

# **Longitudinal Aircraft Dynamics and the Instantaneous Acceleration Center of Rotation**

## **The case of the vanishing zeros**

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Nonminimum-phase zeros, that is, closed-right-half-plane (CRHP) zeros, affect both the open-loop and closed-loop behavior of continuous-time linear systems in undesirable ways [1]. For example, an asymptotically stable linear system with an odd number of positive zeros experiences initial undershoot to a step input (see “Initial Undershoot”). Moreover, under the rules of root locus, zeros in the open-right-half-plane (ORHP) attract closed-loop poles, which limits the controller gain and thus the performance of the closed-loop system. In LQG theory, closed-loop poles are attracted to the reflected locations of the open-loop ORHP zeros in the high-control-authority (that is, cheap-control) limit, thus constraining the achievable closed-loop bandwidth [2, p. 289].

Given the critical role of nonminimum-phase zeros, it is useful to identify physical characteristics that give rise to them. Although spatial separation between sensors and actuators is often postulated as a source of nonminimum-phase zeros, analysis of the transfer functions between separated masses in a serially connected structure shows that this is not necessarily the case [3]. On the other hand, noncollocation in rotational motion typically gives rise to nonminimum-phase zeros [4].

Aside from zero locations, the number of zeros determines the relative degree of the system, which impacts the asymptotic, that is, high-frequency, phase of the transfer function. The relative degree of an asymptotically stable transfer function also plays a role in the initial slope of the step response. This relationship is apparent from the initial value theorem applied to the derivative of the output. When the initial slope of the output is zero, higher order derivatives of the initial response, which determine the initial curvature of the output, can be evaluated to detect the possibility of initial undershoot. The sign of the first nonzero derivative of the output relative to the sign of the dc gain determines whether or not the step response exhibits initial undershoot. The number of derivatives that must be evaluated in order to determine the sign of the first nonzero derivative is equal to the relative degree of the system.

In aircraft dynamics, the IACR of an aircraft is the point on the aircraft that has zero instantaneous acceleration. For an aircraft that is perturbed from steady horizontal flight by an elevator step deflection, the IACR is the point at which the elevator-to-vertical-velocity transfer function and the elevator-to-horizontal-velocity transfer function each have a zero that vanishes. These vanishing zeros play an important role in the aircraft's instantaneous response. As shown in [5, p. 314], the IACR for an F-16 is about 6 feet forward of the center of mass. As an accelerometer is moved forward from the tail to the IACR, a real nonminimum-phase zero moves toward  $\infty$ , where it vanishes. As the accelerometer moves past the IACR, the zero "reappears" at  $-\infty$  and moves toward zero as a minimum-phase zero. Thus, an accelerometer measurement at each point along the aircraft between the tail and the IACR exhibits initial undershoot. This phenomenon plays a role in the literature on aircraft dynamics and control [5]-[14].

In the present article, we demonstrate the relationship between vanishing zeros and the response of the aircraft at the IACR. The IACR of a rigid body is related to, but distinct from, the center of rotation. See "Center of Rotation and Center of Percussion", which discusses the motion of a bar-like rigid body in response to an impact. A bar-like rigid body possesses a point, called the center of percussion, with the property that an impulsive force at this location leads to identically zero translational velocity at another point on the body, called the center of rotation. Another related notion is the instantaneous velocity center of rotation (IVCR), which is discussed in "Instantaneous Velocity Center of Rotation."

The goal of this article is to explain and demonstrate the relationship between vanishing zeros and the response of the aircraft at its IACR. To demonstrate this relationship, we consider both the vertical velocity response and the horizontal velocity response of the aircraft to a step elevator deflection. In particular, we show that, at the IACR, the relative degree of the linearized transfer function from elevator deflection to vertical velocity (and thus to altitude) increases by at least 1, and the relative degree of the linearized transfer function from elevator deflection to horizontal velocity increases by at least 1. Moreover, we provide conditions under which the zeros that vanish at the IACR are nonminimum phase. Furthermore, we characterize the relationship between these vanishing zeros and the potential for initial undershoot in the aircraft's step response. For a business jet example, we show that each point on the aircraft that is aft of the IACR experiences initial undershoot in both vertical and horizontal velocity, whereas each point forward of the IACR does not experience initial velocity undershoot in either the vertical or horizontal directions.

To provide a reasonably self-contained development of the relevant transfer functions, we begin with the nonlinear equations of motion, show how these equations incorporate aerodynamic effects in terms of stability derivatives, and then arrive at the transfer functions for the linearized motion. This development provides an introduction to aircraft dynamics that may be useful to readers who have not had the benefit of a course on flight dynamics. For further details on aircraft dynamics, see [5, 16, 17].

## **Aircraft Kinematics**

The Earth frame  $F_E$ , whose orthogonal axes are labeled  $\hat{i}_E$ ,  $\hat{j}_E$ , and  $\hat{k}_E$ , is assumed to be an inertial frame, that is, a frame with respect to which Newton's second law is valid [18]. A hat denotes a dimensionless unit-length physical vector. The origin  $O_E$  of the Earth frame is any convenient point on the Earth. The axes  $\hat{i}_E$  and  $\hat{j}_E$  are horizontal, while the axis  $\hat{k}_E$  points downward; we assume the Earth is flat. The aircraft frame  $F_{AC}$ , whose axes are labeled  $\hat{i}_{AC}$ ,  $\hat{j}_{AC}$ , is fixed to the aircraft. The center of mass and frame vectors  $\hat{i}_{AC}$  and  $\hat{k}_{AC}$  are shown in Figure 1. The aircraft is assumed to be a three-dimensional rigid body.

In longitudinal flight, the aircraft moves in an inertially nonrotating vertical plane by translating along  $\hat{i}_{AC}$  and  $\hat{k}_{AC}$  and by rotating about  $\hat{j}_{AC}$ . The direction of  $\hat{j}_{AC}$  is thus fixed with respect to  $F_E$ . For

convenience, we assume that  $\hat{j}_{AC} = \hat{j}_E$ . The velocity and acceleration of the aircraft along  $\hat{j}_{AC}$  are thus identically zero for longitudinal flight, as are the roll and yaw components of the angular velocity of the aircraft relative to the Earth frame. The sign of the pitch angle  $\Theta$ , which is the angle from  $\hat{i}_E$  to  $\hat{i}_{AC}$ , is determined by the right hand rule with the thumb pointing along  $\hat{j}_{AC}$  and with the fingers curled around  $\hat{j}_{AC}$ . For example, the pitch angle  $\Theta$ , shown in Figure 1, is negative.

Let  $p$  denote a point in the  $\hat{i}_{AC}$ - $\hat{k}_{AC}$  plane. The position of  $p$  relative to  $O_E$  can be written as

$$\vec{r}_{p/O_E} = r_{ph}\hat{i}_E + r_{pv}\hat{k}_E, \quad (1)$$

where a harpoon denotes a physical vector. The position of  $p$  relative to  $c$  is given by

$$\vec{r}_{p/c} = \vec{r}_{p/O_{AC}} + \vec{r}_{O_{AC}/c} = \vec{r}_{p/O_{AC}} - \vec{r}_{c/O_{AC}}, \quad (2)$$

which can be written as

$$\vec{r}_{p/c} = \ell\hat{i}_{AC} + \eta\hat{k}_{AC},$$

where  $\ell > 0$  indicates that  $p$  is forward of  $c$ , that is, toward the nose, and  $\ell < 0$  denotes that  $p$  is aft of  $c$ , that is, toward the tail. Resolving  $\vec{r}_{p/c}$  in  $F_{AC}$  yields

$$\vec{r}_{p/c}|_{AC} = \begin{bmatrix} \ell \\ 0 \\ \eta \end{bmatrix}. \quad (3)$$

The distance between the aircraft center of mass  $c$  and the point  $p$  is given by

$$|\vec{r}_{p/c}| = \sqrt{\ell^2 + \eta^2}.$$

The orientation matrix, that is, the direction cosine matrix, of  $F_{AC}$  relative to  $F_E$  corresponding to the pitch angle  $\Theta$  is

$$\mathcal{O}_{AC/E} \triangleq \begin{bmatrix} \cos \Theta & 0 & -\sin \Theta \\ 0 & 1 & 0 \\ \sin \Theta & 0 & \cos \Theta \end{bmatrix}.$$

Therefore,

$$\mathcal{O}_{E/AC} = \mathcal{O}_{AC/E}^T = \begin{bmatrix} \cos \Theta & 0 & \sin \Theta \\ 0 & 1 & 0 \\ -\sin \Theta & 0 & \cos \Theta \end{bmatrix}. \quad (4)$$

Hence, using (3) we have

$$\begin{aligned} \vec{r}_{P/c}|_E &= \mathcal{O}_{E/AC} \vec{r}_{P/c}|_{AC} \\ &= \begin{bmatrix} \ell \cos \Theta + \eta \sin \Theta \\ 0 \\ -\ell \sin \Theta + \eta \cos \Theta \end{bmatrix}. \end{aligned} \quad (5)$$

Since, in longitudinal flight, the aircraft rotates about  $\hat{j}_{AC}$ , the angular velocity of  $F_{AC}$  relative to  $F_E$  and resolved in  $F_{AC}$  is given by

$$\vec{\omega}_{AC/E}|_{AC} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix}. \quad (6)$$

Note that  $Q = \dot{\Theta}$  and that  $P$  and  $R$  are identically zero. Resolving  $\vec{\omega}_{AC/E}$  in  $F_E$ , we have

$$\vec{\omega}_{AC/E}|_E = \mathcal{O}_{E/AC} \vec{\omega}_{AC/E}|_{AC} = \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix}. \quad (7)$$

In order to change the frame with respect to which the physical vector  $\vec{x}$  is differentiated, we use the transport theorem, which is given by

$$\frac{A\bullet}{\vec{x}} = \frac{B\bullet}{\vec{x}} + \vec{\omega}_{B/A} \times \vec{x}, \quad (8)$$

where a labeled dot over a physical vector denotes the frame derivative with respect to the indicated frame.

In particular, if  $\vec{x} = x_1 \hat{i}_A + x_2 \hat{j}_A + x_3 \hat{k}_A$ , then  $\frac{A\bullet}{\vec{x}} = \dot{x}_1 \hat{i}_A + \dot{x}_2 \hat{j}_A + \dot{x}_3 \hat{k}_A$ . Hence,

$$\frac{E\bullet}{\vec{\omega}_{AC/E}} = \frac{AC\bullet}{\vec{\omega}_{AC/E}} + \vec{\omega}_{AC/E} \times \vec{\omega}_{AC/E} = \frac{AC\bullet}{\vec{\omega}_{AC/E}}, \quad (9)$$

and thus it follows from (6), (7), and (9) that

$$\left. \begin{matrix} \overrightarrow{\omega}^{AC\bullet} \\ \overrightarrow{\omega}^{AC/E} \end{matrix} \right|_{AC} = \left. \begin{matrix} \overrightarrow{\omega}^{E\bullet} \\ \overrightarrow{\omega}^{AC/E} \end{matrix} \right|_E = \left. \begin{matrix} \overrightarrow{\omega}^{AC\bullet} \\ \overrightarrow{\omega}^{AC/E} \end{matrix} \right|_E = \left. \begin{matrix} \overrightarrow{\omega}^{E\bullet} \\ \overrightarrow{\omega}^{AC/E} \end{matrix} \right|_{AC} = \begin{bmatrix} 0 \\ \ddot{\Theta} \\ 0 \end{bmatrix}.$$

Let  $\overrightarrow{v}_{c/O_E/E}$  and  $\overrightarrow{a}_{c/O_E/E}$  denote the velocity and acceleration of  $c$  relative to  $O_E$  with respect to  $F_E$ , respectively, and let  $\overrightarrow{v}_{p/O_E/E}$  and  $\overrightarrow{a}_{p/O_E/E}$  denote the velocity and acceleration of  $p$  relative to  $O_E$  with respect to  $F_E$ , respectively, that is,

$$\begin{aligned} \overrightarrow{v}_{c/O_E/E} &\triangleq \overrightarrow{r}_{c/O_E}^{E\bullet}, \\ \overrightarrow{a}_{c/O_E/E} &\triangleq \overrightarrow{r}_{c/O_E}^{E\bullet\bullet}, \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{v}_{p/O_E/E} &\triangleq \overrightarrow{r}_{p/O_E}^{E\bullet}, \\ \overrightarrow{a}_{p/O_E/E} &\triangleq \overrightarrow{r}_{p/O_E}^{E\bullet\bullet}. \end{aligned}$$

We resolve  $\overrightarrow{v}_{c/O_E/E}$  in  $F_{AC}$  as

$$\left. \overrightarrow{v}_{c/O_E/E} \right|_{AC} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} U \\ 0 \\ W \end{bmatrix}, \quad (10)$$

and note that  $V$  is identically zero for longitudinal flight.

Next, it follows from (2) that

$$\overrightarrow{r}_{p/O_E} = \overrightarrow{r}_{p/c} + \overrightarrow{r}_{c/O_E},$$

which implies that

$$\overrightarrow{v}_{p/O_E/E} = \overrightarrow{r}_{p/O_E}^{E\bullet} = \overrightarrow{r}_{p/c}^{E\bullet} + \overrightarrow{r}_{c/O_E}^{E\bullet} = \overrightarrow{v}_{p/c/E} + \overrightarrow{v}_{c/O_E/E}, \quad (11)$$

where

$$\vec{v}_{p/c/E} \stackrel{\Delta}{=} \vec{r}_{p/c}^{\bullet E} = \vec{\omega}_{AC/E} \times \vec{r}_{p/c}. \quad (12)$$

Next, it follows from (4)–(7) and (10)–(12) that

$$\begin{aligned} \vec{v}_{p/O_E/E} \Big|_E &= \vec{v}_{c/O_E/E} \Big|_E + \left( \vec{\omega}_{AC/E} \times \vec{r}_{p/c} \right) \Big|_E \\ &= \begin{bmatrix} \cos \Theta & 0 & \sin \Theta \\ 0 & 1 & 0 \\ -\sin \Theta & 0 & \cos \Theta \end{bmatrix} \begin{bmatrix} U \\ 0 \\ W \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \ell \cos \Theta + \eta \sin \Theta \\ 0 \\ -\ell \sin \Theta + \eta \cos \Theta \end{bmatrix} \\ &= \begin{bmatrix} v_{ph} \\ 0 \\ v_{pv} \end{bmatrix}, \end{aligned}$$

where

$$v_{ph} \stackrel{\Delta}{=} (\cos \Theta)U + (\sin \Theta)W - \ell(\sin \Theta)\dot{\Theta} + \eta(\cos \Theta)\dot{\Theta}, \quad (13)$$

$$v_{pv} \stackrel{\Delta}{=} -(\sin \Theta)U + (\cos \Theta)W - \ell(\cos \Theta)\dot{\Theta} - \eta(\sin \Theta)\dot{\Theta}. \quad (14)$$

Next, it follows from (8) and (10) that

$$\begin{aligned} \vec{a}_{c/O_E/E} \Big|_{AC} &= \vec{v}_{c/O_E/E}^{\bullet AC} \Big|_{AC} \\ &= \left( \vec{v}_{c/O_E/E}^{\bullet AC} + \vec{\omega}_{AC/E} \times \vec{v}_{c/O_E/E} \right) \Big|_{AC} \\ &= \begin{bmatrix} \dot{U} \\ 0 \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} U \\ 0 \\ W \end{bmatrix} \\ &= \begin{bmatrix} \dot{U} + \dot{\Theta}W \\ 0 \\ \dot{W} - \dot{\Theta}U \end{bmatrix}. \end{aligned} \quad (15)$$

Differentiating the transport theorem (8) yields

$$\begin{aligned}
\overset{A\bullet\bullet}{\vec{x}} &= \overset{A\bullet}{\vec{x}} + \overset{A\bullet}{\vec{\omega}}_{B/A} \times \overset{A\bullet}{\vec{x}} + \overset{A\bullet}{\vec{\omega}}_{B/A} \times \overset{A\bullet}{\vec{x}} \\
&= \overset{B\bullet\bullet}{\vec{x}} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{x}} + \overset{A\bullet}{\vec{\omega}}_{B/A} \times \overset{A\bullet}{\vec{x}} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \left( \overset{B\bullet}{\vec{x}} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{x}} \right) \\
&= \overset{B\bullet\bullet}{\vec{x}} + 2\overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{x}} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{x}} + \overset{A\bullet}{\vec{\omega}}_{B/A} \times \left( \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{x}} \right), \tag{16}
\end{aligned}$$

which is the double transport theorem. Note that

$$\overset{E\bullet\bullet}{\vec{a}}_{p/O_E/E} \triangleq \overset{E\bullet\bullet}{\vec{r}}_{p/O_E} = \overset{E\bullet\bullet}{\vec{r}}_{p/c} + \overset{E\bullet\bullet}{\vec{r}}_{c/O_E} = \overset{E\bullet\bullet}{\vec{a}}_{p/c/E} + \overset{E\bullet\bullet}{\vec{a}}_{c/O_E/E}, \tag{17}$$

where

$$\overset{E\bullet\bullet}{\vec{a}}_{p/c/E} \triangleq \overset{E\bullet\bullet}{\vec{r}}_{p/c}. \tag{18}$$

Now, using (15)–(18), we have

$$\begin{aligned}
\overset{E\bullet\bullet}{\vec{a}}_{p/O_E/E} \Big|_{AC} &= \overset{E\bullet\bullet}{\vec{a}}_{p/c/E} \Big|_{AC} + \overset{E\bullet\bullet}{\vec{a}}_{c/O_E/E} \Big|_{AC} \\
&= \left( \overset{AC\bullet\bullet}{\vec{r}}_{p/c} + 2\overset{AC\bullet}{\vec{\omega}}_{AC/E} \times \overset{AC\bullet}{\vec{r}}_{p/c} + \overset{AC\bullet}{\vec{\omega}}_{AC/E} \times \overset{AC\bullet}{\vec{r}}_{p/c} + \overset{AC\bullet}{\vec{\omega}}_{AC/E} \times \left( \overset{AC\bullet}{\vec{\omega}}_{AC/E} \times \overset{AC\bullet}{\vec{r}}_{p/c} \right) \right) \Big|_{AC} \\
&\quad + \overset{E\bullet\bullet}{\vec{a}}_{c/O_E/E} \Big|_{AC} \\
&= \overset{AC\bullet}{\vec{\omega}}_{AC/E} \Big|_{AC} \times \overset{AC\bullet}{\vec{r}}_{p/c} \Big|_{AC} + \overset{AC\bullet}{\vec{\omega}}_{AC/E} \Big|_{AC} \times \left( \overset{AC\bullet}{\vec{\omega}}_{AC/E} \Big|_{AC} \times \overset{AC\bullet}{\vec{r}}_{p/c} \Big|_{AC} \right) + \overset{E\bullet\bullet}{\vec{a}}_{c/O_E/E} \Big|_{AC} \\
&= \begin{bmatrix} 0 \\ \ddot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \ell \\ 0 \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \left( \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \ell \\ 0 \\ \eta \end{bmatrix} \right) + \begin{bmatrix} \dot{U} + \dot{\Theta}W \\ 0 \\ \dot{W} - \dot{\Theta}U \end{bmatrix} \\
&= \begin{bmatrix} -\ell\dot{\Theta}^2 + \dot{U} + W\dot{\Theta} + \eta\ddot{\Theta} \\ 0 \\ -\ell\ddot{\Theta} + \dot{W} - U\dot{\Theta} - \eta\dot{\Theta}^2 \end{bmatrix}. \tag{19}
\end{aligned}$$



## Aircraft Dynamics

To apply Newton's second law for translational acceleration, we view  $O_E$  as an unforced particle [18] and all forces as acting at the aircraft's center of mass. We thus have

$$m \vec{a}_{c/O_E/E} = m \vec{g} + \vec{F}_A + \vec{F}_T, \quad (20)$$

where  $m$  is the mass of the aircraft,  $\vec{g} = g \hat{k}_E$  is the acceleration due to gravity,  $\vec{F}_A$  is the aerodynamic force, and  $\vec{F}_T$  is the engine thrust force. Resolving (20) in  $F_{AC}$  yields

$$m \vec{a}_{c/O_E/E} \Big|_{AC} = m \vec{g} \Big|_{AC} + \vec{F}_A \Big|_{AC} + \vec{F}_T \Big|_{AC}, \quad (21)$$

where

$$\vec{g} \Big|_{AC} = \mathcal{O}_{AC/E} \vec{g} \Big|_E = \begin{bmatrix} -g \sin \Theta \\ 0 \\ g \cos \Theta \end{bmatrix}, \quad (22)$$

under longitudinal flight.

