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LONG-WAVELENGTH FLUCTUATION SPECTRA OF CHARGED VERSUS NEUTRAL ONE COMPONENT SYSTEMS

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Synopsis

Microscopic expressions for the long-wavelength excitations of a classical electron gas have been established valid to *all orders* in the density, charge and plasma expansion parameter. The strengths with which these excitations appear in various space–time autocorrelation functions have also been calculated. Throughout, contact with the corresponding results for neutral particles has been established and their differences discussed. Most notably hereby is the difference between the heat mode of neutral- and charged-particle systems as well as the weak strength with which this mode appears in the correlation functions of the latter systems.

1. *Introduction and summary.* Space- and time-dependent autocorrelation functions, such as the well known Van Hove density–density correlation function, play a priviledged role in statistical physics as a bridge between theory and experiment[1]). It is therefore of considerable interest to see how the basic properties of a system are reflected in these correlation functions. One such characteristic property of a system is the way in which a long-wavelength disturbance decays. Of particular interest is the decay of a perturbation of a conserved quantity such as the number, momentum or energy density. Indeed, for systems of neutral particles this decay is correctly described, after a transient time, by hydrodynamics from which a remarkable expression for the Van Hove function, known as the Landau–Placzek formula[1]), can be obtained. Recent microscopic theories[2,3]) have proven the correctness of this result for particles interacting through short-range forces. It is the purpose of this paper to extend these calculations to the case of the Coulomb force whose long-range nature calls for a separate treatment leading to quite a different long-wavelength behaviour of the correlation functions. In fact for illustrative purposes we will treat on a par the case of short- and long-range forces.

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The short-range force case is typical of neutral-particles systems and will henceforth be referred to as such. For the long-range force case we consider the electron gas with neutralizing background and will refer to it as the charged-particle system. More precisely, the separation between the two types of systems will be made on the basis of the behaviour of the spatial Fourier transform of the interaction potential, say $V(k)$, for small wavevectors k. For neutral particles we assume $\lim_{k\to 0} V(k)$ to be *finite*, whereas it is *singular* for charged particles where $V(k) = 4\pi e^2/k^2$, e being the charge on each particle. This singular character of the Coulomb potential will introduce plasma-wave long-wavelength excitations instead of sound wave-like excitations. It will however also modify the heat mode and the strength with which the long-wavelength excitations appear in the correlation functions.

Before summarizing the subsequent sections we call attention to the fact that the extention of the present results to a two-component system is not completely trivial. In fact the electron gas is already some kind of two-component system in which the second component is kept inert. In a real two-component plasma however the fluctuations in charge and mass density are no longer proportional to each other as they are in an electron gas.

In section 2 we present a kinetic theory for the fluctuation spectra embodied in the various correlation functions. We follow hereby quite closely previous microscopic theories designed precisely to compute similar correlation functions in different limiting situations[3-5]). For the reader's facility we keep however the present paper self-contained. The central result of this section is an exact evolution equation [eq. (2.16)] for the equilibrium fluctuation spectra (2.2) of the microscopic phase-space density (2.1). By taking appropriate velocity moments, this equation is then transformed into a matrix equation (2.19) for the hydrodynamic fluctuation spectra of interest to us. In the course of these developments a major role is played by the *invariance properties* of the system for translations, rotations, and space–time reflections.

In section 3 we consider the dispersion equation for the excitations which build up the hydrodynamic correlation functions. In the long-wavelength limit $k \to 0$, or more precisely k smaller than any characteristic wavevector of the system, the weakly damped solutions of the dispersion equation are shown to be the *hydrodynamical modes* of our system. Here a central role is played by the *conservation laws* which permit us to put in evidence a certain number of k factors ensuring the very existence of *weakly* damped modes. The existence of a *finite* small-k limit of the dispersion equation is, on the other hand, shown to be controlled by the small-k behaviour of the equilibrium direct correlation function $c(k)$. In the Coulomb case the existence of this limit is not trivial because $c(k)$ diverges as k^{-2} for small k whereas $c(0)$ is finite for neutral particles. Once these preliminaries are established, we solve the dispersion equation for small k and look for weakly damped modes. For both neutral and charged particles we find five such hydrodynamical modes. There are two transverse shear modes, a heat mode and for neutral particles two sound modes which for charged particles are shifted here into two plasma modes. The singularity of $c(k)$ has produced two important modifications: 1) the heat mode frequencies for charged and neutral particles differ by the specific-heat ratio c_P/c_V; 2) the low-frequency sound modes are converted into high-frequency plasma modes. The various hydrodynamical frequencies are given microscopic expressions valid for arbitrary density, coupling strength and plasma expansion parameter. For the neutral-particle case such general microscopic expressions have been derived earlier[2,3]). For the charged-particle system this is, to our knowledge, done here for the first time. It is also observed that because of the presence of high-frequency modes in the charged-particle case, contrary to what happens for neutral particles, these hydrodynamical frequencies cannot be obtained from the usual linearized hydrodynamic equations[6]). In section 4 we compute the strengths with which these hydrodynamical modes appear in the correlation functions themselves. For neutral particles well known results such as the Landau–Placzek formula are easily recovered. For charged particles we find that the relative strength of the heat mode to the plasma mode is of order k^2 and hence vanishingly small as $k \to 0$ *except* for the excess kinetic-energy autocorrelation function. The latter correlation function is however accessible only to computer calculations.

Conclusions are presented in section 5 while some more technical points are discussed in two appendices.

2. *Kinetic theory of fluctuation spectra.* 2.1. Correlation functions. We will be interested in the space- and time-dependent correlation functions of various fluctuations in charged- and neutral-particle systems. These latter quantities are most easily obtained from the fluctuation of the phase-space density

$$f(\boldsymbol{r}\boldsymbol{p}t;\Gamma) = \sum_{j=1}^{N} \delta\,(\boldsymbol{r} - \boldsymbol{x}_j(t))\,\delta\,(\boldsymbol{p} - \boldsymbol{p}_j(t)), \tag{2.1}$$

of the N particles having positions $\boldsymbol{x}_j$ and momenta $\boldsymbol{p}_j = m\boldsymbol{v}_j$. The dependence of f on the initial phase $\Gamma = \{\boldsymbol{x}_j(0), \boldsymbol{p}_j(0)\}$ will usually be dropped in what follows. For brevity we will denote the equilibrium average over the initial phase simply as $\langle\cdots\rangle = \int \mathrm{d}\Gamma\,\varrho(\Gamma)\ldots$, $\varrho(\Gamma)$ being the normalized equilibrium distribution. The space–time correlation function or fluctuation spectra of f can now be defined as:

$$S\,(\boldsymbol{r} - \boldsymbol{r}', t - t'; \boldsymbol{p}\boldsymbol{p}') = \langle \delta f\,(\boldsymbol{r}\boldsymbol{p}t)\,\delta f\,(\boldsymbol{r}'\boldsymbol{p}'t')\rangle, \tag{2.2}$$

$\delta f = f - \langle f\rangle$ being the fluctuation of f and where moreover the invariance for space–time translations of the equilibrium ensemble has been used. From $S\,(\boldsymbol{r}t; \boldsymbol{p}\boldsymbol{p}')$ we can obtain, *e.g.*, the well known density–density or Van Hove correlation function by integration:

$$G(\boldsymbol{r}t) = (1/n) \int \mathrm{d}\boldsymbol{p} \int \mathrm{d}\boldsymbol{p}' \, S(\boldsymbol{r}t; \boldsymbol{p}\boldsymbol{p}') = (1/n) \langle \delta n(\boldsymbol{r}t) \, \delta n(\boldsymbol{0}0) \rangle . \tag{2.3}$$

Here we have put $n(\boldsymbol{r}t) = \int \mathrm{d}\boldsymbol{p} \, f(\boldsymbol{r}\boldsymbol{p}t)$ and $n = N/\Omega$ as the average number density in the volume Ω enclosing the system. Similarly the momentum–momentum correlation function can be obtained from S by integration:

$$g_{ij}(\boldsymbol{r}t) = \int \mathrm{d}\boldsymbol{p} \int \mathrm{d}\boldsymbol{p}' \, p_i p_j' S(\boldsymbol{r}t; \boldsymbol{p}\boldsymbol{p}'). \tag{2.4}$$

S is clearly a convenient starting point to compute a large class of correlation functions. The Fourier–Fourier transform of $G(\boldsymbol{r}t)$ is often referred to as the dynamic form factor. More generally we introduce the Fourier–Laplace transform of $S(\boldsymbol{r}t, \boldsymbol{p}\boldsymbol{p}')$:

$$S(\boldsymbol{k}z; \boldsymbol{p}\boldsymbol{p}') = \int_{\Omega} \mathrm{d}\boldsymbol{r} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{\mathrm{i}zt} \, S(\boldsymbol{r}t; \boldsymbol{p}\boldsymbol{p}'); \qquad \mathrm{Im}\, z > 0. \tag{2.5}$$

