Longitudinal permeability of spatially periodic rectangular arrays of circular cylinders II. An arbitrary distribution of cylinders inside the unit cell

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Abstract

We study the longitudinal permeability of unidirectional disjoint circular cylinders, when a Newtonian fluid is flowing at low Reynolds number along these cylinders; the longitudinal velocity satisfies the Poisson equation. The cylinders are arranged according to a doubly periodic structure. The number of cylinders in each rectangle can be arbitrary as well as their positions and radii. The method of functional equations yields analytical formulae for permeability in terms of these quantities. These formulae are written also in continuous form to study the flow for large numbers of cylinders. Special attention is paid to the case of the square unit cell, equal radii and lognormal distribution of radii.

1 Introduction

The transport properties of bundles of parallel cylinders attracted the attention of many famous scientists since the 19 th century as reviewed by Landauer (1974). If cylinders arranged according to square or hexagonal arrays, the method of Lord Rayleigh (1892) can be successfully applied. It is based on the reduction of the problem to an infinite set of linear algebraic

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equations which are truncated and solved numerically to get lower - order formulae for the effective tensor. The method of Lord Rayleigh has been extended by McPhedran et al. (1988). Other methods based on integral equations or infinite sets have been applied by Bergman & Dunn (1992), Kolodziej (1987), Sangani and Yao (1988a, 1988b) and others.

The method of functional equations has already been applied to various problems such as rectangular arrays of cylinders (Mityushev&Adler 2000). The major advantage of this method is to provide analytical expression for the effective transport properties in a relatively systematic manner.

The objective of this paper is to provide an analytical expression for the longitudinal permeability of a spatially periodic parttern of parallel cylinders. In the unit cell, the number of cylinders, their radii and their locations are arbitrary, as shown in Figure 1.

This paper is organized as follows. The problem is presented in Section 2, and the boundary value problem which governs the local velocity is derived. We also construct a counterpart of the two-dimensional Hashimoto function. In Section 3 the boundary value problem is reduced to a set of functional equations. An iterative convergent algorithm to solve analytically or numerically the set of functional equations is derived. The expression of permeability is given in Section 4. Approximate analytical formulae for the effective permeability are deduced in Section 5. Section 6 is devoted to the case when discs have equal radii. Large number of discs in the unit cell is discussed by continuous approach in Section 7. The lognormal distribution of the radii is investigated in Section 8. Section 9 presents numerical results and discussion of the general formulae. Appendix contains calculations of the integrals of elliptic functions which are used in the main text.

2 Statement of the problem

Consider a lattice $\mathcal{Q}$ defined by two perpendicular fundamental translation vectors ω_1 and ω_2 in the complex plane $\mathbb{C} \cong \mathbb{R}^2$, where the complex variable z is related to the real ones x and y by the identity $z = x + iy$ $(i^2 = -1)$. It may be assumed that $\omega_1 = \alpha > 0$ and $\omega_2 = i\alpha^{-1}$. The zero unit cell $Q_{(0,0)}$ is displayed in Figure 1; its area $\left|Q_{(0,0)}\right|$ is equal to 1. Let $\mathcal{E} := \cup_j \{e_{\mathbf{j}}\}$ be the set of numbers $e_{\mathbf{j}} := m_1 + im_2$, where $\mathbf{j} = (m_1, m_2)$, m_1 and m_2 are integers. The lattice $\mathcal{Q}$ consists of the cells $Q_{\mathbf{j}} = Q_{(0,0)} + e_{\mathbf{j}}$. Let us consider mutually disjoint disks $D_k := \left\{z \in Q_{(0,0)} : |z - a_k| < r_k\right\}$ in the zero cell $Q_{(0,0)}$. Here $a_k := x_k + iy_k$ is the center, r_k is the radius of D_k. Let $\widetilde{x} + i\widetilde{y} := \frac{1}{n}\sum_{k=1}^{n}(x_k + iy_k)$ denote the center of gravity of $x_k + iy_k$, $D := Q_{(0,0)} \backslash \left(\cup_{k=1}^{n}(D_k \cup \partial D_k)\right)$ be the complement of all disks D_k to the unit cell $Q_{(0,0)}$.

The plane $\mathbb{C}$ is assumed to be the perpendicular cross - section $D_k + e_j$ of an infinite array of parallel circular cylinders. A Newtonian fluid of viscosity μ is flowing at low Reynolds number through this array. When the driving pressure gradient is perpendicular to the plane $\mathbb{C}$, the Stokes equation is reduced to a Poisson equation for the component $w(x, y)$ of the fluid velocity (see Adler, 1992). Since we shall work with dimensionless quantities, it is convenient to assume that the pressure gradient and the viscosity are set equal to 1. Hence, $w(x, y)$ is a solution of the following boundary value problem that we shall study

$$\begin{aligned} \Delta w &= 1 \text{ in } D \\ & w \text{ is doubly periodic} \\ w &= 0 \text{ on the circle } |t - a_k| = r_k,\ k = 1, 2, ..., n, \end{aligned} \tag{2.1}$$

where Δ is the two-dimensional Laplace operator, the position inside a domain is denoted by the complex variable $z = x + iy$; the position along boundaries is denoted by the complex variable t.

The boundary value problem (2.1) with $n = 1$ has been solved by Mityushev&Adler (2000). We shall constantly use this result in the present paper. In particular, the lattice sums S_{2n} and the doubly periodic functions of Weierstrass and Eisenstein are derived in detail by Weil (1976) and Mityushev&Adler (2000). The functional equation equivalent to (2.1) will be obtained in three main steps following Mityushev&Adler (2000). The Poisson equation (2.1) is first reduced to a Laplace equation by a suitable change of unknown. Second the boundary value problem is stated in terms of analytic functions $\psi_k(z)$ whose properties are studied. Third, the boundary value problem for $\psi_k(z)$ is replaced by a set of functional equations (either continuous or discrete) for the unknown functions $\psi_k(z)$.

2.1 Reduction to a Laplace equation

In order to reduce (2.1) to a problem for the Laplace equation, we introduce the function

$$w_0(x, y) := \frac{1}{4\pi} \left(S_2 \left(x - \widetilde{x}\right)^2 + \left(2\pi - S_2\right) \left(y - \widetilde{y}\right)^2 \right) - \frac{1}{2\pi n} \sum_{k=1}^{n} \ln \left|\sigma(z - a_k)\right|, \tag{2.2}$$

where the constant S_2 and the Weierstrass function $\sigma(z)$ are derived in Mityushev&Adler (2000). The function $w_0(x, y)$ has the following properties:

i) $\Delta w_0 = 1$ in $Q_0 \backslash \cup_{k=1}^{n} \{a_k\}$,

ii) $w_0(x,y) \sim -\frac{1}{2\pi n} \ln|z-a_k|$ near $z=a_k$ for each $k=1,2,...,n$.

iii) $w_0(x,y)$ is doubly periodic.

