Longitudinal permeability of spatially periodic rectangular arrays of circular cylinders I. A single cylinder in the unit cell

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Abstract

We study the longitudinal permeability of a spatially periodic rectangular array of circular cylinders, when a Newtonian fluid is flowing at low Reynolds number along the cylinders. The longitudinal component of the velocity obeys a Poisson equation which is transformed into a functional equation. This equation can be solved by the method of successive approximations. The major advantage of this technique is that the permeability of the array can be expressed analytically in terms of the radius of the cylinders and of the aspect ratio of the unit cell.

Key words: functional equation, rectangular array, effective permeability

1 Introduction

The transport properties of the unidirectional cylinders attracted the attention of many scientists since the 19 th century as reviewed by Landauer (1974). If cylinders are arranged according to a square or hexagonal array, the method of Lord Rayleigh (1892) can be successfully applied. It is based on the reduction of the problem to an infinite set of linear algebraic equations which are truncated and solved numerically to get lower - order formulae for the effective tensor. The method of Lord Rayleigh has been extended by McPhedran et al. (1988). Other methods based on integral equations or infinite sets have been applied by Sparrow & Loeffler (1959), Drummond and

Tahir (1984), Bergman & Dunn (1992), Kolodziej (1987), Sangani and Yao (1988a, 1988b) and others.

This paper deals with the longitudinal permeability of rectangular arrays of cylinders, when a Newtonian fluid is flowing at low Reynolds number around the cylinders. We assume that the driving pressure gradient is parallel to the cylinders; the velocity field has only one non-zero component. Stokes equations imply that this component satisfies the Poisson equation. In the present paper the unit cell is a rectangle which contains a single circular disc, which is the section of a cylinder (cf Figure 1). In contrast with the contributions, which were previously mentioned and which can easily deal only with special structures such as square and hexagonal arrays, we apply the method of functional equations (Mityushev&Rogosin, 1999). The major advantage of this method is to provide analytical expressions for the effective transport properties in a relatively systematic manner by using *Mathematica*©. In particular the necessary Hashimoto function can be expanded in terms of the elliptic functions which prove to be very efficient computationally; note that in the previous papers Hashimoto functions were constructed by slowly convergent series. Hence, our final analytical formula involves not only the solid volume fraction ϕ as in the previous papers, but also the aspect ratio of the unit cell α^2.

This paper is organized as follows. The problem is presented in Section 2, and the functional equation which governs the local velocity is derived. The expression of permeability is given in Section 3. The resolution of the functional equation is summarized in Section 4. The analytical and numerical results are presented and discussed in Section 5. Some concluding remarks are proposed in Section 6. Three appendices contain technical details which are useful for the main text.

2 Statement of the problem. Functional equation

Consider a lattice $\mathcal{Q}$ defined by two perpendicular fundamental translation vectors ω_1 and ω_2 in the complex plane $\mathbb{C} \cong \mathbb{R}^2$ where the complex variable z is related to the real ones x and y by the identity $z = x + iy$ ($i^2 = -1$). It may be assumed that $\omega_1 = \alpha > 0$ and $\omega_2 = i\alpha^{-1}$. The zero unit cell $Q_{(0,0)}$ is displayed in Figure 1; its area $\left|Q_{(0,0)}\right|$ is equal to 1. Let $\cup_j \{e_j\}$ be the doubly ordered set of the numbers $e_j := m_1 + im_2$, where $j = (m_1, m_2)$, m_1 and im_2 are integers. The lattice $\mathcal{Q}$ consists of the cells $Q_j = Q_{(0,0)} + e_j$. Let us consider a disk $D_i := \{z \in \mathbb{C} : \ |z| < r_0\}$ in the zero cell $Q_{(0,0)}$. Let

$D_e := Q_{(0,0)} \setminus (D_i \cup \partial D_i)$.

The plane $\mathbb{C}$ is assumed to be the perpendicular cross - section $D_i + e_j$ of an infinite array of parallel circular cylinders. A Newtonian fluid of viscosity μ is flowing at low Reynolds number through this array. When the driving pressure gradient is perpendicular to the plane $\mathbb{C}$, the Stokes equation is reduced to a Poisson equation for the component $w(x, y)$ of the fluid velocity. Since we shall work with dimensionless quantities, it is convenient to assume that the pressure gradient and the viscosity are set equal to 1. Hence, $w(x, y)$ is a solution of the following boundary value problem that we shall study

$$\begin{aligned} \Delta w &= 1 \text{ in } D_e \\ & \quad w \text{ is doubly periodic} \\ w &= 0 \text{ on the circle } |t| = r_0, \end{aligned} \tag{2.1}$$

where Δ is the two-dimensional Laplace operator. Generally speaking, the position inside a domain is denoted by the complex variable $z = x + iy$, but the position along boundaries is denoted by the complex variable t.

The functional equation equivalent to (2.1) will be obtained in three main steps. The Poisson equation (2.1) is first reduced to a Laplace equation by a suitable change of unknown; the boundary value problem is then stated in terms of analytic functions. Second, the boundary conditions can be expressed in terms of an unknown analytic function $\psi(z)$ defined inside D_i whose properties are studied. Third, the boundary value problem for $\psi(z)$ is replaced by a functional equation (either continuous or discrete) for the unknown function $\psi(z)$.

2.1 Reduction to a Laplace equation

Let us start the application of this program. In order to reduce (2.1) to a Laplace equation, we introduce the function

$$w_0(x, y) := -\frac{1}{2\pi} \ln |\sigma(z)| + \frac{1}{4\pi} \left(S_2 x^2 + (2\pi - S_2) y^2 \right), \tag{2.2}$$

where the constant S_2 and the Weierstrass function $\sigma(z)$ are derived in Appendix A. The function $w_0(x, y)$ has the following properties:

i) $\Delta w_0 = 1$ in $Q_{(0,0)} \setminus \{z = 0\}$, since $\ln |\sigma(z)|$ is harmonic in the domain which is considered,

ii) $w_0(x, y) \sim -\frac{1}{2\pi} \ln |z|$ near $z = 0$ (see the first formula (7.3))

iii) $w_0(x, y)$ is doubly periodic.