From which the Fourier–Fourier transform, $\hat{S}(\boldsymbol{k}\omega; \boldsymbol{p}\boldsymbol{p}')$, can be obtained as twice its real part,

$$\hat{S}(\boldsymbol{k}\omega; \boldsymbol{p}\boldsymbol{p}') = 2 \, \mathrm{Re} \, S(\boldsymbol{k}, \omega + \mathrm{i}0; \boldsymbol{p}\boldsymbol{p}'), \tag{2.6}$$

as follows from the invariance under space–time reversal of the equations of motion. The structure factor is then simply obtained as $S(\boldsymbol{k}\omega) = (1/n) \int \mathrm{d}\boldsymbol{p} \int \mathrm{d}\boldsymbol{p}' \times \hat{S}(\boldsymbol{k}\omega; \boldsymbol{p}\boldsymbol{p}')$ and can then be further related to the scattering cross section in the usual way[1]). It is frequently useful to consider the correlation function of the spatial Fourier components of δf, say $\delta f(\boldsymbol{k}\boldsymbol{p}t)$, which is easily related to the spatial Fourier transform of S itself:

$$S(\boldsymbol{k}t; \boldsymbol{p}\boldsymbol{p}') = (\delta f(\boldsymbol{k}\boldsymbol{p}t) \mid \delta f(\boldsymbol{k}\boldsymbol{p}'0)), \tag{2.7}$$

where for further convenience a scalar product in phase space has been introduced according to: $(A \mid B) = \Omega^{-1} \langle AB^* \rangle$, B^* being the complex conjugate of B. The Fourier component $\delta f(\boldsymbol{k}\boldsymbol{p}t)$ itself is given by:

$$\delta f(\boldsymbol{k}\boldsymbol{p}t) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j(t)} \, \delta(\boldsymbol{p} - \boldsymbol{p}_j(t)) - \delta_{\boldsymbol{k}} N \varphi(\boldsymbol{p}), \tag{2.8}$$

where $\varphi(\boldsymbol{p}) = (\beta/2\pi m)^{3/2} \exp(-\beta p^2/2m)$, $\beta^{-1} = k_{\mathrm{B}}T$ is the normalized maxwellian and $\delta_{\boldsymbol{k}}$ the Kronecker delta function ($\delta_{\boldsymbol{k}} = 0$ if $\boldsymbol{k} \neq \boldsymbol{0}$ and $\delta_{\boldsymbol{k}} = 1$ if $\boldsymbol{k} = \boldsymbol{0}$). If moreover we introduce the Liouville operator L according to $f(\boldsymbol{r}\boldsymbol{p}t) = (\exp \mathrm{i}Lt) \times f(\boldsymbol{r}\boldsymbol{p}0)$, the fundamental correlation function can then be written:

$$S(\boldsymbol{k}z; \boldsymbol{p}\boldsymbol{p}') = \mathrm{i} \, (\delta f(\boldsymbol{k}\boldsymbol{p})| \, (z - L)^{-1} \, |\delta f(\boldsymbol{k}\boldsymbol{p}')) \equiv \mathrm{i} \, (\delta f(\boldsymbol{k}\boldsymbol{p}) \mid F(\boldsymbol{k}z\boldsymbol{p}')), \tag{2.9}$$

where $\delta f(\boldsymbol{k}\boldsymbol{p}) \equiv \delta f(\boldsymbol{k}\boldsymbol{p}t = 0)$ and $|F(\boldsymbol{k}z\boldsymbol{p})) \equiv (z - L)^{-1} \, |\delta f(\boldsymbol{k}\boldsymbol{p}))$.

2.2. Kinetic equation. In order to obtain a kinetic equation for S we now use a projection-operator method. From eq. (2.9) we see that $S(\boldsymbol{k}z;\boldsymbol{pp}')$ is the $\delta f(\boldsymbol{kp})$ component of $|F(\boldsymbol{k}z\boldsymbol{p}'))$ which itself evolves according to $(z-L)\,|F(\boldsymbol{k}z\boldsymbol{p}'))$ $= |\delta f(\boldsymbol{kp}'))$. Projecting the latter equation with the aid of two complementary projection operators P and Q ($P+Q = I$, $P^2 = P$, $Q^2 = Q$, $PQ = 0 = QP$) we obtain $zP\,|F) - PLP\,|F) - PLQ\,|F) = P\,|\delta f)$ and a similar equation for $Q\,|F)$. Solving the latter equation for $Q\,|F)$ and substituting this result into the equation for $P\,|F)$ we find

$$[z - PLP - \psi(z)]\,P\,|F(\boldsymbol{k}z\boldsymbol{p}')) = P\,|\delta f(\boldsymbol{kp}')) + D(z)\,Q\,|\delta f(\boldsymbol{kp}')), \tag{2.10}$$

where $\psi(z) = PLQ\,(z - QLQ)^{-1}\,QLP$, whereas $D(z)$ is irrelevant here as we will now choose Q such that $Q\,|\delta f(\boldsymbol{kp}')) = 0$. The appropriate choice for P is thus such as to project any phase-space function $|g(\Gamma))$ onto the one particle state $|\delta f(\boldsymbol{kp}))$. We can take therefore P as

$$P\,|g) = \sum_{\boldsymbol{k}} \int \mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2\,|\delta f(\boldsymbol{kp}_1))\,\mathscr{P}(\boldsymbol{kp}_1\boldsymbol{p}_2)\,(\delta f(\boldsymbol{kp}_2)\,|\,g), \tag{2.11}$$

where $\mathscr{P}(\boldsymbol{kp}_1\boldsymbol{p}_2)$ is the inverse matrix (in the $\boldsymbol{p}_1, \boldsymbol{p}_2$ labels) of $(\delta f(\boldsymbol{kp}_1)\,|\,\delta f(\boldsymbol{kp}_2))$ $= S(\boldsymbol{k}t = 0; \boldsymbol{p}_1\boldsymbol{p}_2) \equiv S_0(\boldsymbol{kp}_1\boldsymbol{p}_2)$, *i.e.*:

$$\int \mathrm{d}\boldsymbol{p}'\,\mathscr{P}(\boldsymbol{kp}_1\boldsymbol{p}')\,S_0(\boldsymbol{kp}'\boldsymbol{p}_2) = \delta(\boldsymbol{p}_1 - \boldsymbol{p}_2) = \int \mathrm{d}\boldsymbol{p}'\,S_0(\boldsymbol{kp}_1\boldsymbol{p}')\,\mathscr{P}(\boldsymbol{kp}'\boldsymbol{p}_2). \tag{2.12}$$

It is easily shown that

$$S_0(\boldsymbol{kpp}') = n\varphi(\boldsymbol{p})\,[\delta(\boldsymbol{p} - \boldsymbol{p}') + \varphi(\boldsymbol{p}')\,h(\boldsymbol{k})], \tag{2.13}$$

where $h(\boldsymbol{k}) = \Omega n\,\langle \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)} - \delta_{\boldsymbol{k}}\rangle$ is the Fourier transform of the binary correlation function, whereas,

$$n\mathscr{P}(\boldsymbol{kpp}') = [\delta(\boldsymbol{p} - \boldsymbol{p}')/\varphi(\boldsymbol{p})] - c(\boldsymbol{k}), \tag{2.14}$$

where $c(\boldsymbol{k})$ is the direct correlation function related to $h(\boldsymbol{k})$ through the Ornstein–Zernike relation

$$1 + h(\boldsymbol{k}) = [1 - c(\boldsymbol{k})]^{-1}. \tag{2.15}$$

It is easily checked that P [eq. (2.11)] is a projection operator. It is the extension to continuous variables of the well-known Mori operator as introduced in the recent literature by Akcasu and Duderstadt[4]).

Returning now to eq. (2.10) we take its scalar product with $(\delta f(\boldsymbol{kp})|$ and obtain after some simple algebra which is given in appendix A,

$$zS(\boldsymbol{k}z;\boldsymbol{pp}') - \int \mathrm{d}\boldsymbol{p}_1\,\Sigma(\boldsymbol{k}z\boldsymbol{pp}_1)\,S(\boldsymbol{k}z;\boldsymbol{p}_1\boldsymbol{p}') = \mathrm{i}S(\boldsymbol{k}t = 0;\boldsymbol{pp}'), \tag{2.16}$$

i.e., a closed equation for S. The evolution operator $\Sigma(\boldsymbol{kzpp}')$ is nonlocal in space and time and splits naturally into three parts $\Sigma = \Sigma^0 + \Sigma^s + \Sigma^c$; the free streaming term

$$\Sigma^0(\boldsymbol{kzpp}') = \boldsymbol{k}\cdot\boldsymbol{v}\,\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{2.17a}$$

the self-consistent field term

$$\Sigma^s(\boldsymbol{kzpp}') = -\boldsymbol{k}\cdot\boldsymbol{v}\,\varphi(\boldsymbol{p})\,c(\boldsymbol{k}) \tag{2.17b}$$

and the nonlocal collision term

$$\Sigma^c(\boldsymbol{kzpp}') = (\delta f(\boldsymbol{kp})|\,LQ\,(z - QLQ)^{-1}\,QL\,|\delta f(\boldsymbol{kp}'))\,[n\varphi(\boldsymbol{p}')]^{-1}. \tag{2.17c}$$

This fundamental kinetic equation (2.16) is, as it stands, still an exact equation which merely separates one-particle dynamics (contained in S) from more-than-one-particle dynamics (contained in Σ which is so to say one-particle "irreducible"). Eq. (2.16) is nevertheless a very convenient starting point for a general (linearized) kinetic theory because, as has been shown in the literature[3-4]), when Σ is expanded in a small parameter characteristic of a kinetic regime, eq. (2.16) reduces precisely to the linearized kinetic equation describing this kinetic regime (*e.g.* the linearized Boltzmann equation for the Boltzmann low-density regime, *etc.* ...). We can therefore use eq. (2.16) to investigate, in full generality, the long-wavelength regime of charged- *versus* neutral-particle systems.