Next, the aerodynamic force  $\vec{F}_A$  is given by

$$\vec{F}_A = -D \hat{i}_W - D_s \hat{j}_W - L \hat{k}_W,$$

where  $\hat{i}_W$ ,  $\hat{j}_W$ , and  $\hat{k}_W$  are the axes of the wind frame, which is a velocity-dependent frame defined such that  $\hat{i}_W$  is aligned with  $\vec{v}_{c/O_E/E}$ , and where  $D$ ,  $D_s$ , and  $L$  denote the magnitudes of the drag, side drag, and lift forces, respectively. For simplicity, we assume  $D_s = 0$ , and thus

$$\vec{F}_A \Big|_W = \begin{bmatrix} -D \\ 0 \\ -L \end{bmatrix}.$$

The stability frame  $F_S$  with axes  $\hat{i}_S$ ,  $\hat{j}_S$ , and  $\hat{k}_S$  is obtained by rotating the wind frame through the sideslip

angle  $\beta$ , which is the angle from the  $\hat{i}_{AC}$ - $\hat{k}_{AC}$  plane to  $\vec{v}_{c/O_E/E}$ . Resolving  $\vec{F}_A$  in the stability frame yields

$$\begin{aligned}\vec{F}_A \Big|_S &= \begin{bmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -D \\ 0 \\ -L \end{bmatrix} \\ &= \begin{bmatrix} -D \cos \beta \\ -D \sin \beta \\ -L \end{bmatrix}.\end{aligned}$$

Furthermore, resolving  $\vec{F}_A$  in the aircraft frame yields

$$\begin{aligned}\vec{F}_A \Big|_{AC} &= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} -D \cos \beta \\ -D \sin \beta \\ -L \end{bmatrix} \\ &= \begin{bmatrix} -D(\cos \beta) \cos \alpha + L \sin \alpha \\ -D \sin \beta \\ -D(\cos \beta) \sin \alpha - L \cos \alpha \end{bmatrix},\end{aligned}$$

where  $\alpha$  is the angle of attack of the aircraft, that is, the angle from  $\hat{i}_S$  to  $\hat{i}_{AC}$ . Since we consider only longitudinal flight, it follows that  $\beta$  is identically zero, and thus

$$\vec{F}_A \Big|_{AC} = \begin{bmatrix} -D \cos \alpha + L \sin \alpha \\ 0 \\ -D \sin \alpha - L \cos \alpha \end{bmatrix}. \quad (23)$$

For the thrust force, we have

$$\begin{aligned}\vec{F}_T \Big|_{AC} &= \begin{bmatrix} \cos \Phi_T & 0 & \sin \Phi_T \\ 0 & 1 & 0 \\ -\sin \Phi_T & 0 & \cos \Phi_T \end{bmatrix} \begin{bmatrix} F_T \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} F_T \cos \Phi_T \\ 0 \\ -F_T \sin \Phi_T \end{bmatrix},\end{aligned} \quad (24)$$

where  $F_T \triangleq |\vec{F}_T|$  is the engine force magnitude and  $\Phi_T$  is the angle from  $\hat{i}_{AC}$  to the engine force direction. We assume that the component of the engine thrust in the direction  $\hat{j}_{AC}$  is zero.

Now, substituting (15), (22), (23), and (24) into (21) yields the surge and plunge equations

$$m(\dot{U} + W\dot{\Theta}) = -mg \sin \Theta - D \cos \alpha + L \sin \alpha + F_T \cos \Phi_T, \quad (25)$$

$$m(\dot{W} - U\dot{\Theta}) = mg \cos \Theta - D \sin \alpha - L \cos \alpha - F_T \sin \Phi_T. \quad (26)$$

The sway equation for  $\dot{V}$  plays no role in longitudinal flight.

Note that the differential equations (25) and (26) involve the variables  $U$ ,  $W$ ,  $\Theta$ , and  $\alpha$ . To eliminate  $W$  from (25) and (26), we derive a relationship among  $W$ ,  $U$ , and  $\alpha$ . Resolving  $\vec{v}_{c/O_E/E}$  in  $F_S$  yields

$$\vec{v}_{c/O_E/E}|_S = \begin{bmatrix} \bar{U} \\ 0 \\ 0 \end{bmatrix},$$

where  $\bar{U} \triangleq \sqrt{U^2 + W^2}$ . Likewise, resolving  $\vec{v}_{c/O_E/E}$  in  $F_{AC}$  yields

$$\begin{aligned} \vec{v}_{c/O_E/E}|_{AC} &= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} \bar{U} \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \bar{U} \cos \alpha \\ 0 \\ \bar{U} \sin \alpha \end{bmatrix}. \end{aligned} \quad (27)$$

It follows from (10) and (27) that

$$\begin{bmatrix} U \\ 0 \\ W \end{bmatrix} = \begin{bmatrix} \bar{U} \cos \alpha \\ 0 \\ \bar{U} \sin \alpha \end{bmatrix}.$$

Hence,

$$\frac{W}{U} = \tan \alpha. \quad (28)$$

For longitudinal flight,  $U$  is nonzero. Thus, it follows from (28) that

$$W = U \tan \alpha, \quad (29)$$

which implies

$$\dot{W} = \dot{U} \tan \alpha + U(\sec^2 \alpha)\dot{\alpha}. \quad (30)$$

Finally, substituting (29) and (30) into (25) and (26) yields

$$m(\dot{U} + U(\tan \alpha)\dot{\Theta}) = -mg \sin \Theta - D \cos \alpha + L \sin \alpha + F_T \cos \Phi_T, \quad (31)$$

$$m(\dot{U} \tan \alpha + U(\sec^2 \alpha)\dot{\alpha} - U\dot{\Theta}) = mg \cos \Theta - D \sin \alpha - L \cos \alpha - F_T \sin \Phi_T. \quad (32)$$

Next, the rotational momentum equation for the aircraft about its center of mass is given by Euler's equation

$$\vec{I}_{AC/c} \overset{AC\bullet}{\vec{\omega}}_{AC/E} + \vec{\omega}_{AC/E} \times \vec{I}_{AC/c} \vec{\omega}_{AC/E} = \vec{M}_{AC/c}, \quad (33)$$

where the physical inertia matrix is defined by

$$\vec{I}_{AC/c} \triangleq \int_{AC} |\vec{r}_{dm/c}|^2 \vec{U} - \vec{r}_{dm/c} \vec{r}'_{dm/c} dm, \quad (34)$$

$\vec{r}_{dm/c}$  is the position of a mass element relative to  $c$ ,  $(\cdot)'$  denotes a physical covector [6, p. 269], and the physical identity matrix  $\vec{U}$  is defined by

$$\vec{U} \triangleq \hat{i}_{AC} \hat{i}'_{AC} + \hat{j}_{AC} \hat{j}'_{AC} + \hat{k}_{AC} \hat{k}'_{AC}. \quad (35)$$

Note that the integral in (34) is evaluated over the aircraft body. In (34) and (35),  $(\cdot)^\times$  denotes the cross-product operator, and the notation  $\vec{x} \vec{y}'$  for vectors  $\vec{x}$  and  $\vec{y}$  denotes a second-order tensor, which operates on a vector  $\vec{z}$  according to  $(\vec{x} \vec{y}') \vec{z} = \vec{x} \vec{y}' \vec{z} = (\vec{y} \cdot \vec{z}) \vec{x}$ . Finally,  $\vec{M}_{AC/c}$  denotes the total thrust and aerodynamic moment acting on the aircraft relative to  $c$ .

Next, resolving  $\vec{I}_{AC/c}$  in  $F_{AC}$  yields

$$\vec{I}_{AC/c} \Big|_{AC} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}, \quad (36)$$

where

$$I_{xx} = \int_{AC} (y^2 + z^2) dm,$$

$$I_{xy} = \int_{AC} xy dm,$$

and likewise for the remaining entries. Assuming that  $\hat{i}_{AC}-\hat{k}_{AC}$  is a plane of symmetry of the aircraft, it follows that

$$I_{xy} = I_{yz} = 0.$$

Thus, (36) becomes

$$\vec{I}_{AC/c} \Big|_{AC} = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}.$$

Now resolving Euler's equation (33) in the aircraft frame, that is,

$$\left( \vec{I}_{AC/c} \frac{AC \bullet}{\vec{\omega}_{AC}} \right) \Big|_{AC} + \left( \vec{\omega}_{AC/E} \times \vec{I}_{AC/c} \vec{\omega}_{AC/E} \right) \Big|_{AC} = \vec{M}_{AC/c} \Big|_{AC},$$

yields

$$\begin{bmatrix} 0 \\ I_{yy} \ddot{\Theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dot{\Theta} \\ 0 & 0 & 0 \\ -\dot{\Theta} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{yy} \dot{\Theta} \\ 0 \end{bmatrix} = \begin{bmatrix} L_{AC} \\ M_{AC} \\ N_{AC} \end{bmatrix},$$

where  $\vec{M}_{AC/c} \Big|_{AC} \triangleq \begin{bmatrix} L_{AC} & M_{AC} & N_{AC} \end{bmatrix}^T$ . The pitch equation is thus given by

$$I_{yy} \ddot{\Theta} = M_{AC}. \quad (37)$$

## Linearizing the Equations of Motion

In steady horizontal longitudinal flight, the aircraft is assumed to fly at constant velocity  $U = U_0$ , constant angle of attack  $\alpha = \alpha_0$ , and constant pitch angle  $\Theta = \Theta_0$ , with  $\vec{v}_{c/O_E/E}$  aligned with  $\hat{i}_E$ . To simplify the aerodynamic analysis, we choose  $F_{AC}$  so that  $\Theta_0 = 0$ . This choice is universally made in the literature; see, for example, [17, p. 67]. Since the steady flight-path angle is zero, this choice of  $F_{AC}$  implies that the steady angle of attack  $\alpha_0$  is zero. Linearizing the surge, plunge, and pitch equations (31), (32), and (37) about  $(U_0, \alpha_0, \Theta_0)$  using the first-order approximations  $U \approx U_0 + u$ ,  $\alpha \approx \alpha_0 + \delta\alpha$ , and  $\Theta \approx \Theta_0 + \theta$ , where  $\alpha_0 = \Theta_0 = 0$ , and dividing the linearized equations by the mass  $m$  and inertia  $I_{yy}$  to solve for the linear and angular acceleration, yields

$$\dot{u} = -g\theta + f_{A_x} + f_{T_x}, \quad (38)$$

$$U_0\delta\dot{\alpha} = U_0q + f_{A_z}, \quad (39)$$

$$\dot{q} = m_{AC}, \quad (40)$$

$$\dot{\theta} = q, \quad (41)$$

where

$$\begin{aligned} f_{A_x} &\triangleq X_{u_0}u + X_{\alpha_0}\delta\alpha + X_{\delta e_0}\delta e, \\ f_{T_x} &\triangleq X_{T_{u_0}}u, \\ f_{A_z} &\triangleq Z_{u_0}u + Z_{\alpha_0}\delta\alpha + Z_{\dot{\alpha}_0}\delta\dot{\alpha} + Z_{q_0}q + Z_{\delta e_0}\delta e, \\ m_{AC} &\triangleq M_{u_0}u + M_{\alpha_0}\delta\alpha + M_{\dot{\alpha}_0}\delta\dot{\alpha} + M_{q_0}q + M_{\delta e_0}\delta e + M_{T_{u_0}}u + M_{T_{\alpha_0}}\delta\alpha, \end{aligned}$$

and  $\delta e$  denotes the elevator perturbation from its trim deflection. Note that  $f_{A_x}$  and  $f_{A_z}$  are the perturbations of  $\vec{F}_A$  in the direction of  $\hat{i}_{AC}$  and  $\hat{k}_{AC}$ , respectively. Furthermore,  $f_{T_x}$  is the perturbation of  $\vec{F}_T$  in the direction of  $\hat{i}_{AC}$ , and  $m_{AC}$  is the perturbation of  $M_{AC}$ . The stability parameters  $X_{u_0}$ ,  $X_{\alpha_0}$ ,  $X_{\delta e_0}$ ,  $X_{T_{u_0}}$ ,  $Z_{u_0}$ ,  $Z_{\alpha_0}$ ,  $Z_{\dot{\alpha}_0}$ ,  $Z_{q_0}$ ,  $Z_{\delta e_0}$ ,  $M_{u_0}$ ,  $M_{\alpha_0}$ ,  $M_{\dot{\alpha}_0}$ ,  $M_{q_0}$ ,  $M_{\delta e_0}$ ,  $M_{T_{u_0}}$ , and  $M_{T_{\alpha_0}}$  are combinations of aerodynamic parameters and stability derivatives, which are defined in Table 1 and Table 2, respectively. The stability parameters are defined in Table 3.

It follows from (38)–(41) that the linearized surge, plunge, and pitch-rate equations are given by

$$\dot{u} = (X_{u_0} + X_{T_{u_0}})u + X_{\alpha_0}\delta\alpha - g\theta + X_{\delta e_0}\delta e, \quad (42)$$

$$U_0\delta\dot{\alpha} = Z_{u_0}u + Z_{\alpha_0}\delta\alpha + (U_0 + Z_{q_0})q + Z_{\dot{\alpha}_0}\delta\dot{\alpha} + Z_{\delta e_0}\delta e, \quad (43)$$

$$\dot{q} = (M_{u_0} + M_{T_{u_0}})u + (M_{\alpha_0} + M_{T_{\alpha_0}})\delta\alpha + M_{q_0}q + M_{\dot{\alpha}_0}\delta\dot{\alpha} + M_{\delta e_0}\delta e. \quad (44)$$

## Laplace Transform Analysis

Taking the Laplace transform of (42), (43), and (44), and assuming that the initial conditions of the perturbations ( $u, \delta\alpha, \theta$ ) are zero yields

$$\begin{bmatrix} s - (X_{u_0} + X_{T_{u_0}}) & -X_{\alpha_0} & g \\ -Z_{u_0} & s(U_0 - Z_{\dot{\alpha}_0}) - Z_{\alpha_0} & -(U_0 + Z_{q_0})s \\ -(M_{u_0} + M_{T_{u_0}}) & -(M_{\dot{\alpha}_0}s + M_{\alpha_0} + M_{T_{\alpha_0}}) & s^2 - M_{q_0}s \end{bmatrix} \begin{bmatrix} \hat{u}(s) \\ \delta\hat{\alpha}(s) \\ \hat{\theta}(s) \end{bmatrix} = \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix} \delta\hat{e}(s),$$

where hat in this context denotes the Laplace transform of a scalar function of time. The transfer functions from  $\delta\hat{e}(s)$  to  $\hat{u}(s)$ ,  $\delta\hat{\alpha}(s)$ , and  $\hat{\theta}(s)$  are thus given by

$$\begin{bmatrix} G_{\hat{u}/\delta\hat{e}}(s) \\ G_{\delta\hat{\alpha}/\delta\hat{e}}(s) \\ G_{\hat{\theta}/\delta\hat{e}}(s) \end{bmatrix} \triangleq \begin{bmatrix} \frac{\hat{u}(s)}{\delta\hat{e}(s)} \\ \frac{\delta\hat{\alpha}(s)}{\delta\hat{e}(s)} \\ \frac{\hat{\theta}(s)}{\delta\hat{e}(s)} \end{bmatrix} = \begin{bmatrix} s - (X_{u_0} + X_{T_{u_0}}) & -X_{\alpha_0} & g \\ -Z_{u_0} & s(U_0 - Z_{\dot{\alpha}_0}) - Z_{\alpha_0} & -(U_0 + Z_{q_0})s \\ -(M_{u_0} + M_{T_{u_0}}) & -(M_{\dot{\alpha}_0}s + M_{\alpha_0} + M_{T_{\alpha_0}}) & s^2 - M_{q_0}s \end{bmatrix}^{-1} \begin{bmatrix} X_{\delta e_0} \\ Z_{\delta e_0} \\ M_{\delta e_0} \end{bmatrix}.$$

Consequently,

$$G_{\hat{u}/\delta\hat{e}}(s) = \frac{A_u s^3 + B_u s^2 + C_u s + D_u}{E s^4 + F s^3 + G s^2 + H s + I}, \quad (45)$$

$$G_{\delta\hat{\alpha}/\delta\hat{e}}(s) = \frac{A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha}{E s^4 + F s^3 + G s^2 + H s + I}, \quad (46)$$

$$G_{\hat{\theta}/\delta\hat{e}}(s) = \frac{A_\theta s^2 + B_\theta s + C_\theta}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \quad (47)$$

where the coefficients of (45), (46), and (47) are defined in tables 4 and 5. Note that the relative degree of (47) is 2. For details, see ‘‘Markov Parameters and Relative Degree’’.

Next, we find the transfer function from the elevator perturbation to the vertical velocity perturbation. It follows from (14) and (29) that

$$v_{pv} = -(\sin \Theta)U + (\cos \Theta)U(\tan \alpha) - \ell(\cos \Theta)\dot{\Theta} - \eta(\sin \Theta)\dot{\Theta}. \quad (48)$$

Letting  $v_{pv_0}$  denote the vertical velocity in steady horizontal longitudinal flight, it follows from (48) that

$$v_{pv_0} = 0.$$

Linearizing (48) about  $(U_0, \alpha_0, \Theta_0)$  using the first-order approximations  $v_{pv} \approx v_{pv_0} + \delta v_{pv}$ ,  $U \approx U_0 + u$ ,  $\alpha \approx \delta\alpha$ , and  $\Theta \approx \theta$  yields

$$v_{pv_0} + \delta v_{pv} = -(\sin \theta)(U_0 + u) + (\cos \theta)(U_0 + u)(\tan \delta\alpha) - \ell(\cos \theta)\dot{\theta} - \eta(\sin \theta)\dot{\theta},$$

where  $\delta v_{pv}$  is the first-order approximation of the vertical velocity perturbation. Neglecting products of perturbation variables, and approximating  $\cos \theta \approx 1$ ,  $\sin \theta \approx \theta$ , and  $\tan \delta\alpha \approx \delta\alpha$  yields

$$\delta v_{pv} = U_0\delta\alpha - U_0\theta - \ell\dot{\theta}. \quad (49)$$

Next, taking the Laplace transform of (49) and assuming that the initial conditions of the perturbations  $(u, \delta\alpha, \theta)$  are zero yields

$$\delta\hat{v}_{pv}(s) = U_0\delta\hat{\alpha}(s) - (U_0 + \ell s)\hat{\theta}(s). \quad (50)$$

It follows from (46), (47), and (50) that the transfer function from  $\delta\hat{e}(s)$  to  $\delta\hat{v}_{pv}(s)$  is given by

$$G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s) = \frac{A_v s^3 + B_v s^2 + C_v s + D_v}{Es^4 + Fs^3 + Gs^2 + Hs + I}, \quad (51)$$

where the numerator coefficients are defined in Table 4 and the denominator coefficients are defined in Table 5.



Next, to find the transfer function from the elevator perturbation to the horizontal velocity perturbation, it follows from (13) and (29) that

$$v_{\text{ph}} = (\cos \Theta)U + (\sin \Theta)(\tan \alpha)U - \ell(\sin \Theta)\dot{\Theta} + \eta(\cos \Theta)\dot{\Theta}. \quad (52)$$

Letting  $v_{\text{ph}_0}$  denote the horizontal velocity in steady horizontal longitudinal flight, it follows from (52) that

$$v_{\text{ph}_0} = U_0.$$

Linearizing (52) about  $(U_0, \alpha_0, \Theta_0)$  using the first-order approximations  $v_{\text{ph}} \approx v_{\text{ph}_0} + \delta v_{\text{ph}}$ ,  $U \approx U_0 + u$ ,  $\alpha \approx \delta\alpha$ , and  $\Theta \approx \theta$  yields

$$v_{\text{ph}_0} + \delta v_{\text{ph}} = (\cos \theta)(U_0 + u) + (\sin \theta)(U_0 + u)(\tan \delta\alpha) - \ell(\sin \theta)\dot{\theta} + \eta(\cos \theta)\dot{\theta},$$

where  $\delta v_{\text{ph}}$  is the first-order approximation of the horizontal velocity perturbation. Neglecting products of perturbation variables, and approximating  $\cos \theta \approx 1$ ,  $\sin \theta \approx \theta$ , and  $\tan \delta\alpha \approx \delta\alpha$  yields

$$\delta v_{\text{ph}} = u + \eta\dot{\theta}. \quad (53)$$

Next, taking the Laplace transform of (53) and assuming that the initial conditions of the perturbations  $(u, \delta\alpha, \theta)$  are zero yields

$$\delta \hat{v}_{\text{ph}}(s) = \hat{u}(s) + \eta s \hat{\theta}(s). \quad (54)$$

It follows from (45), (47), and (54) that the transfer function from  $\delta \hat{e}(s)$  to  $\delta \hat{v}_{\text{ph}}(s)$  is given by

$$G_{\delta \hat{v}_{\text{ph}}/\delta \hat{e}}(s) = \frac{A_{\text{h}}s^3 + B_{\text{h}}s^2 + C_{\text{h}}s + D_{\text{h}}}{E s^4 + F s^3 + G s^2 + H s + I}, \quad (55)$$

where the numerator coefficients are defined in Table 4, and the denominator coefficients are defined in Table 5.

## Instantaneous Velocity Center of Rotation

The point  $p_{IVCR}$  is an *instantaneous velocity center of rotation (IVCR)* of the aircraft at time  $t_0$  if  $p_{IVCR}$  is fixed relative to the aircraft, and, at time  $t_0$ , the angular velocity of the aircraft relative to  $F_E$  is not zero and the velocity of  $p_{IVCR}$  relative to  $O_{AC}$  with respect to  $F_E$  is zero. For details, see “Instantaneous Velocity Center of Rotation”. For longitudinal flight, it follows from (S11) that the location of the unique  $p_{IVCR}$  whose coordinate along  $\hat{j}_{AC}$  is zero, if it exists, is given by

$$\vec{r}_{p_{IVCR}/c} = \frac{1}{|\vec{\omega}_{AC/E}|^2} \vec{\omega}_{AC/E} \times \vec{v}_{c/O_E/E} + \frac{\vec{\omega}_{AC/E} \cdot \vec{r}_{p/O_E/E}}{|\vec{\omega}_{AC/E}|^2} \vec{\omega}_{AC/E}. \quad (56)$$

Note that the second term in (56) is zero since the  $\hat{j}_{AC}$  component of  $p_{IVCR}$  is zero. Thus, (56) can be written as

$$\begin{aligned} \vec{r}_{p_{IVCR}/c} &= \frac{1}{|\vec{\omega}_{AC/E}|^2} \vec{\omega}_{AC/E} \times \vec{v}_{c/O_E/E} \\ &= \frac{1}{\dot{\Theta}^2} [\dot{\Theta} \hat{j}_{AC} \times (U \hat{i}_{AC} + W \hat{k}_{AC})] \\ &= \frac{W}{\dot{\Theta}} \hat{i}_{AC} - \frac{U}{\dot{\Theta}} \hat{k}_{AC}. \end{aligned}$$

Therefore,

$$\vec{r}_{p_{IVCR}/c} \Big|_{AC} = \begin{bmatrix} \ell_{IVCR} \\ 0 \\ \eta_{IVCR} \end{bmatrix}, \quad (57)$$

where

$$\ell_{IVCR} \triangleq \frac{W}{\dot{\Theta}} = \frac{U \tan \alpha}{\dot{\Theta}},$$

and

$$\eta_{IVCR} \triangleq -\frac{U}{\dot{\Theta}}.$$

Since  $\dot{\Theta}_0 = 0$ , it follows that  $\ell_{IVCR}$  and  $\eta_{IVCR}$  are infinite for steady flight, and thus no IVCR exists in steady flight.