Let us prove the third property, for instance, for the x - direction. Denote by $[w_0]_x$ the jump of $w_0(x,y)$ along the cell Q_0 in the x - direction. By virtue of the properties of the σ-function (see Mityushev&Adler (2000))

$$[w_0]_x = \frac{S_2}{4\pi}\left[(x-\widetilde{x})^2\right]_x - \frac{1}{2\pi n}\sum_{k=1}^{n}\left[\ln|\sigma(z-a_k)|\right]_x = 0.$$

We now are ready to rewrite problem (2.1) in terms of harmonic functions. Introduce the new unknown function $u(x,y)$ by

$$u(x,y) = w(x,y) - w_0(x,y) \text{ in } D. \tag{2.3}$$

$u(x,y)$ is a solution of the following boundary value problem

$$\begin{aligned} \Delta u &= 0 \text{ in } D \\ & u \quad \text{is doubly periodic} \\ u &= -w_0,\ |t-a_k| = r_k,\ k=1,2,...,n. \end{aligned} \tag{2.4}$$

2.2 Expression of the boundary conditions in terms of complex potentials

In order to represent the boundary value $-w_0$ as a real part of an analytic function in each $|t-a_k| = r_k$, we introduce the functions

$$f_k(z) := -\frac{1}{4\pi}[(S_2-\pi)(z-a_k)^2 + 2\left(S_2\alpha_k - i(2\pi - S_2)\beta_k\right)(z-a_k) + \tag{2.5}$$

$$S_2\alpha_k^2 + (2\pi - S_2)\beta_k^2 + \pi r_k^2] +$$

$$\frac{1}{2\pi n}\sum_{m\neq k}^{n} \ln\sigma(z-a_m) + \frac{1}{2\pi n}\ln\frac{\sigma(z-a_k)}{z-a_k} + \frac{1}{2\pi n}\ln r_k,\ k=1,2,...,n,$$

where $\alpha_k := x_k - \widetilde{x},\ \beta_k = y_k - \widetilde{y}$. The function $f_k(z)$ is analytic in $|z-a_k| < r_k$, Hölder continuous in $|z-a_k| \le r_k$ and satisfies the relation

$$Re\ f_k(t) = -w_0(x,y),\ |t-a_k| = r_k \text{ for each } k=1,2,...,n,$$

since the function $-w_0(x,y)$ can be represented as

$$-w_0(x,y) = -\frac{1}{4\pi}\left[S_2\left(Re\ (t-a_k)+\alpha_k\right)^2 + (2\pi - S_2)\left(Im\ (t-a_k)+\beta_k\right)^2\right]$$

$$+\frac{1}{2\pi n}\sum_{m=1}^{n}\ln|\sigma(z-a_m)|\,, \quad |t-a_k|=r_k.$$

It is known that each harmonic function is the real part of a complex potential in the complex plane $\mathbb{C}$. If a function $U(x,y)$ is harmonic in a multiply connected domain G of the complex plane, it can be expressed as

$$U(x,y)=Re\ \left[\Phi(z)+\sum_{k=1}^{n}A_k\ln(z-z_k)\right],\ z=x+iy\in G$$

according to the decomposition theorem (see Axler, 1996 et al.). Here, the function $\Phi(z)$ is analytic and single-valued in G, and A_k are real numbers. If we assume that $\infty\in G$, and G_k $(k=1,2,...,n)$ are connected components of the complement of G to $\mathbb{C}$, z_k is a point of G_k, the connectivity of G is equal to $n-1$ and

$$\sum_{k=1}^{n}A_k=0. \tag{2.6}$$

We now extend the decomposition theorem to the torus represented by the lattice $\mathcal{Q}$. We assert that the function $u(x,y)$ which is harmonic in D and doubly periodic, can be written as

$$u(x,y)=Re\ \left\{\varphi(z)+\sum_{k=1}^{n}A_k\left[\ln\sigma(z-a_k)+a_k\zeta(z-a_k)\right]\right\},\ z\in D, \tag{2.7}$$

where σ and ζ are Weierstrass' functions; A_k are real constants satisfying relation (2.6). The function $\varphi(z)$ is analytic in D and quasi-periodic. According to the terminology of the elliptic function theory, we define a quasi-periodic function as a function satisfying the relations

$$\varphi(z+\alpha)-i\gamma_1=\varphi(z)=\varphi(z+i\alpha^{-1})-i\gamma_2, \tag{2.8}$$

where γ_1 and γ_2 are real constants. The choice of a branch of the logarithm does not impact on the value $u(x,y)$ because we actually deal with $Re\ \ln z$ in (2.7). Let us choose an arbitrary branch of $\ln(z-a_k)$ and suppose that the cut corresponding to this fixed branch is doubly periodic and has no common points with D_m for each $m\neq k$.

The local form of the representation (2.7) in D follows from the decomposition theorem on the plane $\mathbb{C}$, where each bracket of (2.7) yields the increment of the complex potential along $|t-a_k|=r_k$. We only have to prove that $\varphi(z)$ is quasi-periodic; for this purpose calculate the jump (see

Mityushev&Adler (2000))

$$0 = [u]_x = Re\ \left\{ [\varphi(z)]_x + \sum_{k=1}^{n} A_k \left[\pi i + S_2\alpha \left(z - a_k + \frac{\alpha}{2} \right) + S_2\alpha a_k \right] \right\} =$$

$$Re\ [\varphi(z)]_x + S_2\alpha \sum_{k=1}^{n} A_k \left(x + \frac{\alpha}{2} \right) = Re\ [\varphi(z)]_x\,,$$

where relation (2.6) is used. Hence, $Re\ [\varphi(z)]_x = 0$ and $\varphi(z)$ has a purely imaginary jump along the $x-$direction. Similarly $\varphi(z)$ has a purely imaginary jump along the $y-$direction. The elliptic function theory implies that these jumps are constants and therefore (2.7) is valid.

We now can rewrite the problem (2.4) in terms of complex functions. Let us look for a function $\varphi(z)$ analytic in D, continuous in $D \cup \partial D$ and quasi-periodic with the following boundary condition

$$Re\ \varphi(t) = Re\ g_k(t), \ \ |t - a_k| = r_k,\ k = 1, 2, ..., n, \tag{2.9}$$

where the function $g_k(t)$ is defined by

$$g_k(t) := f_k(t) - \sum_{m \neq k}^{n} A_m \left[\ln \sigma(t - a_m) + a_m \zeta(t - a_m) \right] - \tag{2.10}$$

$$A_k \left[\ln \frac{\sigma(t - a_k)}{t - a_k} + a_k \zeta(t - a_k) - \frac{a_k}{t - a_k} \right] - A_k \left[\ln r_k + \frac{\overline{a_k}}{r_k^2}(t - a_k) \right].$$

$g_k(t)$ contains n unknown real constants A_k satisfying relation (2.6). For sake of convenience, we represent the data $Re\ g_k(t)$ in such a way that $g_k(z)$ is analytic in $|z - a_k| < r_k$ and Hölder continuous in $|z - a_k| \leq r_k$.

A few words about the general theory of problem (2.9) are needed. Problem (2.9) is a Riemann-Hilbert problem for a multiply connected domain D on the Riemann surface torus represented by the lattice $\mathcal{Q}$ (see Mityushev&Rogosin, 1999; Zverovich, 1971). Our statement of problem (2.9) corresponds to the modified Dirichlet problem studied by Mikhlin (1964) and completely solved by Mityushev&Rogosin (1999) in the complex plane. In the present paper we extend the method of functional equations presented by Mityushev&Rogosin (1999) from the complex plane to the torus $\mathcal{Q}$. As a first step, let us write (2.9) as a $\mathbb{R}-$linear conjugation problem

$$\varphi(t) = \psi_k(t) - \overline{\psi_k(t)} + g_k(t) + i\omega_k, \ \ |t - a_k| = r_k,\ k = 1, 2, ..., n, \tag{2.11}$$

where the unknown function $\psi_k(z)$ is analytic in $|z - a_k| < r_k$ and Hölder continuous in $|z - a_k| \leq r_k,\ \psi_k(a_k) = 0$. The real constants ω_k are unknown as well as A_k. But the form of ω_k does not impact on the required function $Re\ \varphi(z)$. Hence, we shall not calculate ω_k as discussed in Section 3.