The properties i) - ii) are obvious. In order to prove the third property, for instance for the x - direction, let us denote by $[w_0]_x$ the jump of $w_0(x, y)$

along the cell $Q_{(0,0)}$ in the x - direction. Application of (7.5) yields

$$[w_0]_x = \frac{S_2}{4\pi}\left[x^2\right]_x - \frac{1}{2\pi}\left[Re\ \nu(z)\right]_x = 0.$$

The function $w_0(x,y)$ is a two-dimensional Hashimoto's function S_1 (Hashimoto (1959)). More precisely, $S_1 = 4\pi w_0(x,y)$. The function $w_0(x,y)$ is easily calculated with the help of (7.4). Note that formulae (7.1) and (7.4) are valid for an arbitrary lattice; moreover, the function $w_0(x,y)$ defined by (2.2) satisfies i) - iii) in the general case. It is easily seen that $\frac{S_2-\pi}{4\pi}Re\ t^2 + \frac{r_0^2}{4} = \frac{1}{4\pi}\left(S_2x^2 + (2\pi - S_2)\,y^2\right)$ for $|t|^2 = x^2 + y^2 = r_0^2$. Then (2.2) implies

$$-w_0 = \frac{1}{2\pi}\ln|\sigma(t)| - \frac{S_2-\pi}{4\pi}Re\ t^2 - \frac{r_0^2}{4},\ \ |t| = r_0. \tag{2.3}$$

We are now ready to rewrite problem (2.1) in terms of harmonic functions. Introduce the new unknown function $u(x,y)$ by

$$u(x,y) = w(x,y) - w_0(x,y) \text{ in } D_e. \tag{2.4}$$

It is easily checked that $u(x,y)$ satisfies the following boundary value problem

$$\begin{aligned} \Delta u &= 0 \text{ in } D_e \\ & u \quad \text{is doubly periodic} \\ u &= -w_0 \text{ on the circle } |t| = r_0. \end{aligned} \tag{2.5}$$

2.2 Expression of the boundary conditions in terms of $\psi(z)$

Now that the first part of the program is completed, let us start the second part and express the boundary condition in terms of an unknown analytic function $\psi(z)$.

2.2.1 Introduction of the analytic potential $\varphi(z)$ defined on D_e

In order to represent the boundary value $-w_0$ as a real part of an analytic function, we introduce the function $f(z)$ which is analytic in the unit cell $Q_{(0,0)}$

$$f(z) := \frac{1}{2\pi}\ln\frac{\sigma(z)}{z} - \frac{S_2-\pi}{4\pi}z^2, \tag{2.6}$$

$f(z)$ is equal to 0 at $z=0$. The function $f(z)$ can be considered as a complex potential of the real function $-w_0$, because the real part of $\frac{1}{2\pi}\ln\sigma(z)$ is equal to $\frac{1}{2\pi}\ln|\sigma(z)|$. The term $-\frac{1}{2\pi}\ln z$ is added to $f(z)$ to exclude the singularity inside the disc $|z| \leq r_0$. Therefore, w_0 and $f(t)$ are related by

$$-w_0 = Re\ f(t) + \frac{1}{2\pi}\ln r_0 + r_0^2/4,\ |t| = r_0. \tag{2.7}$$

A complex potential $\varphi(z)$ can be introduced which is expressed as

$$Re\ \varphi(z) = u(x,y) - \frac{1}{2\pi}\ln r_0 + \frac{r_0^2}{4},\ z \in D_e. \tag{2.8}$$

It may be useful to insist that $\varphi(z)$ is defined outside of all discs $D_i + e_j$. Then we arrive at the boundary value problem

$$Re\ \varphi(t) = Re\ f(t),\ |t| = r_0 \tag{2.9}$$

with respect to the function $\varphi(z)$ analytic in D_e and continuous in $D_e \cup \partial D_e$. The known function $f(z)$ has the form (2.6). Relation (2.8) implies that the function $Re\ \varphi(z)$ is doubly periodic in $\mathbb{C}$ as is $u(x,y)$. However, the imaginary part of $\varphi(z)$ can have jumps along $Q_{(0,0)}$. It follows from the elliptic function theory (Hurwitz (1964)) that these jumps can be only constants. Hence,

$$\varphi(z+\alpha) - i\gamma_1 = \varphi(z) = \varphi(z+i\alpha^{-1}) - i\gamma_2, \tag{2.10}$$

where γ_1 and γ_2 are real constants. According to Hurwitz (1964) the functions satisfying (2.10) are called quasi-periodic in $\mathbb{C}$. Actually, it will be shown that $\varphi(z)$ is doubly periodic. Problem (2.9) can be considered as a Dirichlet (Schwarz) problem or Hilbert-Riemann problem (Zverovich 1971, Mityushev 1997b) in the class of doubly periodic functions (i.e., on a torus). We have written the problem (2.9) in such a form that the real part of the unknown function $\varphi(z)$ analytic in D_e is in the left-hand side and the real part of the known function $f(z)$ analytic in D_i is in the right-hand side.

2.2.2 Introduction of the analytic function $\psi(z)$ defined on D_i

Now let us deal with the interior of D_i. In general the function $\varphi(z)$ is not continued analytically inside $|z| \leq r_0$; instead we introduce a function $\psi(z)$ which is analytic in $|z| < r_0$ and continuous in $|z| \leq r_0$; it is defined by the conjugation condition

$$\varphi(t) = \psi(t) - \overline{\psi(t)} + f(t),\ |t| = r_0. \tag{2.11}$$

The condition (2.9) is known as an $\mathbb{R}$-linear problem of the analytic function theory (see (Mityushev&Rogosin, 1999)).