Next, for the elevator step deflection  $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$ , where  $\varepsilon \neq 0$ , we approximate  $\ell_{\text{IVCR}}$  and  $\eta_{\text{IVCR}}$  at  $t_0^+$  using the first-order approximations  $U \approx U_0 + u$ ,  $\alpha \approx \delta\alpha$ , and  $\Theta \approx \theta$ . Thus,

$$\ell_{\text{IVCR}}(t_0^+) \approx \frac{(U_0 + u(t_0^+))(\tan \delta\alpha(t_0^+))}{\dot{\theta}(t_0^+)}, \quad (58)$$

$$\eta_{\text{IVCR}}(t_0^+) \approx -\frac{U_0 + u(t_0^+)}{\dot{\theta}(t_0^+)}, \quad (59)$$

where it follows from the initial value theorem that

$$\begin{aligned} \theta(t_0^+) &= \lim_{s \rightarrow \infty} s\hat{\theta}(s) \\ &= \lim_{s \rightarrow \infty} sG_{\hat{\theta}/\delta e}(s) \frac{\varepsilon}{s} \\ &= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_\theta s^2 + B_\theta s + C_\theta)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned} \quad (60)$$

$$\begin{aligned} \dot{\theta}(t_0^+) &= \lim_{s \rightarrow \infty} s[s\hat{\theta}(s) - \theta(t_0^+)] \\ &= \lim_{s \rightarrow \infty} s^2 G_{\hat{\theta}/\delta e}(s) \frac{\varepsilon}{s} \\ &= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_\theta s^3 + B_\theta s^2 + C_\theta s)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned} \quad (61)$$

$$\begin{aligned} \delta\alpha(t_0^+) &= \lim_{s \rightarrow \infty} s\delta\hat{\alpha}(s) \\ &= \lim_{s \rightarrow \infty} sG_{\delta\hat{\alpha}/\delta e}(s) \frac{\varepsilon}{s} \\ &= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0, \end{aligned} \quad (62)$$

$$\begin{aligned} u(t_0^+) &= \lim_{s \rightarrow \infty} s\hat{u}(s) \\ &= \lim_{s \rightarrow \infty} sG_{\hat{u}/\delta e}(s) \frac{\varepsilon}{s} \\ &= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_u s^3 + B_u s^2 + C_u s + D_u)}{Es^4 + Fs^3 + Gs^2 + Hs + I} \\ &= 0. \end{aligned} \quad (63)$$

Thus it follows from (58)–(63) that

$$\begin{aligned}\ell_{\text{IVCR}}(t_0^+) &\approx \frac{U_0 \tan \alpha_0}{\dot{\theta}(t_0^+)} = \infty, \\ \eta_{\text{IVCR}}(t_0^+) &\approx -\frac{U_0}{\dot{\theta}(t_0^+)} = \infty.\end{aligned}$$

Therefore, no IVCR exists for an elevator step deflection.

## Instantaneous Acceleration Center of Rotation

The point  $p_{\text{IACR}}$  is an *instantaneous acceleration center of rotation (IACR)* of the aircraft at time  $t_0$  if  $p_{\text{IACR}}$  is fixed relative to the aircraft, and, at time  $t_0$ , the acceleration of  $p_{\text{IACR}}$  relative to  $O_{\text{AC}}$  with respect to  $F_{\text{E}}$  is zero. For details, see “Instantaneous Acceleration Center of Rotation”. It follows from (3) that the location of the unique  $p_{\text{IACR}}$  whose coordinate along  $\hat{j}_{\text{AC}}$  is zero, if it exists, has the form

$$\vec{r}_{p_{\text{IACR}}/c} \Big|_{\text{AC}} = \begin{bmatrix} \ell_{\text{IACR}} \\ 0 \\ \eta_{\text{IACR}} \end{bmatrix}. \quad (64)$$

It thus follows from (19) and the definition of the IACR that

$$\vec{a}_{p_{\text{IACR}}/O_{\text{E}}/E} \Big|_{\text{AC}} = \begin{bmatrix} -\ell_{\text{IACR}} \dot{\Theta}^2 + \dot{U} + W \dot{\Theta} + \eta_{\text{IACR}} \ddot{\Theta} \\ 0 \\ -\ell_{\text{IACR}} \ddot{\Theta} + \dot{W} - U \dot{\Theta} - \eta_{\text{IACR}} \dot{\Theta}^2 \end{bmatrix} = 0,$$

which implies

$$\ell_{\text{IACR}} = \frac{W \dot{\Theta}^3 + \dot{U} \dot{\Theta}^2 - U \dot{\Theta} \ddot{\Theta} + \dot{W} \ddot{\Theta}}{\dot{\Theta}^4 + \ddot{\Theta}^2}, \quad (65)$$

$$\eta_{\text{IACR}} = \frac{-U \dot{\Theta}^3 + \dot{W} \dot{\Theta}^2 + W \dot{\Theta} \ddot{\Theta} - \dot{U} \ddot{\Theta}}{\dot{\Theta}^4 - \ddot{\Theta}^2}. \quad (66)$$

Alternatively, using (S25) yields

$$\begin{aligned}\vec{r}_{\text{PIACR}/c} &= \frac{|\vec{\omega}_{\text{AC}/\text{E}}|^2 \vec{a}_{c/\text{O}_\text{E}/\text{E}} + \overset{\text{B}\bullet}{\vec{\omega}}_{\text{AC}/\text{E}} \times \vec{a}_{c/\text{O}_\text{E}/\text{E}}}{|\vec{\omega}_{\text{AC}/\text{E}}|^4 + |\overset{\text{B}\bullet}{\vec{\omega}}_{\text{AC}/\text{E}}|^2} \\ &= \frac{\dot{\Theta}^2 \vec{a}_{c/\text{O}_\text{E}/\text{E}} + \overset{\text{B}\bullet}{\vec{\omega}}_{\text{AC}/\text{E}} \times \vec{a}_{c/\text{O}_\text{E}/\text{E}}}{\dot{\Theta}^4 + \ddot{\Theta}^2}.\end{aligned}$$

Therefore,

$$\vec{r}_{\text{PIACR}/c}|_{\text{AC}} = \frac{1}{\dot{\Theta}^4 + \ddot{\Theta}^2} \left( \dot{\Theta}^2 \begin{bmatrix} \dot{U} + W\dot{\Theta} \\ 0 \\ \dot{W} - U\dot{\Theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \ddot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{U} + W\dot{\Theta} \\ 0 \\ \dot{W} - U\dot{\Theta} \end{bmatrix} \right) = \begin{bmatrix} \frac{W\dot{\Theta}^3 + \dot{U}\dot{\Theta}^2 - U\dot{\Theta}\ddot{\Theta} + \dot{W}\ddot{\Theta}}{\dot{\Theta}^4 + \ddot{\Theta}^2} \\ 0 \\ \frac{-U\dot{\Theta}^3 + \dot{W}\dot{\Theta}^2 + W\dot{\Theta}\ddot{\Theta} - \dot{U}\ddot{\Theta}}{\dot{\Theta}^4 + \ddot{\Theta}^2} \end{bmatrix},$$

which agrees with (64), (65), and (66).

Next, it follows from (29), (30), (65), and (66) that

$$\ell_{\text{IACR}} = \frac{U(\tan \alpha)\dot{\Theta}^3 + \dot{U}\dot{\Theta}^2 - U\dot{\Theta}\ddot{\Theta} + \left(\dot{U} \tan \alpha + U(\sec^2 \alpha)\dot{\alpha}\right) \ddot{\Theta}}{\dot{\Theta}^4 + \ddot{\Theta}^2}, \quad (67)$$

$$\eta_{\text{IACR}} = \frac{-U\dot{\Theta}^3 + \left(\dot{U} \tan \alpha + U(\sec^2 \alpha)\dot{\alpha}\right) \dot{\Theta}^2 + U(\tan \alpha)\dot{\Theta}\ddot{\Theta} - \dot{U}\ddot{\Theta}}{\dot{\Theta}^4 + \ddot{\Theta}^2}. \quad (68)$$

Since  $\dot{\Theta}_0 = 0$  and  $\ddot{\Theta}_0 = 0$ , it follows that  $\ell_{\text{IACR}}$  and  $\eta_{\text{IACR}}$  are infinite for steady flight.

Next, for the elevator step deflection  $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$ , where  $\varepsilon \neq 0$ , we approximate  $\ell_{\text{IACR}}$  and  $\eta_{\text{IACR}}$  at  $t_0^+$  using the first-order approximations  $U \approx U_0 + u$ ,  $\alpha \approx \delta\alpha$ , and  $\Theta \approx \theta$ . Thus,

$$\begin{aligned}\ell_{\text{IACR}}(t_0^+) &\approx \frac{1}{\ddot{\theta}^2(0^+) + \dot{\theta}^4(0^+)} \left( [U_0 + u(t_0^+)](\tan \delta\alpha(t_0^+))\dot{\theta}^3(t_0^+) + \dot{u}(t_0^+)\dot{\theta}^2(t_0^+) \right. \\ &\quad + [\dot{u}(t_0^+)(\tan \delta\alpha(t_0^+)) + [U_0 + u(t_0^+)](\sec^2 \delta\alpha(t_0^+))\delta\dot{\alpha}(t_0^+)] \ddot{\theta}(t_0^+) \\ &\quad \left. - [U_0 + \dot{u}(t_0^+)]\dot{\theta}(t_0^+)\ddot{\theta}(t_0^+) \right), \quad (69)\end{aligned}$$

$$\begin{aligned}\eta_{\text{IACR}}(t_0^+) &\approx \frac{1}{\ddot{\theta}^2(0^+) + \dot{\theta}^4(0^+)} \left( [U_0 + u(t_0^+)](\tan \delta\alpha(t_0^+))\dot{\theta}(t_0^+)\ddot{\theta}(t_0^+) - \dot{u}(t_0^+)\ddot{\theta}(t_0^+) \right. \\ &\quad + [\dot{u}(t_0^+)(\tan \delta\alpha(t_0^+)) + [U_0 + u(t_0^+)](\sec^2 \delta\alpha(t_0^+))\delta\dot{\alpha}(t_0^+)] \dot{\theta}^2(t_0^+) \\ &\quad \left. - [U_0 + u(t_0^+)]\dot{\theta}^3(t_0^+) \right), \quad (70)\end{aligned}$$

where the initial value theorem implies that

$$\begin{aligned}
\delta\dot{\alpha}(t_0^+) &= \lim_{s \rightarrow \infty} s[s\delta\hat{\alpha}(s) - \delta\alpha(t_0^+)] \\
&= \lim_{s \rightarrow \infty} s^2 G_{\delta\hat{\alpha}/\delta\hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_\alpha s^4 + B_\alpha s^3 + C_\alpha s^2 + D_\alpha)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= \frac{\varepsilon A_\alpha}{E},
\end{aligned} \tag{71}$$

$$\begin{aligned}
\ddot{\theta}(t_0^+) &= \lim_{s \rightarrow \infty} s[s^2\hat{\theta}(s) - s\theta(t_0^+) - \dot{\theta}(t_0^+)] \\
&= \lim_{s \rightarrow \infty} s^3 G_{\hat{\theta}/\delta\hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_\theta s^4 + B_\theta s^3 + C_\theta s^2)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= \frac{\varepsilon A_\theta}{E},
\end{aligned} \tag{72}$$

$$\begin{aligned}
\dot{u}(t_0^+) &= \lim_{s \rightarrow \infty} s[s\hat{u}(s) - u(t_0^+)] \\
&= \lim_{s \rightarrow \infty} s^2 G_{\delta\hat{u}/\delta\hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow \infty} \frac{\varepsilon(A_u s^4 + B_u s^3 + C_u s^2 + D_u s)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= \frac{\varepsilon A_u}{E}.
\end{aligned} \tag{73}$$

It thus follows from (60)–(63), (69)–(73), and the expressions given in Table 4 that

$$\begin{aligned}
\ell_{\text{IACR}}(t_0^+) &\approx \frac{U_0 A_\alpha}{A_\theta} \\
&= \frac{U_0 Z_{\delta e_0}}{Z_{\delta e_0} M_{\dot{\alpha}_0} + M_{\delta e_0} (U_0 - Z_{\dot{\alpha}_0})}
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
\eta_{\text{IACR}}(t_0^+) &\approx -\frac{A_u}{A_\theta} \\
&= -\frac{X_{\delta e_0} (U_0 - Z_{\dot{\alpha}_0})}{Z_{\delta e_0} M_{\dot{\alpha}_0} + M_{\delta e_0} (U_0 - Z_{\dot{\alpha}_0})}.
\end{aligned} \tag{75}$$

## Initial Slope and Quadratic Curvature of the Vertical and Horizontal Velocity Perturbations at the IACR

The vertical velocity perturbation  $\delta v_{pv}(t_0^+)$  and the initial slope  $\delta \dot{v}_{pv}(t_0^+)$  of the vertical velocity perturbation at p due to the elevator step deflection  $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$ , where  $\varepsilon \neq 0$ , are given by

$$\begin{aligned} \delta v_{pv}(t_0^+) &= \lim_{s \rightarrow \infty} s \delta \hat{v}_{pv}(s) \\ &= \lim_{s \rightarrow \infty} s G_{\delta \hat{v}_{pv}/\delta \hat{e}}(s) \frac{\varepsilon}{s} \\ &= \lim_{s \rightarrow \infty} \frac{\varepsilon (A_v s^3 + B_v s^2 + C_v s + D_v)}{E s^4 + F s^3 + G s^2 + H s + I} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \delta \dot{v}_{pv}(t_0^+) &= \lim_{s \rightarrow \infty} s [s \delta \hat{v}_{pv}(s) - \delta v_{pv}(t_0^+)] \\ &= \lim_{s \rightarrow \infty} s^2 G_{\delta \hat{v}_{pv}/\delta \hat{e}}(s) \frac{\varepsilon}{s} \\ &= \lim_{s \rightarrow \infty} \frac{\varepsilon (A_v s^4 + B_v s^3 + C_v s^2 + D_v s)}{E s^4 + F s^3 + G s^2 + H s + I} \\ &= \frac{\varepsilon A_v}{E}. \end{aligned} \tag{76}$$

Next, it follows from the expression for  $A_v$  given in Table 4 that

$$A_v = -\ell A_\theta + U_0 A_\alpha. \tag{77}$$

Therefore,  $A_v = 0$  if and only if

$$\ell = \frac{U_0 A_\alpha}{A_\theta}. \tag{78}$$

Hence, it follows from (76) that  $\delta \dot{v}_{pv}(t_0^+) = 0$  if and only if  $\ell$  satisfies (78). For details, see ‘‘The Initial Curvature Theorem and Unit-Step Response’’.

Similarly, the horizontal velocity perturbation  $\delta v_{ph}(t_0^+)$  at p due to the elevator step deflection  $\delta e(t) =$

$\varepsilon \mathbf{1}(t - t_0)$ , where  $\varepsilon \neq 0$ , is given by

$$\begin{aligned}
\delta v_{\text{ph}}(t_0^+) &= \lim_{s \rightarrow \infty} s \delta \hat{v}_{\text{ph}}(s) \\
&= \lim_{s \rightarrow \infty} s G_{\delta \hat{v}_{\text{ph}}/\delta \hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow \infty} \frac{\varepsilon (A_h s^3 + B_h s^2 + C_h s + D_h)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= 0,
\end{aligned}$$

while the initial slope  $\delta \dot{v}_{\text{ph}}(t_0^+)$  of the horizontal velocity perturbation is given by

$$\begin{aligned}
\delta \dot{v}_{\text{ph}}(t_0^+) &= \lim_{s \rightarrow \infty} s [s \delta \hat{v}_{\text{ph}}(s) - \delta v_{\text{ph}}(t_0^+)] \\
&= \lim_{s \rightarrow \infty} s^2 G_{\delta \hat{v}_{\text{ph}}/\delta \hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow \infty} \frac{\varepsilon (A_h s^4 + B_h s^3 + C_h s^2 + D_h s)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= \frac{\varepsilon A_h}{E}.
\end{aligned} \tag{79}$$

Next, it follows from the expression for  $A_h$  given in Table 4 that

$$A_h = \eta A_\theta + A_u. \tag{80}$$

Therefore,  $A_h = 0$  if and only if

$$\eta = -\frac{A_u}{A_\theta}. \tag{81}$$

Hence, it follows from (79) that  $\delta \dot{v}_{\text{ph}}(t_0^+) = 0$  if and only if  $\eta$  satisfies (81).

Next, it follows from (74) and (75) that  $p_{\text{IACR}}$  satisfies both (78) and (81). Therefore,  $A_v = 0$  and  $A_h = 0$  if and only if  $(\ell, \eta) = (\ell_{\text{IACR}}, \eta_{\text{IACR}})$ . Thus, evaluating (76) and (79) at the IACR  $(\ell_{\text{IACR}}, \eta_{\text{IACR}})$  for the elevator step deflection  $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$ , where  $\varepsilon \neq 0$ , yields  $\delta \dot{v}_{\text{pv}}(t_0^+) = 0$  and  $\delta \dot{v}_{\text{ph}}(t_0^+) = 0$ . Therefore, at the IACR, the initial slopes of the vertical and horizontal velocity perturbations are zero.

Since  $A_v = 0$  at the IACR, it follows that the transfer function  $G_{\delta \hat{v}_{\text{pv}}/\delta \hat{e}}(s)$  at the IACR becomes

$$G_{\delta \hat{v}_{\text{pv}}/\delta \hat{e}}(s) = \frac{B_v s^2 + C_v s + D_v}{E s^4 + F s^3 + G s^2 + H s + I}.$$



Next, at the IACR, it follows from the expression for  $B_v$  given in Table 4 that

$$\begin{aligned} B_v &= -\ell_{\text{IACR}}B_\theta - U_0A_\theta + U_0B_\alpha \\ &= -\left(\frac{A_\alpha B_\theta}{A_\theta} + A_\theta - B_\alpha\right)U_0. \end{aligned}$$

Consequently, if  $B_v \neq 0$ , then the relative degree of  $G_{\delta\hat{v}_{\text{pv}}/\delta\hat{e}}(s)$  increases from 1 to 2, and thus one of the zeros of  $G_{\delta\hat{v}_{\text{pv}}/\delta\hat{e}}(s)$  vanishes at the IACR.

Similarly, at the IACR,  $A_h = 0$ . Thus, if  $B_h \neq 0$ , then the relative degree of  $G_{\delta\hat{v}_{\text{ph}}/\delta\hat{e}}(s)$  increases from 1 to 2, and thus one of the zeros of  $G_{\delta\hat{v}_{\text{ph}}/\delta\hat{e}}(s)$  vanishes at the IACR. The vanishing zeros are a consequence of the fact that the initial slope of the vertical velocity perturbation and the horizontal velocity perturbation are zero at the IACR. Note that  $\ell_{\text{IACR}}$  and  $\eta_{\text{IACR}}$  depend on the speed  $U_0$  and the stability derivatives  $Z_{\delta e_0}$ ,  $Z_{\dot{\alpha}_0}$ ,  $X_{\delta e_0}$ ,  $M_{\dot{\alpha}_0}$ , and  $M_{\delta e_0}$ . Vanishing zeros are discussed in [19].

## Initial Undershoot of the Vertical Velocity for an Elevator Deflection

Let  $G(s) \triangleq \frac{\beta(s)}{s^r \alpha(s)}$  be a strictly proper transfer function with relative degree  $d > 0$ , where  $r \geq 0$  and  $\alpha(s)$  is asymptotically stable. Let  $y(t)$  denote the response of  $G$  to the step command  $\mathbf{1}(t - t_0)$ . Then *initial undershoot* occurs at time  $t_0$  if the step response initially moves in the direction opposite to its asymptotic direction, that is,

$$y^{(d)}(t_0^+)y^{(r)}(\infty) < 0. \quad (82)$$

To determine whether the vertical velocity perturbation  $\delta v_{\text{pv}}(t)$  to the elevator step deflection  $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$  exhibits initial undershoot, we investigate (82) with  $G(s) = G_{\delta\hat{v}_{\text{pv}}/\delta\hat{e}}(s)$ ,  $r = 0$ , and  $y(t) = \delta v_{\text{pv}}(t)$ .

First, the asymptotic direction of the step response is given by the sign of

$$\begin{aligned}
\delta v_{\text{pv}}(\infty) &= \lim_{s \rightarrow 0} s \delta \hat{v}_{\text{pv}}(s) \\
&= \lim_{s \rightarrow 0} s G_{\delta \hat{v}_{\text{pv}} / \delta \hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow 0} \frac{\varepsilon (A_v s^3 + B_v s^2 + C_v s + D_v)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= \frac{\varepsilon D_v}{I}.
\end{aligned} \tag{83}$$

It follows from Table 4 and Table 5 that  $\delta v_{\text{pv}}(\infty)$  does not depend on the location of  $p$ , that is, the value of  $(\ell, \eta)$ .