Boundary value problems of the type (2.11) were reduced to integral equations by Zverovich (1971) and Mikhlin (1964). But such equations which can be solved numerically do not provide analytic formulae with explicit structure parameters. In the next section, we apply the method of functional equations to obtain analytic lower-ordered formulae for the effective permeability.

3 Functional equations

3.1 Costruction of functional equations

Let us represent the unknown functions $\psi_k(z)$ by their Taylor expansion

$$\psi_k(z) = \sum_{l=1}^{\infty} \psi_{lk}(z-a_k)^l,\ |z-a_k| \le r_k,\ k = 1, 2, ..., n.$$

Following Mityushev (1997), we introduce the operators $\mathbf{W}_k$ and $\mathbf{V}_k$

$$\mathbf{W}_k\psi_k(z) := \sum_{\mathbf{j}} \sum_{l=1}^{\infty} \overline{\psi_{lk}} \frac{r_k^{2l}}{(z-a_k-e_j)^l} = \sum_{l=1}^{\infty} \overline{\psi_{lk}} r_k^{2l} E_l(z-a_k),$$

$$\mathbf{V}_k\psi_k(z) := \sum_{l=1}^{\infty} \overline{\psi_{lk}} r_k^{2l} \sigma_l(z-a_k),$$

where $E_l(z)$ is the Eisenstein's function of order l (see Weil (1976) and Mityushev&Adler (2000)) and $\sigma_l(z)$ is the modified Eisenstein's function of order l. We introduce the function $\Phi(z)$ which is analytic in $|z-a_k| < r_k$ and D

$$\Phi(z) := \begin{cases} \psi_k(z) + \sum_{m \ne k}^{n} \mathbf{W}_m\psi_m(z) + \mathbf{V}_k\psi_k(z) + g_k(z) + i\omega_k,\ |z-a_k| \le r_k, \\ k = 1, 2, ..., n, \\ \varphi(z) + \sum_{m=1}^{n} \mathbf{W}_m\psi_m(z),\ z \in D. \end{cases}$$

Calculate the jump of $\Phi(z)$ along each circle $|t-a_k| = r_k$

$$\Delta_k := \Phi^+(t) - \Phi^-(t) = \varphi(t) + \mathbf{W}_k\psi_k(t) - \mathbf{V}_k\psi_k(t) - \psi_k(t) - g_k(t) - i\omega_k.$$

Use of the relation $E_l(z) - \sigma_l(z) = z^{-l}$ yields

$$\mathbf{W}_k\psi_k(t) - \mathbf{V}_k\psi_k(t) = \sum_{l=1}^{\infty} \overline{\psi_{lk}} r_k^{2l}(t-a_k)^{-l} = \overline{\psi_k(t)},\ |t-a_k| = r_k.$$

Here we apply the equality $r_k^2(t-a_k)^{-1} = \overline{t-a_k}$ on $|t-a_k| = r_k$. (2.11) implies that $\Delta_k = 0$; hence, $\Phi(z)$ is analytic in Q_0 by the principle of analytic continuation. Let us find the jumps of Φ along Q_0. We have

$$[\Phi]_x = [\varphi]_x + \sum_{m=1}^{n} [\mathbf{W}_m \psi_m(z)]_x = [\varphi]_x \, ,$$

since $[E_l]_x = 0$ for each $l = 1, 2, ...,$

$$[\Phi]_y = [\varphi]_y + \sum_{m=1}^{n} \overline{\psi_{1m}} r_m^2 \, [E_1]_y = [\varphi]_y - 2\pi i \alpha^{-1} \sum_{m=1}^{n} \overline{\psi_{1m}} r_m^2,$$

because $[E_l]_y = 0$ for $l = 2, 3, ...$ and $[E_1]_y = -2\pi i \alpha^{-1}$. Application of the generalized Liouville's theorem on the lattice $\mathcal{Q}$ implies

$$\Phi(z) = \Phi_0 + \Phi_1 z.$$

Let us determine the constants Φ_0 and Φ_1. It follows from the quasi-periodicity of φ that

$$[\Phi]_x = \Phi_1 \alpha \text{ and } Re \ [\Phi]_x = 0 \Longrightarrow Re \ \Phi_1 = 0,$$

$$[\Phi]_y = i \Phi_1 \alpha^{-1} \text{ and } Re \ [\Phi]_y = 0 \Longrightarrow Im \ \Phi_1 = -2\pi Im \ \sum_{m=1}^{n} \overline{\psi_{1m}} r_m^2.$$

Therefore,

$$\Phi_1 = 2\pi i Im \ \sum_{m=1}^{n} \overline{\psi_{1m}} r_m^2. \tag{3.1}$$

It follows from the definition of Φ in the disks $|z-a_k| \leq r_k$ that

$$\begin{aligned} \psi_k(z) &= -\sum_{m \neq k}^{n} \mathbf{W}_m \psi_m(z) - \mathbf{V}_k \psi_k(z) - g_k(z) - i\omega_k + \Phi_0 + \Phi_1 z \quad (3.2) \\ |z-a_k| &\leq r_k, \ k = 1, 2, ..., n. \end{aligned}$$

The equalities (3.2) can be considered as a set of n functional equations with respect to n functions $\psi_k(z)$ analytic in $|z-a_k| < r_k$ and continuous in $|z-a_k| \leq r_k$. The set (3.2) contains also undetermined constants ω_k. The general theory of such functional equations is discussed by Mityushev&Rogosin (1999) and Mityushev (1997). We note that (3.2) does not contain any integral term which requires difficult numerical calculations. Here we deal with compositions of functions instead of integrals.

Write (3.2) in discrete form

$$\psi_k(z) = -\sum_{m\neq k}^{n}\sum_{l=1}^{\infty}\overline{\psi_{lm}}r_m^{2l}E_l(z-a_m) - \sum_{l=1}^{\infty}\overline{\psi_{lk}}r_k^{2l}\sigma_l(z-a_k) - \tag{3.3}$$

$$-g_k(z) - i\omega_k + \Phi_0 + 2\pi i z Im\sum_{m=1}^{n}\overline{\psi_{1m}}r_m^2,\ |z-a_k|\leq r_k,\ k=1,2,...,n.$$

Substitute $z=a_k$ in (3.3) and take the real part:

$$0 = -\sum_{m\neq k}^{n}\sum_{l=1}^{\infty}r_m^{2l}Re\ \left[\overline{\psi_{lm}}E_l(a_k-a_m)\right] - \sum_{l=1}^{\infty}r_k^{4l}S_{2l}Re\ \overline{\psi_{2l,k}} - \tag{3.4}$$

$$-Re\ g_k(a_k) + q + Re\ [2\pi i a_k]\, Im\ \sum_{m=1}^{n}\psi_{1m}r_m^2,\ k=1,2,...,n,$$

where $q := Re\ \Phi_0$. Note that $Im\ \Phi_0$ does not impact on the final form of $Re\ \varphi(z)$. Hence, we shall calculate only q. Since values $Re\ g_k(a_k)$ contain the unknown constants A_k, we may consider (2.6) and (3.4) as $n+1$ real linear algebraic equations with respect to the $n+1$ real unknowns $A_1, A_2, ..., A_n, q$. Thus, (3.3), (3.4) and (2.6) generate a couple of equations, namely the functional equations (3.3) and the linear algebraic equations (3.4) and (2.6).

