We consider an equation that Hocherman and Rosenau conjectured to blow up in finite time:

$$h_t = -(h^4 h_{xxx})_x - 138(h^{5.9} h_x)_x = -(h^4 h_{xxx})_x - 20(h^{6.9})_{xx}$$

with initial data (4.3). This equation is subcritical in the sense that it satisfies Theorem 3.3, its solutions remaining uniformly bounded for all time. However,

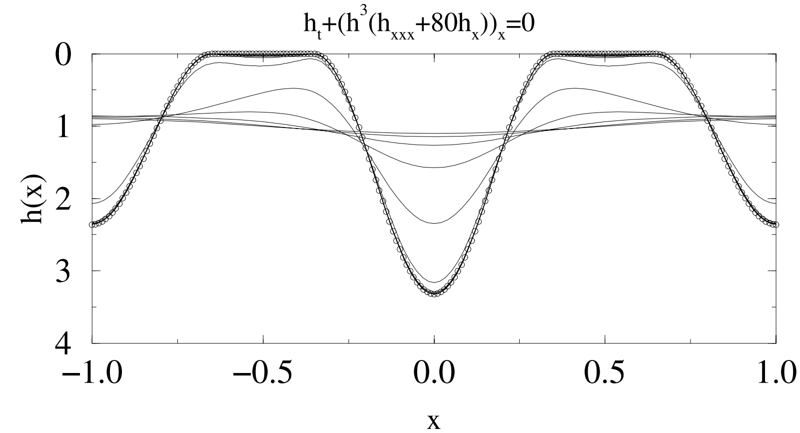

Figure 4.1. Instability and droplet formation in a thin hanging film of liquid. The circles denote the weak solution (4.4).

if the exponent 5.9 were a 6, the equation would be in the critical case for which there are no analytical results.

The solid line in Figure 4.2 shows the growth of the maximum of the solution. There is an initial rapid growth, followed by saturation. This behavior is to be contrasted with the apparent lack of saturation shown by the dashed line. The dashed line presents the growth of the maximum for the same initial data in the critical case $h_t = -(h^4 h_{xxx})_x - 140(h^6 h_x)_x$.

4.3 Preliminary Evidence of Blowup in the Critical Case

We present preliminary numerical results that suggest a finite-time blowup for the critical case of Conjecture 2. We consider an equation with critical exponents

$$h_t = -(h^4 h_{xxx})_x - 140(h^6 h_x)_x = -(h^4 h_{xxx})_x - 20(h^7)_{xx}$$

with initial data (4.3). The heuristic scaling argument presented in the introduction suggests that blowup is possible for the critical case. As $m > \frac{n}{2}$, the exponents do not preclude a finite-time blowup of the H^1-norm.

The computations show that the positive solution simultaneously approaches infinity and zero as $t \uparrow 0.00042$. The solution appears to go to zero at two points,

[10] For this reason, we compute solutions of the original equation (4.2) rather than solutions of the approximate equation used in Section 3 to prove existence of nonnegative weak solutions.