Next, the initial direction of the step response is given by the sign of

$$\begin{aligned}
\delta v_{\text{pv}}^{(d)}(t_0^+) &= \lim_{s \rightarrow \infty} s [s^d \delta \hat{v}_{\text{pv}}(s) - s^{d-1} \delta v_{\text{pv}}(t_0^+) - \dots - \delta v_{\text{pv}}^{(d-1)}(t_0^+)] \\
&= \lim_{s \rightarrow \infty} s^{d+1} \delta \hat{v}_{\text{pv}}(s) \\
&= \lim_{s \rightarrow \infty} s^{d+1} G_{\delta \hat{v}_{\text{pv}} / \delta \hat{e}}(s) \frac{\varepsilon}{s} \\
&= \varepsilon s^d \left( \frac{A_v s^3 + B_v s^2 + C_v s + D_v}{E s^4 + F s^3 + G s^2 + H s + I} \right) \\
&= \begin{cases} \frac{\varepsilon A_v}{E}, & \text{if } d = 1, \quad (\text{that is, } A_v \neq 0) \\ \frac{\varepsilon B_v}{E}, & \text{if } d = 2, \quad (\text{that is, } A_v = 0, B_v \neq 0) \\ \frac{\varepsilon C_v}{E}, & \text{if } d = 3, \quad (\text{that is, } A_v = B_v = 0, C_v \neq 0) \\ \frac{\varepsilon D_v}{E}, & \text{if } d = 4. \quad (\text{that is, } A_v = B_v = C_v = 0, D_v \neq 0) \end{cases}
\end{aligned} \tag{84}$$

Thus, for  $d = 1$ ,  $\delta v_{\text{pv}}(t)$  exhibits initial undershoot if and only if  $\delta \dot{v}_{\text{pv}}(t_0^+) \delta v_{\text{pv}}(\infty) = \frac{A_v D_v}{E I} < 0$ ; for  $d = 2$ ,  $\delta v_{\text{pv}}(t)$  exhibits initial undershoot if and only if  $\delta \ddot{v}_{\text{pv}}(t_0^+) \delta v_{\text{pv}}(\infty) = \frac{B_v D_v}{E I} < 0$ ; for  $d = 3$ ,  $\delta v_{\text{pv}}(t)$  exhibits initial undershoot if and only if  $\delta v_{\text{pv}}^{(3)}(t_0^+) \delta v_{\text{pv}}(\infty) = \frac{C_v D_v}{E I} < 0$ . Furthermore, for  $d = 4$ ,  $\delta v_{\text{pv}}(t)$  does not exhibit initial undershoot since  $\delta v_{\text{pv}}^{(4)}(t_0^+) \delta v_{\text{pv}}(\infty) = \frac{D_v^2}{E I} \geq 0$ ;

The following results follow from (78), (82), (83), and (84) along with Proposition S1.

**Proposition 1** Assume that  $\ell$  does not satisfy (78). Then the following statements hold:

- (i) The relative degree of  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  is 1, and thus  $A_v \neq 0$ .
- (ii)  $\delta v_{pv}(t)$  exhibits initial undershoot if and only if  $\frac{A_v D_v}{EI} < 0$ .
- (iii)  $\delta v_{pv}(t)$  exhibits initial undershoot if and only if  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  has either exactly one or exactly three real nonminimum-phase zeros.

**Proposition 2** Assume that  $\ell$  satisfies (78) and  $B_v \neq 0$ . Then the following statements hold:

- (i) The relative degree of  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  is 2, and thus  $A_v = 0$ .
- (ii)  $\delta v_{pv}(t)$  exhibits initial undershoot if and only if  $\frac{B_v D_v}{EI} < 0$ .
- (iii)  $\delta v_{pv}(t)$  exhibits initial undershoot if and only if  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  has exactly one real nonminimum-phase zero.

Following the same procedure for  $\delta r_{pv}(t)$  yields identical results, that is,  $\delta r_{pv}(t)$  exhibits initial undershoot if and only if  $\delta v_{pv}(t)$  exhibits initial undershoot.

## Initial Undershoot of the Horizontal velocity for an Elevator Step Deflection

To determine whether the horizontal velocity perturbation  $\delta v_{ph}(t)$  to the elevator step deflection  $\delta e(t) = \varepsilon \mathbf{1}(t - t_0)$  exhibits initial undershoot, we investigate (82) with  $G(s) = G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$ ,  $r = 0$ , and  $y(t) = \delta v_{ph}(t)$ .

First, the asymptotic direction of the step response is given by the sign of

$$\begin{aligned}
\delta v_{ph}(\infty) &= \lim_{s \rightarrow 0} s \delta\hat{v}_{ph}(s) \\
&= \lim_{s \rightarrow 0} s G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s) \frac{\varepsilon}{s} \\
&= \lim_{s \rightarrow 0} \frac{\varepsilon (A_h s^3 + B_h s^2 + C_h s + D_h)}{E s^4 + F s^3 + G s^2 + H s + I} \\
&= \frac{\varepsilon D_h}{I}.
\end{aligned} \tag{85}$$

It follows from Table 4 and Table 5 that  $\delta v_{\text{ph}}(\infty)$  does not depend on the location of  $p$ , that is, the value of  $(\ell, \eta)$ .

Next, the initial direction of the step response is given by the sign of

$$\begin{aligned}
\delta v_{\text{ph}}^{(d)}(t_0^+) &= \lim_{s \rightarrow \infty} s[s^d \delta \hat{v}_{\text{ph}}(s) - s^{d-1} \delta v_{\text{ph}}(t_0^+) - \dots - \delta v_{\text{ph}}^{(d-1)}(t_0^+)] \\
&= \lim_{s \rightarrow \infty} s^{d+1} \delta \hat{v}_{\text{ph}}(s) \\
&= \lim_{s \rightarrow \infty} s^{d+1} G_{\delta \hat{v}_{\text{ph}}/\delta \hat{e}}(s) \frac{\varepsilon}{s} \\
&= \varepsilon s^d \left( \frac{A_h s^3 + B_h s^2 + C_h s + D_h}{E s^4 + F s^3 + G s^2 + H s + I} \right) \\
&= \begin{cases} \frac{\varepsilon A_h}{E}, & \text{if } d = 1, \quad (\text{that is, } A_h \neq 0) \\ \frac{\varepsilon B_h}{E}, & \text{if } d = 2, \quad (\text{that is, } A_h = 0, B_h \neq 0) \\ \frac{\varepsilon C_h}{E}, & \text{if } d = 3, \quad (\text{that is } A_h = B_h = 0, C_h \neq 0) \\ \frac{\varepsilon D_h}{E}, & \text{if } d = 4. \quad (\text{that is, } A_h = B_h = C_h = 0, D_h \neq 0) \end{cases} \quad (86)
\end{aligned}$$

Thus, for  $d = 1$ ,  $\delta v_{\text{ph}}(t)$  exhibits initial undershoot if and only if  $\delta \dot{v}_{\text{ph}}(t_0^+) \delta v_{\text{ph}}(\infty) = \frac{A_h D_h}{EI} < 0$ ; for  $d = 2$ ,  $\delta v_{\text{ph}}(t)$  exhibits initial undershoot if and only if  $\delta \ddot{v}_{\text{ph}}(t_0^+) \delta v_{\text{ph}}(\infty) = \frac{B_h D_h}{EI} < 0$ ; for  $d = 3$ ,  $\delta v_{\text{ph}}(t)$  exhibits initial undershoot if and only if  $\delta v_{\text{ph}}^{(3)}(t_0^+) \delta v_{\text{ph}}(\infty) = \frac{C_h D_h}{EI} < 0$ . Furthermore, for  $d = 4$ ,  $\delta v_{\text{ph}}(t)$  does not exhibit initial undershoot since  $\delta v_{\text{ph}}^{(4)}(t_0^+) \delta v_{\text{ph}}(\infty) = \frac{D_h^2}{EI} \geq 0$ .

The following results follow from (81), (82), (85), and (86) along with Proposition S1.

**Proposition 4** Assume that  $\eta$  does not satisfy (81). Then the following statements hold:

- (i) The relative degree of  $G_{\delta \hat{v}_{\text{ph}}/\delta \hat{e}}(s)$  is 1, and thus  $A_h \neq 0$ .
- (ii)  $\delta v_{\text{ph}}(t)$  exhibits initial undershoot if and only if  $\frac{A_h D_h}{EI} < 0$ .
- (iii)  $\delta v_{\text{ph}}(t)$  exhibits initial undershoot if and only if  $G_{\delta \hat{v}_{\text{ph}}/\delta \hat{e}}(s)$  has either exactly one or exactly three real nonminimum-phase zeros.

**Proposition 5** Assume that  $\eta$  satisfies (81) and  $B_h \neq 0$ . Then the following statements hold:

- (i) The relative degree of  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  is 2, and thus  $A_h = 0$ .
- (ii)  $\delta v_{ph}(t)$  exhibits initial undershoot if and only if  $\frac{B_h D_h}{Eh} < 0$ .
- (iii)  $\delta v_{ph}(t)$  exhibits initial undershoot if and only if  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  has exactly one real nonminimum-phase zero.

The following result is a special case of Proposition 2 and 5, where we consider the response at the IACR.

**Proposition 6** Assume that  $(\ell, \eta) = (\ell_{IACR}, \eta_{IACR})$ ,  $B_v \neq 0$ , and  $B_h \neq 0$ . Then the following statements hold:

- (i) The relative degrees of  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  and  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  are 2. Thus,  $A_v = 0$  and  $A_h = 0$ .
- (ii)  $\delta v_{pv}(t)$  exhibits initial undershoot if and only if  $\frac{B_v D_v}{EI} < 0$ .
- (iii)  $\delta v_{ph}(t)$  exhibits initial undershoot if and only if  $\frac{B_h D_h}{EI} < 0$ .
- (iv)  $\delta v_{pv}(t)$  exhibits initial undershoot if and only if  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  has exactly one real nonminimum-phase zero.
- (v)  $\delta v_{ph}(t)$  has initial undershoot if and only if  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  has exactly one real nonminimum-phase zero.

## Business Jet Example

To illustrate the instantaneous acceleration center of rotation, the initial slope of the vertical velocity and horizontal velocity, and vanishing zeros, we consider a business jet in cruise whose numerical data are given in Table 6 [17, p. 330].

For all expressions below, the units of  $\ell$  and  $\eta$  are feet. Using the data given in Table 6 as well as the expressions given in Table 4 and (45), (46), (47), and (51), the transfer functions from  $\delta\hat{e}(s)$  to  $\hat{u}(s)$ ,  $\delta\hat{\alpha}(s)$ , and  $\hat{\theta}(s)$  are

$$G_{\hat{u}/\delta\hat{e}}(s) = \frac{-378.85s^2 + 2.72e5s + 2.40e5}{675.99(s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068)} \text{ ft/(s-rad)},$$

$$G_{\delta\hat{\alpha}/\delta\hat{e}}(s) = \frac{42.20s^3 + 11939.02s^2 + 88.5773s + 79.30}{675.99(s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068)},$$

$$G_{\hat{\theta}/\delta\hat{e}}(s) = \frac{-11930.17s^2 - 7652.06s - 78.52}{675.99(s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068)}.$$

Furthermore, the transfer functions from  $\delta\hat{e}(s)$  to  $\delta\hat{v}_{pv}$  and  $\delta\hat{v}_{ph}$  are

$$G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s) = \frac{(42.15 + 17.65\ell)s^3 + (23854.0 + 11.3\ell)s^2 + (7740.6 + 0.1\ell)s + 157.2}{s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068} \text{ ft/(s-rad)}, \quad (87)$$

$$G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s) = -\frac{17.65\eta s^3 + (11.32\eta - 0.56)s^2 - (402.4 - 0.12\eta)s - 355.0}{s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068} \text{ ft/(s-rad)}. \quad (88)$$

Next, with  $U_0 = 675.12$  ft/s,  $A_\alpha = -42.20$  1/s,  $A_u = 0$  m/s<sup>2</sup>,  $E = 675.99$  1/s,  $\varepsilon = 1$  deg-s = 0.017 rad-s, and  $A_\theta = 11930.17$  1/s<sup>2</sup>, it follows from (74) and (75) that

$$\ell_{\text{IACR}} \approx -\frac{(675.12)(42.20)}{11930.17} \text{ ft} = -2.3881 \text{ ft},$$

$$\eta_{\text{IACR}} \approx -\frac{0}{11930.17} \text{ ft} = 0 \text{ ft}.$$

Next, using (87), the initial vertical velocity slope response due the 1-deg elevator deflection  $\delta e(t) = (0.017)\mathbf{1}(t - t_0^+)$  is given by

$$\delta\dot{v}_{pv}(t_0^+) = 42.15 + 17.65\ell.$$

It follows that, at  $\ell = \ell_{\text{IACR}}$ ,  $\delta\dot{v}_{pv}(t_0^+) = 0$ , and the number of zeros of the transfer function  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$  decreases from three to two.

Likewise, using (88), the initial horizontal velocity slope response due to the 1-deg step elevator deflection  $\delta e(t) = (0.017)\mathbf{1}(t - t_0^+)$  is given by

$$\delta\dot{v}_{ph}(t_0^+) = 17.65\eta.$$

It follows that  $\eta = \eta_{\text{IACR}}$ ,  $\delta\dot{v}_{\text{ph}}(t_0^+) = 0$ , and the number of zeros of the transfer function  $G_{\delta\dot{v}_{\text{ph}}/\delta\hat{e}}(s)$  decreases from three to two.

To demonstrate the initial vertical velocity perturbation  $\delta v_{\text{pv}}$  and initial horizontal velocity perturbation  $\delta v_{\text{ph}}$  forward and aft of the IACR, we simulate  $\delta v_{\text{pv}}$  and  $\delta v_{\text{ph}}$  with the 1-deg step elevator deflection  $\delta e(t) = (0.017)\mathbf{1}(t - t_0^+)$  for several values of  $\ell$  and  $\eta$ . Figure 2 shows that, for  $\ell = -20$  ft,  $\delta v_{\text{pv}}$  experiences initial undershoot, whereas, for  $\eta = 20$  ft,  $\delta v_{\text{ph}}$  experiences initial undershoot, as defined in [1] and “Initial Undershoot”. This initial undershoot is a consequence of the fact that, for all  $\ell < \ell_{\text{IACR}}$ , the transfer function  $G_{\delta\dot{v}_{\text{pv}}/\delta\hat{e}}(s)$  has one nonminimum-phase zero; for all  $\eta > \eta_{\text{IACR}}$ , the transfer function  $G_{\delta\dot{v}_{\text{ph}}/\delta\hat{e}}(s)$  has one nonminimum-phase zero. On the other hand, for all  $\ell > \ell_{\text{IACR}}$ , the initial slope  $\delta\dot{v}_{\text{pv}}(0^+)$  is in the direction of the asymptotic vertical velocity; for all  $\eta < \eta_{\text{IACR}}$ , the initial slope  $\delta\dot{v}_{\text{ph}}(0^+)$  is in the direction of the asymptotic horizontal velocity. Finally, for all  $\ell = \ell_{\text{IACR}}$ , the initial slope  $\delta\dot{v}_{\text{pv}}(0^+)$  is zero; for all  $\eta = \eta_{\text{IACR}}$ , the initial slope  $\delta\dot{v}_{\text{ph}}(0^+)$  is zero. Note that at the IACR, the initial slopes of both  $\delta\dot{v}_{\text{pv}}(0^+)$  and  $\delta\dot{v}_{\text{ph}}(0^+)$  are zero, as a consequence of the definition of the IACR. Simulations over a longer time interval are shown in Figure 3.

Next, we apply the Routh test to determine the locations of the poles and zeros of (87); for details, see “Routh Test for Third- and Fourth-Order Polynomials”. Note that following the same procedure for the horizontal velocity perturbation transfer function (88) yields the similar results. Thus, we analyze the vertical velocity perturbation transfer function (87) as an example. Writing the denominator of (87) as  $p(s)$ , where  $p(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$  is defined by

$$p(s) = s^4 + 2.01s^3 + 8.05s^2 + 0.085s + 0.068,$$

it follows that

$$a_1a_2a_3 - a_0a_3^2 - a_1^2 = 1.2353 [1/s^6] > 0.$$

Consequently, all of the poles of  $G_{\delta\dot{v}_{\text{pv}}/\delta\hat{e}}$  are in the open left half plane (OLHP).

To determine the zeros of the transfer function from the elevator deflection  $\delta\hat{e}(s)$  to the vertical velocity perturbation  $\delta\dot{v}_{\text{pv}}(s)$ , we apply the Routh test to the numerator of (87). Defining the polynomial

$q(s) = s^3 + a_2s^2 + a_1s + a_0$  by

$$q(s) \triangleq s^3 + \frac{177307 + 84.13\ell}{313.3 + 131.2\ell}s^2 + \frac{57535.6 + 0.8608\ell}{313.3 + 131.2\ell}s + \frac{1168.6}{313.3 + 131.2\ell},$$

it follows that

$$\begin{aligned} a_1a_2 - a_0 &= \left( \frac{57535.6 + 0.8608\ell}{313.3 + 131.2\ell} \right) \left( \frac{177307 + 84.13\ell}{313.3 + 131.2\ell} \right) - \frac{1168.6}{313.3 + 131.2\ell} \\ &= \frac{g(\ell)}{(313.3 + 131.2\ell)(0.11\ell + 0.27)} \text{ ft/s}, \end{aligned} \quad (89)$$

where  $g(\ell) \triangleq \ell^2 + 457.36\ell + 0.88 \text{ ft}^2$ . For  $\ell > \ell_{\text{IACR}}$ , it follows that  $313.3 + 131.2\ell$ ,  $0.11\ell + 0.27$ , and  $g(\ell)$  are positive, and thus (89) is positive. Therefore, for all  $\ell > \ell_{\text{IACR}}$ , all of the roots of  $q(s)$  are in the OLHP. On the other hand, for all  $\ell < \ell_{\text{IACR}}$ , one zero of  $G_{\delta\hat{v}_{\text{pv}}/\delta\hat{e}}(s)$  is in the ORHP and two zeros are in the OLHP. This result follows from the first row of the Routh table, where one sign change appears. Figure 4 shows that a real zero approaches  $\infty$  as  $\ell$  increases toward  $\ell_{\text{IACR}}$ , whereas a real zero approaches  $-\infty$  as  $\ell$  decreases toward  $\ell_{\text{IACR}}$ . This zero thus vanishes at the IACR. For  $\ell \in [-25, 25]$  ft, Figure 5 shows the locations of the two remaining zeros of  $G_{\delta\hat{v}_{\text{pv}}/\delta\hat{e}}(s)$ , which are real and do not vanish at the IACR.

For the horizontal velocity perturbation  $\delta v_{\text{ph}}$ , Figure 6(a) shows that, as  $\eta$  increases toward  $\eta_{\text{IACR}}$ , one zero approaches  $-\infty$ , one zero approaches  $\infty$ , and the remaining zero approaches  $0.88 \text{ rad/s}$ . Figure 6(b) shows that, as  $\eta$  decreases toward  $\eta_{\text{IACR}}$ , one zero approaches  $-\infty$ , one zero approaches  $\infty$ , and the remaining zero approaches  $0.88 \text{ rad/s}$ . Consequently, two zeros of  $G_{\delta\hat{v}_{\text{ph}}/\delta\hat{e}}(s)$  vanish at the IACR.

## Conclusions

In this article, we used Laplace techniques to analyze the response of an aircraft to an elevator step deflection. We showed that the aircraft's initial response to an elevator step command is characterized by the instantaneous acceleration center of rotation, which is the point along the aircraft at which the acceleration of the aircraft is zero. This point, which depends on the inertia and aerodynamics of the aircraft, is determined by deriving the linearized longitudinal equations of motion and evaluating the location of the instantaneous acceleration center of rotation to first order. The initial vertical velocity and horizontal velocity response



requires an increase in relative degree of the associated transfer functions at the instantaneous acceleration center of rotation. This increase in relative degree requires that zeros must vanish at the instantaneous acceleration center of rotation.

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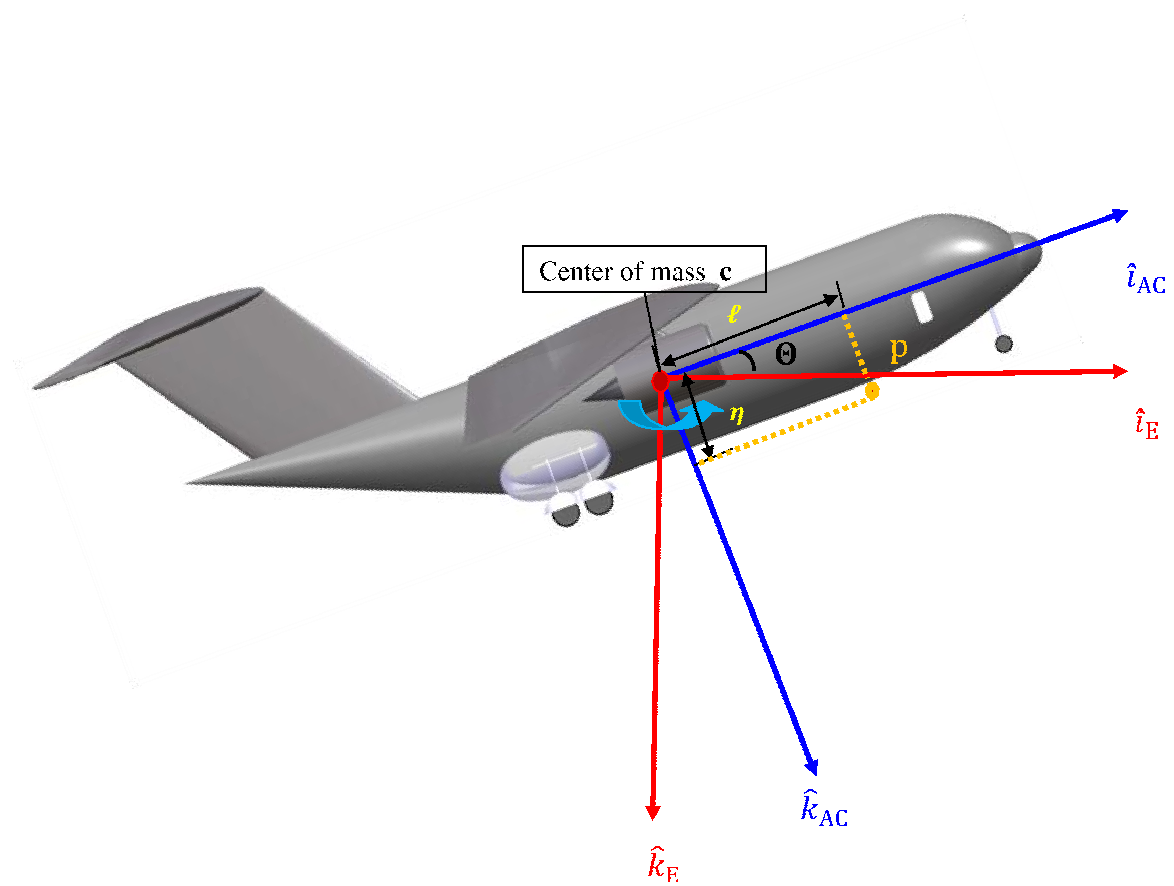


Figure 1: Aircraft and Earth frames. The aircraft frame is fixed to the aircraft, while the Earth frame is assumed to be an inertial frame. The signed quantities  $\ell$  and  $\eta$  are determined by the location of the point  $p$  at which the output is defined relative to the center of mass  $c$ . The pitch angle  $\Theta$ , as shown, is positive, as determined by the right hand rule about the axis  $\hat{j}_{AC} = \hat{j}_E$ , which is not shown but which is directed out of the page.