2.3. Hydrodynamic projection. In a hydrodynamic description the interest is shifted from the full phase-space distribution $f(\boldsymbol{r}\boldsymbol{p}t)$ to its first five moments: the number density $n(\boldsymbol{r}t) = \int \mathrm{d}\boldsymbol{p}\, f(\boldsymbol{r}\boldsymbol{p}t)$, the momentum density $\boldsymbol{g}(\boldsymbol{r}t) = \int \mathrm{d}\boldsymbol{p}\,\boldsymbol{p}\,f(\boldsymbol{r}\boldsymbol{p}t)$ and the kinetic energy density $\varepsilon(\boldsymbol{r}t) = \int \mathrm{d}\boldsymbol{p}\,(p^2/2m)\,f(\boldsymbol{r}\boldsymbol{p}t)$. More precisely we will be interested here in the fluctuation spectra of these conserved quantities. The fact that the kinetic energy alone is not a conserved quantity will be taken into account subsequently. It is convenient to denote the velocity moments in scalar-product form. In contradistinction with the scalar product in phase space (2.7) we will denote the scalar product in momentum space by triangular brackets [$A(\boldsymbol{pp}')$ is a matrix in momentum space]:

$$\langle f|\,A\,|g\rangle = \int \mathrm{d}\boldsymbol{p} \int \mathrm{d}\boldsymbol{p}'\, f(\boldsymbol{p})\,A(\boldsymbol{pp}')\,n\varphi(\boldsymbol{p}')\,g^*(\boldsymbol{p}'), \tag{2.18a}$$

$$\langle f|g\rangle = \int \mathrm{d}\boldsymbol{p}\, f(\boldsymbol{p})\,n\varphi(\boldsymbol{p})\,g^*(\boldsymbol{p}). \tag{2.18b}$$

In order to extract from eq. (2.16) the information about the hydrodynamic moments of S, we introduce an orthonormalized basis $\langle i| = u_i(\boldsymbol{p})/a_i$, $\langle i|j\rangle = \delta_{ij}$ containing as its five first members a density state, $i = 1 = n$, $u_n = 1$, $a_n^2 = \langle u_n|u_n\rangle$

$= n$, three momentum states $i = g_i$ $(i = 2, 3, 4)$, $u_{g_i} = p_i$, $a_{g_i}^2 = \langle u_{g_i} | u_{g_i} \rangle = a_g^2$ and an energy state $i = 5 = \varepsilon$, $u = (p^2/2m) - \langle p^2/2mn | u_n \rangle$, $a_\varepsilon^2 = \langle u_\varepsilon | u_\varepsilon \rangle$. With the aid of this basis we construct a projection operator $\bar{P} = \sum_{i=1}^{5} |i\rangle \langle i|$, $(\bar{Q} = I - \bar{P})$, in momentum space, not to be confused with P of eq. (2.11) which projects in phase space, repeat the operations leading to eq. (2.10) but now starting from eq. (2.16) multiplied with $[n\varphi(p')]^{-1}$. The resulting projection of eq. (2.16) onto the hydrodynamical subspace can be written in the following compact matrix form $(i, i' = 1$ to $5)$:

$$\sum_{j=1}^{5} (z\delta_{ij} - \Omega_{ij}(kz))\, G_{ji'}(kz) = \mathrm{i} G_{ii'}^0(k). \tag{2.19}$$

Here $G_{ij} = \langle i| S |j\rangle$ are the hydrodynamic components of S [*e.g.*, G_{11} is n times the Fourier–Laplace transform of the Van Hove correlation function introduced in eq. (2.3)] and similarly $G_{ij}^0 = \langle i| S^0 |j\rangle$, whereas $\Omega_{ij} = \langle i| \bar{\Sigma} |j\rangle + \langle i| \bar{\psi} |j\rangle$, $\bar{\Sigma}(kz)$ being the operator matrix in momentum space whose p, p' components are given by $\Sigma(kzpp')$ of eq. (2.17) whereas $\bar{\psi} = \bar{P}\bar{\Sigma}\bar{Q}(z - \bar{Q}\bar{\Sigma}\bar{Q})^{-1}\bar{Q}\bar{\Sigma}\bar{P}$ is the same functional of $\bar{P}$ and $\bar{\Sigma}$ as the ψ of eq. (2.10) is of P and L. More explicitly we have:

$$G_{ij}^0(k) = \delta_{ij}\,[1 + \delta_{in} h(k)], \tag{2.20a}$$

$$\langle i| \bar{\Sigma}^0 |j\rangle = (\delta_{in}\delta_{j1} + \delta_{i1}\delta_{jn}) \frac{ka_g}{ma_n} + (\delta_{i\varepsilon}\delta_{j1} + \delta_{i1}\delta_{j\varepsilon}) \frac{2ka_\varepsilon}{3a_g}, \tag{2.20b}$$

$$\langle i| \bar{\Sigma}^s |j\rangle = -\delta_{i1}\delta_{jn}(ka_g/ma_n)\, c(k), \tag{2.20c}$$

$$\langle i| \bar{\Sigma}^c |j\rangle = (1/a_i a_j)\,(v_i| Q\,(QLQ - z)^{-1}\, Q\, |v_j). \tag{2.20d}$$

In eq. (2.20b) the state index 1 refers to the longitudinal momentum state, *i.e.*, $u_{g_i}(p)$ is separated into a component along the wave vector k, $u_l(p) = \hat{k} \cdot p$, and two transverse components, orthogonal to k, $u_{t_i}(p) = \varepsilon_i \cdot p$ $[k = k\hat{k}$, $\hat{k} \cdot \varepsilon_i = 0$, $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$; $(i, j) = (1, 2)]$. After using the explicit form of Σ^c (2.17c) and L, the v_i $(i = 1, \ldots, 5)$ functions introduced in (2.20d) can be identified as $v_i = \sum_{j=1}^{N} \times\, \mathrm{e}^{-\mathrm{i}k \cdot x_j} F_j \cdot (\partial u_i(p_j)/\partial p_j)$ where F_j is the force acting on particle j $(\dot{p}_j = F_j)$. Finally the following properties of $\langle i| \bar{\psi} |j\rangle$ should be noticed. As $\bar{\Sigma}^s$ operates only in the hydrodynamic subspace [*cf.* (2.17b) and (2.20c)] we have $\bar{\Sigma}^s\bar{Q} = 0 = \bar{Q}\bar{\Sigma}^s$ and consequently $\bar{\Sigma}^s$ drops completely out of $\bar{\psi}$, a very fortunate property in the plasma case as will be seen subsequently. As moreover $\langle n| \bar{\Sigma}^0 \bar{Q} = 0 = \bar{\Sigma}\bar{Q}^0 |n\rangle$ and $v_n = 0$ it follows that $\langle i| \bar{\psi} |n\rangle$, $\langle n| \bar{\psi} |i\rangle$, $\langle i| \bar{\Sigma}^c |n\rangle$, $\langle n| \bar{\Sigma}^c |i\rangle$ all vanish identically for all $i = 1, \ldots, 5$. We now possess a closed set of *exact* equations (2.19) for the space–time correlation functions of interest to us $G_{ij}(kz)$, in terms of their initial values $G_{ij}^0(k)$ (2.20a) and the matrix $\Omega_{ij}(kz)$ which itself has been given explicit expressions in terms of equilibrium and momentum averages.

3. *Long-wavelength limit of the dispersion equation.* Before considering the behaviour of the correlation functions $G_{ij}(\boldsymbol{k}z)$ in the long-wavelength limit it is possible to reduce, without approximation, the system of equations (2.19) still further. Indeed, because of the invariance under rotations of the equilibrium distribution, a function such as $S(\boldsymbol{k}z\boldsymbol{p}\boldsymbol{p}')$ which is an equilibrium average [see (2.9)] can only depend on the three vectors $\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{p}'$ through rotationally invariant combinations: $S = S(z, \boldsymbol{k}^2, \boldsymbol{p}^2, \boldsymbol{p}'^2, \boldsymbol{k}\cdot\boldsymbol{p}, \boldsymbol{k}\cdot\boldsymbol{p}', \boldsymbol{p}\cdot\boldsymbol{p}')$. Consequently the matrix elements $G_{t_i j} = \langle t_i | S | j \rangle$ or $G_{j t_i}$ vanish whenever $j \neq t_i$ because its integrand is an odd function of the transverse momentum component $\boldsymbol{\varepsilon}_i \cdot \boldsymbol{p}$ or $\boldsymbol{\varepsilon}_i \cdot \boldsymbol{p}'$. As, by virtue of the same argument, the three matrices Ω, G and G^0 of eq. (2.19) can only couple the transverse component t_i to itself, we can decompose eq. (2.19) into a set of two identical *transverse equations*,

$$[z - \Omega_t(\boldsymbol{k}z)]\, G_t(\boldsymbol{k}z) = i, \qquad t = t_1 \text{ or } t_2, \tag{3.1}$$

where we have put $\Omega_{t_1 t_1} = \Omega_{t_2 t_2} = \Omega_t$ and $G_{t_1 t_1} = G_{t_2 t_2} = G_t$, and a set of three *longitudinal equations* which after inspection of (2.20) can be written:

$$\begin{aligned} &zG_{nj} - \Omega_{nl}G_{lj} = \mathrm{i}\,[1 + h(k)]\,\delta_{jn}, \\ &zG_{lj} - \Omega_{ln}G_{nj} - \Omega_{ll}G_{lj} - \Omega_{l\varepsilon}G_{\varepsilon j} = \mathrm{i}\delta_{jl}, \qquad j = n, l, \varepsilon, \\ &zG_{\varepsilon j} - \Omega_{\varepsilon l}G_{lj} - \Omega_{\varepsilon\varepsilon}G_{\varepsilon j} = \mathrm{i}\delta_{j\varepsilon}, \end{aligned} \tag{3.2}$$

and constitute an exact closed set of algebraic equations from which G_{ij} can be computed in terms of Ω_{ij} and $h(\boldsymbol{k})$.