Let us summarize some of the properties of $\psi(z)$. If the function $\varphi(z)$ is known from (2.9), then $\psi(z)$ is constructed by the following Dirichlet (Schwarz) problem which is a trivial sequence of (2.11)

$$2Im\,\psi(t) = Im\ (\varphi(t) - f(t))\,,\ |t| = r_0. \tag{2.12}$$

The following lemma follows from the properties of symmetry of $f(z)$ and $\varphi(z)$

Lemma 2.1. *The function $\psi(z)$ satisfies the relations*

$$\psi(z) = \overline{\psi(\overline{z})} \text{ and } \psi(z) = \psi(-z),\ \ |z| \leq r_0.$$

It follows from the lemma that the Taylor expansion of $\psi(z)$ has the form

$$\psi(z) = \sum_{m=0}^{\infty} \psi_{2m} z^{2m}, \text{ where } \psi_{2m} \in \mathbb{R}. \tag{2.13}$$

Remark 2.2. The problem (2.12) with respect to ψ has a unique solution up to a purely imaginary additive constant. This constant may be equated to zero, and $Re\,\psi(0) = 0$. Since the lemma implies that ψ_0 is real, $\psi_0 = 0$.

The function $\psi(z)$ corresponds to the perturbation of the zero-th disc to the uniform flow in an infinite medium without any other cylinder. In order to tentatively take into account the influence of all the cylinders, this perturbation is made spatially periodic in the following way

$$\sum_j \overline{\psi\left(\frac{r_0^2}{\overline{\overline{z - e_j}}}\right)} = \sum_j \sum_{m=1}^{\infty} \psi_{2m} \frac{r_0^{4m}}{(z - e_j)^{2m}} = \sum_{m=1}^{\infty} \psi_{2m} r_0^{4m} E_{2m}(z). \tag{2.14}$$

Actually we sum over all the cylinders the inversion function $\overline{\psi\left(\frac{r_0^2}{\overline{z}}\right)}$ which is analytic outside of the disc $|z| \leq r_0$. We use here the doubly periodic Eisenstein's functions $E_{2m}(z)$ (see Appendix A) and the Eisenstein's summation (7.6) on $j = (m_1, m_2)$ for $E_2(z)$. The series (2.14) converges absolutely and almost uniformly in $Q_{(0,0)} \setminus \{0\}$(Mityushev 1997c).

2.2.3 Introduction of the analytic function $\Phi(z)$ defined in D_e and D_i

Let us now go back to problem (2.11). In order to reduce it to a functional equation, introduce the function $\Phi(z)$ which is analytic in the domains D_e and $|z| < r_0$

$$\Phi(z) := \begin{cases} \psi(z) + \sum_j{}' \overline{\psi\left(\frac{r_0^2}{\overline{\overline{z - e_j}}}\right)} + f(z), & |z| \leq r_0, \\ \varphi(z) + \sum_j \overline{\psi\left(\frac{r_0^2}{\overline{\overline{z - e_j}}}\right)}, & z \in D_e. \end{cases} \tag{2.15}$$

The sum $\sum_j{}'$ contains all cylinder centres e_j, except the origin $j = (0,0)$. Actually, it will be shown that $\Phi(z)$ is a constant.

Roughly speaking the function $\Phi(z)$ can be treated as the sum of all complex potentials arising in the problem. In D_e, the function $\Phi(z)$ is the sum of the potential $\varphi(z)$ inside D_e and of the potentials $\overline{\psi\left(\frac{r_0^2}{\overline{z-e_j}}\right)}$ induced by all the other cylinders; in D_i, $\Phi(z)$ is the sum of the potential $\psi(z)$ inside $|z| \leq r_0$, of the potentials of the other cylinders and of the external potential $f(z)$. Calculate the jump of Φ along $|t| = r_0$

$$[\Phi]_{|t|=r_0} = \psi(t) + f(t) - \varphi(t) - \overline{\psi(r_0^2/\overline{t})} = 0. \tag{2.16}$$

Here, we used the relation $t = r_0^2/\overline{t}$ on $|t| = r_0$ and (2.11). The principle of analytic continuation implies that $\Phi(z)$ is analytic in $\mathbb{C}$ and quasi-periodic as $\varphi(z)$. Liouville's theorem in the class of doubly periodic functions (Hurwitz (1964)) applied to $\Phi(z)$ implies that

$$\Phi(z) = B_1 z + B, \tag{2.17}$$

where B_1 and B are arbitrary constants. B_1 can be determined in the following way. Calculate the jumps of $\Phi(z)$ along $Q_{(0,0)}$ using the definition of $\Phi(z)$

$$\Phi(z+\alpha) - \Phi(z) = \varphi(z+\alpha) - \varphi(z) = i\gamma_1,\ \Phi(z+i\alpha^{-1}) - \Phi(z) = i\gamma_2.$$

since the functions $E_{2m}(z)$ $(m = 1, 2, ...)$ are doubly periodic. We now calculate the same jumps with the help of (2.17)

$$\Phi(z+\alpha) - \Phi(z) = B_1\alpha,\ \Phi(z+i\alpha^{-1}) - \Phi(z) = B_1 i\alpha^{-1}.$$

Then $B_1\alpha = i\gamma_1$ and $B_1 i\alpha^{-1} = i\gamma_2$. Hence, B_1 is simultaneously purely real and purely imaginary; this is only possible for $B_1 = 0$. Therefore,

$$\Phi(z) = B = \textit{constant}. \tag{2.18}$$

The definition of $\Phi(z)$ in $|z| \leq r_0$ implies that

$$\psi(z) = -\sum_j{}' \overline{\psi\left(\frac{r_0^2}{\overline{z-e_j}}\right)} - f(z) + B,\ |z| \leq r_0. \tag{2.19}$$

The constant B can be expressed as

$$B = \Phi(0) = \psi(0) + \sum_j{}' \overline{\psi\left(\frac{r_0^2}{\overline{-e_j}}\right)} + f(0) = \sum_{m=1}^{\infty} \psi_{2m} S_{2m}. \tag{2.20}$$

2.2.4 The functional equation for ψ

We are now close to the goal since (2.19) and (2.20) can be added to yield

$$\psi(z) = -\sum_j{}' \left[\overline{\psi\left(\frac{r_0^2}{\overline{z - e_j}}\right)} - \overline{\psi\left(\frac{r_0^2}{\overline{-e_j}}\right)} \right] - f(z), \; |z| \leq r_0. \tag{2.21}$$

or

$$\psi(z) = -\sum_{m=1}^{\infty} \psi_{2m} r_0^{4m} \left[\sigma_{2m}(z) - S_{2m}\right] - f(z), \; |z| \leq r_0, \tag{2.22}$$

where $\sigma_{2m}(z)$ is the modified Eisenstein's function (see Appendix A); the forcing term for the equation $f(z)$ is recalled to be known and given by (2.6). The constant B in (2.19) and (2.20) is irrelevant.