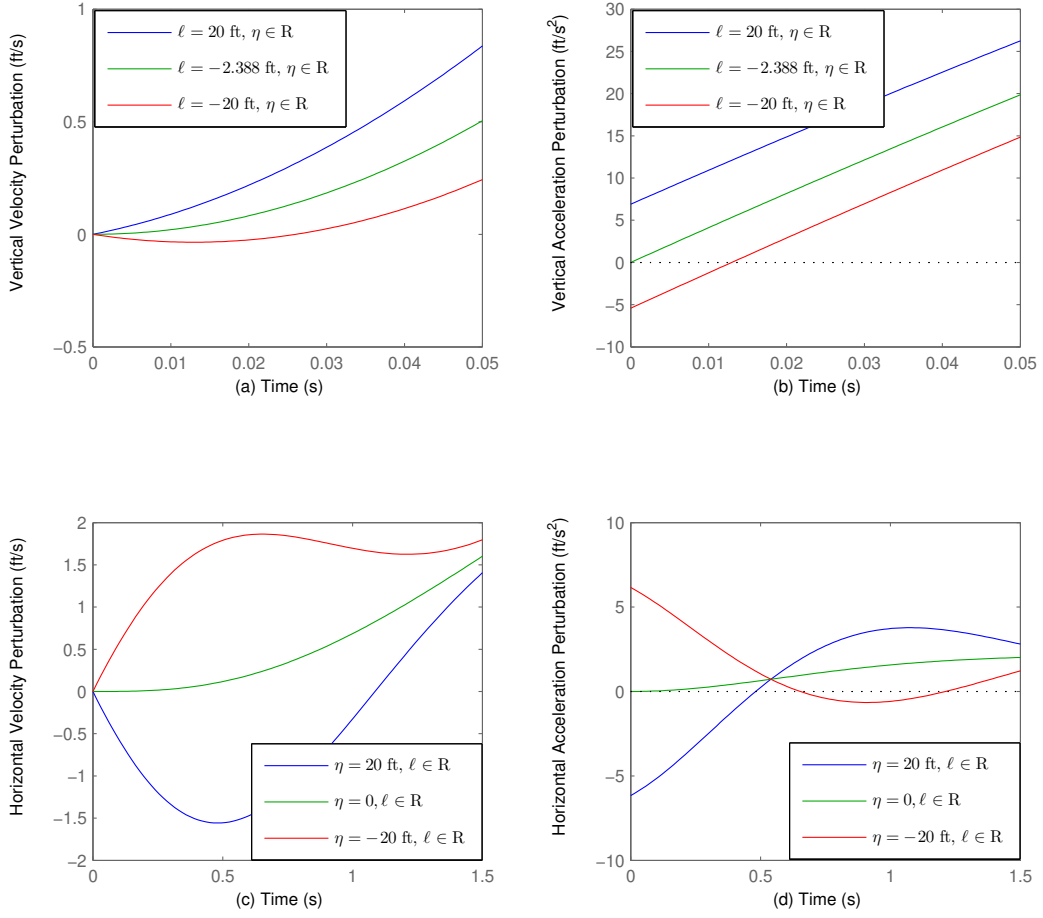


Figure 2: The responses of the vertical velocity perturbation  $\delta v_{pv}(t)$  and the horizontal velocity perturbation  $\delta v_{ph}(t)$  of a typical business jet to the 1-deg step elevator deflection  $\delta e(t) = 0.0171(t - t_0)$  at  $t_0 = 0$  based on the aircraft parameters given in [17]. In (a) and (b), for all  $\ell < \ell_{\text{IACR}} = -2.388$  ft and  $\eta \in \mathbb{R}$ , where  $\ell_{\text{IACR}}$  is the component along  $\hat{k}_{\text{AC}}$  of the location of the IACR, the transfer function  $G_{\delta v_{pv}/\delta e}(s)$  has one positive zero. For  $\ell = \ell_{\text{IACR}}$  and all  $\eta \in \mathbb{R}$ , the initial slope of the vertical velocity perturbation is zero, that is, the vertical acceleration at  $t_0^+$  is zero. In (c) and (d), for all  $\ell \in \mathbb{R}$  and  $\eta > \eta_{\text{IACR}} = 0$  ft, where  $\eta_{\text{IACR}}$  is the component along  $\hat{i}_{\text{AC}}$  of the location of the IACR, the transfer function  $G_{\delta v_{ph}/\delta e}(s)$  has one positive zero. For all  $\ell \in \mathbb{R}$  and  $\eta = \eta_{\text{IACR}}$ , the initial slope of the horizontal velocity perturbation is zero, that is, the horizontal acceleration at  $t_0^+$  is zero, which indicates that  $(\ell, \eta) = (\ell_{\text{IACR}}, \eta_{\text{IACR}})$  is the location of the IACR. This point is characterized by the vanishing zero, which, because of the increase in relative degree, yields zero initial slopes in both directions  $\hat{i}_{\text{AC}}$  and  $\hat{k}_{\text{AC}}$ . Figure 3 shows the same simulations over a longer time interval.

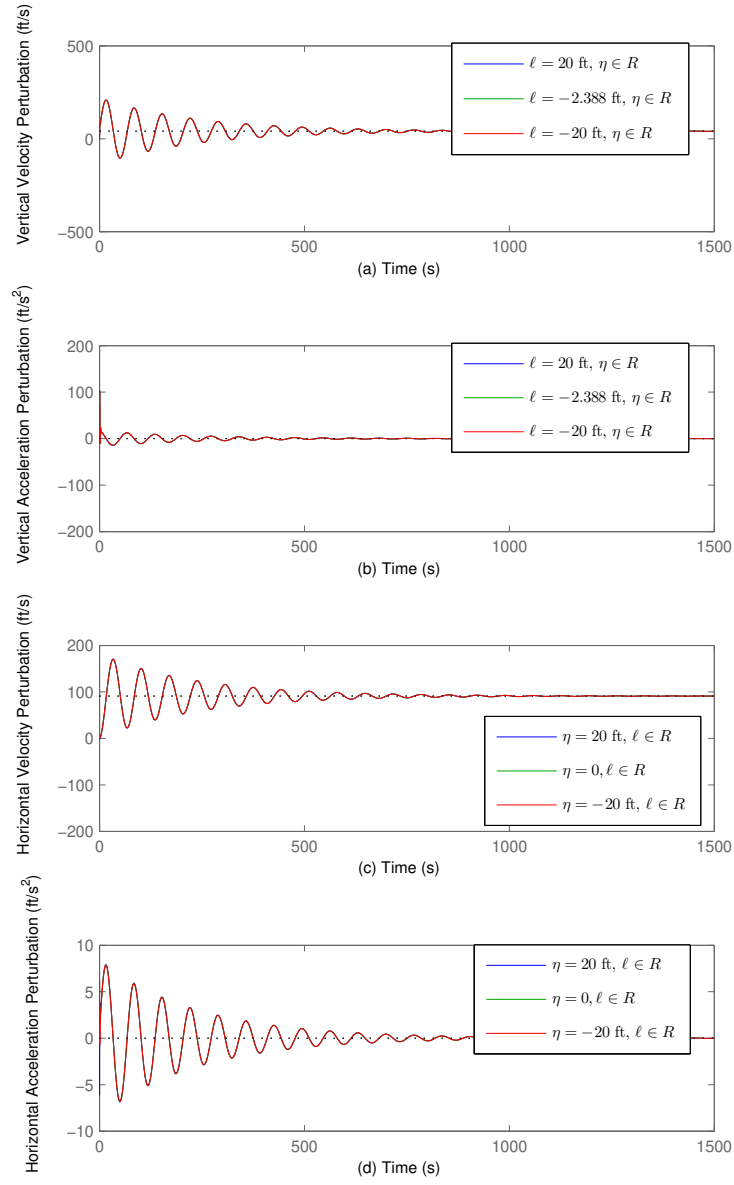


Figure 3: The responses of the vertical velocity perturbation  $\delta v_{pv}(t)$ , the vertical acceleration perturbation  $\delta \dot{v}_{pv}(t)$ , the horizontal velocity perturbation  $\delta v_{ph}(t)$ , and the horizontal acceleration perturbation  $\delta \dot{v}_{ph}(t)$  of a typical business jet to the 1-deg step elevator deflection  $\delta e(t) = 0.0171(t - t_0^+)$  at  $t_0 = 0$  based on the aircraft parameters given in [17]. Note that, for all values of  $(\ell, \eta)$ , the poles in (87) and (88) are close to the imaginary axis. Thus,  $\delta v_{pv}(t)$ ,  $\delta \dot{v}_{pv}(t)$ ,  $\delta v_{ph}(t)$ , and  $\delta \dot{v}_{ph}(t)$  reach their steady states values slowly. As shown in Figure 2, the initial curvatures of  $\delta v_{pv}(t)$  and  $\delta v_{ph}(t)$  are different for different values of  $(\ell, \eta)$ . However, for all values of  $(\ell, \eta)$ , the vertical velocity perturbation and the horizontal velocity perturbation approach nonzero constants, and both acceleration perturbations approach zero.

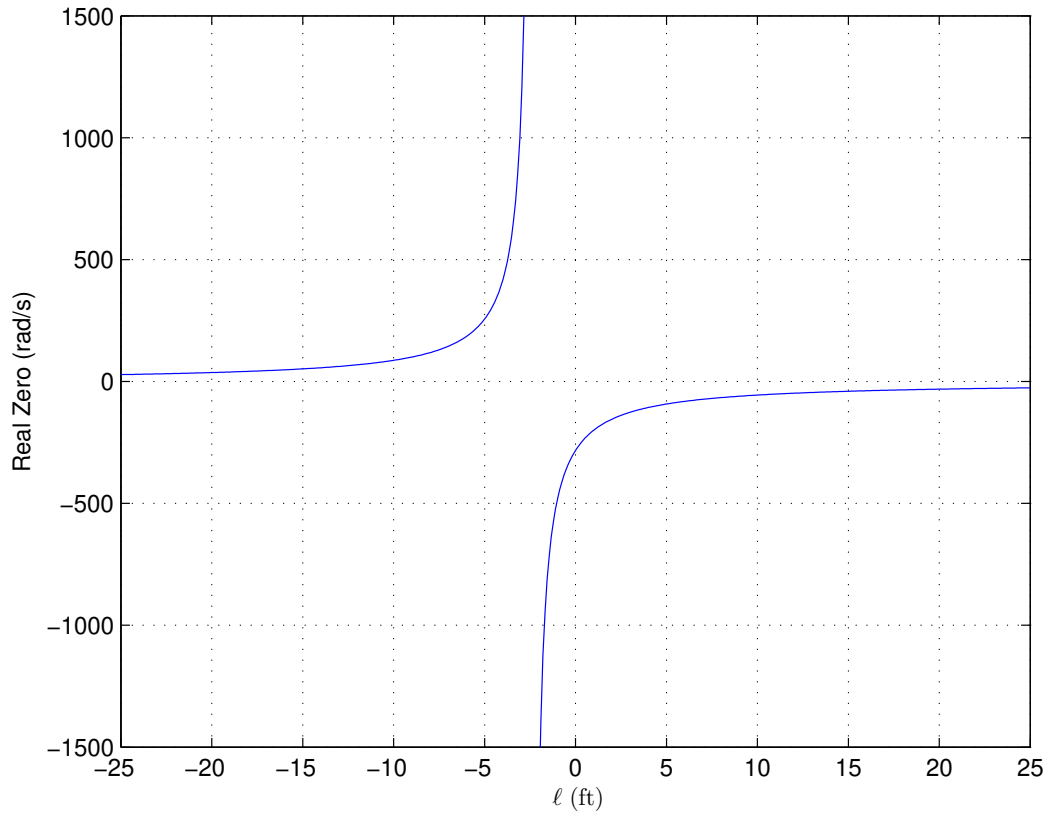


Figure 4: The real zero of a business jet based on data given in [17]. This plot shows the location of one of the real zeros of the numerator of the transfer function  $G_{\delta v_{pv}/\delta e}(s)$  from the elevator input  $\delta e$  to the vertical velocity  $\delta v_{pv}$  of the aircraft at p as a function of the component  $\ell$  along the direction  $\hat{k}_{AC}$  of the location of p. Note that negative values of  $\ell$  correspond to locations of p aft of the aircraft's center of mass, that is, toward the tail of the aircraft. The asymptotic values of the real zero are  $1.349 \times 10^{-4}$  rad/s as  $\ell$  approaches  $-\infty$ , and  $-1.366 \times 10^{-4}$  rad/s as  $\ell$  approaches  $\infty$ . Figure 5 shows the locations of the remaining real zeros.

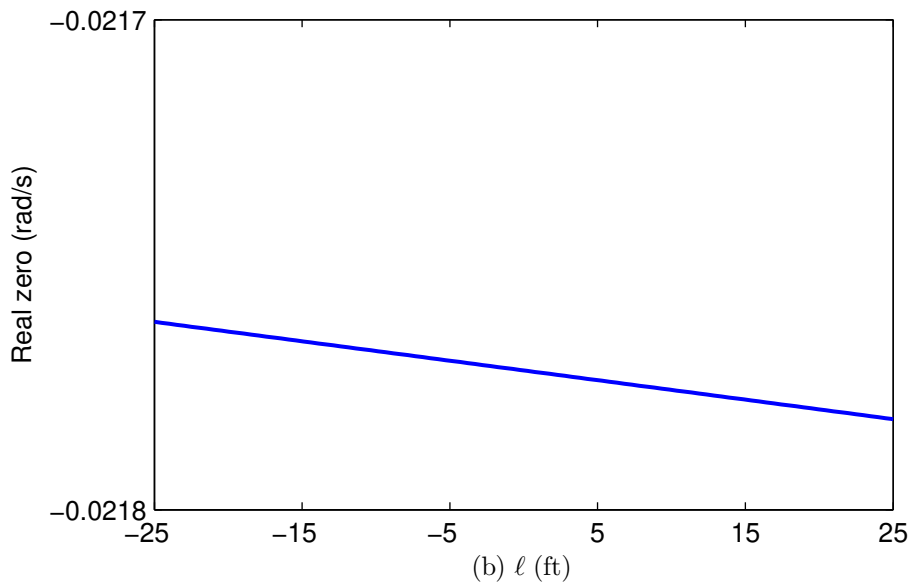
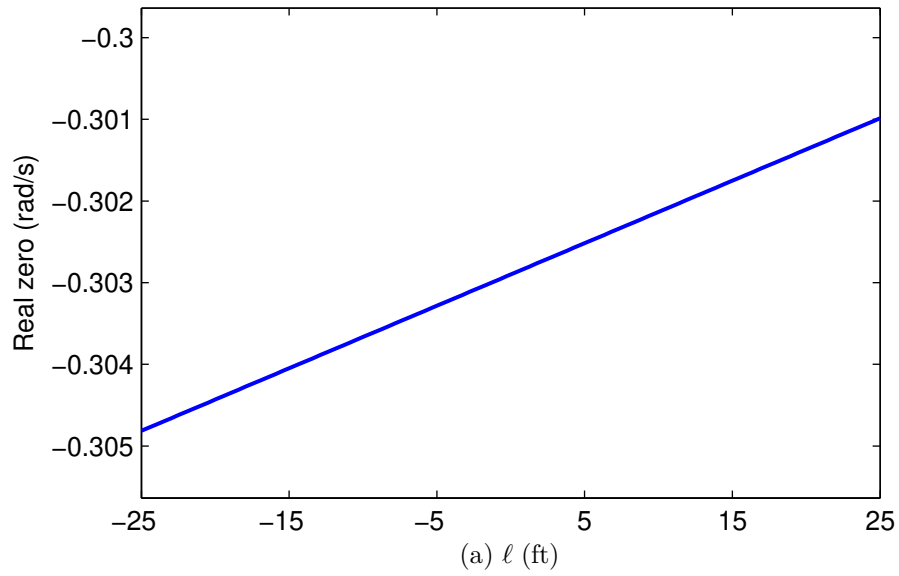
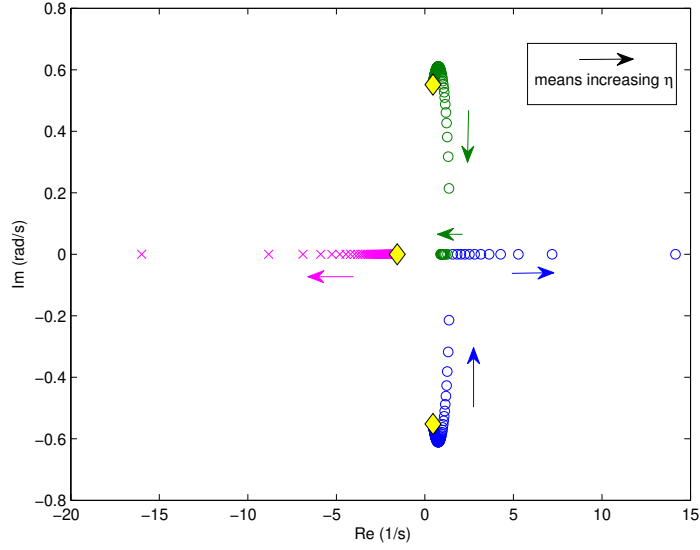
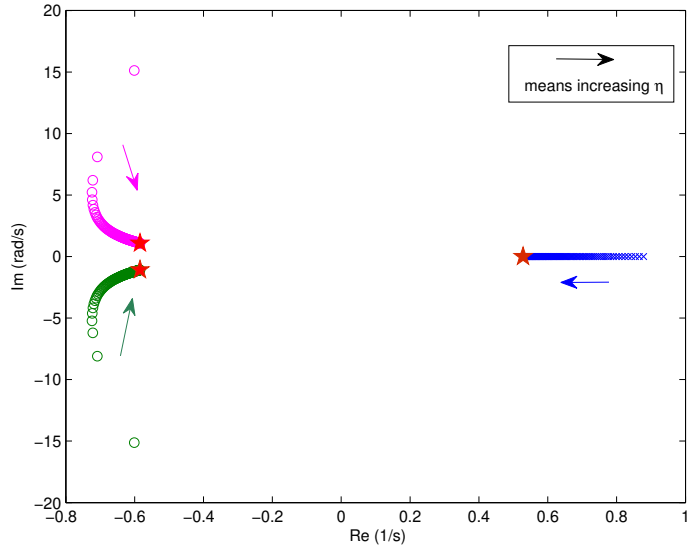


Figure 5: Zeros of the transfer function  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$ . For  $\ell \in [-25, 25]$  ft, these plots show the locations of the two remaining zeros of  $G_{\delta\hat{v}_{pv}/\delta\hat{e}}(s)$ , which are real and do not vanish at the IACR.





(a)



(b)

Figure 6: Zeros of the transfer function  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$ . (a) shows the locations of the zeros of  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  for each location of  $p$  along  $\hat{k}_{AC}$  parameterized by  $\eta \in [-25 \text{ ft}, 0.1 \text{ ft}]$ , where  $\eta_{IACR} = 0 \text{ ft}$ . The diamonds denote the zeros for  $\eta = -25 \text{ ft}$ . The zero denoted by crosses approaches  $-\infty$  as  $\eta$  increases toward  $\eta_{IACR}$ ; one of the zeros denoted by circles approaches  $0.88 \text{ rad/s}$  as  $\eta$  increases toward  $\eta_{IACR}$ , while the remaining zero approaches  $\infty \text{ rad/s}$  as  $\eta$  increases toward  $\eta_{IACR}$ . (b) shows the locations of the zeros of  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  for each location of  $p$  along  $\hat{k}_{AC}$  parameterized by  $\eta \in [0.1 \text{ ft}, 25 \text{ ft}]$ , where  $\eta_{IACR} = 0 \text{ ft}$ . The stars denote the zeros for  $\eta = 25 \text{ ft}$ . One of the zeros denoted by circles approaches  $j\infty \text{ rad/s}$  as  $\eta$  decreases toward  $\eta_{IACR}$ , while the remaining zero approaches  $-j\infty \text{ rad/s}$  on the real axis as  $\eta$  decreases toward  $\eta_{IACR}$ . The zero denoted by crosses approaches  $0.88 \text{ rad/s}$  as  $\eta$  decreases toward  $\eta_{IACR}$ . Consequently, two zeros of  $G_{\delta\hat{v}_{ph}/\delta\hat{e}}(s)$  vanish at the IACR.

$S$	wing area
$b$	wing tip-to-tip distance
$\bar{c}$	wing mean chord
$\rho$	air density
$V_{AC}$	aircraft speed
$p_d$	dynamic pressure $\frac{1}{2}\rho V_{AC}^2$

Table 1: Aerodynamic parameters. These parameters characterize the basic features of the aircraft for steady longitudinal flight.