3.2 Iterative algorithm to solve functional equations

Let us construct an iterative algorithm to determine q, A_k and ψ_{lk}. We introduce the auxiliary values

$$P_k := -\sum_{m\neq k}^{n}\sum_{l=1}^{\infty}(-1)^l\, r_m^{2l}Re\ \left[\overline{\psi_{lk}}E_{mk}^l\right] - \sum_{l=1}^{\infty}r_k^{4l}S_{2l}Re\ \psi_{2l,k} - Re\ f_k(a_k) -$$

$$2\pi y_k\sum_{m=1}^{n}r_m^2 Im\ \overline{\psi_{1m}} + \sum_{m\neq k}^{n}A_m\eta_{km},\ k=1,2,...,n, \tag{3.5}$$

where η_{km} are defined as follows $\eta_{km} := \ln|\sigma(a_k-a_m)| - Re\ a_k\zeta(a_k-a_m)$. Then (3.4) becomes

$$A_k = (\ln r_k)^{-1}(P_k+q),\ k=1,2,...,n. \tag{3.6}$$

It follows from (2.6)

$$q = -L\sum_{m=1}^{n}\frac{P_m}{\ln r_m}, \tag{3.7}$$

where $L := \left(\sum_{m=1}^{n} \frac{1}{\ln r_m}\right)^{-1}$. The iterative algorithm is based on (3.3) - (3.7), since the right-hand part of (3.3) has an accuracy order in r_m^2 higher than the left-hand part. The right-hand part of (3.6) has an order on $(\ln r_m)^{-1}$ higher than the left-hand part.

First we take as zero-th approximation

$$A_k^0 = 0 \text{ and } \psi_k^{(0)}(z) = 0 \tag{3.8}$$

Then,

$$P_k^{(0)} = -f_k(a_k),\ q^{(0)} = -L\sum_{m=1}^{n} \frac{P_m^{(0)}}{\ln r_m} \tag{3.9}$$

We can propose a finite iterative algorithm modifying the algorithm (3.3) - (3.7). The finite algorithm is summarized as follows. Let us choose a natural number N, corresponding to calculation accuracy which is $O\left(\max(r_1^{2m_1} r_2^{2m_2} \cdots r_s^{2m_s})\right)$, where $m_1 + m_2 + ... + m_s = N + 1$, as $r :=\max_k r_k \to 0$. We may assume that N is even.

The first step of the algorithm consists of using (3.8) and calculating $P_k^{(0)}$ and $q^{(0)}$ by (3.9).

The p-th step consists of the sequence of operations. $\psi_{lk}^{(p-1)}$ and $A_k^{(p-1)}$ are known as the $(p-1)-$th approximations of the values required. Using (3.5) we introduce the values

$$P_k^{(p)} := -\sum_{m\neq k}^{n}\sum_{l=1}^{N} (-1)^l\, r_m^{2l} Re\ \left[\overline{\psi_{lk}^{(p-1)}} E_{mk}^l\right] - \sum_{l=1}^{N/2} r_k^{4l} S_{2l} Re\ \psi_{2l,k}^{(p-1)} - Re\ f_k(a_k) +$$

$$2\pi y_k \sum_{m=1}^{n} r_m^2 Im\ \psi_{1m}^{(p-1)} + \sum_{m\neq k}^{n} A_m^{(p-1)} \eta_{km},\ k = 1, 2, ..., n,$$

$$q^{(p)} = -L\sum_{m=1}^{n} \frac{P_m^{(p)}}{\ln r_m}, \tag{3.10}$$

$$A_k^{(p)} = (\ln r_k)^{-1}\left(P_k^{(p)} + q^{(p)}\right),\ k = 1, 2, ..., n. \tag{3.11}$$

The next approximation for $\psi_k(z)$ is derived from (3.3)

$$\psi_k^{(p)}(z) = -\sum_{m\neq k}^{n}\sum_{l=1}^{N-1} \overline{\psi_{lm}^{(p-1)}} r_m^{2l} E_l(t-a_m) - \sum_{l=1}^{N} \overline{\psi_{lk}^{(p-1)}} r_k^{2l} \sigma_l(t-a_k) - \tag{3.12}$$

$$-g_k(z) + q^{(p)} - 2\pi i z Im\ \sum_{m=1}^{n} \overline{\psi_{1m}^{(p-1)}} r_m^2,\ |z-a_k| \leq r_k,\ k = 1, 2, ..., n.$$

Here, the functions $g_k(z)$ contain the values $A_k^{(p)}$ according to (2.10).

If ψ_{lk} are known then

$$Re\ \phi(z) = q - \sum_{k=1}^{n}\sum_{l=1}^{\infty} r_m^2 Re\ \left[\overline{\psi_{lk}} E_l(z-a_m)\right] + Re\ [2\pi i z]\, Im\ \left[\sum_{k=1}^{n} r_k^2 \overline{\psi_{1k}}\right] \tag{3.13}$$

In the next section, we present the finite algorithm in discrete form. Moreover, we shall do a modification which considerably reduces the amount of required calculations.

4 Effective permeability

We now proceed to calculate the effective permeability K_{II} by integrating the velocity $w(x,y)$ over the unit cell

$$K_{II} = -\int_D w(x,y)d\sigma_z,$$

where $w(x,y)$ is the solution of problem (2.1). Application of (2.3) and (2.7) yields

$$K_{II} = -\int_D \left[w_0(x,y) + Re\ \varphi(z) + \sum_{k=1}^{n} A_k \left(\ln\left|\sigma\left(z-a_k\right)\right| + a_k\zeta\left(z-a_k\right)\right)\right] d\sigma_z.$$

According to Adler (1992), K_{II} can be expressed by $k_c^* := 4\pi K_{II}$. Then

$$k_c^* = -I_1 - \sum_{k=1}^{n}\left(4\pi A_k - \frac{2}{n}\right) J_k - 4\pi\sum_{k=1}^{n} A_k Re\ \left(a_k Y_k\right) - 4\pi I_3 \tag{4.1}$$

where

$$I_1 := \int_D \left(S_2\left(x-\widetilde{x}\right)^2 + \left(2\pi - S_2\right)\left(y-\widetilde{y}\right)^2\right) d\sigma_z,\ J_k := \int_D \ln\left|\sigma\left(z-a_k\right)\right| d\sigma_z,$$
$$Y_k := \int_D \zeta\left(z-a_k\right) d\sigma_z,\ I_3 := \int_D Re\ \varphi(z) d\sigma_z, \tag{4.2}$$

where $\int_D F(z)d\sigma_z := \int\int_D F(x+iy)dxdy$ denotes a double integral. Without loss of generality, we assume that $\widetilde{x} + i\widetilde{y} = 0$, i.e., the center of gravity of all a_k is equal to zero.

Let us substitute (3.13) in I_3

$$I_3 = q\,|D| - \sum_{m=1}^{n}\sum_{l=1}^{\infty} r_m^{2l} Re\ \left[\overline{\psi_{lm}} F_l^m\right] + 2\pi \sum_{m=1}^{n} \pi r_m^2 y_m \sum_{m=1}^{n} r_m^2 Im\ \psi_{1m}, \quad (4.3)$$

where $F_l^m := \int_D E_l\,(z-a_k)\,d\sigma_z,\ l=1,2,...$ are calculated in Appendix (see (10.7), (10.9) and (10.10)). Hence, in order to calculate k_c^* from (4.1), we need the constants $q,\ A_k,\ \psi_{lk}\ (l=1,2,...;\ k=1,2,...,n)$.