Equation (2.21) is the equation that we were looking for; it replaces the initial boundary value problem (2.1) for the Laplace equation. Equation (2.21) is called the functional equation for the unknown function $\psi(z)$ which is analytic in $|z| < r_0$ and continuous in $|z| \leq r_0$. (2.22) is called the discrete form of (2.21). Such functional equations have been studied by Mityushev (1997b, 1998) and in references cited therein.

If the solution $\psi(z)$ of (2.21) is known, the definition (2.15) of $\Phi(z)$ implies that the complex potential $\varphi(z)$ can be expressed as

$$\varphi(z) = -\sum_j \overline{\psi\left(\frac{r_0^2}{\overline{z - e_j}}\right)} + B = -\sum_{m=1}^{\infty} \psi_{2m} r_0^{4m} \left[E_{2m}(z) - S_{2m}\right]. \tag{2.23}$$

Hence, it is seen on this relation that $\varphi(z)$ is doubly periodic.

3 Effective permeability

We now proceed to calculate the effective permeability K_{II} (Adler 1992) which is defined as the integral of the flow velocity over the unit cell

$$-K_{II} = \int_{D_e} w(x,y) d\sigma_z = \int_{D_e} u d\sigma_z + \int_{D_e} w_0 d\sigma_z, \tag{3.1}$$

where $\int_{D_e} w d\sigma_z$ is the double integral $\int\int_{D_e} w(x,y)dxdy$. Introduction of the complex potential into (3.1) yields

$$-K_{II} = \frac{1}{4\pi} I_1 + \left(\frac{1}{2\pi} \ln r_0 - \frac{r_0^2}{4}\right)\left(1 - \pi r_0^2\right) - \frac{1}{2\pi} I_2 + I_3, \tag{3.2}$$

where the double integrals I_1 and I_2 are derived in Appendix B. They can be expressed as

$$I_1 := \int_{D_e} \left(S_2 x^2 + (2\pi - S_2)\, y^2 \right) d\sigma_z = \frac{1}{12} \left(S_2 \alpha^2 + (2\pi - S_2)\, \alpha^{-2} \right) - \frac{1}{2} \left(\pi r_0^2 \right)^2 , \tag{3.3}$$

$$I_2 := \int_{D_e} \ln |\sigma(z)|\, d\sigma_z = T(0) + \frac{1}{2} \pi r_0^2 - \pi r_0^2 \ln r_0, \tag{3.4}$$

where the constant S_2 is recalled to be given in Appendix A. The constant $T(0)$ is given by (8.3). By application of (8.4), (8.5) and (8.6), the double integral I_3 is defined as

$$I_3 := \int_{D_e} Re\ \varphi(z) d\sigma_z \tag{3.5}$$

It can be expressed in terms of ψ_{2m} (cf Appendix B)

$$I_3 = \sum_{m=1}^{\infty} \psi_{2m} S_{2m} r_0^{4m} - \psi_2 \pi r_0^4. \tag{3.6}$$

Therefore, the permeability K_{II}, or equivalently

$$k_c^* := 4\pi K_{II}\ , \tag{3.7}$$

can be calculated by (3.2), where I_1, I_2, I_3 are given by (3.3), (3.4), (3.6), respectively. In order to apply (3.6), we have to determine ψ_{2m} by solving the functional equation (2.22). The other terms are only functions of the lattice sums S_{2m}.

4 Solution to the functional equation

4.1 Exact solution to the functional equation in the form of series

In the present section we study the functional equation (2.21) and its discrete form (2.22). Following Mityushev (1997c), we briefly recall the main properties of this functional equation. Let us consider the Banach space $\mathcal{C}^+$ consisting of the functions analytic in $|z| < r_0$ and continuous in $|z| \leq r_0$ with the norm $\|\psi\| := \max_{|z|\leq r_0} |\psi(z)|$. Convergence in $\mathcal{C}^+$ corresponds to uniform convergence in $|z| \leq r_0$. Let us write (2.21) in an operator form $\psi = A\psi + f$, where the operator A is defined by the right-hand side of (2.21).

The operator A is compact in $\mathcal{C}^+$, and the equation $\psi = \rho A\psi + f$ has a unique solution for each ρ satisfying the inequality $|\rho| < 1$. This solution can be found by the method of successive approximations with an expansion of the form $\psi = \sum_{n=0}^{\infty} \rho^n A^n f$. For $\rho = 1$, it is known (Mityushev (1998)) that equation (2.21) has a unique solution for sufficiently small r_0 which can also be found by the method of successive approximations. This last property allows us to find approximate solutions up to $O\left(r_0^{4N}\right)$ for an arbitrary fixed natural number N.

The exact solution of (2.21) can be represented in the form

$$\begin{aligned} \psi(z) &= -f(z) + \sum_{j_1}{}' \left[\overline{f\left(\frac{r_0^2}{\overline{z-e_{j_1}}}\right)} - \overline{f\left(\frac{r_0^2}{\overline{-e_{j_1}}}\right)}\right] + \\ &\quad \sum_{j_1}{}' \sum_{j_2}{}' \left[f\left(\frac{r_0^2}{\overline{\frac{r_0^2}{\overline{z-e_{j_1}}} - e_{j_2}}}\right) - f\left(\frac{r_0^2}{\overline{\frac{r_0^2}{\overline{-e_{j_1}}} - e_{j_2}}}\right)\right] + \ldots . \end{aligned} \tag{4.1}$$

However, another form of this solution will be used, namely, a discrete form which involves the coefficients ψ_{2m} in order to compute I_3 from (3.6), and k_c^* from (3.7).