3.1. Long-wavelength transverse modes. The only nonvanishing correlation function involving the transverse momentum component is $G_t = G_{t_i t_i}$ which is readily obtained from eq. (3.1) as $G_t(\boldsymbol{k}z) = \mathrm{i}\,[z - \Omega_t(\boldsymbol{k}z)]^{-1}$ where Ω_t is according to (2.20) entirely due to $\bar{\Sigma}^{\mathrm{c}}$ and $\bar{\psi}$:

$$\Omega_t(\boldsymbol{k}z) = \langle t | \bar{\Sigma}^{\mathrm{c}} | t \rangle + \langle t | (\bar{\Sigma}^0 + \bar{\Sigma}^{\mathrm{c}})\, \bar{Q}\, [z - \bar{Q}(\bar{\Sigma}^0 + \bar{\Sigma}^{\mathrm{c}})\bar{Q}]^{-1} \times \bar{Q}(\bar{\Sigma}^0 + \bar{\Sigma}^{\mathrm{c}}) | t \rangle, \tag{3.3}$$

where $t = t_1$ or t_2. The transverse excitations or modes can be obtained by solving the dispersion equation $z - \Omega_t(\boldsymbol{k}z) = 0$ for z as a function of $\boldsymbol{k}$. As a consequence of momentum conservation this dispersion equation takes on a particular form. Indeed from (2.20d) we see that $\langle t | \bar{\Sigma}^{\mathrm{c}}$ or $\bar{\Sigma}^{\mathrm{c}} | t \rangle$ does involve the function $v_t(\boldsymbol{k}) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j} \boldsymbol{F}_j \cdot \boldsymbol{\varepsilon}$ which vanishes as k goes to zero, $v_t(\boldsymbol{k} = 0) = (\sum_{j=1}^{N} \boldsymbol{F}_j)\cdot\boldsymbol{\varepsilon} = (\sum_j \dot{\boldsymbol{p}}_j)\cdot\boldsymbol{\varepsilon}$, because the total momentum $\boldsymbol{P} = \sum_j \boldsymbol{p}_j$ is conserved, $\dot{\boldsymbol{P}} = 0$. Consequently we can write $v_t(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_t(\boldsymbol{k})$ with $\boldsymbol{I}_t(\boldsymbol{k} = 0)$ finite, $\boldsymbol{I}_t(\boldsymbol{k})$ being related to the microscopic stress tensor as explained in appendix B. Counting the k factors in (3.3) we can now give it the suggestive form $\Omega_t(\boldsymbol{k}z) = -\mathrm{i}k^2 D_t(\boldsymbol{k}z)$, $D_t(\boldsymbol{k}=0\ z)$ being finite. Solving the dispersion relation $z(\boldsymbol{k}) + \mathrm{i}k^2 D_t(\boldsymbol{k}z(\boldsymbol{k})) = 0$ for small k values we arrive at once at the shear mode

$$z_t(\boldsymbol{k}) = -\mathrm{i}k^2 D_t(\boldsymbol{0}0) + \mathcal{O}(k^3) = -\mathrm{i}k^2(\eta/nm) + \mathcal{O}(k^3); \quad t = t_1 \text{ or } t_2, \quad (3.4)$$

where the shear diffusivity D_t has been expressed in terms of the shear viscosity through $\eta = nmD_t$. The explicit expression of the viscosity can be obtained from (3.3): $\eta = \lim_{k\to 0}\lim_{z\to 0}(nm/-\mathrm{i}k^2)\,\Omega_t(\boldsymbol{k}z)$. It naturally splits into a sum of two contributions, one for each term of (3.3); $\eta = \eta^{\Sigma} + \eta^{\Psi}$:

$$\eta^{\Sigma} = mn \lim_{k\to 0}(1/k^2)\,\langle t|\,\mathrm{i}\bar{\Sigma}^{\mathrm{c}}(\boldsymbol{k}z=0)\,|t\rangle$$
$$\equiv (mn/a_t^2)\,(\hat{\boldsymbol{k}}\cdot\boldsymbol{I}_t(\boldsymbol{k}=0)|\,Q\,(QLQ-\mathrm{i}0)^{-1}\,Q\,|\hat{\boldsymbol{k}}\cdot\boldsymbol{I}_t(\boldsymbol{k}=0)), \quad (3.5\mathrm{a})$$

$$\eta^{\Psi} = mn \lim_{k\to 0}\langle t|\,(1/k)\,[\bar{\Sigma}^0 + \bar{\Sigma}^{\mathrm{c}}(\boldsymbol{k},0)]\,\bar{Q}\,(\mathrm{i}\bar{\Sigma}^{\mathrm{c}}(\boldsymbol{0}0))^{-1}$$
$$\times\ \bar{Q}\,[\bar{\Sigma}^0 + \bar{\Sigma}^{\mathrm{c}}(\boldsymbol{k}0)]\,(1/k)\,|t\rangle. \quad (3.5\mathrm{b})$$

The first term η^{Σ} is a purely potential contribution whereas η^{Ψ} contains both kinetic and potential contributions to the viscosity. It can also be shown that each term of η is separately positive[3]) and moreover that (3.5) compares well with other expressions of η found from Green–Kubo formulae[2]) or, for specific kinetic regimes, from standard kinetic theories. To conclude this section we observe that both for charged- and neutral-particle systems there is a twofold-degenerate transverse shear mode (3.4). Only the collision term with the aid of which the viscosity has to be calculated according to (3.5) will differ from one system to the other.

3.2.1. Longitudinal modes: long-wavelength limit of the dispersion equation. We now turn to the more interesting case of the longitudinal modes. These do show up for example in Van Hove's correlation function or its Fourier transform $S(\boldsymbol{k}\omega) = 2\,\mathrm{Re}\,G_{nn}(\boldsymbol{k}\ \omega + \mathrm{i}0)$, the structure factor. For a general correlation function G_{ij} we can write $G_{ij}(\boldsymbol{k}z) = \Delta_{ij}(\boldsymbol{k}z)/\Delta(\boldsymbol{k}z)$, $(i, j = n, l, \varepsilon)$, where $\Delta(\boldsymbol{k}z)$ is the determinant of the algebraic system (3.2), $\Delta = |z - \Omega(\boldsymbol{k}z)|$, and Δ_{ij} the cofactor of G_{ij} in this system. More explicitly we have

$$\Delta(\boldsymbol{k}z) = (z - \Omega_{\varepsilon\varepsilon})\,[z(z - \Omega_{ll}) - \Omega_{nl}\Omega_{ln}] - z\Omega_{\varepsilon l}\Omega_{l\varepsilon} \quad (3.6)$$

and the solutions of the dispersion equation $\Delta(\boldsymbol{k}\ z(\boldsymbol{k})) = 0$ define a set of longitudinal excitations or modes which will show up as poles of the various longitudinal correlation functions $G_{ij}(i, j = n, l, \varepsilon)$. In a later section we will compute explicitly such correlation functions in the long-wavelength limit. Here we first consider the dispersion equation $\Delta = 0$ in more detail.

To do so we need some additional information about the Ω_{ij} elements as they appear in Δ [eq. (3.6)]. From the general properties considered in section 2.3 we obtain at once that $\Omega_{nl} = ka_g/ma_n$ and $\Omega_{ln} = \Omega_{nl}\,[1 - c(\boldsymbol{k})]$. If we now invoke momentum conservation we can write, in complete analogy with the transverse case considered in section 3.1, $\Omega_{ll}(\boldsymbol{k}z) = -\mathrm{i}k^2 D_l(\boldsymbol{k}z)$ and $\Omega_{l\varepsilon}(\boldsymbol{k}z) = \Omega_{\varepsilon l}(\boldsymbol{k}z) = kD_{l\varepsilon}(\boldsymbol{k}z)$, where D_l and $D_{l\varepsilon}$ have finite real limits for vanishing k and z (the consequences of the conservation laws for Ω_{ij} are more fully discussed in appendix B). In appendix B we also show that energy conservation leads to the form $\Omega_{\varepsilon\varepsilon}(\boldsymbol{k}z) = -\mathrm{i}k^2 D_\varepsilon(\boldsymbol{k}z) + zB_\varepsilon(\boldsymbol{k}z)$ where both D_ε and B_ε have finite real limits as $k \to 0$ and $z \to \mathrm{i}0$. The appearance of a new type of contribution, B_ε, expresses the fact that the kinetic energy alone is only conserved asymptotically as z goes to zero; at finite z only the total energy is conserved.