We now discuss the form of the calculated coefficients and the accuracy. One can see that each iteration of (3.12) increases the order of accuracy by a factor r_k^2 for the coefficients ψ_{lk} and each iteration of (3.11) increases the order of accuracy by a factor $(\ln r_k)^{-1}$. This shows that the coefficients ψ_{lk}, A_k and q can be represented as series with the basic elements $L\Theta_{t_1...t_n}^{s_1...s_n}$ defined as

$$\Theta_{t_1...t_n}^{s_1...s_n} := \frac{r_1^{2s_1} r_2^{2s_2} \cdots r_n^{2s_n}}{\ln^{t_1} r_1 \ln r_2 \cdots \ln^{t_n} r_n}. \quad (4.4)$$

The form of k_c^* preserves the same structure. Hence,

$$k_c^* = -2L + L \sum_{s_j,t_j} k^* \left[{s_1...s_n \atop t_1...t_n}\right] \Theta_{t_1...t_n}^{s_1...s_n}, \quad (4.5)$$

where the constants $k^* \left[{s_1...s_n \atop t_1...t_n}\right]$ depend only on the lattice parameter α and on the locations of the centers of the discs a_k $(k=1,2,...,n)$. The exceptional term $-2L$ appears in the zero-th approximation of q. Actually, in the zero-th approximation

$$P_k = -Re\ f(a_k) = -\frac{1}{2\pi n}\ln r_k + O(\textit{quasi-constant}),\ \text{as}\ r \to 0,$$

where terms of the form $L \ln^{-1} r_k$ are called quasi-constant. Then, it follows from (3.10) that

$$q = \frac{L}{2\pi} + O(\textit{quasi-constant}).$$

If equal radii $r_k = r$ are taken, the basic functions become

$$\Theta_t^s := \frac{r^{2s}}{\ln^t r},\ s = 0,1,...;\ t = -1,0,1,2,...\ ,$$

since $L = \frac{1}{n}\ln r$. In this case quasi-constants become usual constants and the leading term in (4.1) takes the form

$$k_c^* = -\frac{2}{n}\ln r + O(r^0),\ \text{as}\ r \to 0.$$

Let us come back to the general case with various r_k. Consider the set of divisors of the basic elements $\Theta_{t_1 \dots t_m}^{s_1 \dots s_n}$. We say that an accuracy is given, if a set $\mathcal{A}$ of mutually prime divisors is given. A divisor $\mathfrak{a}$ is called maximal if it can be divided by each divisor from $\mathcal{A}$ and if it is minimal from all such divisors. The maximal power in $\mathfrak{a}$ is called the index of $\mathfrak{a}$ and is designated by κ. For instance, if the accuracy is given by $\mathcal{A} = \left\{\Theta_{1,0}^{1,1},\ \Theta_{0,2}^{0,0}\right\}$, then

$$k_c^* = L\left[c_0 + \left(\ln^{-1} r_1 + c_5\right)\left(c_1 + c_2 r_1^2 + c_3 r_2^2 + c_4 r_1^2 r_2^2\right) + c_6 \ln^{-1} r_2 + c_6 \ln^{-2} r_2 - 2\right].$$

In this case $\mathfrak{a} = \Theta_{1,2}^{1,1}$ and $\kappa = 2$. The required accuracy can be reached by κ steps of the iterative method previously described. However, the number of iterations can be reduced. Let us note that ψ_{ls} enters in k_c^* with the factor r_s^{2l}. Hence, the accuracy can be reached by the reduced accuracy $\mathcal{A}_l$ for ψ_{ls}, where $\mathcal{A}_l$ is obtained from $\mathcal{A}$ by changing the power s_l by $s_l - l$. Moreover, if the accuracy for a term is reached in the algorithm, this term does not change in the next iterations.

For equal radii the accuracy is defined by the divisors $r^{2s}\left(\ln^{-1} r\right)^t$. For instance, if we take $\mathcal{A} = \left\{r^4,\ \left(\ln^{-1} r\right)^1\right\}$, we mean that k_c^* is calculated as

$$k_c^* = -\frac{2}{n}\ln r + c_0 + c_1 r^2 + c_2 r^4 + c_3 (\ln r)^{-1} + O\left(\frac{r^{2s}}{(\ln r)^{-t}}\right), \text{ as } r \to 0 \text{ for } s+t > 1.$$

According to the algorithm, the constants A_k have the form

$$A_k = (\ln r)^{-1}\left(c_4 + O\left(\frac{r^{2s}}{(\ln r)^{-t}}\right)\right).$$

The ψ_{lk} are involved into k_c^* with r^{2l}; hence, we need to calculate ψ_{1k} up $O(r^2)$ and ψ_{2k} up $O(r^0)$.

We now proceed to write the algorithm to calculate the ψ_{lk} from (3.12), i.e., we write (3.12) in a discrete form. The coefficients ψ_{1k} and ψ_{2k} have to be calculated separately. We select the terms with $(z - a_k)^1$ in (3.12)

$$\psi_{1k}^{(p)} = \sum_{m \neq k}^{n} \sum_{l=2}^{N} \overline{\psi_{lm}^{(p-1)}} r_m^{2l} l E_{mk}^{l+1} + \tag{4.6}$$

$$\sum_{m=1}^{n} r_m^2 \left(\left(S_2 + \rho_{mk}\right)\psi_{1m}^{(p-1)} + 2\pi i Im\, \overline{\psi_{1m}^{(p-1)}}\right) +$$

$$+\sum_{l=2}^{N/2} \overline{\psi_{2l-1,k}^{(p-1)}} r_k^{4l-2} (2l-1) S_{2l} + \frac{1}{2\pi}\left(S_2 x_k - i\left(2\pi - S_2\right) y_k\right) +$$

$$\frac{1}{2\pi n}\sum_{m\neq k}^{n}\zeta_{mk}+\sum_{m\neq k}^{n}A_m^{(p)}\left(\zeta_{mk}+a_k\rho_{mk}\right)+\frac{A_k^{(p)}\overline{a_k}}{r_k^2}.$$

Let us note that ψ_{1k} enters with the factor r_k^2. Hence, the term $A_k^{(p)}\overline{a_k}r_k^{-2}$ from (4.6) contains a removable singularity at $r_k=0$. Selecting the terms with $(z-a_k)^s$ in (3.12), we obtain

$$\psi_{sk}^{(p)}=(-1)^{s+1}\sum_{m\neq k}^{n}\sum_{l=1}^{N}\overline{\psi_{lm}^{(p-1)}}r_m^{2l}C_{l+s-1}^{s}E_{mk}^{l+s}- \tag{4.7}$$

$$\sum_{l=1}^{N}\overline{\psi_{l,k}^{(p-1)}}r_k^{4(l-1)}C_{l+s-1}^{s}S_{l+s}-f_{sk}+\sum_{m=1}^{n}A_m^{(p)}w_{ms},\ k=1,2,...,n;\ s=2,3,4,...\ ,$$

where

$$w_{ms}=\frac{1}{s!}\left[\ln\sigma\left(z-a_m\right)+a_m\zeta\left(z-a_m\right)\right]_{z=a_k}^{(s)}\ \text{ if } m\neq k,$$

$$w_{ms}=\frac{1}{s!}\left[\ln\frac{\sigma\left(z-a_k\right)}{z-a_k}+a_k\left(\zeta\left(z-a_k\right)-\frac{1}{z-a_k}\right)\right]_{z=a_k}^{(s)}\ \text{ if } m=k.$$

Here (s) is the derivative of order s, $C_n^k=\frac{n!}{k!(n-k)!}$.