Substitute the expansion (2.13) for $\psi(z)$ into (2.22)

$$\psi_{2n} = -\sum_{m=1}^{\infty} \psi_{2m} r_0^{4m} C_{2(m+n)-1}^{2n} S_{2(m+n)} + p_n, \tag{4.2}$$

where

$$p_n := \frac{S_2 - \pi}{4\pi} \text{ for } n = 1 \text{ and } p_n := \frac{S_{2n}}{4\pi n} \text{ for } n = 2, 3, \ldots. \tag{4.3}$$

The relations (4.2) can be considered as an infinite set of linear algebraic equations with respect to ψ_{2n}. Frequently, infinite sets of equations are hard to solve even numerically. Here, it is very easy to obtain an analytical expression for ψ_{2n}, because the method of successive approximations is applicable to (4.2). Actually, the continuous equation (2.21) in the space $\mathcal{C}^+$ is represented as the discrete equation (4.2) in an appropriate space c^+ of sequences which is isomorphic to $\mathcal{C}^+$. Hence, convergence of the successive approximations for (2.21) implies convergence for (4.2). Thus, an exact form of ψ_{2n} up to $O(r_0^{4N})$ can be determined for an arbitrary fixed natural number N. We shall do it in the next section.

4.2 Calculation of ψ_{2n} with a given accuracy

Let us apply this method. The zero-th approximation is $\psi_{2n}^0 = p_n$, where p_n has the form (4.3). Substitution of ψ_{2n}^0 into (4.2) yields ψ_{2n}^1 and so on. If we restrict ourselves to the accuracy $O(r_0^{4N})$, where N is an arbitrary fixed natural number, a simpler algorithm can be derived. First note that we need to determine ψ_{2m} up to $O(r_0^{4(N-m)}),\ m=1,2,...,N$. Then, (4.2) is simplified as

$$\psi_{2n} = -\sum_{m=1}^{N-m} \psi_{2m} r_0^{4m} C_{2(m+n)-1}^{2n} S_{2(m+n)} + p_n,\ n=1,2,...,N. \tag{4.4}$$

The system (4.4) is finite, triangular and it can be solved as follows. Let us write (4.4) in an explicit form

$$\begin{gathered}
\psi_2 = -C_3^2 S_4 r_0^4 \psi_2 - C_5^2 S_6 r_0^8 \psi_4 - ... - C_{2N+1}^2 S_{2N+2} r_0^{4N} \psi_{2N} + \frac{S_2-\pi}{4\pi}, \\
\psi_4 = -C_5^4 S_6 r_0^4 \psi_2 - C_7^4 S_8 r_0^8 \psi_4 - ... - C_{2N+1}^4 S_{2N+2} r_0^{4(N-1)} \psi_{2(N-1)} + \frac{S_4}{16\pi}, \\
... \\
\psi_{2(N-1)} = -C_{2N-1}^{2(N-1)} S_{2N} r_0^4 \psi_2 + \frac{S_{2(N-1)}}{4\pi\,(N-1)}, \\
\psi_{2N} = \frac{S_{2N}}{4\pi N}.
\end{gathered} \tag{4.5}$$

In order to solve (4.5), we first determine the zero-th approximation

$$\psi_2^0 = \frac{S_2-\pi}{4\pi},\ \psi_4^0 = \frac{S_4}{16\pi}, ...,\ \psi_{2(N-1)}^0 = \frac{S_{2(N-1)}}{4\pi\,(N-1)},\ \psi_{2N}^0 = \psi_{2N} = \frac{S_{2N}}{4\pi N}. \tag{4.6}$$

It easily seen from (4.5) that this approximation is enough to exactly determine ψ_{2N}. Substituting (4.6) into (4.5), we obtain the next approximation up to $O(r_0^4)$:

$$\psi_2^1 = \frac{S_2-\pi}{4\pi}\left(1 - C_3^2 S_4 r_0^4\right),\ \psi_4^1 = \frac{S_4}{16\pi} - C_5^4 S_6 r_0^4 \frac{S_2-\pi}{4\pi}, ..., \tag{4.7}$$

$$\psi_{2(N-1)}^1 = \psi_{2(N-1)} = \frac{S_{2(N-1)}}{4\pi\,(N-1)} - C_{2N-1}^{2(N-1)} S_{2N} r_0^4 \frac{S_2-\pi}{4\pi},$$

ψ_{2N} has been calculated in the previous iteration. The approximation (4.7) for $n = N-1$ provides the desired exact formula for $\psi_{2(N-1)}$ up to $O(r_0^4)$. Further we substitute ψ_{2n}^1 into (4.5) and obtain the next approximation ψ_{2n}^2 for $n = 1,2,...,N-2$. ψ_{2N} and $\psi_{2(N-1)}$ have been calculated in the previous iterations. And so forth. In the last N - th step, we obtain the desired formula for ψ_2.

We have derived the algorithm to calculate ψ_{2n} up to $O(r_0^{4N})$ with fixed N. It is possible to modify this algorithm and to calculate ψ_{2n} without any restriction by application of this technique to (4.2). The zeroth approximation is

$$\psi_2^0 = \frac{S_2 - \pi}{4\pi}, \ \psi_{2n}^0 = \frac{S_{2n}}{4\pi n}, n = 2, 3, ... \tag{4.8}$$

Substituting (4.8) into (4.2) yields the first approximation

$$\psi_2^1 = \frac{S_2 - \pi}{4\pi}\left(1 - C_3^2 S_4 r_0^4\right), \ \psi_{2n}^1 = \frac{S_{2n}}{4\pi n} - C_{2n+1}^{2n} S_{2n+2} r_0^4 \frac{S_2 - \pi}{4\pi}, n = 2, 3, ... \tag{4.9}$$

and so on.

5 Analytical and numerical results

Symbolic computations were performed by $Mathematica^{©}$ to apply the method derived in Sec.4 and the formulae of Appendices A and B. The programs summarized in Appendix C yield the following analytical results.

If $\alpha = 1$ (square array), then

$$k_c^* = \ln \phi^{-1} - 1.47644 + 2\phi - 0.5\phi^2 - 0.0509713\phi^4 + 0.077465\phi^8 -$$

$$0.109757\phi^{12} + 0.122794\phi^{16} - 0.146135\phi^{20} + 0.244536\phi^{24} - 0.322667\phi^{28} + \tag{5.1}$$

$$0.310566\phi^{32} - 0.541237\phi^{36} + 0.820399\phi^{40} + O(\phi^{44}),$$

where $\phi = \pi r_0^2$. The terms up to ϕ^8 are identical to the terms of Drummond & Tahir (1984) who also provided a formula for k_c^* up to ϕ^8 for hexagonal arrays of cylinders.