As we are interested in the long-wavelength or $k \to 0$ limit, one more item of information will be needed, namely the behaviour of $c(\boldsymbol{k})$ for small k. This thermodynamic information will turn out to be crucial as, here, for the first time an explicit difference between charged- and neutral-particle systems will show up. Indeed we have:

$$\lim_{k\to 0}\,[1 + h(\boldsymbol{k})] = \begin{cases} (n/\beta)\,\chi_T + \mathcal{O}(k^2), & \text{neutral particles},\\ k^2/k_\mathrm{D}^2 = \mathcal{O}(k^2), & \text{charged particles}, \end{cases} \tag{3.7a}$$

where $\chi_T = (1/n)\,(\mathrm{d}n/\mathrm{d}p)_T = (nmc^2)^{-1}$ is the isothermal compressibility and c the isothermal sound speed of a system of particles of mass m, density n and thermal speed $v_0 = (m\beta)^{-\frac{1}{2}}$, $\beta^{-1} = k_\mathrm{B}T$. For the charged-particle case eq. (3.7a) does not point to a vanishing compressibility of the election gas but on the contrary (3.7a) is a consequence of the singular nature of the Coulomb potential (k_D is the Debye wavevector, $k_\mathrm{D}^2 = 4\pi e^2 n\beta$, for electrons of charge e). To see this it is more interesting to consider $1 - c(\boldsymbol{k}) = [1 + h(\boldsymbol{k})]^{-1}$:

$$\lim_{k\to 0}\,[1 - c(\boldsymbol{k})] = \begin{cases} (c^2/v_0^2) + \mathcal{O}(k^2), & \text{neutral particles},\\ (k_\mathrm{D}^2/k^2) + (k_\mathrm{D}^2/k_\mathrm{s}^2) + \mathcal{O}(k^2), & \text{charged particles}. \end{cases} \tag{3.7b}$$

For the neutral case (3.7b) is simply an alternative expression of (3.7a). For the charged case (3.7b) results from the well known expression[7]) of the static-electron structure factor $S_0(k) = n\,[1 + h(\boldsymbol{k})] = n\,(k^2/k_\mathrm{D}^2)\,\mathrm{Re}\,[1 - 1/\varepsilon(k,0)]$ where $\varepsilon(k,0)$ is the static dielectric constant of an electron gas which behaves as $\varepsilon(k,0) = 1 + k_\mathrm{s}^2/k^2$ for small k. Here we have introduced the inverse screening length k_s defined as $k_\mathrm{s}^2 = \omega_\mathrm{P}^2/c^2 = 4\pi e^2 n^2 \chi_T$, where c and χ_T are the isothermal sound speed and compressibility of an electron gas of density n, charge e, mass m and plasma frequency ω_P ($\omega_\mathrm{P}^2 = 4\pi e^2 n/m$). In the perfect-gas limit of free electrons, this exact screening wave vector k_s reduces to the Debye wave vector k_D as is easily verified ($k_\mathrm{s}^2/k_\mathrm{D}^2 = v_0^2/c^2$). From (3.7b) we see that in the case of charged particles $c(\boldsymbol{k}) \sim k^{-2}$

for small k. We see thus that by going from neutral to charged particles the direct correlation function $c(\boldsymbol{k})$ has become singular in the long-wavelength limit. Some care has thus to be exercised when taking the $k \to 0$ limit of expressions such as (3.6). The singular factor $c(\boldsymbol{k})$ has been introduced into our exact expressions through eq. (2.14). It appears in two different manners. First there is an implicit or functional dependence because $\bar{\Sigma}^c$ of (2.17c) depends on Q which itself depends on a k integral involving $c(\boldsymbol{k})$ through (2.14) and (2.11). This functional dependence of $\bar{\Sigma}^c$ on $c(\boldsymbol{k})$ should not concern us here as we will assume that a collision operator *can* be constructed which remains finite in the asymptotic limit of vanishing k and z. It is fair to say, however, that in the present stage of plasma kinetic theory this goal has only been partially achieved since the Balescu–Guernsey–Lenard collision operator still presents a small-distance divergence which is not easily removed in a completely satisfactory way[8]). There remains nevertheless the explicit dependence of our expressions on $c(\boldsymbol{k})$ through the self-consistent field term $\bar{\Sigma}^s$ (2.17b). At this place we are very fortunate to recall that, as it only operates in the hydrodynamic subspace, $\bar{\Sigma}^s$ did drop out of $\bar{\psi}$ completely and that moreover $\bar{\Sigma}^s$ appears in the final expressions such as (3.6) only *via* the combination $\Omega_{nl}\Omega_{ln} = (ka_g/ma_n)^2 \times [1 - c(\boldsymbol{k})]$. Now, for small k we have $\Omega_{nl}\Omega_{ln} = k^2 (v_0^2 - c^2) + \mathcal{O}(k^4)$ for neutral particles and $\Omega_{nl}\Omega_{ln} = \omega_P^2 [1 + k^2/k_s^2 + \mathcal{O}(k^4)]$ for electrons of plasma frequency $\omega_P = k_D v_0$. Therefore even the explicit dependence on $c(\boldsymbol{k})$ does not introduce any singularity because the final expressions are *finite* in both cases for small k. Clearly this will no longer be the case if $c(\boldsymbol{k})$, or the interaction potential, would be more singular than k^{-2} for small k values. Having established that, notwithstanding the appearance of a singular term ($\bar{\Sigma}^s$) in the kinetic equation (2.16), the exact expressions obtained for the correlation functions $G_{ij} = \Delta_{ij}/\Delta$ are well defined in the long-wavelength limit, we can now turn our attention to the solutions of the dispersion equation in the latter limit.

3.2.1. Long-wavelength longitudinal modes. The dispersion relation for longitudinal modes $\Delta\,(\boldsymbol{k}z) = 0$ has in general an infinite number of roots $z = z(\boldsymbol{k})$ each of which defines a branch of the dispersion equation. We will consider here only a restricted class of solutions of the dispersion equation which we will call hydrodynamical modes. These roots are characterised by the fact that: 1) they are analytic in k for small k, *i.e.* they can be written $z(\boldsymbol{k}) = z_0 + z_1 + z_2 + \mathcal{O}(k^3)$ where $z_n \sim \mathcal{O}(k^n)$ for $n = 0, 1, 2$; and 2) they are weakly damped so as to have a macroscopic relevance, *i.e.* $\mathrm{Im}\, z(\boldsymbol{k}) \sim \mathcal{O}(k^2)$ and $|\mathrm{Im}\, z(\boldsymbol{k})| \ll |\mathrm{Re}\, z(\boldsymbol{k})|$ for small k whenever $\mathrm{Re}\, z(\boldsymbol{k})$ is different from zero for small k. To select these branches we substitute $z(\boldsymbol{k}) = z_0 + z_1 + z_2 + \mathcal{O}(k^3)$ in the dispersion equation $\Delta\,(\boldsymbol{k}\, z(\boldsymbol{k})) = 0$ and collect the various powers of k.

To zeroth order we obtain from eq. (3.6):

$$\Delta\,(\boldsymbol{0}z_0) = z_0\,[1 - B_c(\boldsymbol{0}\,z_0)] \times \begin{bmatrix} z_0^2, & \text{neutral particles}, \\ (z_0^2 - \omega_P^2), & \text{charged particles}, \end{bmatrix} \tag{3.8}$$

i.e., as discussed in the previous section, a well defined dispersion equation. For both neutral and charged particles there is a thermal branch starting at $z_0 = 0$ and is due to the thermal fluctuations. The density and longitudinal momentum fluctuations in turn lead to a double branch starting at $z_0 = 0$ for neutral particles and at the nonzero real values $z_0 = \pm\omega_P$ for charged particles as is seen from eq. (3.8). The finiteness of z_0 in the latter case is entirely due to the singular nature of the direct correlation function $c(\boldsymbol{k})$ for electrons. Apart from these three branches (3.8) also admits the roots of $1 - B_\varepsilon(0\,z_0) = 0$ which presumably correspond to relaxation frequencies having a nonzero imaginary part for $k = 0$ and hence are strongly damped[5]). Henceforth we will assume thus that for the hydrodynamical modes $B_\varepsilon(0\,z_0) \neq 1$. To *first order* we obtain, persuing the small-k expansion of (3.6), for the neutral case

$$z_1\,\{[1 - B_\varepsilon(\mathit{00})]\,(z_1^2 - k^2c^2) - k^2D_{l\varepsilon}^2(\mathit{00})\} = 0 \tag{3.9}$$

and for charged particles

$$z_1\,\{[1 - B_\varepsilon(\mathit{0}\,z_0)]\,(-\omega_P^2)\} = 0. \tag{3.10}$$

From eq. (3.9) we see that we can distinguish between a thermal branch for which $z_1 = 0$ and two oppositely propagating sound waves for which

$$z_1^2 = k^2\bar{c}^2; \qquad \bar{c}^2 = c^2 + D_{l\varepsilon}^2(\mathit{00})/[1 - B_\varepsilon(\mathit{00})], \tag{3.11}$$

i.e., the sound speed has been shifted from its isothermal value c to its isentropic value $\bar{c}$ because of the coupling between the energy and momentum fluctuations contained in $D_{l\varepsilon}$. More precisely, it can be shown[3]) that $D_{l\varepsilon}^2(\mathit{00})/[1 - B\,(\mathit{00})] = (T/c_V)\,(\mathrm{d}p/\mathrm{d}nmT)_n^2$, c_V being the specific heat at constant volume, so that $\bar{c}^2 = (\mathrm{d}p/\mathrm{d}nmT)_S^2$. For the charged-particle case eq. (3.10) indicates that $z_1 = 0$ for each of the three z_0 values solution of (3.8) and for which we have already assumed that $B_\varepsilon(0\,z_0) \neq 1$.