$C_L$	$\frac{L}{\rho_d S}$
$C_D$	$\frac{D}{\rho_d S}$
$C_L(u, q, r, \delta\alpha, \delta\dot{\alpha}, \delta e)$	$C_{L_0} + \frac{1}{U_0} C_{L_{u_0}} u + \frac{\bar{c}}{2U_0} C_{L_{q_0}} q + \frac{b}{2U_0} C_{L_{r_0}} r$ $+ C_{L_{\alpha_0}} \delta\alpha + \frac{\bar{c}}{2U_0} C_{L_{\dot{\alpha}_0}} \delta\dot{\alpha} + C_{L_{\delta e_0}} \delta e$
$C_D(u, q, r, \delta\alpha, \delta\dot{\alpha}, \delta e)$	$C_{D_0} + \frac{1}{U_0} C_{D_{u_0}} u + \frac{\bar{c}}{2U_0} C_{D_{q_0}} q + \frac{b}{2U_0} C_{D_{r_0}} r$ $+ C_{D_{\alpha_0}} \delta\alpha + C_{D_{\dot{\alpha}_0}} \delta\dot{\alpha} + C_{D_{\delta e_0}} \delta e$
$C_{L_{u_0}}$	$\left. \frac{\partial C_L}{\partial \left(\frac{u}{U_0}\right)} \right _0$
$C_{L_{q_0}}$	$\left. \frac{\partial C_L}{\partial \left(\frac{\bar{c}q}{2U_0}\right)} \right _0$
$C_{L_{r_0}}$	$\left. \frac{\partial C_L}{\partial \left(\frac{br}{2U_0}\right)} \right _0$
$C_{L_{\alpha_0}}$	$\left. \frac{\partial C_L}{\partial \delta\alpha} \right _0$
$C_{L_{\dot{\alpha}_0}}$	$\left. \frac{\partial C_L}{\partial \left(\frac{\bar{c}\delta\dot{\alpha}}{2U_0}\right)} \right _0$
$C_{D_{u_0}}$	$2K C_{L_0} C_{L_{u_0}}$
$C_{D_{q_0}}$	$2K C_{L_0} C_{L_{q_0}}$
$C_{D_{r_0}}$	$2K C_{L_0} C_{L_{r_0}}$
$C_{D_{\alpha_0}}$	$2K C_{L_0} C_{L_{\alpha_0}}$
$C_{D_{\dot{\alpha}_0}}$	$2K C_{L_0} C_{L_{\dot{\alpha}_0}}$
$C_{D_{\delta e_0}}$	$2K C_{L_0} C_{L_{\delta e_0}}$
$C_y$	$\frac{F_{A_y}}{\rho_d S}$
$C_{x_0}$	$-C_{D_0}$
$C_{x_{u_0}}$	$\left. \frac{\partial C_x}{\partial \left(\frac{u}{U_0}\right)} \right _0$

Table 2: Stability derivatives. The aerodynamic parameters are given in Table 1. These lift, drag, force, and moment derivatives model the aerodynamic forces and moments applied to the aircraft due to perturbations away from steady longitudinal flight. This table is based on Table 6.1 of [16].

Stability parameter	Definition	Units
$X_{u_0}$	$-\frac{p_{d_0} S}{m U_0} (2C_{D_0} + C_{D_{u_0}})$	1/sec
$X_{T_{u_0}}$	$\frac{p_{d_0} S}{m U_0} (2C_{T_{x_0}} + C_{T_{x_{u_0}}})$	1/sec
$X_{\alpha_0}$	$\frac{p_{d_0} S}{m} (C_{L_0} - C_{D_{\delta\alpha_0}})$	ft/sec <sup>2</sup> -rad
$X_{\delta e_0}$	$\frac{p_{d_0} S}{m} C_{D_{\delta e_0}}$	ft/sec <sup>2</sup> -rad
$Z_{u_0}$	$-\frac{p_{d_0} S}{m U_0} (2C_{L_0} + C_{L_{u_0}})$	1/sec
$Z_{\alpha_0}$	$\frac{p_{d_0} S}{m} (C_{L_{\alpha_0}} - C_{D_0})$	ft/sec <sup>2</sup> -rad
$Z_{\dot{\alpha}_0}$	$-\frac{p_{d_0} S \bar{c}}{2m U_0} C_{L_{\dot{\alpha}_0}}$	ft/sec-rad
$Z_{q_0}$	$\frac{p_{d_0} S \bar{c}}{2m U_0} C_{L_{q_0}}$	ft/sec-rad
$Z_{\delta e_0}$	$\frac{p_{d_0} S}{m} C_{L_{\delta e_0}}$	ft/sec <sup>2</sup> -rad
$M_{u_0}$	$\frac{p_{d_0} S \bar{c}}{I_{yy} U_0} (2C_{m_0} + C_{m_{u_0}})$	rad/ft-sec
$M_{T_{u_0}}$	$\frac{p_{d_0} S \bar{c}}{I_{yy} U_0} (2C_{T_{m_0}} + C_{T_{m_{u_0}}})$	1/ft-sec
$M_{\alpha_0}$	$\frac{p_{d_0} S \bar{c}}{I_{yy}} C_{m_{\alpha_0}}$	1/sec <sup>2</sup>
$M_{T_{\alpha_0}}$	$\frac{p_{d_0} S \bar{c}}{I_{yy}} C_{T_{m_{\alpha_0}}}$	1/sec <sup>2</sup>
$M_{\dot{\alpha}_0}$	$\frac{p_{d_0} S \bar{c}^2}{2I_{yy} U_0} C_{m_{\dot{\alpha}_0}}$	1/sec
$M_{q_0}$	$\frac{p_{d_0} S \bar{c}^2}{2I_{yy} U_0} C_{m_{q_0}}$	1/sec
$M_{\delta e_0}$	$\frac{p_{d_0} S \bar{c}}{I_{yy}} C_{m_{\delta e_0}}$	1/sec <sup>2</sup>

Table 3: Stability parameters. These parameters are functions of the aircraft parameters and stability derivatives given in Table 2. This table is based on Table 6.3 of [16].

$A_u$	$X_{\delta e_0}(U_0 - Z_{\dot{\alpha}_0})$
$B_u$	$-X_{\delta e_0} [(U_0 - Z_{\dot{\alpha}_0})M_{q_0} + Z_{\alpha_0} + M_{\dot{\alpha}_0}(U_0 + Z_{q_0}) + Z_{\delta e_0}X_{\alpha_0}]$
$C_u$	$X_{\delta e_0} [M_{q_0}Z_{\alpha_0} - (M_{\alpha_0} + M_{T_{\alpha_0}})(U_0 + Z_{q_0})]$ $-Z_{\delta e_0} [M_{\dot{\alpha}_0}g + X_{\alpha_0}M_{q_0}] + M_{\delta e_0} [X_{\dot{\alpha}_0}(U_0 + Z_{q_0}) - (U_0 - Z_{\dot{\alpha}_0})g]$
$D_u$	$-Z_{\delta e_0}M_{\alpha_0}g + M_{\delta e_0}Z_{\alpha_0}g$
$A_\alpha$	$Z_{\delta e_0}$
$B_\alpha$	$X_{\delta e_0}Z_{u_0} + Z_{\delta e_0} [-M_{q_0} - (X_{u_0} + X_{T_{u_0}})] + M_{\delta e_0}(U_0 + Z_{q_0})$
$C_\alpha$	$X_{\delta e_0} [(U_0 + Z_{q_0})(M_{u_0} + M_{T_{u_0}}) - M_{q_0}Z_{u_0}] + Z_{\delta e_0}M_{q_0}(X_{u_0} + X_{T_{u_0}})$ $-M_{\delta e_0}(U_0 + Z_{q_0})(X_{u_0} + X_{T_{u_0}})$
$D_\alpha$	$Z_{\delta e_0}(M_{u_0} + M_{T_{u_0}})g - M_{\delta e_0}Z_{u_0}g$
$A_\theta$	$M_{\delta e_0}(U_0 - Z_{\dot{\alpha}_0}) + Z_{\delta e_0}M_{\dot{\alpha}_0}$
$B_\theta$	$X_{\delta e_0} [Z_{u_0}M_{\dot{\alpha}_0} + (U_0 - Z_{\dot{\alpha}_0})(M_{u_0} + M_{T_{u_0}})]$ $+Z_{\delta e_0} [(M_{\alpha_0} + M_{T_{\alpha_0}}) - M_{\dot{\alpha}_0}(X_{u_0} + X_{T_{u_0}})]$ $+M_{\delta e_0} [-Z_{\alpha_0} - (U_0 - Z_{\dot{\alpha}_0})(X_{u_0} + X_{T_{u_0}})]$
$C_\theta$	$X_{\delta e_0} [(M_{\alpha_0} + M_{T_{\alpha_0}})Z_{u_0} - Z_{\alpha_0}(M_{u_0} + M_{T_{u_0}})]$ $+M_{\delta e_0} [Z_{\alpha_0}(X_{u_0} + X_{T_{u_0}}) - X_{\alpha_0}Z_{u_0}]$ $+Z_{\delta e_0} [-(M_{\alpha_0} + M_{T_{\alpha_0}})(X_{u_0} + X_{T_{u_0}}) + X_{\alpha_0}(M_{u_0} + M_{T_{u_0}})]$
$A_v$	$-\ell A_\theta + U_0 A_\alpha$
$B_v$	$-\ell B_\theta - U_0 A_\theta + U_0 B_\alpha$
$C_v$	$-\ell C_\theta - U_0 B_\theta + U_0 C_\alpha$
$D_v$	$-U_0 C_\theta + U_0 D_\alpha$
$A_h$	$\eta A_\theta + A_u$
$B_h$	$\eta B_\theta + B_u$
$C_h$	$\eta C_\theta + C_u$
$D_h$	$D_u$

Table 4: Transfer function numerator coefficients. These coefficients appear in the transfer functions from the elevator deflection  $\delta\hat{e}(s)$  to  $\hat{u}(s)$ ,  $\delta\hat{\alpha}(s)$ ,  $\hat{\theta}(s)$ ,  $\delta\hat{v}_{ph}(s)$ , and  $\delta\hat{v}_{pv}(s)$ .

$E$	$U_0 - Z_{\dot{\alpha}_0}$
$F$	$-(U_0 - Z_{\dot{\alpha}_0})(X_{u_0} - X_{T_{u_0}} + M_{q_0}) - Z_{\alpha_0} - M_{\dot{\alpha}_0}(U_0 + Z_{q_0})$
$G$	$(X_{u_0} - X_{T_{u_0}}) [M_{q_0}(U_0 - Z_{\dot{\alpha}_0}) + Z_{\alpha_0} - M_{\dot{\alpha}_0}(U_0 + Z_{q_0})] + M_{q_0}Z_{\alpha_0}$ $-Z_{u_0}X_{\alpha_0} - (M_{\alpha_0} + M_{T_{\alpha_0}})(U_0 + Z_{q_0})$
$H$	$g [Z_{u_0}M_{\dot{\alpha}_0} + (M_{u_0} + M_{T_{u_0}})(U_0 - Z_{\dot{\alpha}_0})]$ $+ (M_{u_0} + M_{T_{u_0}}) [-X_{\alpha_0}(U_0 + Z_{q_0})] + Z_{u_0}X_{\alpha_0}M_{q_0}$ $+ (X_{u_0} + X_{T_{u_0}}) [(M_{\alpha_0} + M_{T_{\alpha_0}})(U_0 + Z_{q_0}) - M_{q_0}Z_{\alpha_0}]$
$I$	$g [(M_{\alpha_0} + M_{T_{\alpha_0}})Z_{u_0} - Z_{\alpha_0}(M_{u_0} + M_{T_{u_0}})]$

Table 5: Transfer function denominator coefficients. These coefficients appear in the transfer functions from the elevator deflection  $\delta\hat{e}(s)$  to  $\hat{u}(s)$ ,  $\delta\hat{\alpha}(s)$ ,  $\hat{\theta}(s)$ ,  $\delta\hat{v}_{ph}(s)$ , and  $\delta\hat{v}_{pv}(s)$ .

Stability parameter	Value	Units
$\Theta_0$	0.0000	rad
$U_0$	400.0000	kt
$X_{u_0}$	-0.0074	1/s
$X_{\Gamma_{u_0}}$	0.0000	1/s
$X_{\alpha_0}$	8.9782	ft-rad/s <sup>2</sup>
$X_{\delta e_0}$	0.0000	ft-rad/s <sup>2</sup>
$Z_{u_0}$	0.1390	1/s
$Z_{\alpha_0}$	-445.7224	ft-rad/s <sup>2</sup>
$Z_{\dot{\alpha}_0}$	-0.8705	ft-rad/s
$Z_{q_0}$	-1.8598	ft-rad/s
$Z_{\delta e_0}$	-42.1968	ft-rad/s <sup>2</sup>
$M_{u_0}$	0.0011	rad/ft-s
$M_{\Gamma_{u_0}}$	-0.0002	1/ft-s
$M_{\alpha_0}$	-7.4416	1/s <sup>2</sup>
$M_{\Gamma_{\alpha_0}}$	0.0000	1/s <sup>2</sup>
$M_{\dot{\alpha}_0}$	-0.4062	1/s
$M_{q_0}$	-0.9397	1/s
$M_{\delta e_0}$	-17.6737	1/s <sup>2</sup>

Table 6: Stability parameter values. These data for a business jet are given in [17, p. 330].

## Sidebar 1

### Center of Rotation and Center of Percussion

Consider the free rigid body shown in Figure S1, with concentrated masses  $m_1, \dots, m_n$  at distances of  $\ell_1, \dots, \ell_n$ , respectively, from the point  $O_B$ , which is the origin of the body-fixed frame  $F_B$ . The frame  $F_A$  is assumed to be an inertial frame. Consider a force  $\vec{F}$  that impacts the structure at point P and perpendicular to the body, and assume that R is the point on the body at which the velocity  $\vec{v}_{R/O_A/A}$  of R relative to  $O_A$  with respect to  $F_A$  is identically zero following the impact. The point R is the *center of rotation relative to P*; equivalently, P is the *center of percussion relative to R*. Let  $\ell_R$  and  $\ell_P$  denote the distances from the upper end of the body to R and P, respectively. The distance  $\ell_c$  from the upper end of the body to the center of mass c is given by

$$\ell_c = \frac{\sum_{i=1}^n m_i \ell_i}{m_{\text{total}}}, \quad (\text{S1})$$

where  $m_{\text{total}} \triangleq \sum_{i=1}^n m_i$  is the total mass of the body.

Next, the velocity  $\vec{v}_{R/O_A/A}$  of R relative to  $O_A$  with respect to  $F_A$  can be written as

$$\begin{aligned} \vec{v}_{R/O_A/A} &= \overset{A\bullet}{\vec{r}}_{R/O_A} \\ &= \overset{A\bullet}{\vec{r}}_{R/c} + \overset{A\bullet}{\vec{r}}_{c/O_A} \\ &= \vec{v}_{c/O_A/A} + \overset{B\bullet}{\vec{r}}_{R/c} + \vec{\omega}_{B/A} \times \vec{r}_{R/c} \\ &= \vec{v}_{c/O_A/A} + \vec{\omega}_{B/A} \times \vec{r}_{R/c}, \end{aligned} \quad (\text{S2})$$

where  $\vec{v}_{c/O_A/A}$  is the velocity of c relative to  $O_A$  with respect to  $F_A$ ,  $\vec{\omega}_{B/A}$  is the angular velocity of  $F_B$  relative to  $F_A$ , and  $\vec{r}_{R/c}$  is the position of R relative to c. Note that  $\overset{B\bullet}{\vec{r}}_{R/c} = 0$  since R and c are fixed in the



body. These vectors can be resolved in  $F_B$  as

$$\begin{aligned}\vec{v}_{R/O_A/A} &= v_R \hat{j}_A, \\ \vec{v}_{c/O_A/A} &= v_c \hat{j}_A, \\ \vec{\omega}_{B/A} &= \omega \hat{k}_A, \\ \vec{r}_{R/c} &= (\ell_R - \ell_c) \hat{i}_A.\end{aligned}$$

Thus, (S2) implies that

$$\begin{bmatrix} 0 \\ v_R \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v_c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} \ell_R - \ell_c \\ 0 \\ 0 \end{bmatrix},$$

that is,

$$v_R = v_c + (\ell_R - \ell_c)\omega. \quad (\text{S3})$$

Next, viewing  $O_A$  as an unforced particle, Newton's second law implies

$$\vec{F} = m_{\text{total}} \frac{A \bullet}{v_{c/O_A/A}},$$

where  $\vec{F} = F_0 \delta(t) \hat{j}_B$ . Thus, since  $\frac{A \bullet}{v_{c/O_A/A}} = \dot{v}_c \hat{j}_A$ , it follows that

$$F_0 \delta(t) = m_{\text{total}} \dot{v}_c(t). \quad (\text{S4})$$

Next, the moment  $\vec{M}_{P/c}$  on P about c due to  $\vec{F}$  is given by

$$\vec{M}_{P/c} = \vec{r}_{P/c} \times \vec{F} = I_c \frac{A \bullet}{\omega_{B/A}},$$

where  $I_c \triangleq \sum_{i=1}^n m_i (\ell_i - \ell_c)^2$  is the moment of inertia of the body relative to c. The position of P relative to c is given by  $\vec{r}_{P/c} = (\ell_P - \ell_c) \hat{i}_A$ . Therefore,

$$\begin{bmatrix} \ell_P - \ell_c \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ F_0 \delta(t) \\ 0 \end{bmatrix} = I_c \begin{bmatrix} 0 \\ 0 \\ \dot{\omega}(t) \end{bmatrix},$$

that is,

$$F_0(\ell_P - \ell_c)\delta(t) = I_c\dot{\omega}(t). \quad (\text{S5})$$

Using (S4) and (S5), and differentiating (S3), it follows that

$$\dot{v}_R(t) = \left( \frac{F_0}{m_{\text{total}}} + (\ell_R - \ell_c) \frac{F_0(\ell_P - \ell_c)}{I_c} \right) \delta(t).$$

Since R is the center of rotation, we have, for all  $t \geq 0$ ,

$$v_R(t) = \left( \frac{1}{m_{\text{total}}} + \frac{(\ell_R - \ell_c)(\ell_P - \ell_c)}{I_c} \right) F_0 = 0.$$

Therefore,

$$\frac{1}{m_{\text{total}}} + \frac{(\ell_R - \ell_c)(\ell_P - \ell_c)}{I_c} = 0.$$

It follows that

$$\ell_R = \ell_c - \frac{I_c}{m_{\text{total}}(\ell_P - \ell_c)}. \quad (\text{S6})$$

Consequently, if the force impacts the body at the center of percussion P located at  $\ell_P$ , then the translational velocity  $v_R$  at the center of rotation located at  $\ell_R$  given by (S6) is identically zero.

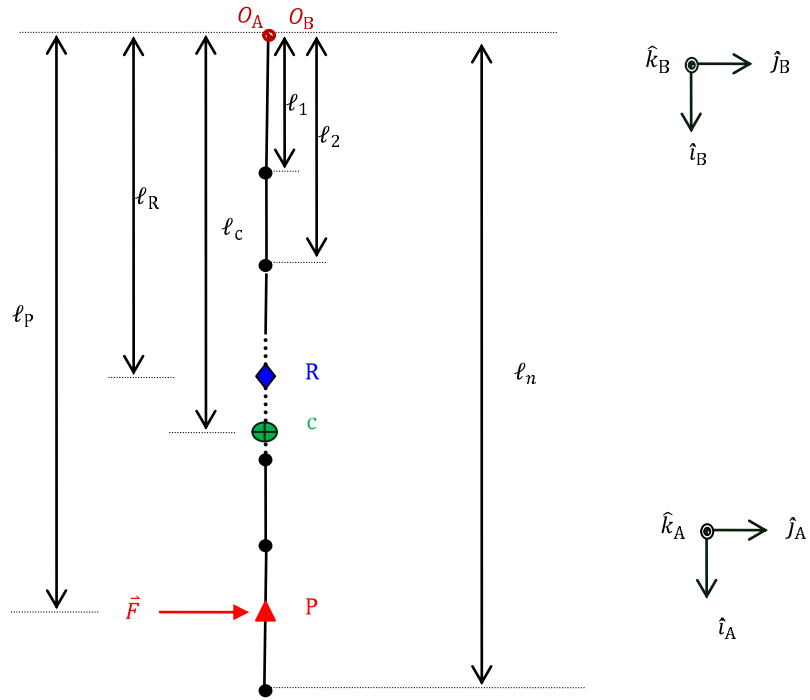


Figure S1: A free rigid body with nonuniform concentrated masses  $m_1, \dots, m_n$  at distances of  $l_1, \dots, l_n$  from the upper end  $O_B$  of the structure. The point  $R$  is the center of rotation relative to  $P$ , while the point  $P$  is the center of percussion relative to  $R$ .

## Sidebar 2

### Instantaneous Velocity Center of Rotation

Let  $\mathcal{B}$  be a rigid body with body-fixed frame  $F_B$ , let  $F_A$  be a frame with origin  $O_A$ , and let  $\vec{\omega}_{B/A}$  be the angular velocity of  $F_B$  relative to  $F_A$ . A point  $p$  that is fixed relative to  $\mathcal{B}$  is an *instantaneous velocity center of rotation* (IVCR) of  $\mathcal{B}$  at time  $t$  if  $\vec{\omega}_{B/A}(t) \neq 0$  and  $\vec{v}_{p/O_A/A}(t) = 0$  [S1, pp. 147-149], [S2, pp. 49-52]. The motion of  $\mathcal{B}$  can be viewed as instantaneously rotating about  $p$ . See Figure S2.