For rectangular array an analogous formula can be written, but it is too long. Hence, we only provide the text of the program written in $Mathematica^{©}$ in Appendix C and the following formula up to ϕ^{10} calculated by this program

$$\begin{aligned} k_c^* \ = \ & -I_1 + 2I_2 + \phi - 2\phi^2 - \ln\phi/\pi + \phi\ln\phi/\pi + 2\phi^2 S_2\pi^{-1} - \phi^2 S_2^2/\pi^2 + \\ & 3\phi^4 S_4\pi^{-2} - 6\phi^4 S_2 S_4\pi^{-3} + 3\phi^4 S_2^2 S_4\pi^{-4} - \frac{1}{2}\phi^4 S_4^2\pi^{-4} - 9\phi^6 S_4^2\pi^{-4} + \\ & 18\phi^6 S_2 S_4^2\pi^{-5} - 9\phi^6 S_2^2 S_4^2\pi^{-6} + 27\phi^8 S_4^3\pi^{-6} - 54\phi^6 S_2 S_4^3\pi^{-7} + \\ & 27\phi^8 S_2^2 S_4^3\pi^{-8} - 81\phi^{10} S_4^4\pi^{-8} + 162\phi^{10} S_2 S_4^4\pi^{-9} - 81\phi^{10} S_2^2 S_4^4\pi^{-10} - \end{aligned}$$

$$10\phi^6 S_4 S_6\pi^{-5} + 10\phi^6 S_2 S_4 S_6\pi^{-6} + 30\phi^8 S_4^2 S_6\pi^{-7} - 30\phi^8 S_2 S_4^2 S_6\pi^{-8} -$$

$$90\phi^{10} S_4^3 S_6\pi^{-9} + 90\phi^{10} S_2 S_4^3 S_6\pi^{-10} - \frac{1}{3}\phi^6 S_6^2\pi^{-6} - 50\phi^8 S_6^2\pi^{-6} + \tag{5.2}$$

$$100\phi^8 S_2 S_6^2 \pi^{-7} - 50\phi^8 S_2^2 S_6^2 \pi^{-8} + 300\phi^{10} S_4 S_6^2 \pi^{-8} - 600\phi^{10} S_2 S_4 S_6^2 \pi^{-10} +$$
$$300\phi^{10} S_2^2 S_4 S_6^2 \pi^{-10} - 25\phi^{10} S_4^2 S_6^2 \pi^{-10} + \frac{35}{2}\phi^8 S_4^2 S_8 \pi^{-8} - 14\phi^8 S_6 S_8 \pi^{-7} +$$
$$14\phi^8 S_2 S_6 S_8 \pi^{-8} + 392\phi^{10} S_4 S_6 S_8 \pi^{-9} - 392\phi^{10} S_2 S_4 S_6 S_8 \pi^{-10} - \frac{1}{4}\phi^8 S_8^2 \pi^{-8} -$$
$$147\phi^{10} S_8^2 \pi^{-8} + 294\phi^{10} S_2 S_8^2 \pi^{-9} - 147\phi^{10} S_2^2 S_8^2 \pi^{-10} + 84\phi^{10} S_4 S_6 S_{10} \pi^{-10} -$$
$$18\phi^{10} S_8 S_{10} \pi^{-9} + 18\phi^{10} S_2 S_8 S_{10} \pi^{-10} + O(\phi^{10}),$$

where I_1 and I_2 have the forms (3.3) and (3.4); the Rayleigh sums S_{2n} are calculated with (7.1) and (7.2). Let us note that the term $\phi \ln \phi/\pi$ from (5.2) is reduced with the corresponding term of $2I_2$ (see (3.4)). The function $k_c^* = k_c^*(\alpha)$ is displayed in Figure 2.

6 Conclusion

We have studied the longitudinal permeability of rectangular arrays of circular cylinders, when a Newtonian fluid is flowing at low Reynolds number along the cylinders. We have reduced the problem to a functional equation and solved it. For square arrays the series of Drummond & Tahir (1984) were extended up to ϕ^8; we have computed the terms up to ϕ^{40} in (5.1). The next terms can be computed by the algorithm derived in Appendix C.

The major advantage of the method of functional equations is to provide analytical expressions for the longitudinal permeability up to a given precision in a relatively systematic manner in terms of the solid concentration ϕ and of the aspect ratio of the unit cell α^2. Numerical applications of (5.2) with various ϕ and α show that the permeability reaches a minimum for the square unit cell when ϕ is kept fixed.

The slow convergence for the two-dimensional Hashimoto's function was also solved. It is constructed in the form (2.2) which is very efficient computationally.

The present paper can be considered as an introduction to flow around cylinders and is exploited in the second part where more complex boundary value problems are discussed.

7 Appendix A

This Appendix recalls the basic functions of the elliptic function theory, namely the Weierstrass' functions due to Hurwitz (1964) and the Eisenstein's functions due to Weil (1976). We do not use the traditional parameters of this theory. As assumed in previous works devoted to periodic problems of porous media and composite materials, the lattice sums are used. Modified Eisenstein's functions are also needed, as well as some formulae applied in our calculations.