To *second order* we obtain, continuing the expansion of the thermal mode ($z_0 = 0$), for the neutral case,

$$\{z_2\,[1 - B_\varepsilon(\mathit{00})] + \mathrm{i}k^2D_\varepsilon(\mathit{00})\}\,c^2k^2 + z_2k^2D_{l\varepsilon}^2(\mathit{00}) = 0 \tag{3.12}$$

and

$$\{z_2\,[1 - B_\varepsilon(\mathit{00})] + \mathrm{i}k^2D_\varepsilon(\mathit{00})\}\,\omega_P^2 = 0, \tag{3.13}$$

for the charged case. Once more the coupling between different fluctuations ($D_{l\varepsilon}$) cannot come into play in the charged case because the latter effect is $\mathcal{O}(k^2)$ with respect to the ω_P^2 contribution as seen by comparing (3.13) with (3.12) as well as

(3.10) with (3.9). Solving (3.12)–(3.13) we obtain the following thermal mode:

$$z_T(k) = -\mathrm{i}k^2 \{D_\varepsilon(\mathit{00})/[1 - B_\varepsilon(\mathit{00})]\} \times \begin{bmatrix} (1 + D^2_{l\varepsilon}(\mathit{00})/c^2\,[1 - B_\varepsilon(\mathit{00})])^{-1} \\ 1 \end{bmatrix} + \mathcal{O}(k^3) \quad \begin{matrix} \text{neutral particles}, & (3.14a) \\ \text{charged particles}. & (3.14b) \end{matrix}$$

Using (3.11) we see that the two thermal modes differ by a factor $\{1 + D^2_{l\varepsilon}(\mathit{00})/c^2 \times [1 - B_\varepsilon(\mathit{00})]\}^{-1} = c^2/\bar{c}^2 \equiv c_V/c_P$. In a similar fashion we obtain, after some algebra, for the sound modes of the neutral system:

$$z_\pm(k) = \pm k\bar{c} - \mathrm{i}\,\frac{k^2}{2}\left[\frac{D_\varepsilon(\mathit{00})}{1 - B_\varepsilon(\mathit{00})}\left(1 - \frac{c^2}{\bar{c}^2}\right) + \frac{D_l(\mathit{00})}{1 - B_\varepsilon(\mathit{00})} + \mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{D^2_{\varepsilon l}(kz)}{1 - B_\varepsilon(kz)}\right)\Bigg|_{\substack{k=0\\ z=+\mathrm{i}0}}\right] + \mathcal{O}(k^3), \qquad (3.15)$$

whereas the plasma modes of the charged system yield:

$$z^{\mathrm{P}}_\pm(k) = \pm\omega_{\mathrm{P}}\left[1 + \frac{k^2c^2}{2\omega^2_{\mathrm{P}}}\left(1 + \frac{D^2_{\varepsilon l}(\mathit{0}, \pm\omega_{\mathrm{P}})}{c^2\,[1 - B_\varepsilon(\mathit{0}, \pm\omega_{\mathrm{P}})]}\right)\right] - \mathrm{i}\tfrac{1}{2}k^2 D_l(\mathit{0}, \pm\omega_{\mathrm{P}}) + \mathcal{O}(k^3), \qquad (3.16)$$

or equivalently:

$$z^{\mathrm{P}}_\pm(k) = \pm\omega_{\mathrm{P}}(1 + k^2/2k_s^2) + \frac{k^2}{2}\lim_{z\to\pm\omega_{\mathrm{P}}}\lim_{k\to 0}\left(\frac{\Omega^2_{\varepsilon l}(kz) + [z - \Omega_{\varepsilon\varepsilon}(kz)]\,\Omega_{ll}(kz)}{k^2\,[z - \Omega_{\varepsilon\varepsilon}(kz)]}\right) + \mathcal{O}(k^3). \qquad (3.17)$$

3.3. Hydrodynamic frequencies. Let us recollect the results of sections 3.1 and 3.2 and introduce a more standard notation. In the limit of long wavelengths we found a set of five collective excitations which, as is easily checked, are analytic in k and weakly damped. They reflect the existence of five independent conservation laws for our system. They consist of two transverse shear modes $z_{t_i} = -\mathrm{i}k^2 D_t$ $(i = 1, 2)$ (3.4), one thermal mode $z_T = -\mathrm{i}k^2 D_T$ (3.14) and two longitudinal propagating modes which are the sound modes $z_\pm = \pm k\bar{c} - \mathrm{i}\tfrac{1}{2}k^2\Gamma$ (3.15) for the neutral system and the plasma modes $z^{\mathrm{P}}_\pm = \pm\omega_{\mathrm{P}}(1 + \tfrac{1}{2}k^2\gamma) - \mathrm{i}\tfrac{1}{2}k^2\Gamma_{\mathrm{P}}$ (3.16) for the charged system. The shear diffusivity D_t is usually expressed in terms of the shear viscosity $\eta = nmD_t$. Instead of the thermal diffusivity D_T one can also use a thermal conductivity $\varkappa$, however the relation between D_T and $\varkappa$ will be different for the neutral and charged systems. For neutral particles we write as usual $\varkappa = mnc_P D_T$ but for charged particles we have then on the contrary $\varkappa = mnc_V D_T^P$ (P refers to plasma). For the propagating modes the difference between the neutral and charged systems is even stronger. In the neutral system we have two oppositely propagating sound waves with $\bar{c}$ as phase and group velocity and a damping Γ which is usually written $\Gamma = D_T\,[(c_P/c_V) - 1] + \phi/nm$ introducing a longitudinal viscosity ϕ consisting of a shear contribution $\frac{4}{3}\eta$ and the bulk viscosity ξ according to $\phi = \frac{4}{3}\eta + \xi$. In the charged case we have two oppositely propagating waves of infinite phase velocity ω_P/k and a small group velocity $\omega_P k\gamma$, for small k, as well as a small damping Γ_P. The various coefficients entering the hydrodynamical frequencies can be identified from the explicit expressions given in (3.4), (3.11) and (3.14)–(3.17). The latter expressions are *exact* in the sense that they involve the collision operator and thermodynamic functions computed to arbitrary order in the coupling constant, density and temperature. In the neutral-particle case it is seen that the transport coefficients so defined involve knowledge of the collision operator $\bar{\Sigma}^c(\boldsymbol{k}z)$ in the vicinity of $k = 0$ and $z = 0$. In the charged-particle case we see from (3.16) that the plasma modes require moreover knowledge of the collision operator at the nonzero frequency ω_P. Therefore $z_{\pm}^P$ is not readily expressible in terms of transport coefficients unless one uses nonzero-frequency transport coefficients. The latter however are complex quantities and a separation of (3.16) into real and imaginary parts does indeed lead to:

$$\gamma = k_S^{-2} + \mathrm{Re}\left(\frac{D_{\varepsilon l}^2(\boldsymbol{0}\omega_P)}{\omega_P^2\,[1 - B_\varepsilon(\boldsymbol{0}\omega_P)]}\right) + \mathrm{Im}\left(\frac{D_l(\boldsymbol{0}\omega_P)}{\omega_P}\right), \tag{3.18a}$$

$$\Gamma_P = \mathrm{Re}\, D_l(\boldsymbol{0}\omega_P) - \mathrm{Im}\left(\frac{D_{\varepsilon l}^2(\boldsymbol{0}\omega_P)}{\omega_P\,[1 - B_\varepsilon(\boldsymbol{0}\omega_P)]}\right), \tag{3.18b}$$

where we have taken into account that the real (imaginary) part of $D_l(\boldsymbol{0}\omega_P)$ and $D_{\varepsilon l}^2(\boldsymbol{0}\omega_P)/[1 - B_\varepsilon(\boldsymbol{0}\omega_P)]$ is an even (odd) function of ω_P. Eqs. (3.18) constitute a definition of the thermal correction, γ, to the plasma frequency ω_P and its damping, Γ_P, given here, to our knowledge, for the first time to *all orders* in the parameters characterising the state of the electron gas (e^2, n, T). To compare with known results one can expand, *e.g.*, γ and Γ_P of (3.18) with respect to the plasma expansion parameter k_D^3/n. For a low-density – high-temperature plasma we obtain in the limit of vanishing plasma parameter from (3.18):

$$\gamma = 3k_D^{-2} + \mathcal{O}(k_D^3/n), \tag{3.19a}$$

$$\Gamma_P = \mathcal{O}(k_D^3/n), \tag{3.19b}$$

i.e., a result in agreement with the collisionless Vlassov equation as it should be to this order. The $\mathcal{O}(k_D^3/n)$ terms appearing in (3.19) have been calculated in the literature in the framework of the high-frequency conductivity and will not be considered here[9]). Instead we want to comment on (3.18). If we were to substitute *incorrectly* D_l, $D_{\varepsilon l}$ and B_ε in (3.18) by their value at $\omega_P = 0$ the plasma mode would read $z'_\pm = \pm\omega_P(1 + k^2\bar{c}^2/2\omega_P^2) - i\frac{1}{2}k^2\eta/nm$, *i.e.* precisely what one would obtain from the usual linearised hydrodynamic equations applied to the electron gas[10]). This incorrect result stems from the fact that the usual hydrodynamic equations cannot describe adequately the rapid oscillations occurring at ω_P but only the low-frequency, long-wavelength motions such as described by the thermal mode (3.14) which is correctly given by hydrodynamics as $z_T^P = -ik^2(\varkappa/mnc_V)$. The difference between $z_\pm^P$ and $z'_\pm$ is observable even in the nondissipative part γ. Indeed, from $z'_\pm$ we would obtain $\gamma' = \bar{c}^2/\omega_P^2 = (c_P/c_V)\,k_S^{-2} \simeq \frac{5}{3}k_D^{-2}$ in the perfect-gas limit instead of $3k_D^{-2}$ (3.19a). To correct for this discrepancy one has to modify the hydrodynamic equations in the high-frequency region by introducing relations between the currents and the thermodynamic driving forces which are nonlocal in time. This has been discussed in more detail in a previous paper[6]). An unusual although interesting situation which can be discussed on the basis of (3.18) would be one in which the collision frequency, say ω_c, largely exceeds the plasma frequency ω_P so that a small ω_P/ω_c expansion of (3.18) becomes interesting. This however is not yet sufficient to reduce $z_\pm^P$ to $z'_\pm$ because even to zeroth order in ω_P/ω_c, the nonzero frequency ω_P will appear as argument in $z_\pm^P$ The very existence of a situation in which $z'_\pm$ would be a good approximation to $z_\pm^P$ is not obvious for the electron system. *If* we also were to take into account the presence of mobile ions and neutral particles together with the electrons then the situation just described will be considerably modified as we now have mechanisms which allow for mass fluctuations with reduced charge fluctuations and hence are able to reduce the importance of the Coulomb singularity. These interesting questions will however be deferred to future investigations. We now continue with a study of the small-k limit of the correlation functions G_{ij} for the one-component neutral- *versus* charged-particle systems under consideration.