Let  $q$  be a point that is fixed relative to  $\mathcal{B}$ . It follows from the definition of an IVCR and the transport theorem that  $p$  is an IVCR of  $\mathcal{B}$  if and only if  $\vec{\omega}_{B/A} \neq 0$  and

$$\vec{v}_{p/O_A/A} = \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A} = 0. \quad (S7)$$

Resolving  $\vec{v}_{q/O_A/A}$ ,  $\vec{\omega}_{B/A}$ , and  $\vec{r}_{p/q}$  in  $F_B$  as

$$v \triangleq \vec{v}_{q/O_A/A} \Big|_B, \quad \omega \triangleq \vec{\omega}_{B/A} \Big|_B, \quad r \triangleq \vec{r}_{p/q} \Big|_B, \quad (S8)$$

(S7) can be rewritten as

$$\omega^\times r + v = 0. \quad (S9)$$

Note that the existence of an IVCR depends on the existence of a solution to (S9). Since  $\omega^\times$  is singular, (S9) has either zero or infinitely many solutions. Let  $\mathcal{R}$  denote range.

**Fact S1.** The following statements hold:

- i) If  $v \notin \mathcal{R}(\omega^\times)$ , then  $\mathcal{B}$  has no IVCR.
- ii) If  $v \in \mathcal{R}(\omega^\times)$ , then  $\mathcal{B}$  has infinitely many IVCRs.
- iii) Suppose  $v \in \mathcal{R}(\omega^\times)$ . Then  $p$  is an IVCR if and only if there exists  $\alpha \in \mathbb{R}$  such that

$$r = \alpha \omega - \frac{1}{|\omega|^2} \omega \times v. \quad (S10)$$

Note that if  $\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} = \omega^T v = -\omega^T(\omega^\times r) = 0$ . Hence, if  $\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} \neq 0$ , then  $\mathcal{B}$  has no IVCR. This situation occurs, for example, in bullet flight, where the translational velocity is parallel to its angular velocity.

**Fact S2.**  $p$  is an IVCR of  $\mathcal{B}$  if and only if  $p$  satisfies the following conditions:

$$\begin{aligned} i) \quad & \vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} = 0. \\ ii) \quad & \vec{\omega}_{B/A} \times \left( \vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} \right) = 0. \end{aligned}$$

In this case,

$$\vec{r}_{p/q} = \frac{1}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} + \frac{\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A}. \quad (\text{S11})$$

**Proof** Assume that  $p$  is an IVCR of  $\mathcal{B}$ . Then it follows from (S7) that

$$\vec{\omega}_{B/A} \cdot \vec{v}_{q/O_A/A} = \vec{\omega}_{B/A} \cdot \left( -\vec{\omega}_{B/A} \times \vec{r}_{p/q} \right) = 0,$$

which proves *i*). To prove *ii*), it follows from (S7) that

$$\vec{\omega}_{B/A} \times \left( \vec{r}_{p/c} - \frac{1}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A} \times \vec{v}_{c/O_A/A} \right) = \vec{\omega}_{B/A} \times \vec{r}_{p/c} + \vec{v}_{c/O_A/A} = 0.$$

Hence, *ii*) holds.

Conversely, it follows from *ii*) that there exists  $\alpha \in \mathbb{R}$  such that  $\vec{r}_{p/q} = \frac{1}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} + \alpha \vec{\omega}_{B/A}$ . Using *i*) and *ii*), it follows that

$$\begin{aligned} \vec{v}_{p/O_A/A} &= \vec{v}_{p/q/A} + \vec{v}_{q/O_A/A} \\ &= \vec{v}_{p/q/B} + \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A} \\ &= \vec{\omega}_{B/A} \times \left( \frac{1}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} + \alpha \vec{\omega}_{B/A} \right) + \vec{v}_{q/O_A/A} \\ &= -\vec{v}_{q/O_A/A} + \vec{v}_{q/O_A/A} \\ &= 0. \end{aligned}$$

To show (S11), assume  $p$  is an IVCR of  $\mathcal{B}$ . It follows from (S7) that

$$\vec{\omega}_{B/A} \times \vec{v}_{p/O_A/A} = \vec{\omega}_{B/A} \times \left( \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{v}_{q/O_A/A} \right) = 0,$$

which implies that

$$(\vec{\omega}_{B/A} \cdot \vec{r}_{p/q})\vec{\omega}_{B/A} - |\vec{\omega}_{B/A}|^2\vec{r}_{p/q} + \vec{\omega}_{B/A} \times \vec{v}_{q/O_A/A} = 0. \quad (\text{S12})$$

Hence, solving for  $\vec{r}_{p/q}$  in (S12) yields (S11). □

## References

[S1] H. Josephs and R. L. Huston, *Dynamics of Mechanical Systems*, CRC Press, 2002.

[S2] A. K. Mallik, A. Ghosh, and G. Dittrich, *Kinematic Analysis and Synthesis of Mechanisms*, CRC Press, 1994.

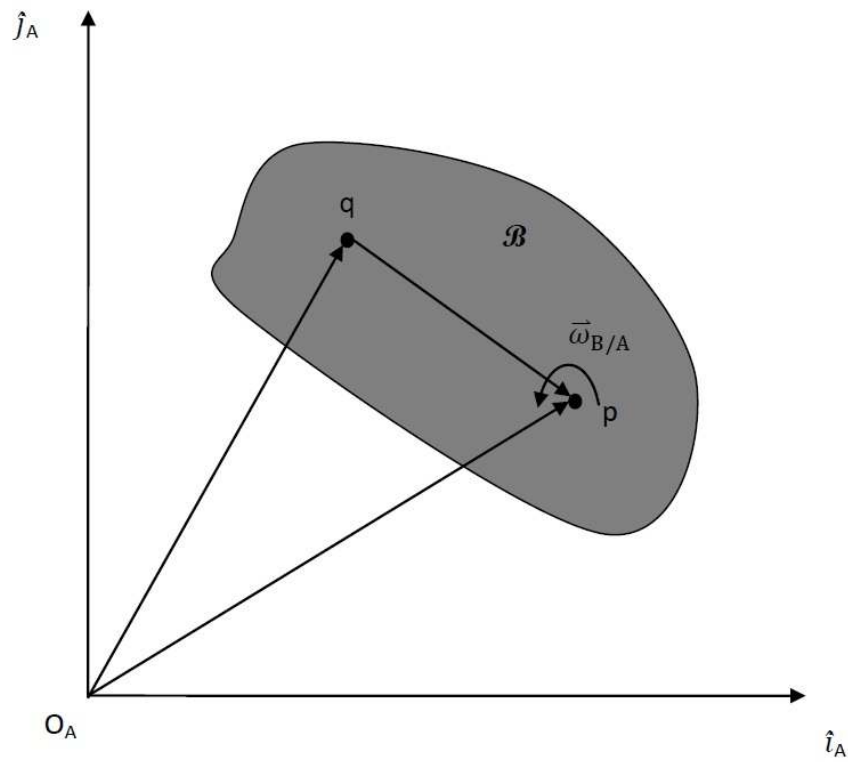


Figure S2: Instantaneous velocity center of rotation.  $\mathcal{B}$  is a rigid body. The point  $q$ , which is fixed relative to  $\mathcal{B}$ .  $F_A$  is a frame with origin  $O_A$ ,  $\vec{\omega}_{B/A}$  is the angular velocity of  $F_B$  relative to  $F_A$ , and it is assumed that  $\vec{\omega}_{B/A} \neq 0$ . The point  $p$ , which is fixed relative to  $\mathcal{B}$ , has the property that, at time  $t$ , the velocity of  $p$  relative to  $O_A$  with respect to the frame  $F_A$  is zero. Thus  $\mathcal{B}$  is instantaneously rotating about  $p$ .

### Sidebar 3

## Instantaneous Acceleration Center of Rotation

Let  $\mathcal{B}$  be a rigid body with body-fixed frame  $F_B$ , let  $F_A$  be a frame with origin  $O_A$ , and let  $\vec{\omega}_{B/A}$  be the angular velocity of  $F_B$  relative to  $F_A$ . A point  $p$  that is fixed relative to  $\mathcal{B}$  is an *instantaneous acceleration center of rotation* (IACR) at time  $t$  if  $\vec{a}_{p/O_A/A}(t) = 0$  [S1, pp. 150-155], [S3, pp. 336-338].

To characterize this property, let  $q$  be a point fixed relative to the rigid body  $\mathcal{B}$ . It follows from the definition of an IACR and the transport theorem that  $p$  is an IACR if and only if

$$\vec{a}_{p/O_A/A} = \vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} = 0. \quad (\text{S13})$$

Resolving  $\vec{a}_{q/O_A/A}$ ,  $\vec{\omega}_{B/A}$ , and  $\vec{r}_{p/q}$  in  $F_B$  as

$$a \triangleq \vec{a}_{q/O_A/A}|_B, \quad \omega \triangleq \vec{\omega}_{B/A}|_B, \quad \dot{\omega} \triangleq \vec{\omega}_{B/A}^{\bullet}|_B, \quad r \triangleq \vec{r}_{p/q}|_B, \quad (\text{S14})$$

(S13) can be rewritten as

$$(\dot{\omega}^\times + \omega^{\times 2})r + a = 0. \quad (\text{S15})$$

Note that there exists an IACR if and only if there exists  $r$  satisfying (S15). Furthermore, (S15) can yield zero, one, or infinitely many IACRs.

Note that the determinant of  $\dot{\omega}^\times + \omega^{\times 2}$  is given by

$$\begin{aligned} \det(\dot{\omega}^\times + \omega^{\times 2}) &= (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}^{\bullet})^2 - (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}^{\bullet})(\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}^{\bullet}) \\ &= -|\vec{\omega}_{B/A}|^2 |\vec{\omega}_{B/A}^{\bullet}|^2 \sin^2 \theta, \end{aligned} \quad (\text{S16})$$

where

$$\theta \triangleq \cos^{-1} \frac{\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}^{\bullet}}{|\vec{\omega}_{B/A}| |\vec{\omega}_{B/A}^{\bullet}|}. \quad (\text{S17})$$



**Fact S3.** There exists a unique IACR if and only if  $\theta/\pi$  is not an integer,  $\vec{\omega}_{B/A} \neq 0$ , and  $\vec{\omega}_{B/A}^{\bullet} \neq 0$ .

**Proof** Suppose (S15) has a unique solution. Therefore,  $\dot{\omega}^\times + \omega^{\times 2}$  is nonsingular, and thus the determinant of  $\dot{\omega}^\times + \omega^{\times 2}$  is nonzero. Hence, it follows from (S16) that

$$\det(\dot{\omega}^\times + \omega^{\times 2}) = -|\vec{\omega}_{B/A}|^2 |\vec{\omega}_{B/A}^{\bullet}|^2 \sin^2 \theta \neq 0,$$

which implies that  $\theta/\pi$  is not an integer,  $\vec{\omega}_{B/A} \neq 0$ ,  $\vec{\omega}_{B/A}^{\bullet} \neq 0$ , and  $\vec{\omega}_{B/A}^{\bullet} \neq 0$ .

Conversely, since  $\theta/\pi$  is not an integer,  $\vec{\omega}_{B/A} \neq 0$ , and  $\vec{\omega}_{B/A}^{\bullet} \neq 0$ , it follows from (S16) that  $\det(\dot{\omega}^\times + \omega^{\times 2}) = -|\vec{\omega}_{B/A}|^2 |\vec{\omega}_{B/A}^{\bullet}|^2 \sin^2 \theta \neq 0$ , which implies that (S15) has a unique solution.  $\square$

**Fact S4.** Assume  $\vec{\omega}_{B/A} = 0$ ,  $\vec{\omega}_{B/A}^{\bullet} \neq 0$ , and  $\vec{a}_{q/O_A/A} \neq 0$ . Then p is an IACR if and only if p satisfies the following conditions:

$$\begin{aligned} i) \quad & \vec{\omega}_{B/A}^{\bullet} \cdot \vec{a}_{q/O_A/A} = 0. \\ ii) \quad & \vec{\omega}_{B/A}^{\bullet} \times \left( \vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{q/O_A/A} \right) = 0. \end{aligned}$$

In this case, p satisfies

$$\vec{r}_{p/1} = \frac{1}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{q/O_A/A} + \frac{\vec{\omega}_{B/A}^{\bullet} \cdot \vec{r}_{p/q}}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet}. \quad (\text{S18})$$

**Proof** Assume p is an IACR. Since  $\vec{\omega}_{B/A} = 0$ , it follows from (S13) that

$$\begin{aligned} \vec{\omega}_{B/A}^{\bullet} \cdot \vec{a}_{q/O_A/A} &= \vec{\omega}_{B/A}^{\bullet} \cdot \left( -\vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} - \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) \right) \\ &= -\vec{\omega}_{B/A}^{\bullet} \cdot \left( \vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} \right) \\ &= 0, \end{aligned}$$

which proves *i*). To prove *ii*), it follows from (S13) that

$$\vec{\omega}_{B/A}^{\bullet} \times \left( \vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{q/O_A/A} \right) = \vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} + \vec{a}_{q/O_A/A} = 0.$$

Hence, *ii*) holds.

Conversely, it follows from *ii*) that there exists  $\alpha \in \mathbb{R}$  such that

$$\vec{r}_{p/q} = \frac{1}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{q/O_A/A} + \alpha \vec{\omega}_{B/A}^{\bullet}. \quad (\text{S19})$$

Using *i*) and (S19), it follows that

$$\begin{aligned} \vec{a}_{p/O_A/A} &= \vec{r}_{p/O_A}^{\bullet\bullet} \\ &= \vec{r}_{p/c}^{\bullet\bullet} + \vec{r}_{c/O_A}^{\bullet\bullet} \\ &= \vec{r}_{p/q}^{\bullet\bullet} + 2\vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} + \vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} + \vec{\omega}_{B/A}^{\bullet} \times (\vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} \\ &= \vec{\omega}_{B/A}^{\bullet} \times \left( \frac{1}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{q/O_A/A} + \alpha \vec{\omega}_{B/A}^{\bullet} \right) + \vec{a}_{q/O_A/A} \\ &= \frac{\vec{\omega}_{B/A}^{\bullet} \cdot \vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}^{\bullet}|^2} \vec{\omega}_{B/A}^{\bullet} - \vec{a}_{q/O_A/A} + \vec{a}_{q/O_A/A} \\ &= 0. \end{aligned}$$

To show (S18), assume  $p$  is an IACR. It follows from (S13) that

$$\vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{p/O_A/A} = \vec{\omega}_{B/A}^{\bullet} \times \left( \vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q} + \vec{\omega}_{B/A}^{\bullet} \times (\vec{\omega}_{B/A}^{\bullet} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} \right) = 0,$$

which implies that

$$\left( \vec{\omega}_{B/A}^{\bullet} \cdot \vec{r}_{p/q} \right) \vec{\omega}_{B/A}^{\bullet} - \left( \vec{\omega}_{B/A}^{\bullet} \cdot \vec{\omega}_{B/A}^{\bullet} \right) \vec{r}_{p/q} + \vec{\omega}_{B/A}^{\bullet} \times \vec{a}_{q/O_A/A} = 0. \quad (\text{S20})$$

Hence, solving for  $\vec{r}_{p/q}$  in (S20) yields (S18). □

**Fact S5.** Assume  $\overset{B\bullet}{\vec{\omega}}_{B/A} = 0$ ,  $\vec{\omega}_{B/A} \neq 0$ , and  $\vec{a}_{q/O_A/A} \neq 0$ . Then  $p$  is an IACR if and only if  $p$  satisfies the following conditions:

$$\begin{aligned} i) \quad & \vec{\omega}_{B/A} \cdot \vec{a}_{q/O_A/A} = 0. \\ ii) \quad & \vec{\omega}_{B/A} \times \left( \vec{r}_{p/q} - \frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2} \right) = 0. \end{aligned}$$

In this case,

$$\vec{r}_{p/q} = \frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2} + \frac{\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}}{|\vec{\omega}_{B/A}|^2} \vec{\omega}_{B/A}. \quad (S21)$$

**Proof** Assume  $p$  is an IACR. Since  $\overset{B\bullet}{\vec{\omega}}_{B/A} = 0$ , it follows from (S13) that

$$\begin{aligned} \vec{\omega}_{B/A} \cdot \vec{a}_{q/O_A/A} &= \vec{\omega}_{B/A} \cdot \left( -\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} - \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) \right) \\ &= -\vec{\omega}_{B/A} \cdot (\vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q})) \\ &= 0, \end{aligned}$$

which proves *i*). To prove *ii*), it follows from (S13) that

$$\begin{aligned} \vec{\omega}_{B/A} \times \left( \vec{r}_{p/q} - \frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2} \right) &= \vec{\omega}_{B/A} \times \vec{r}_{p/q} - \vec{\omega}_{B/A} \times \frac{-\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} - \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q})}{|\vec{\omega}_{B/A}|^2} \\ &= \vec{\omega}_{B/A} \times \vec{r}_{p/q} + \vec{\omega}_{B/A} \times \frac{\vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q})}{|\vec{\omega}_{B/A}|^2} \\ &= \vec{\omega}_{B/A} \times \vec{r}_{p/q} - \vec{\omega}_{B/A} \times \vec{r}_{p/q} \\ &= 0. \end{aligned} \quad (S22)$$

Hence, *ii*) holds.

Conversely, it follows from *ii*) that there exists  $\alpha \in \mathbb{R}$  such that

$$\vec{r}_{p/q} = \frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2} + \alpha \vec{\omega}_{B/A}. \quad (\text{S23})$$

Using *i*) and (S23), it follows that

$$\begin{aligned} \vec{a}_{p/O_A/A} &= \overset{A\bullet\bullet}{\vec{r}}_{p/O_A} \\ &= \overset{A\bullet\bullet}{\vec{r}}_{p/q} + \overset{A\bullet\bullet}{\vec{r}}_{q/O_A} \\ &= \overset{B\bullet\bullet}{\vec{r}}_{p/q} + 2\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q}) + \vec{a}_{q/O_A/A} \\ &= \vec{\omega}_{B/A} \times \left( \vec{\omega}_{B/A} \times \left( \frac{\vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^2} + \alpha \vec{\omega}_{B/A} \right) \right) + \vec{a}_{q/O_A/A} \\ &= -\vec{a}_{q/O_A/A} + \vec{a}_{q/O_A/A} \\ &= 0. \end{aligned}$$

To show (S21), assume  $p$  is an IACR. It follows from (S13) that

$$\begin{aligned} \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q}) + \vec{a}_{q/O_A/A} \\ = \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q}) + \vec{a}_{q/O_A/A} \\ = (\vec{\omega}_{B/A} \cdot \overset{B\bullet}{\vec{r}}_{p/q}) \vec{\omega}_{B/A} - (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}) \overset{B\bullet}{\vec{r}}_{p/q} + \vec{a}_{q/O_A/A} \\ = 0. \end{aligned} \quad (\text{S24})$$

Solving for  $\overset{B\bullet}{\vec{r}}_{p/q}$  in (S24) yields (S21).  $\square$

**Fact S6.** Assume  $\overset{B\bullet}{\vec{\omega}}_{B/A} = 0$  and  $\vec{\omega}_{B/A} = 0$ . Then every point  $p$  that is fixed relative to  $\mathcal{B}$  is an IACR if and only if

$$\vec{a}_{q/O_A/A} = 0.$$

**Proof** Assume  $p$  is an IACR, it follows from (S13) that

$$\begin{aligned} 0 &= \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q}) + \vec{a}_{q/O_A/A} \\ &= \vec{a}_{q/O_A/A}. \end{aligned}$$

Conversely,

$$\begin{aligned}
\vec{a}_{p/O_A/A} &= \vec{r}_{p/O_A}^{A\bullet\bullet} \\
&= \vec{r}_{p/q}^{A\bullet\bullet} + \vec{r}_{q/O_A}^{A\bullet\bullet} \\
&= \vec{r}_{p/q}^{B\bullet\bullet} + 2\vec{\omega}_{B/A} \times \vec{r}_{p/q}^{B\bullet} + \vec{\omega}_{B/A} \times \vec{r}_{p/q}^{B\bullet} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} \\
&= \vec{a}_{q/O_A/A} \\
&= 0.
\end{aligned}$$

□

**Fact S7.** Assume  $\vec{\omega}_{B/A}^{B\bullet}$  and  $\vec{\omega}_{B/A}$  are colinear, and let  $\kappa \triangleq \frac{\vec{\omega}_{B/A}^{B\bullet} \cdot \vec{\omega}_{B/A}}{|\vec{\omega}_{B/A}|^2}$ . Then p is an IACR if and only if p satisfies the following conditions:

- i)  $\vec{\omega}_{B/A} \cdot \vec{a}_{q/O_A/A} = 0$ .
- ii)  $\vec{\omega}_{B/A}$  and  $\vec{r}_{p/q} - \frac{|\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \vec{\omega}_{B/A}^{B\bullet} \times \vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^4 + |\vec{\omega}_{B/A}^{B\bullet}|^2}$  are colinear.