First, we consider the lattice sums $S_{2n} := \sum_j{}' e_j^{-2n}$ introduced by Rayleigh (1892). The theory of elliptic functions and Mityushev (1997a) provide the following formulae which are computationally efficient

$$
\begin{gathered}
S_2 = \left(\frac{\pi}{\alpha}\right)^2 \left(\frac{1}{3} - 8 \sum_{m=1}^{\infty} \frac{m h^{2m}}{1 - h^{2m}}\right), \text{ where } h = \exp\left(-\frac{\pi}{\alpha^2}\right), \\
S_4 = \frac{1}{3}\left(\frac{\pi}{\alpha}\right)^4 \left(\frac{1}{15} + 16 \sum_{m=1}^{\infty} \frac{m^3 h^{2m}}{1 - h^{2m}}\right), \\
S_6 = \frac{1}{15}\left(\frac{\pi}{\alpha}\right)^6 \left(\frac{2}{63} - 16 \sum_{m=1}^{\infty} \frac{m^5 h^{2m}}{1 - h^{2m}}\right).
\end{gathered} \tag{7.1}
$$

The other sums are calculated by the recursion formula

$$
S_{2k} = \frac{3}{(2k+1)(2k-1)(k-3)} \sum_{m=2}^{k-2} (2m-1)(2k-2m-1) S_{2m} S_{2(k-m)}.
$$

Let us write the first sums as functions of S_4 and S_6

$$
\begin{aligned}
S_8 &= \frac{3}{7} S_4^2, \; S_{10} = \frac{5}{11} S_4 S_6, \; S_{12} = \frac{1}{143}\left(18 S_4^3 + 25 S_6^2\right), && (7.2) \\
S_{14} &= \frac{30}{143} S_4^2 S_6, \; S_{16} = \frac{3 S_4 \left(33 S_4^3 + 100 S_6^2\right)}{2431}, \\
S_{18} &= \frac{5 S_6}{46189}\left(783 S_4^3 + 275 S_6^2\right), \; S_{20} = \frac{3 S_4^2}{508079}\left(2178 S_4^3 + 12125 S_6^2\right).
\end{aligned}
$$

The following Weierstrass' functions can be expressed as Taylor expansions

$$
\begin{gathered}
\ln \sigma(z) = \ln z - \sum_{n=2}^{\infty} \frac{S_{2n}}{2n} z^{2n}, \\
\zeta(z) = \frac{1}{z} - \sum_{n=2}^{\infty} S_{2n} z^{2n-1}, \\
\mathcal{P}(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1) S_{2n} z^{2n-2}.
\end{gathered} \tag{7.3}
$$

The branch of $\ln z$ is chosen so that $\ln z = \ln |z| + i \arg z$, where $\arg z \in [0, 2\pi)$; the cut of $\ln z$ is the positive real half-axis. It is easily seen that $(\ln \sigma(z))^{/} = \zeta(z)$, $\zeta^{/}(z) = -\mathcal{P}(z)$. The formulae (7.3) are not used here for calculating the Weierstrass' functions. For instance, $\sigma(z)$ is better computed by

$$\sigma(z) = \alpha \frac{\theta_1(z\alpha^{-1})}{\theta_1^{/}(0)} \exp\left(\frac{S_2}{2} z^2\right), \tag{7.4}$$

where $\theta_1(z) = i \sum_{m=-\infty}^{\infty} (-1)^m h^{(m-0.5)^2} \exp\left[i\pi (2m-1) z\right]$ is the Jacobi θ-function. Calculations with the formulae (7.1) and (7.4) are very fast because the coefficient h is smaller than 0.04324 for $0 < \alpha \leq 1$.

The function $\mathcal{P}(z)$ is doubly periodic. The functions $\zeta(z)$ and $\nu(z) = \ln \sigma(z)$ have the following jumps along $Q_{(0,0)}$

$$\zeta(z+\alpha) - \zeta(z) = \alpha S_2, \ \zeta(z + i\alpha^{-1}) - \zeta(z) = -i\alpha^{-1}\left(2\pi - S_2\right),$$

$$\nu(z+\alpha) - \nu(z) = \pi i + \alpha S_2 \left(z + \frac{\alpha}{2}\right), \tag{7.5}$$

$$\nu(z + i\alpha^{-1}) - \nu(z) = \pi i - i\alpha^{-1}\left(2\pi - S_2\right)\left(z + \frac{i}{2\alpha}\right).$$

The Eisenstein's functions (Weil (1976)) are also useful

$$E_m(z) := \sum_j \left(z - e_j\right)^{-m}.$$

These series are absolutely and almost uniformly convergent in $Q_{(0,0)} \setminus \{0\}$ for $m \geq 3$. When $m = 1$ or $m = 2$, a special method of summation is applied (Weil (1976))

$$\sum_j := \lim_{N\to\infty} \sum_{m_2=-N}^{-N} \left(\lim_{M\to\infty} \sum_{m_1=-M}^{-M} \right) \tag{7.6}$$

The Eisenstein's and Weierstrass' functions are related by the identities

$$E_1(z) = \zeta(z) - S_2 z, \ E_2(z) = \mathcal{P}(z) + S_2.$$

The modified Eisenstein's functions are defined by

$$\sigma_l(z) = E_l(z) - z^{-l}, \ l = 1, 2, \ldots .$$

They are analytic in the domain $Q_{(0,0)}$; $\sigma_l(0) = S_l$, where $S_l = 0$ for odd l.

8 Appendix B

This Appendix derives the formulae (3.3), (3.4) and (3.6). The double integral in the domain G and the ordinary integral along the curve ∂G are related by Green's formula

$$\int_G Q_x d\sigma_z = \int_{\partial G} Q dy \tag{8.1}$$

Using (8.1) we calculate the integral

$$I_1 = \left(\int_{Q_{(0,0)}} - \int_{D_i}\right) \left(S_2 x^2 + (2\pi - S_2)\, y^2\right) d\sigma_z =$$

$$\left(\int_{\partial Q_{(0,0)}} - \int_{\partial D_i}\right) \left(\frac{1}{3} S_2 x^3 + (2\pi - S_2)\, x y^2\right) dy =$$

$$\frac{1}{12}\left(S_2 \alpha^2 + (2\pi - S_2)\, \alpha^{-2}\right) - \frac{1}{2}\left(\pi r_0^2\right)^2 .$$

The integral I_2 has the form

$$I_2 = \int_{Q_{(0,0)}} \ln|\sigma(z)|\, d\sigma_z - \int_{D_i} \ln|\sigma(z)|\, d\sigma_z. \tag{8.2}$$