4. *Long-wavelength limit of the correlation functions.* It has been known for years that the density–density correlation function $G_{nn}(kz)$ of neutral systems has a long-wavelength limit which can be simply expressed in terms of transport coefficients and thermodynamic coefficients so as to corroborate its calculation from hydrodynamics plus static fluctuation theory (Landau–Placzek theory[1])). We now establish a similar expression for the electron gas. As k goes to zero we can write $\Delta(kz) = [z - z_T(k)]\,[z - z_+(k)]\,[z - z_-(k)]\,[1 - B_\varepsilon(0z)]\,\bar{\Delta}(kz)$, where z_T, $z_\pm$ are given by (3.14), (3.16) for neutral particles and by (3.15), (3.17) for charged particles and where $\bar{\Delta}(kz)$ is a function which tends to unity as k tends to zero, for all values of z. Each of the longitudinal modes will contribute to the

autocorrelation function $G_{jj}(\boldsymbol{k}z)$ $(j = n, l, \varepsilon)$ with a strength given according to:

$$G_{jj}(\boldsymbol{k}z) = \frac{\mathrm{i}a_T^j}{z - z_T(\boldsymbol{k})} + \sum_{\pm} \frac{\mathrm{i}a_\pm^j}{z - z_\pm(\boldsymbol{k})} + \bar{G}_{jj}(\boldsymbol{k}z), \tag{4.1}$$

where $\mathrm{i}a_T^j$ and $\mathrm{i}a_\pm^j$ denote the residue of $G_{jj}(\boldsymbol{k}z)$ at $z = z_T(\boldsymbol{k})$ and $z = z_\pm(\boldsymbol{k})$ respectively, whereas $\bar{G}_{jj}(\boldsymbol{k}z)$ denotes the contribution from the remaining non-hydrodynamical roots of the dispersion equation. As the latter roots have a finite damping at $k = 0$ they do not contribute to $G_{jj}(\boldsymbol{k}t)$ for $t \sim \mathcal{O}(k^{-2})$ and we can forget about $\bar{G}_{jj}(\boldsymbol{k}t)$ on such time scales. The explicit values of a_T^j, $a_\pm^j$, for small k, are easily computed from the explicit expressions given in section 3. We have

TABLE I

Strengths of the various modes as they appear in the various autocorrelation functions

	Neutral particles	Charged particles
a_T^n	$(v_0^2/c^2)(1 - c^2/\bar{c}^2)$	$(k^2/k_D^2)[k^2(\bar{c}^2 - c^2)/\omega_P^2] = \mathcal{O}(k^4)$
$a_\pm^n$	$(v_0^2/2c^2)(c^2/\bar{c}^2)$	$\frac{1}{2}(k^2/k_D^2)$
a_T^l	$\mathcal{O}(k^2)$	$\mathcal{O}(k^4)$
$a_\pm^l$	$\frac{1}{2}$	$\frac{1}{2}$
a_T^ε	$[(v_0^2 - c^2)/\bar{c}^2]\, c_V^0/c_V$	c_V^0/c_V
$a_\pm^\varepsilon$	$\frac{1}{2}[1 - (v_0^2 - c^2)/\bar{c}^2]\, c_V^0/c_V$	$\mathcal{O}(k^2)$

summarised the results in table I where we took into account that, as has been shown elsewhere[3]), $1 - B_\varepsilon(\boldsymbol{0}, 0) = c_V/c_V^0$, c_V^0 being the perfect-gas limit of the specific heat c_V. From the results listed in table I one can read off several interesting properties of the Fourier transform $\hat{G}_{jj}(\boldsymbol{k}\omega) = 2\,\mathrm{Re}\, G_{jj}(\boldsymbol{k}z = \omega + \mathrm{i}0)$ [see (2.6)] as a function of the real frequency ω. Each mode appears in $\hat{G}_{jj}(\boldsymbol{k}\omega)$ as a peak with a given height and width. The relative heights of these peaks are given by

$$\lim_{k\to 0} [\hat{G}_{jj}(\boldsymbol{k}\omega = 0)/\hat{G}_{jj}(\boldsymbol{k}\omega = \mathrm{Re}\, z_\pm)] = (a_T^j/a_\pm^j)(\mathrm{Im}\, z_\pm/\mathrm{Im}\, z_T),$$

while the relative area under them is

$$\lim_{k\to 0}\Big[\int_{\omega\simeq 0} \mathrm{d}\omega\, \hat{G}_{jj}(\boldsymbol{k}\omega) \Big/ \int_{\omega\simeq \mathrm{Re}\, z_\pm} \mathrm{d}\omega\, \hat{G}_{jj}(\boldsymbol{k}\omega)\Big] = a_T^j/a_\pm^j.$$

whereas the widths are determined by Im z_T and Im $z_\pm$ themselves. The total area verifies furthermore the sum rule

$$\lim_{k\to 0} \int d\omega\, \hat{G}_{JJ}(k\omega) = \lim_{k\to 0} \pi G_{JJ}(kt = 0)$$

or

$$\lim_{k\to 0} \left(a_T^J + \sum_{\pm} a_\pm^J\right) = \lim_{k\to 0} [1 + \delta_{Jn} h(k)],$$

with one remarkable exception, namely $G_{\varepsilon\varepsilon}$. Indeed, as has been observed earlier for the neutral-gas case[11]), the kinetic energy not being a conserved quantity, part of its long-wavelength fluctuations is not correctly described by hydrodynamics. As a result the above sum rule is satisfied for $G_{\varepsilon\varepsilon}$ only up to a remainder which is shown here to be equal to $(c_V^0 - c_V)/c_V$ both for charged- and neutral-particle systems.

Another important observation which one can read off from table I is that for the plasma case the thermal mode $z_T^P(k)$ only shows up in $G_{\varepsilon\varepsilon}$ whereas G_{nn} and G_{ll} are dominated by the plasma modes $z_\pm^P(k)$. As $G_{\varepsilon\varepsilon}$ is not readily observable in a light-scattering experiment, such as is G_{nn}, *e.g.*, it will turn out difficult to check the absence of the c_V/c_P factor in z_T^P as compared to z_T [see (3.14)–(3.15)]. Such an effect is presumably accessible only to computer calculations of $G_{\varepsilon\varepsilon}$.

5. *Conclusions.* The long-wavelength fluctuations and correlation functions of a charged-particle one-component system turn out to be quite different from their neutral-particle equivalents because the singular Coulomb potential results in a lack of coupling between the low-frequency temperature fluctuations and the high-frequency density fluctuations. Consequently the heat mode of a plasma lacks a c_P/c_V factor while the low-frequency sound modes are shifted into the much higher plasma modes. Moreover the heat mode disappears from the Landau–Placzek formula as $k \to 0$ and persists only in $G_{\varepsilon\varepsilon}$.

The method used here did permit us to establish these results for arbitrary density, interaction strength and plasma expansion parameter. Except for the important role played by the invariance and conservation properties, our calculations only involve the long-wavelength limit $k \to 0$.

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APPENDIX A

Derivation of the kinetic equation (2.16). The algebraic steps leading from eq. (2.10) to (2.16) involve a calculation of $(\delta f(\boldsymbol{kp})|\, P\, |F(\boldsymbol{kzp}'))$, $(\delta f(\boldsymbol{kp})|\, PLP \times |F(\boldsymbol{kzp}'))$, $(\delta f(\boldsymbol{kp})|\, \psi\, |F(\boldsymbol{kzp}'))$, $(\delta f(\boldsymbol{kp})|\, P\, |\delta f(\boldsymbol{kp}'))$ and $(\delta f(\boldsymbol{kp})|\, DQ\, |\delta f(\boldsymbol{kp}'))$.