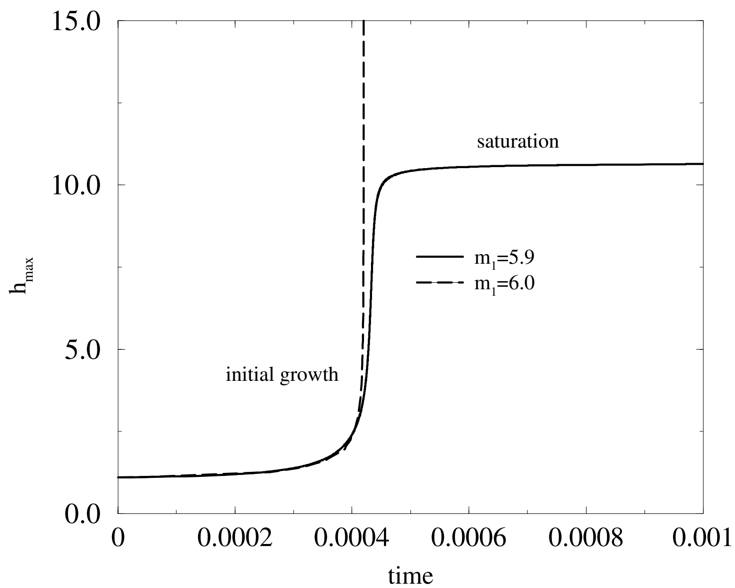

Figure 4.2. Initial growth and saturation of the height of the solution for $h_t = -(h^4 h_{xxx})_x - 20(h^{6.9})_{xx}$. The dashed line represents the solution to the same equation with the 6.9 replaced by 7.

one on each side of the point at which it is blowing up. Figure 4.3 presents a logarithmic-scale plot of h near the point where the blowup occurs.

5 Summary and Conclusions

This paper considers a class of 1-D long-wave, unstable, degenerate diffusion equations arising largely in the context of surface-tension-driven interface motion.

For equations of the form

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x \,, \tag{5.1}$$

we show that for a class of nonlinear diffusion coefficients f and g, positive smooth solutions remain uniformly bounded. Equations with such degenerate diffusion coefficients were conjectured in [38] to yield finite-time singularities. Specifically, it was conjectured that only if $g(h)/f(h)$ decays as $h \uparrow \infty$ will solutions remain bounded. Here we prove that it suffices if $g(h)/(h^2 f(h))$ decays

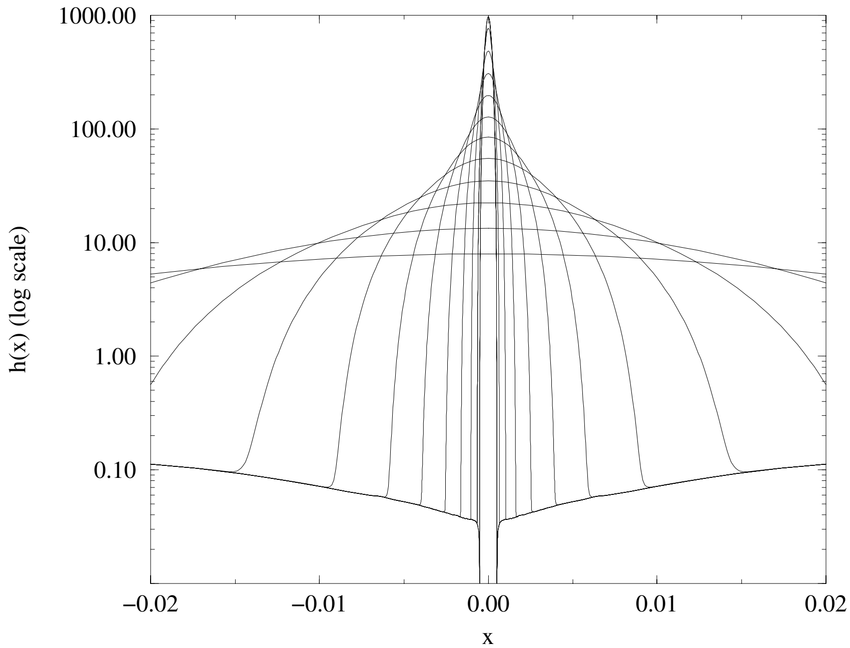

Figure 4.3. Apparent singularity occurring in finite time for solution to $h_t = -(h^4 h_{xxx})_x - 20(h^7)_{xx}$.

as a power law as $h \uparrow \infty$ to preclude blowup.

Positive solutions are a natural class of solutions when $f(y)$ vanishes sufficiently fast as $y \downarrow 0$, since we show that such equations often preserve positivity ($h_0 > 0 \Longrightarrow h > 0$). When $f(y)$ vanishes at a rate too slow to permit the positivity-preserving property, we show that it is still possible that f vanishes fast enough to yield a weak maximum principle ($h_0 \geq 0 \Longrightarrow h \geq 0$).