5 Discrete lower-order formulae for the effective permeability

5.1 Non equal radii

The algorithm derived in the previous section is used to calculate an analytic expression of k_c^* by symbolic calculation of the coefficients of the series (4.5) corresponding to constants and quasi-constants, $(\ln r_k)^{-1}$, πr_k^2 and $(\pi r_k^2)^2$. We also calculate the terms of the type $c\,(\pi r_k^2)^2$, where c is a quasi-constant. In the present section the symbol ,,$\approx$ " is employed in equalities with an appropriate accuracy. Final formulae use integrals which were calculated analytically in the Appendix for the square cell ($\alpha=1$).

Introduce the values

$$c_{k0}:=-\frac{1}{4\pi}\left(S_2x_k^2+\left(2\pi-S_2\right)y_k^2\right)+\frac{1}{2\pi n}\sum_{m\neq k}^{n}\sigma_{mk},\ k=1,2,...,n, \tag{5.1}$$

which appear in the expression

$$Re\ f\left(a_k\right)=c_{k0}+\frac{1}{2\pi n}\ln r_k-\frac{r_k^2}{4}.$$

Formula (4.1) can be expressed as

$$k_c^* = -I_1 + \frac{2}{n}\sum_{k=1}^{n} J_k - 4\pi \sum_{k=1}^{n} A_k T_k - 4\pi I_3^1 - 4\pi I_3^2, \tag{5.2}$$

where I_1 is given by (10.1) - (10.2),

$$\frac{2}{n}\sum_{k=1}^{n} J_k \approx \frac{\pi}{n}\sum_{k=1}^{n} |a_k|^2 + 2T(0) - \frac{2}{n}\sum_{k=1}^{n}\sum_{m\neq k}^{n} \pi r_m^2 \sigma_{mk} + \frac{1}{n}\sum_{k=1}^{n} \pi r_k^2, \tag{5.3}$$

$$T_k := J_k + Re\ [a_k Y_k] \approx 2T(0) - \frac{\pi}{2}|a_k|^2, \tag{5.4}$$

The term $T(0) \approx -1.048576$ according to (10.4).

We introduce (10.5) into (5.3) and omit the term $\pi r_k^2 \ln r_k$ in (5.3) which is reduced with the same term in $4\pi I_3^1$. Hence,

$$I_3^1 = q\left(1 - \sum_{k=1}^{n} \pi r_k^2\right),$$

$$I_3^2 \approx -\sum_{m=1}^{n}\left[r_m^2 Re\ \left(\psi_{1m} F_1^m\right) + r_m^4 Re\ \left(\psi_{2m} F_2^m\right)\right] + 2\sum_{k=1}^{n} \pi r_k^2 y_k Im\ \psi_{1k},$$

where

$$q \approx \frac{L}{2\pi} + L\sum_{k=1}^{n}\frac{c_{k0}}{\ln r_k} - \frac{L}{4\pi}\sum_{k=1}^{n}\frac{\pi r_k^2}{\ln r_k} + L\sum_{k=1}^{n}\frac{\Psi_k}{\ln r_k}, \tag{5.5}$$

$$\Psi_k = \sum_{m\neq k}^{n}\left[r_m^2 Re\ \left(\overline{\psi_{1m}^{(1)}}\zeta_{mk}\right) + r_m^4 Re\ \left(\overline{\psi_{2m}^{(0)}}\rho_{mk}\right)\right] + S_2 \sum_{m=1}^{n} r_m^4 Re\ \psi_{1m}^{(0)} - \tag{5.6}$$

$$\sum_{m=1}^{n} r_m^2\left(S_2 Re\ \left(\psi_{1m}^{(0)}(a_k - a_m)\right) + 2\pi x_k Im\ \psi_{1m}^{(0)}\right),$$

F_1^m has the form (10.7), $F_2^m \approx \pi$. Here, the formulae (10.7) - (10.9) are used with a given accuracy. According to the general algorithm, the coefficients ψ_{lm} are needed in the form

$$\psi_{1m}^{(1)} \approx \tilde{\psi}_{1m}^{(-1)} r_m^2 \ln^{-1} r_m + \tilde{\psi}_{1m}^{(0)} + \tilde{\psi}_{1m}^{(1)} r_m^2,\ \psi_{2m}^{(2)} \approx \tilde{\psi}_{2m}^{(0)},$$

where $\tilde{\psi}_{lm}^{(j)}$ does not depend on r_k^2. We have to calculate A_k and q using only one iteration in Step 2 from Section 3. Tedious calculations imply

$$A_k \approx \frac{1}{2\pi n}\left(1 - \frac{nL}{\ln r_k}\right) + \left(c_{k0} - L\sum_{m=1}^{n}\frac{c_{m0}}{\ln r_m}\right)\frac{1}{\ln r_k}. \tag{5.7}$$

The formulae (4.6), (4.7) yield the required approximation for ψ_{1k} and ψ_{2k}

$$\psi_{1k}^{(0)} = \frac{1}{2\pi}\left(S_2 x_k - i\left(2\pi - S_2\right) y_k\right) - \frac{1}{2\pi n}\sum_{m\neq k}^{n} \zeta_{mk},\ \psi_{2k}^{(0)} = \frac{S_2-\pi}{4\pi} + \frac{1}{4\pi n}\sum_{m\neq k}^{n}\rho_{mk}, \tag{5.8}$$

$$\psi_{1k}^{(1)} = \sum_{m\neq k}^{n}\overline{\psi_{2m}^{(0)}} r_m^4 2E_{mk}^3 + \sum_{m=1}^{n} r_m^2 \left(S_2 Re\ \psi_{1m}^{(0)} - i\left(2\pi - S_2\right) Im\ \psi_{1m}^{(0)}\right) + \psi_{1k}^{(0)},\ k = 1,2,...,n.$$

We have presented k_c^* in the form (5.2), where I_1 is given by (10.2), the sum $\frac{2}{n}\sum_{k=1}^{n} J_k$ by (5.3), and so on (see formulae (5.3) - (5.6)). Let us write k_c^* in the extended form for the square array

$$k_c^* \approx -2L + c_0 + c_1 + c_2 + c_3, \tag{5.9}$$

where c_0 contains constants and quasi-constants, c_1 - logarithms, c_2 - the terms πr_k^2, c_3 - the terms $\pi r_k^2 \pi r_m^2$.