In this case, p satisfies

$$\vec{r}_{p/q} = \frac{|\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \vec{\omega}_{B/A}^{B\bullet} \times \vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^4 + |\vec{\omega}_{B/A}^{B\bullet}|^2} + \frac{|\vec{\omega}_{B/A}|^2 (\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}) + \kappa \vec{\omega}_{B/A}^{B\bullet} \cdot \vec{r}_{p/q}}{|\vec{\omega}_{B/A}|^4 + |\vec{\omega}_{B/A}^{B\bullet}|^2} \vec{\omega}_{B/A}. \quad (S25)$$

**Proof** Assume p is an IACR. It follows from (S13) that  $\vec{\omega}_{B/A} \cdot \vec{a}_{q/O_A/A} = 0$ , which proves i). To prove ii), note that p is an IACR, it follows from (S13) that

$$\begin{aligned}
0 &= \vec{\omega}_{B/A}^{B\bullet} \times \vec{r}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \vec{a}_{q/O_A/A} \\
&= \vec{\omega}_{B/A}^{B\bullet} \times \vec{r}_{p/q} + (\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}) \vec{\omega}_{B/A} - (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}^{B\bullet}) \vec{r}_{p/q} + \vec{a}_{q/O_A/A}.
\end{aligned} \quad (S26)$$

Next, the cross product of  $\overset{B\bullet}{\vec{\omega}}_{B/A}$  and (S26) can be expressed as

$$\begin{aligned} 0 &= \overset{B\bullet}{\vec{\omega}}_{B/A} \times \left( \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} + (\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}) \vec{\omega}_{B/A} - (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}) \vec{r}_{p/q} + \vec{a}_{q/O_A/A} \right) \\ &= (\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{r}_{p/q}) \overset{B\bullet}{\vec{\omega}}_{B/A} - |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2 \vec{r}_{p/q} - |\vec{\omega}_{B/A}|^2 (\vec{\omega}_{B/A} \times \vec{r}_{p/q}) + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A}. \end{aligned} \quad (S27)$$

It follows from (S26) that  $\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q}$  is given by

$$\overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{r}_{p/q} = -(\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}) \vec{\omega}_{B/A} + (\vec{\omega}_{B/A} \cdot \vec{\omega}_{B/A}) \vec{r}_{p/q} - \vec{a}_{q/O_A/A}. \quad (S28)$$

Substituting (S28) into (S27) yields

$$\begin{aligned} 0 &= (\overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{r}_{p/q}) \overset{B\bullet}{\vec{\omega}}_{B/A} - |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2 \vec{r}_{p/q} + |\vec{\omega}_{B/A}|^2 (\vec{\omega}_{B/A} \cdot \vec{r}_{p/q}) \vec{\omega}_{B/A} \\ &\quad - |\vec{\omega}_{B/A}|^4 \vec{r}_{p/q} + |\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} \\ &= [\kappa \overset{B\bullet}{\vec{\omega}}_{B/A} \cdot \vec{r}_{p/q} + (\vec{\omega}_{B/A} \cdot \vec{r}_{p/q})] \vec{\omega}_{B/A} + |\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} \\ &\quad - (|\overset{B\bullet}{\vec{\omega}}_{B/A}|^2 + |\vec{\omega}_{B/A}|^4) \vec{r}_{p/q}. \end{aligned} \quad (S29)$$

Hence, *ii*) holds.

Conversely, it follows from *ii*) that there exists  $\alpha \in \mathbb{R}$  such that

$$\vec{r}_{p/q} = \frac{|\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A}}{|\vec{\omega}_{B/A}|^4 + |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2} + \alpha \overset{B\bullet}{\vec{\omega}}_{B/A}.$$

Using *i*) and *ii*),  $\vec{a}_{p/O_A/A}$  is given by

$$\begin{aligned}
\vec{a}_{p/O_A/A} &= \overset{A\bullet\bullet}{\vec{r}}_{p/O_A} \\
&= \overset{A\bullet\bullet}{\vec{r}}_{p/q} + \overset{A\bullet\bullet}{\vec{r}}_{q/O_A} \\
&= \overset{B\bullet\bullet}{\vec{r}}_{p/q} + 2\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q} + \vec{\omega}_{B/A} \times (\vec{\omega}_{B/A} \times \overset{B\bullet}{\vec{r}}_{p/q}) + \vec{a}_{q/O_A/A} \\
&= \overset{B\bullet}{\vec{\omega}}_{B/A} \times \left( \frac{|\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} + \alpha \vec{\omega}_{B/A}}{|\vec{\omega}_{B/A}|^4 + |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2} \right) \\
&\quad + \vec{\omega}_{B/A} \times \left( \vec{\omega}_{B/A} \times \left( \frac{|\vec{\omega}_{B/A}|^2 \vec{a}_{q/O_A/A} + \overset{B\bullet}{\vec{\omega}}_{B/A} \times \vec{a}_{q/O_A/A} + \alpha \vec{\omega}_{B/A}}{|\vec{\omega}_{B/A}|^4 + |\overset{B\bullet}{\vec{\omega}}_{B/A}|^2} \right) \right) + \vec{a}_{q/O_A/A} \\
&= -\vec{a}_{q/O_A/A} + \vec{a}_{q/O_A/A} \\
&= 0.
\end{aligned}$$

Finally, (S29) implies (S25). □

## References

- [S3] G. N. Sandor and A. G. Erdman, *Advanced Mechanism Design: Analysis and Synthesis*, Prentice Hall, 1984.
- [S4] J. M. R. Martinez and J. Duffy, "Determination of the acceleration center of a rigid body in spatial motion," *Eur. J. Mech. A/Solids*, Vol. 17, pp. 969–977, 1998.
- [S5] K. E. Bisshopp, "Note on Acceleration in Laminar Motion," *J. Mechanisms*, Vol. 4, pp. 235–242, 1969.

## Sidebar 4

### The Initial Curvature Theorem and the Unit-Step Response

**Initial Slope Theorem** Let  $\hat{y}(s)$  denote the Laplace transform of  $y(t)$ . Then the initial slope of  $y(t)$  is given by

$$y'(0^+) \triangleq \lim_{t \rightarrow 0^+} y'(t) = \lim_{s \rightarrow \infty} s[s\hat{y}(s) - y(0^+)].$$

To illustrate the initial slope theorem, we consider the unit-step response of the asymptotically stable, strictly proper transfer function  $G$  with relative degree  $d \geq 1$ . The unit-step response has the initial value  $y(0^+) \triangleq \lim_{t \rightarrow 0^+} y(t) = \lim_{s \rightarrow \infty} s(G(s)\frac{1}{s}) = G(\infty) = 0$ . The initial slope of  $y(t)$  is thus given by

$$y'(0^+) = \lim_{s \rightarrow \infty} s^2\hat{y}(s) = \lim_{s \rightarrow \infty} sG(s).$$

Consequently, if  $d = 1$ , then  $y'(0^+) \neq 0$ , whereas, if  $d \geq 2$ , then  $y'(0^+) = 0$ . These results are illustrated in Figure S3 and Figure S4.

**Initial Curvature Theorem** Let  $\hat{y}(s)$  denote the Laplace transform of the output  $y(t)$ . Then, the initial curvature of  $y(t)$  is given by

$$y^{(d)}(0^+) \triangleq \lim_{t \rightarrow 0^+} y^{(d)}(t) = \lim_{s \rightarrow \infty} s^{d+1}\hat{y}(s),$$

where  $y^{(d)}$  denotes the  $d$ th derivative of  $y$ , and  $d$  is the relative degree of  $G(s)$ .

Let us consider the unit-step response of the asymptotically stable, strictly proper transfer function  $G$  with relative degree  $d \geq 1$ , where

$$G(s) = \frac{\beta_{n-d}s^{n-d} + \beta_{n-d-1}s^{n-d-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}.$$



The initial curvature of the unit step response is thus given by

$$\begin{aligned}y^{(i)}(0^+) &= \lim_{s \rightarrow \infty} s^{i+1} \hat{y}(s) \\ &= \lim_{s \rightarrow \infty} s^{i+1} G(s) \frac{1}{s} \\ &= \lim_{s \rightarrow \infty} s^i G(s) \\ &= \begin{cases} 0, & i = 1, \dots, d-1, \\ \beta_{n-d}, & i = d. \end{cases}\end{aligned}$$

Therefore, the initial curvature of the unit step response depends on the sign of the  $d$ th derivative  $y^{(d)}(0^+) = \beta_{n-d}$ .

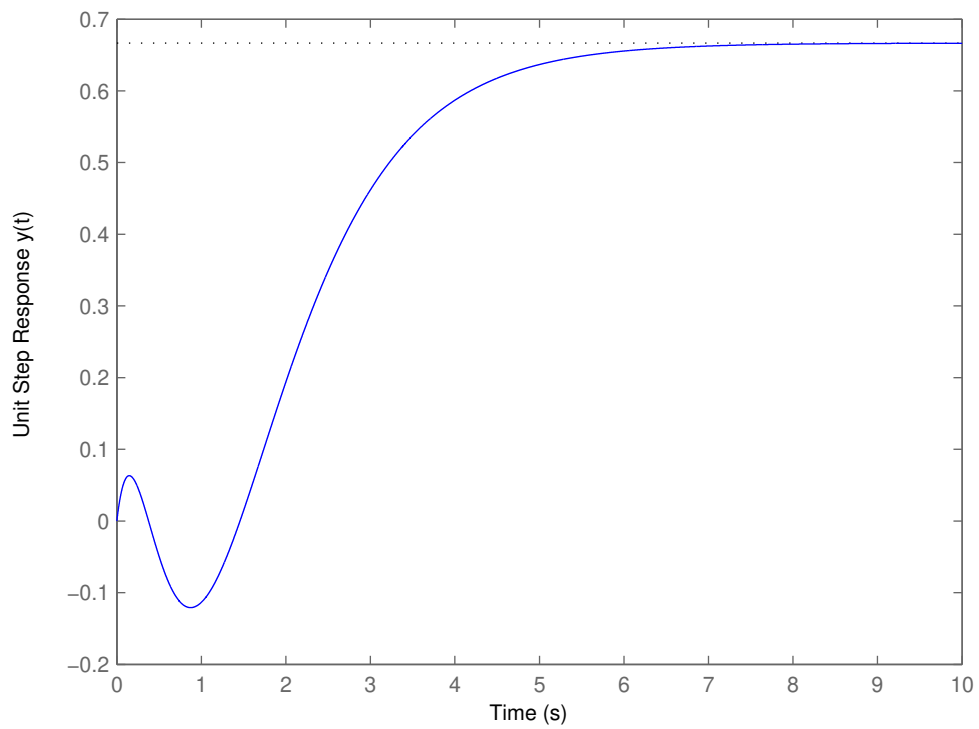


Figure S3: The unit step response of the asymptotically stable transfer function  $G(s) = \frac{(s-2)^2}{(s+1)(s+2)(s+3)}$  with relative degree  $d = 1$ . The initial slope  $y'(0^+)$  of the unit step response is 1.

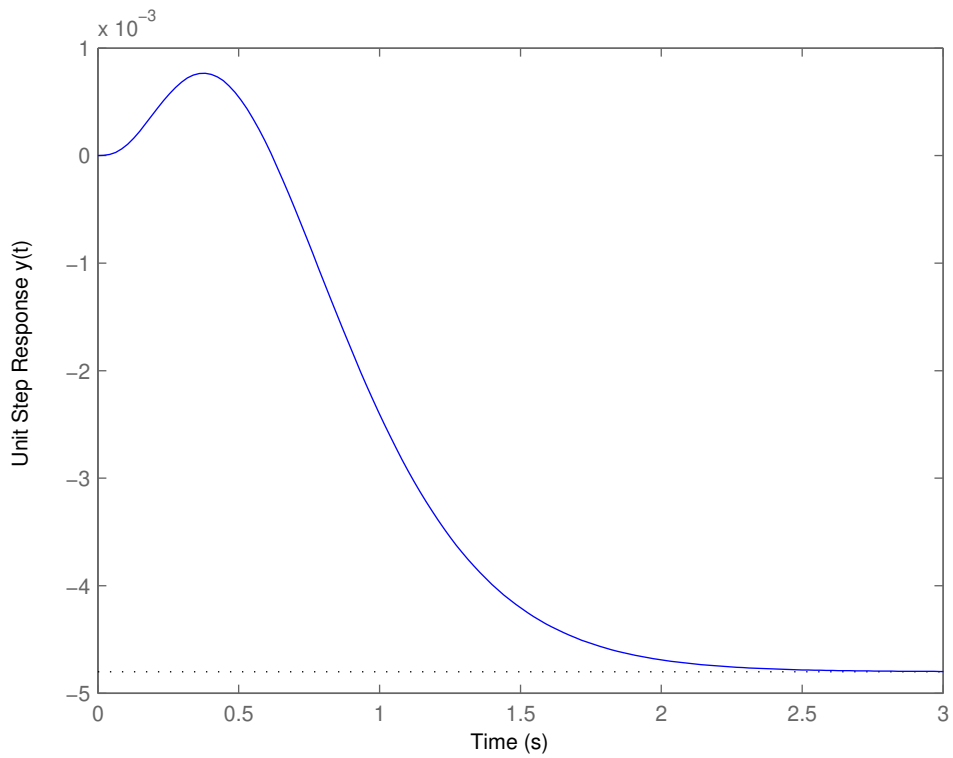


Figure S4: The unit step response of the asymptotically stable transfer function  $G(s) = \frac{s-3}{(s+5)^4}$ , whose relative degree is 3. The initial slope  $y'(0^+)$  of the unit step response is 0, whereas the initial curvature  $y''(0^+)$  of the unit step response is 1.

## Sidebar 5 Initial Undershoot

*Initial undershoot* occurs when the step response of a transfer function initially moves in the direction opposite to the direction of its asymptotic value.

Let  $G(s) \triangleq \frac{\beta(s)}{s^r \alpha(s)}$  be a strictly proper transfer function with relative degree  $d > 0$ , where  $r \geq 0$  and  $\alpha(s)$  is asymptotically stable. Let  $y(t)$  be the unit-step response of  $G$ . Then *initial undershoot* occurs at  $t = 0$  if

$$y^{(d)}(0^+)y^{(r)}(\infty) < 0,$$

where  $y^{(d)}(0^+) \triangleq \lim_{t \rightarrow 0^+} y^{(d)}(t)$  and  $y^{(r)}(\infty) \triangleq \lim_{t \rightarrow \infty} y^{(r)}(t)$ . The unit-step response has the initial curvature

$$y^{(d)}(0^+) = \lim_{t \rightarrow 0^+} y^{(d)}(t) = \lim_{s \rightarrow \infty} s(s^d \hat{y}(s)) = \lim_{s \rightarrow \infty} s^{d+1} \left( G(s) \frac{1}{s} \right) = \frac{\beta(\infty)}{\alpha(\infty)},$$

as well as the asymptotic curvature

$$y^{(r)}(\infty) \triangleq \lim_{t \rightarrow \infty} y^{(r)}(t) = \lim_{s \rightarrow 0} s^{(r+1)} \left( G(s) \frac{1}{s} \right) = \frac{\beta(0)}{\alpha(0)}.$$

The initial direction of the step response depends on the sign of the product of the initial curvature  $y^{(d)}(0^+)$  and the asymptotic curvature  $y^{(r)}(\infty)$ . The following result is discussed in [3].

**Proposition S1** Let  $G \triangleq \frac{\beta(s)}{s^r \alpha(s)}$  be a strictly proper transfer function, where  $r \geq 0$  and  $\alpha(s)$  is asymptotically stable. Then the unit step response has initial undershoot if and only if  $G(s)$  has an odd number of positive zeros.

As an example, consider the transfer function  $G(s) = -\frac{(s-1)(s-2)(s-3)}{s(s+1)(s+2)(s+3)(s+4)}$ . The unit impulse response exhibits initial undershoot with three direction reversals due to the three positive zeros, as shown in Figure S5.

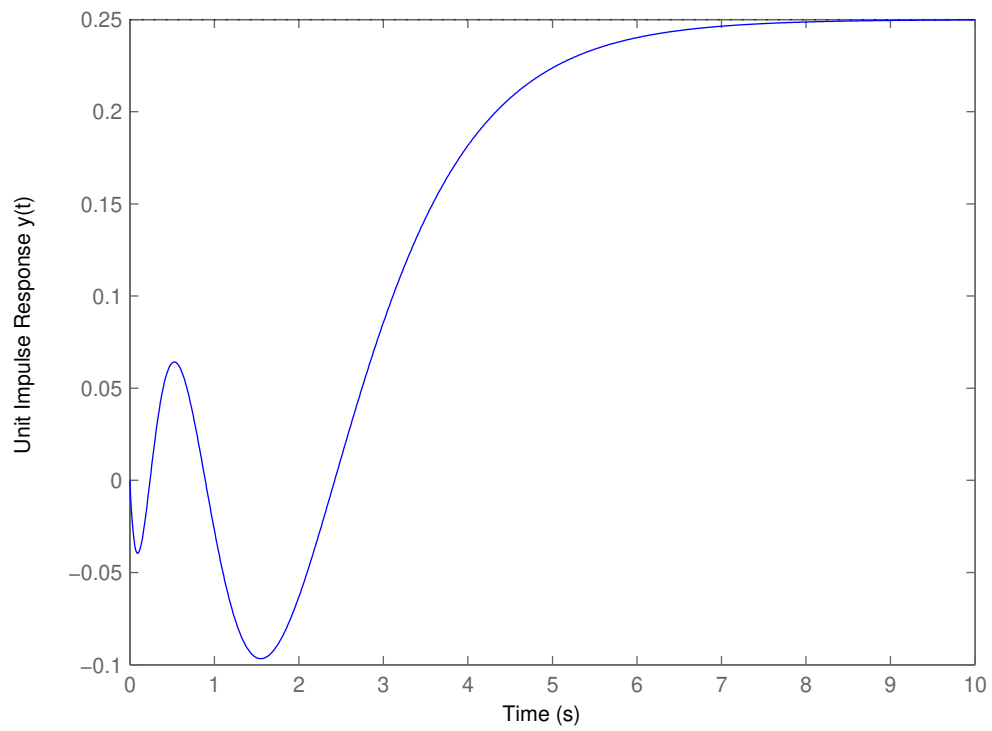


Figure S5: Unit impulse response of the transfer function  $G(s) = -\frac{(s-1)(s-2)(s-3)}{s(s+1)(s+2)(s+3)(s+4)}$ . The impulse response of this system exhibits initial undershoot with three direction reversals due to the three positive zeros.

## Sidebar 6

### Markov Parameters and Relative Degree

Consider

$$\begin{aligned}\dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), \\ \dot{y}(t) &= \tilde{C}x(t) + \tilde{D}u(t),\end{aligned}$$

whose Laplace form is given by

$$\begin{aligned}s\hat{x}(s) - x(0) &= \tilde{A}\hat{x}(s) + \tilde{B}\hat{u}(s), \\ \hat{y}(s) &= \tilde{C}\hat{x}(s) + \tilde{D}\hat{u}(s).\end{aligned}$$

Then,

$$\hat{y}(s) = \tilde{C}(sI - \tilde{A})^{-1}x(0) + [\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}]\hat{u}(s),$$

where

$$G(s) \triangleq \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}.$$

Expanding  $G(s)$  in a Laurent series about infinity yields

$$\begin{aligned}G(s) &= \frac{1}{s}\tilde{C}\left(I - \frac{1}{s}\tilde{A}\right)^{-1}\tilde{B} + \tilde{D} \\ &= \tilde{D} + \frac{1}{s}\tilde{C}\tilde{B} + \frac{1}{s^2}\tilde{C}\tilde{A}\tilde{B} + \dots.\end{aligned}\tag{S30}$$

We now consider  $G_{\hat{\theta}/\delta\hat{e}}(s)$  given by (47). Using (S30), we obtain

$$\lim_{s \rightarrow \infty} sG_{\hat{\theta}/\delta\hat{e}}(s) = \tilde{C}\tilde{B},$$

We can now write (42)–(44) in state space form with elevator-deflection input and setting  $X_{\dot{\alpha}_0} = 0$ ,  $Z_{\dot{\alpha}_0} = 0$ , and  $M_{\dot{\alpha}_0} = 0$  for convenience yields

$$\begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \delta\dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_{u_0} + X_{T_{u_0}} & X_{\alpha_0} & X_{q_0} & -g \\ \frac{Z_{u_0}}{U_0} & \frac{Z_{\alpha_0}}{U_0} & \frac{U_0 + Z_{q_0}}{U_0} & 0 \\ M_{u_0} + M_{T_{u_0}} & M_{\alpha_0} + M_{T_{\alpha_0}} & M_{q_0} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \delta\alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} X_{\delta e_0} \\ \frac{Z_{\delta e_0}}{U_0} \\ M_{\delta e_0} \\ 0 \end{bmatrix} \delta e.\tag{S31}$$

where

$$\tilde{B} \triangleq \begin{bmatrix} X_{\delta e_0} \\ \frac{Z_{\delta e_0}}{U_0} \\ M_{\delta e_0} \\ 0 \end{bmatrix}, \quad \tilde{C} \triangleq \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that

$$\tilde{C}\tilde{B} = 0.$$

Since  $\tilde{D} = 0$  and  $\tilde{C}\tilde{B} = 0$ , it follows from (S30) that

$$\lim_{s \rightarrow \infty} s^2 G(s) = \tilde{C}\tilde{A}\tilde{B}, \quad (\text{S32})$$

where we consider

$$\tilde{A} \triangleq \begin{bmatrix} \dot{u} \\ \delta \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_{u_0} + X_{T_{u_0}} & X_{\alpha_0} & X_{q_0} & -g \\ \frac{Z_{u_0}}{U_0} & \frac{Z_{\alpha_0}}{U_0} & \frac{U_0 + Z_{q_0}}{U_0} & 0 \\ M_{u_0} + M_{T_{u_0}} & M_{\alpha_0} + M_{T_{\alpha_0}} & M_{q_0} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which is given by (S31). Therefore,

$$\lim_{s \rightarrow \infty} s^2 G_{\hat{\theta}/\delta \hat{e}}(s) = \frac{A_\theta}{E}, \quad (\text{S33})$$

where  $A_\theta$  is the coefficient of  $s^2$  in the numerator of (47). From (S32) and (S33) it follows that  $A_\theta/E = \tilde{C}\tilde{A}\tilde{B} = M_{\delta e_0}$  for  $X_{\dot{\alpha}_0} = 0$ ,  $Z_{\dot{\alpha}_0} = 0$ , and  $M_{\dot{\alpha}_0} = 0$ . It thus follows that the numerator of  $G_{\hat{\theta}/\delta \hat{e}}(s)$  in (47) is of second order.

## Sidebar 7

### Routh Test for Third- and Fourth-Order Polynomials

All three roots of the cubic polynomial  $p(s) = s^3 + a_2s^2 + a_1s + a_0$  are in the open left half plane if and only if

$$a_0, a_1, a_2 > 0$$

and

$$a_0 < a_1a_2.$$

All four roots of the quartic polynomial  $p(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$  are in the OLHP if and only if

$$a_0, a_1, a_2, a_3 > 0$$

and

$$a_0a_3^2 + a_1^2 < a_1a_2a_3.$$



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