We prove the existence of nonnegative weak solutions for a range of equations of type (5.1). The weak existence theory builds upon previous theory [3, 9, 13] for related equations with $g \leq 0$ or $g = 0$—equations that do not have a long-wave instability. We extend recent work on finite speed of propagation of the support for the $g = 0$ case to prove that in the $g \geq 0$ case the nonnegative weak solutions also have finite speed of propagation.

We present preliminary numerical evidence of finite-time blowup in a "critical case" where $g(h) = ch^2 f(h)$, suggesting that our conjecture is sharp. The numerics show h blowing up at one point and touching down at a pair of points, one to each side. The observed singularities appear to have interesting structure involving matched asymptotics and second-type self-similarity. Such behavior is found in the $h \downarrow 0$ singularities in the $g = 0$ case [1, 11, 12, 21]. The supercritical case, which we do not present here, has singularities with the expected dimensional scaling. We will pursue the details of the structure of the singularities in a separate paper.

Another model mentioned by Hocherman and Rosenau [38] is the Benny equation [46], which in the nonconvective case takes the form of (5.1) with $f(h) = c_1 h^3$ and $g(h) = c_2 h^6 + c_3 h^3$, $c_1, c_2 > 0$. Numerical computations in [46] supported Conjecture 1. It is interesting to note that, were they to have considered a "modified Benny equation" with $g(h) = c_2 h^m$, Hocherman and Rosenau would have predicted finite-time singularities for $m > 3$; however, our results here prove that blowup is not possible for all $m < 5$ nor in the critical case $m = 5$ for initial data of sufficiently small mean.

We note that a subclass of problems, included in the possible blowup scenario of Conjecture 2, are those with "negative exponents" n_1, m_1. Consider, for example, the equation

$$h_t = -\left(\left(\frac{h^4}{1+h^7}\right) h_{xxx}\right)_x - \left(\left(\frac{h^4}{1+h^5}\right) h_x\right)_x,$$

where the behavior of the coefficients f and g near $h = 0$ insures positivity of the solution whenever it is bounded. If a blowup were to occur for this equation, it would necessarily involve a singularity in higher derivatives since both diffusion exponents decay as $h \to 0$.

There remain unsolved theoretical problems. For example, the numerical evidence suggests that Conjecture 2 is sharp. Can one prove this? There are conjectures related to pinching singularities that are still unproven for the $g = 0$ and $g \leq 0$ cases. For example, pinching singularities were numerically observed for equation (5.1) with $f(h) = h$, $g(h) = 0$ [1, 11, 12, 21]. There is no analytical proof. The question of uniqueness of nonnegative weak solutions also remains an open problem.

Variational methods have been applied to $h_t = -(h h_{xxx})_x$, proving the existence of solutions with *nonzero* contact angles [44]. Such approaches have yet to be brought to bear on nonquadratic nonlinearities or equations with a second-order term.

Furthermore, there are related long-wave unstable equations to which the energy methods of this paper do not directly apply. One example is motion by the Laplacian of mean curvature [8, 16]. Other examples are addressed in the paper [38] in which Hocherman and Rosenau made their conjecture.

In higher space dimensions, we expect a different type of scaling to occur. Consider nonnegative solutions of the evolution equation

$$h_t = -\nabla \cdot (f(h)\nabla\Delta h + g(h)\nabla h)\,, \quad x \in \mathbb{R}^D\,, \tag{5.2}$$

and suppose that the solution blows up in finite time with $h_{\max} \uparrow \infty$. Applying the scaling argument presented in the introduction, finite-time blowup can only happen if

$$\lim_{h\to\infty} \frac{g(h)}{h^{2/D} f(h)} = \infty\,.$$

The methods of Section 2 do not immediately extend to higher dimensions. This is due to the Sobolev embedding lemma that states that the H^1-norm controls L^∞ in one space dimension but not in higher dimensions. Various analytical results have been proven in higher dimensions. We refer the reader to a recent paper of Dal Passo et al. for further references [18].

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