These four values can be represented as

$$c_0 = -\frac{\pi}{6} + \frac{\pi}{n}\sum_{k=1}^{n}\left|a_k\right|^2 + 2T(0) - 4\pi L\sum_{k=1}^{n}\frac{c_{k0}}{\ln r_k}, \tag{5.10}$$

where the constants c_{k0} are given by (5.1). Moreover,

$$c_1 = -4\pi\sum_{k=1}^{n} A_k\left[2T(0) - \frac{\pi}{2}\left|a_k\right|^2\right] - 4\pi q \tag{5.11}$$

where A_m is given by (5.7), and q by (5.5). Then,

$$c_2 = -\pi\sum_{k=1}^{n}\pi r_k^2\left|a_k\right|^2 - \frac{2}{n}\sum_{k=1}^{n}\sum_{m\neq k}^{n}\pi r_m^2\sigma_{mk} + \frac{1}{n}\sum_{k=1}^{n}\pi r_k^2 -$$

$$4\pi L\sum_{k=1}^{n}\frac{1}{\ln r_k}\left(-\frac{r_k^2}{4} + \Psi_k^{(0)}\right) + 4\pi\sum_{k=1}^{n}\pi r_k^2 L\sum_{k=1}^{n}\frac{c_{k0}}{\ln r_k} - \tag{5.12}$$

$$2\pi\sum_{m=1}^{n} r_m^2 Re\ \left[a_m\left(\pi\overline{a_m} - \frac{1}{n}\sum_{m\neq k}^{n}\zeta_{mk}\right)\right],$$

where

$$\Psi_k^{(0)} = \sum_{m\neq k}^{n} r_m^2 Re\ \left[\frac{1}{2}\left(\overline{a_m} - \frac{1}{n}\sum_{l\neq m}^{n}\overline{\zeta_{lm}}\right)E_{mk}^1\right].$$

Finally,

$$c_3 = \frac{1}{2}\sum_{k=1}^{n}\left(\pi r_k^2\right)^2 + 4\pi\sum_{m=1}^{n}\left[r_m^2\left(Re\ \psi_{1m}^{(1)}\left(\pi a_m + \sum_{s=1}^{n}\pi r_s^2\left(\zeta_{sm} - (a_m - a_s)\right)\right)\right) - \right.$$

$$\left.\pi r_m^4 Re\ \psi_{2m}^{(0)}\right] - 8\pi\sum_{k=1}^{n}\pi r_k^2 y_k \sum_{k=1}^{n}\pi r_k^2 Im\ \psi_{1k}^{(0)},$$

where $\psi_{1k}^{(j)}$, $\psi_{2k}^{(0)}$ have the forms (4.6) and (4.7), respectively. In this paper, the arguments of the elliptic functions are the differences $a_k - a_m$. For the sake of brevity, we introduce the following notations

$$\sigma_{mk} := \ln\left|\sigma\left(a_k - a_m\right)\right|,\ \zeta_{mk} := \zeta\left(a_k - a_m\right),\ \eta_{mk} := \sigma_{mk} - a_k\zeta_{mk},$$

$$\rho_{mk} := \mathcal{P}\left(a_k - a_m\right),\ E_{mk}^{l} := E_l\left(a_k - a_m\right). \tag{5.13}$$

5.2 Equal radii

Let us consider the case where $r_k = r$ for each $k = 1, 2, ..., n$ and the square array. The formulae from the previous sections become

$$k_c^* \approx -\frac{2}{n}\ln r + c_0^0 + c_1^0\frac{n}{\ln r} + c_2^0 n\pi r^2 + c_3^0\left(n\pi r^2\right)^2. \tag{5.14}$$

The coefficients of the various terms $\ln r$ and $\phi := n\pi r^2$ of (5.14) are given by

$$c_0^0 = -\frac{\pi}{6} + 2T(0) - \frac{2}{n}\sum_{k=1}^{n}\sum_{m\neq k}^{n}\sigma_{mk}, \tag{5.15}$$

$$c_1^0 = -\frac{4\pi}{n}\sum_{k=1}^{n}B_k\left[2T(0) - \frac{\pi}{2}\left|a_k\right|^2\right], \tag{5.16}$$

where

$$B_k = c_{k0} - \frac{1}{n}\sum_{m=1}^{n}c_{m0}.$$

We have

$$c_2^0 = \frac{\pi}{n}\sum_{k=1}^{n}\left|a_k\right|^2 - \frac{2}{n^2}\sum_{k=1}^{n}\sum_{m\neq k}^{n}\sigma_{mk} + \frac{2}{n} -$$

$$\frac{2}{\pi n^2}\sum_{k=1}^{n}\sum_{m\neq k}^{n}Re\ \left[\left(\pi\overline{a_k} - \frac{1}{n}\sum_{s\neq m}^{n}\overline{\zeta_{mk}}\right)E_{mk}^{1}\right] + \tag{5.17}$$

$$\frac{4\pi}{n}\sum_{k=1}^{n} c_{k0} + \frac{2}{n}\sum_{k=1}^{n} Re\ \left[a_k\left(\pi a_k - \frac{1}{n}\sum_{m\neq k}^{n}\overline{\zeta_{mk}}\right)\right],$$

$$c_3^0 = \frac{1}{2n} + \frac{4}{n^2}\sum_{m=1}^{n}\sum_{s=1}^{n} Re\ \left[a_m \pi \psi_{1m}^0\right] + \frac{4}{n^2}\sum_{m=1}^{n} Re\ \left[a_m \psi_{1m}^{(0)}\right] + \frac{4}{n^2}\sum_{m=1}^{n} Re\ \psi_{2m}^{(0)}, \tag{5.18}$$

where $\psi_{1k}^{(0)}$ and $\psi_{2k}^{(0)}$ given by (4.6) and (4.7), respectively. This completes the terms in (5.14).

Hence, the great advantage of the method of functional equations is visible in formula (5.14), since as claimed at the beginning, the coefficients have explicit expressions (5.15) - (5.18) as functions of the centers a_k.

6 Statistical lower-order formulae for the effective permeability

The formulae derived in the previous section admit a statistical interpretation. Let us consider (5.9) and (5.14) from the statistical point of view. If the number of holes is sufficiently large, we may suppose that the centers of holes a_k and the radii of holes r_k are distributed according to given distribution functions. For the sake of simplicity, we assume that the distributions of a_k and r_k are independent. Let us introduce the distribution measures $d\chi(z)$ defined in $Q_{(0,0)}$ for a_k and $d\eta(r)$ defined in $(0, +\infty)$ for r_k. For instance, $\int_G d\chi$ denotes the probability that $n\,|G|$ points a_k are in the domain $G \subset Q_{(0,0)}$. recall that $\left|Q_{(0,0)}\right| = 1$. Since $d\eta(r)$ does not depend on $d\chi(z)$, it gives the distribution of the radii r_k. In particular the normalization conditions hold, i.e.,

$$\int_0^{+\infty} d\eta(r) = 1,\ \int_{Q_{(0,0)}} d\chi(z) = 1$$

We stress here that only non-overlapping discs are considered. Therefore, the distribution $d\chi(z)$ must satisfy some conditions. For instance, in Section 9 we discuss an example where each center a_k may be shifted within in a prescribed region and where each r_k is bounded by a fixed value.

Let us study the form of k_c^* by calculating the statistical average

$$\langle k_c^* \rangle = \int_0^{+\infty} d\eta(r) \int_{Q_{(0,0)}} k_c^* d\chi(z). \tag{6.1}$$

Here we assume that the measures $d\chi$ and $d\eta$ are such that the integral (6.1) converges. Because of the independence of the measures $d\chi$ and $d\eta$, we separately calculate the statistical average of the coefficients of (4.5) by the measure $d\chi(z)$ and the statistical average of the basis functions L, $L\Theta_{t_1...t_n}^{s_1...s_n}$. (4.5) yields

$$\langle k_c^* \rangle = \left[\int_0^{+\infty} L d\eta(r) \right] \left\{ -2 + \sum_{s_j,t_j} \left[\int_0^{+\infty} \Theta_{t_1...t_n}^{s_1...s_n} d\eta(r) \right] \left[\int_{Q_{(0,0)}} k^* \left[{s_1...s_n \atop t_1...t_n} \right] d\chi(z) \right] \right\}. \tag{6.2}$$