From the very definition of P in eq. (2.11) it follows that: $(\delta f(\boldsymbol{kp})|\, P\,|\delta f(\boldsymbol{kp}'))$ $= S_0(\boldsymbol{kpp}')$ (2.13), $(\delta f(\boldsymbol{kp})|\, P\,|F(\boldsymbol{k}z\boldsymbol{p}')) = -\mathrm{i}S(\boldsymbol{k}z\boldsymbol{pp}')$ and $(\delta f(\boldsymbol{kp})|\, DQ\,|\delta f(\boldsymbol{kp}'))$ $= 0$ because $Q\,|\delta f(\boldsymbol{kp}')) = 0$. As we also have $(\delta f(\boldsymbol{kp})|\, Q = 0$, we can put $(\delta f|\, PLP\,|F) = (\delta f|\, LP\,|F)$ and $(\delta f|\, \psi\,|F) = (\delta f|\, LQ\,(z - QLQ)^{-1}\, QLP\,|F)$. According to eq. (2.11) we write $(\delta f|\cdots P\,|F) = \sum_l \int \mathrm{d}\boldsymbol{p}_1\, \mathrm{d}\boldsymbol{p}_2\, (\delta f|\cdots|\delta f(\boldsymbol{lp}_1))\, \mathscr{P}(\boldsymbol{lp}_1\boldsymbol{p}_2)$ $\times\, (\delta f(\boldsymbol{lp}_2)\,|\,F)$. Because of translational invariance of the equilibrium averages all matrix elements of the form $(\delta f(\boldsymbol{kp})|\cdots|\delta f(\boldsymbol{lp}'))$ vanish except when $\boldsymbol{l} = \boldsymbol{k}$. Therefore we have $(\delta f(\boldsymbol{kp})|\cdots P\,|F(\boldsymbol{k}z\boldsymbol{p}')) = \int \mathrm{d}\boldsymbol{p}_1\, \mathrm{d}\boldsymbol{p}_2\, (\delta f(\boldsymbol{kp})|\cdots|\delta f(\boldsymbol{kp}_1))\, \mathscr{P}$ $\times\, (\boldsymbol{kp}_1\boldsymbol{p}_2)\,(-\mathrm{i})\, S(\boldsymbol{k}z\boldsymbol{p}_2\boldsymbol{p}')$ and we arrive at once at the general structure of eq. (2.16) with $\Sigma(\boldsymbol{k}z\boldsymbol{pp}_2) = \int \mathrm{d}\boldsymbol{p}_1\, (\delta f(\boldsymbol{kp})|\, (L + \psi)\,|\delta f(\boldsymbol{kp}_1))\, \mathscr{P}(\boldsymbol{kp}_1\boldsymbol{p}_2)$. The contribution of L to this Σ naturally splits into a free streaming term, Σ^0, and an average force or self-consistent field term Σ^s. Indeed if we decompose $L = L_0 + \delta L$ into a free particle (L_0) and interaction (δL) term, and observe that $(\delta f(\boldsymbol{kp})|$ is a left eigenfunction of L_0 with eigenvalue $\boldsymbol{k}\cdot\boldsymbol{p}/m$ we obtain in view of (2.12) immediately Σ^0 as given by (2.17a) for this contribution of L_0 to Σ. For the contribution of δL it is easily shown that $(\delta f(\boldsymbol{kp})|\, \delta L\,|\delta f(\boldsymbol{kp}_1)) = -\boldsymbol{k}\cdot\boldsymbol{v}n\varphi(\boldsymbol{p})\,\varphi(\boldsymbol{p}_1)$ $\times\, h(\boldsymbol{k})$ and its contribution to Σ will read then $-\boldsymbol{k}\cdot\boldsymbol{v}\,\varphi(\boldsymbol{p})\, h(\boldsymbol{k})\,[1 - c(\boldsymbol{k})]$ which on using (2.15) reduces to the Σ^s of (2.17b). Finally, for the contribution of ψ to Σ we see that the first term of $\mathscr{P}(\boldsymbol{kp}_1\boldsymbol{p}_2)$ [see eq. (2.14)] precisely leads to the collision term Σ^c (2.17c) whereas the second term of $\mathscr{P}$ (2.14) yields a contribution proportional to $QL\,|\delta n(\boldsymbol{k}))$ which vanishes because $L\,|\delta n)$ is, by virtue of the continuity equation, nothing but the longitudinal momentum density which lies entirely in the one-particle phase space to which the Q space is orthogonal by definition and hence $QL\,|\delta n) = 0$.

APPENDIX B

Consequences of the conservation laws. Consider the matrix elements of the collision operator (2.17c) between two arbitrary states g and h:

$$\langle h|\, \bar{\Sigma}^c(\boldsymbol{k}z)\,|g\rangle = \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{p}'\, h(\boldsymbol{p})\, (\delta\dot{f}(\boldsymbol{kp})|\, A(z)\,|\delta\dot{f}(\boldsymbol{kp}'))\, g^*(\boldsymbol{p}')$$

$$\equiv (\dot{H}(\boldsymbol{k})|\, A(z)\,|\dot{G}(\boldsymbol{k})), \tag{B.1}$$

where, *e.g.*, $\dot{H}(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{p}\, h(\boldsymbol{p})\, \dot{f}(\boldsymbol{kp})$, $\dot{f}(\boldsymbol{kp}) = \partial_t f(\boldsymbol{kp}t)|_{t=0}$ and $A(z) \equiv Q\,(z - QLQ)^1\, Q$. If we take $h(\boldsymbol{p})$ or $g(\boldsymbol{p})$ to correspond to a conserved quantity, the time derivative, say $\dot{H}(\boldsymbol{k})$, appearing in eq. (B.1) can be replaced by the divergence of a current according to the conservation law $\dot{H}(\boldsymbol{k}) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_H(\boldsymbol{k}) = 0$. Therefore a matrix element involving the number density $u_n(\boldsymbol{p}) = 1$ can be written $\langle u_n|\, \bar{\Sigma}^c\,|g\rangle$ $= (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_n(\boldsymbol{k})|\, A\,|\dot{G}(\boldsymbol{k}))$ because of the number conservation law $\dot{n}(\boldsymbol{k}) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_n(\boldsymbol{k})$ $= 0$. Moreover, $\boldsymbol{I}_n$ is a one-particle expression which is orthogonal to Q space and hence $\langle u_n| \bar{\Sigma}^c \sim (I_n| A \sim (I_n| Q = 0$. For a momentum state we obtain $\langle u_l|\bar{\Sigma}^c|h\rangle = (-\mathrm{i}\boldsymbol{k}\cdot I_l \,|\, A\,|\dot{H})$ and $\langle u_{t_i}| \bar{\Sigma}^c\,|h\rangle = (-\mathrm{i}\boldsymbol{k}\cdot I_{t_i}|\, A\,|\dot{H})$ where $I_l = \boldsymbol{\pi}\cdot\hat{\boldsymbol{k}}$ and $I_{t_i} = \boldsymbol{\pi}\cdot\boldsymbol{\varepsilon}_i$ with $\boldsymbol{\pi}$ being the microscopic stress tensor, a fact which has been used in the main text.

For the kinetic-energy state u_ε the situation is somewhat different because only the total energy is conserved. We can however always write $\langle u_\varepsilon|\, \bar{\Sigma}^c\,|g\rangle = (\dot{E}_K|\,A\,|\dot{G}) = -(\dot{E}_P + \mathrm{i}\boldsymbol{k}\cdot \boldsymbol{I}_E|\, A\,|\dot{G})$ by invoking the energy conservation law $\dot{E}_K + \dot{E}_P + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_E = 0$ where $\boldsymbol{I}_E$ is the microscopic energy current, E_K the kinetic-energy density and $E_P(\boldsymbol{k}) = \frac{1}{2}\sum_{i,j} V_{ij}\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_j)$ the potential-energy density due to the interaction potential $V_{ij} = V_{ij}(|\boldsymbol{x}_i - \boldsymbol{x}_j|)$. We thus have: $\langle u_\varepsilon|\, \bar{\Sigma}^c\,|g\rangle = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_E| \times A\,|\dot{G}) - \mathrm{i}\,(E_P|\,(Q+P)\,LA\,|\dot{G})$ or because of the identity $QLA = zA + Q$, $\langle u_\varepsilon|\,\bar{\Sigma}^c\,|g\rangle = (-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{I}_E|\,A\,|\dot{G}) + \mathrm{i}z\,(E_P|\,A\,|\dot{G}) - \mathrm{i}\,(E_P|\,Q + PLA\,|\dot{G})$. As $(E_P|\,PLA \times |\dot{G})$ contains $(\delta n|\,LQ$ it vanishes (see appendix A) and similarly $(E_P|\,Q\,|\dot{G})$ vanishes identically because of (2.15) so that finally we remain with

$$\langle u_\varepsilon|\,\bar{\Sigma}^c\,|g\rangle = -\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{I}_E|\,A\,|\dot{G}) - \mathrm{i}z\,(E_P|\,A\,|\dot{G}). \tag{B.2}$$

Eq. (B.2) shows explicitly that the kinetic energy u_ε becomes a collisional invariant only in the asymptotic limit of vanishing $\boldsymbol{k}$ and z.

The above considerations do have some important implications for the Ω_{ij} matrix elements. For example, momentum conservation leads now immediately to the form $\Omega_t(\boldsymbol{k}z) = -\mathrm{i}k^2 D_t(\boldsymbol{k}z)$ and $\Omega_{ll}(\boldsymbol{k}z) = -\mathrm{i}k^2 D_l(\boldsymbol{k}z)$ (the i factors have been introduced for later convenience). The energy conservation in turn leads to $\Omega_{\varepsilon\varepsilon}(\boldsymbol{k}z) = -\mathrm{i}k^2 D_\varepsilon(\boldsymbol{k}z) + zB_\varepsilon(\boldsymbol{k}z)$ whereas we also have $\Omega_{l\varepsilon}(\boldsymbol{k}z) = \Omega_{\varepsilon l}(\boldsymbol{k}z) = kD_{l\varepsilon}(\boldsymbol{k}z)$.

Finally, the invariance properties of the equilibrium distribution imply that $\Omega_{ij}(\boldsymbol{k}z) = -\Omega_{ij}(-\boldsymbol{k}, -z) = \Omega^*_{ij}(\boldsymbol{k}z^*)$ as is easily checked from the explicit expressions given in (2.17). These properties in turn imply that the even (odd) part in $\boldsymbol{k}$ of $\Omega_{ij}(\boldsymbol{k}z)$ is odd (even) in z.

Gathering these results it is easily established[3]) that D_t, D_l, D_ε, $D_{l\varepsilon}$ and B_ε have finite real limits as $k \to 0$ and $z \to +\mathrm{i}0$.

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