The first integral in (8.2) is calculated by (8.1)

$$T(0) := \int_{Q_{(0,0)}} \ln|\sigma(z)|\, d\sigma_z = \left(\alpha^2 + \alpha^{-2}\right)\left(-\frac{3}{4} + \frac{\pi}{8} - \frac{1}{4}\ln 2\right) + \frac{\alpha^2 - \alpha^{-2}}{2}\ln\alpha - P_0, \tag{8.3}$$

where

$$P_0 := \sum_{k=2}^{\infty} \frac{S_{2k}}{2k(2k+1)} Re \int_{-0.5\alpha^{-2}}^{0.5\alpha^2} \left[\left(\frac{\alpha}{2} + iy\right)^{2k+1} - \left(-\frac{\alpha}{2} + iy\right)^{2k+1}\right] dy =$$

$$\left(-\frac{1}{96} + \frac{1}{320}\left(\alpha^4 + \alpha^{-4}\right)\right) S_4 + \left(\frac{1}{2688}\left(\alpha^6 - \alpha^{-6}\right) - \frac{1}{384}\left(\alpha^2 - \alpha^{-2}\right)\right) S_6 +$$

$$\left(\frac{7}{5120} + \frac{1}{18432}\left(\alpha^8 + \alpha^{-8}\right) - \frac{1}{1536}\left(\alpha^4 + \alpha^{-4}\right)\right) S_8 +$$

$$\left(\frac{1}{112640}\left(\alpha^{10} - \alpha^{-10}\right) - \frac{1}{6144}\left(\alpha^6 - \alpha^{-6}\right) + \frac{1}{5120}\left(\alpha^2 - \alpha^{-2}\right)\right) S_{10} + \dots .$$

The second integral in (8.2) can be calculated in polar coordinates

$$\int_{D_i} \ln|\sigma(z)|\, d\sigma_z = -\frac{\pi r_0^2}{2} + \pi r_0^2 \ln r_0.$$

Therefore, I_2 can be expressed with the help of $T(0)$ given by (8.3)

$$I_2 = T(0) + \frac{\pi r_0^2}{2} - \pi r_0^2 \ln r_0.$$

We now proceed to deduce formula (3.6). (3.5) and (2.23) imply

$$I_3 = -\sum_{m=1}^{\infty} \psi_{2m} r_0^{4m} F_{2m}, \tag{8.4}$$

where

$$F_{2m} = \int_{D_e} Re\ [E_{2m} - S_{2m}]\, d\sigma_z,\ m = 1, 2, \dots .$$

F_{2m} is constructed by means of Green's formula (8.1). Moreover, the following formula is applied (Weil (1976))

$$E_{2m}(z) = -\frac{1}{2m-1} E'_{2m-1}(z)$$

The cases $m = 1$ and $m > 1$ need to be distinguished. For $m = 1$,

$$F_2 = -\int_{\partial Q_{(0,0)}} Re\ E_1(t) dy + \int_{|t|=r_0} Re\ E_1(t) dy - S_2 \left(1 - \pi r_0^2\right).$$

The Eisenstein's function $E_1(z)$ has a zero jump along $Q_{(0,0)}$ in the x - direction. Hence, $\int_{\partial Q_{(0,0)}} Re\ E_1(t) dy = 0$. The next integral is calculated in polar coordinates

$$\int_{|t|=r_0} Re\ E_1(t) dy = r_0 \int_0^{2\pi} Re\ \left[r_0^{-1} e^{-i\theta} - \sum_{n=1}^{\infty} S_{2n} r_0^{2n-1} e^{i\theta(2n-1)} \right] \cos\theta d\theta = \pi \left(1 - \pi r_0^2\right).$$

Therefore,

$$F_2 = \pi - S_2. \tag{8.5}$$

The case $m > 1$ is similarly processed

$$F_{2m} = -\frac{1}{2m-1} \int_{\partial Q_{(0,0)}} Re\ E_{2m-1}(t) dy + \frac{1}{2m-1} \int_{|t|=r_0} Re\ E_{2m-1}(t) dy - S_{2m} \left(1 - \pi r_0^2\right).$$

The first integral is equal to zero. The second integral is determined with the help of

$$E_{2m-1}(z) = z^{-2m+1} - \sum_{n=1}^{\infty} C_{2(m+n)-3}^{2n-1} S_{2(m-1+n)} z^{2n-1},$$

since

$$\frac{1}{2m-1} \int_{|t|=r_0} Re\ E_{2m-1}(t) dy = -\pi r_0^2 S_{2m}.$$

Therefore,

$$F_{2m} = -S_{2m},\ m = 2, 3, \dots . \tag{8.6}$$

Substitution of (8.5) and (8.6) into (8.4) yields the formula (3.6).

9 Appendix C

This Appendix summarizes the program in $Mathematica^{©}$ which calculates k_c^* for any rectangular array of cylinders with an arbitrary prescribed accuracy. In order to compute k_c^* in agreement with (3.7) and (3.2), we calculate I_1 and I_2 with (3.3) and (3.4). The crucial part of the calculations is the integral I_3 calculated with (3.6) in terms of the coefficients ψ_{2n}. We only give the central part of the program which computes the analytic expression of ψ_{2n} with a prescribed accuracy $O(r_0^n)$

```
c[p,q]:=q!/(p!*(q-p)!)
psi[r,n,0]:=If[n==1,S[2]-Pi,S[2*n]/n]
psi[r,n,p]:=psi[r,n,0]-Sum[c[2*n,2*(n+m)-1]*
S[2*(n+m)]*psi[r,m,p-m]*r^4*m),m,1,p]
r:=Sqrt[v/Pi]
p:=20
I3:=Sum[psi[r,m,p+1-m]*S[2*m]*r^4*m),m,1,p]-psi[r,1,p]*Pi*r^4
kc:=I1-2*I2+(2*Log[r]-Pi*r^2)*(1-Pi*r^2)+I3
```

Here, we take $n = p = 20$. psi[r,n,p] corresponds to ψ_{2n}^p, S[2*n] to S_{2n}, and v to ϕ, respectively; I1, I2 and I3 correspond to I_1, I_2 and I_3, respectively. kc corresponds to k_c^*.

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Figure 1: The unit cell of the rectangular array of cylinders

Figure 2: The effective permeability k_c^* as a function of the parameter α of the rectangle for various solid concentrations ϕ. It is calculated with (5.2)