Define

$$R(s,m) := \int_0^{+\infty} r^{2s} \ln^{-m} r d\eta(r), \; s = 0,1,...; \; \; m = 0,1,... \,. \tag{6.3}$$

Then

$$\int_0^{+\infty} L d\eta(r) = [nR(0,1)]^{-1} \,.$$

Using (4.4) we have

$$\int_0^{+\infty} \Theta_{t_1...t_n}^{s_1...s_n} d\eta(r) = R(s_1,t_1) R(s_2,t_2) ... R(s_n,t_n). \tag{6.4}$$

6.1 Equal radii

First, we consider the discrete formula (5.14) for equal radii and derive $\langle k_c^* \rangle$ in this case with an appropriate accuracy. We have

$$\langle k_c^* \rangle = -\frac{2}{nR(0,1)} + \langle c_0 \rangle + \langle c_1 \rangle nR(0,1) + \langle c_2 \rangle \phi + \langle c_3 \rangle \phi^2, \tag{6.5}$$

where $\langle c_j \rangle$ is the statistical average of c_j^0. Considering sums in c_j^0 as Riemannian sums of integrals, we obtain

$$\langle c_0 \rangle = -\frac{\pi}{6} + 2T(0) - 2 \int_{Q_{(0,0)}} \int_{Q_{(0,0)}} \ln |\sigma(z-w)| \, d\chi_w d\chi_z, \tag{6.6}$$

where as always $z = x + iy$. Introduce the function $B(x,y) := c(x,y) - \int_{Q_{(0,0)}} c(x,y) d\chi$. Application of the relations $E_1(z) = \zeta(z) - S_2 z$ and $\int_{Q_{(0,0)}} B(z) d\chi = 0$ yields

$$\langle c_1 \rangle = -\pi \int_{Q_{(0,0)}} B(x,y) \left(x^2 + y^2 \right) d\chi. \tag{6.7}$$

The two next coefficients are

$$\langle c_2 \rangle = \pi \int_{Q_{(0,0)}} \left(x^2 + y^2 \right) d\chi - 2 \int_{Q_{(0,0)}} \int_{Q_{(0,0)}} \ln |\sigma(z-w)| \, d\chi_z d\chi_w -$$

$$\frac{2}{\pi}\int_{Q_{(0,0)}}\int_{Q_{(0,0)}} Re\ \left[\left(\pi\overline{z}-\int_{Q_{(0,0)}}\overline{\zeta(w-\varsigma)}d\chi_\zeta\right)(\zeta(z-w)-\pi(z-w))\right]d\chi_z d\chi_w+$$

$$4\pi\int_{Q_{(0,0)}} c(x,y)d\chi+2\int_{Q_{(0,0)}} Re\ \left[z\left(\pi z-\int_{Q_{(0,0)}}\overline{\zeta(z-w)}d\chi_w\right)\right]d\chi_z. \tag{6.8}$$

$$\langle c_3\rangle = 2\pi\int_{Q_{(0,0)}} Re\ \left[z\left(z-\frac{1}{\pi}\int_{Q_{(0,0)}}\overline{\zeta(z-w)}d\chi_w\right)\right]d\chi_z, \tag{6.9}$$

Let us note that n tends toward $+\infty$ in the calculations of $\langle c_j\rangle$; $R(0,1)$ tends to zero if $n\to+\infty$ (or $r\to 0$). Hence, the undetermined term $nR(0,1)$ in (6.5) should be calculated, i.e., the relation between n and $R(0,1)$ should be precisely described in each particular case.

The formula (6.5) can be considered as the average value of k_c^* for a given distribution $d\chi$. It gives a good estimation if the number of holes n is sufficiently large, or more precisely, if the Riemannian sums from (5.15) - (5.18) provide good approximations of the integrals from (6.6) - (6.9). Here, we are close to the question of substituting an ensemble of holes by a large hole, but we shall not discuss it; in a sense, it is directly related to the approximation of the Riemannian sums by the corresponding integrals.

6.2 Non equal radii

We now proceed to study the case of different radii r_k. It follows from (5.10) that $\langle c_0\rangle$ contains in addition to some constants the term

$$-4\pi\frac{1}{nR(0,1)}\sum_{k=1}^{n}c_{k0}\int_0^{+\infty}\frac{d\eta\,(r_k)}{\ln r_k}=-\frac{4\pi}{n}\sum_{k=1}^{n}c_{k0}.$$

One can see that the result does not depend on r_k and $d\eta\,(r)$. Hence, $\langle c_0\rangle$ has the same form (6.6) as in the case of equal radii. Along similar lines, we calculate the term $\langle c_1\rangle$ corresponding to c_1 from (5.11). First, we calculate the statistical average

$$\left\langle\frac{1}{n}\sum_{m\neq k}^{n}\frac{c_{m0}}{\ln r_m}\frac{1}{\ln r_k}+\frac{c_{k0}}{n\ln^2 r_k}\right\rangle = R^2(0,1)\left\langle\frac{1}{n}\sum_{m\neq k}^{n}c_{m0}\right\rangle+\left\langle\frac{c_{k0}}{n}\right\rangle R(0,2),$$

where the statistical average in the left-hand part is determined with the measure $d\eta(r)d\chi(z)$ and the statistical average in the right-hand part is determined with $d\chi(z)$. After calculation of the statistical average with $d\chi\,(z)$

$(n \to +\infty)$, the term $\left\langle \frac{c_{k0}}{n} \right\rangle R(0,2)$ vanishes. Then, the statistical average of A_k from (5.7) can be calculated as in the case of equal radii

$$\langle A_k \rangle \approx \left\langle c_{k0} - \frac{1}{n} \sum_{m=1}^{n} c_{m0} \right\rangle R(0,1).$$

The statistical average of q from (5.5) becomes

$$\langle q \rangle \approx \frac{1}{2\pi n R(0,1)} + \left\langle \frac{1}{n} \sum_{m=1}^{n} c_{m0} \right\rangle - \frac{R(1,1)}{4\pi R(0,1)}. \tag{6.10}$$

Here, the difference between the cases of equal and non-equal radii can be clearly seen. For equal radii, the term $\frac{R(1,1)}{R(0,1)}$ in (6.10) is replaced by $R(1,0)$. Therefore, the first two terms of (6.5) are valid for different radii; the next terms are changed as it is shown.

In general, an average value can be very different from the value obtained for a typical realization and it can be estimated by the standard deviation which can be expressed by

$$D(v_k) = \frac{1}{n} \sum_{k=1}^{n} \left(v_k^2 - \widetilde{v}^2 \right),$$

where $\widetilde{v} := \frac{1}{n} \sum_{k=1}^{n} v_k$. The statistical average is

$$\langle D(v_k) \rangle = \int_{Q_{(0,0)}} v^2(x,y) d\chi - \left(\int_{Q_{(0,0)}} v(x,y) d\chi \right)^2$$

with an appropriate function $v(x,y)$ corresponding to v_k. The standard deviation of k_c^* can be derived as follows

$$\left\langle (k_c^*)^2 - \langle k_c^* \rangle^2 \right\rangle = \left\langle L^2 \left(-2 + \sum_{s_j, t_j} k^* \begin{bmatrix} s_1 \ldots s_n \\ t_1 \ldots t_n \end{bmatrix} \Theta_{t_1 \ldots t_n}^{s_1 \ldots s_n} \right)^2 \right\rangle - \langle k_c^* \rangle^2,$$

where $\langle k_c^* \rangle$ is calculated with (6.2).

7 Lognormal distribution of the radii

In the previous section, we discuss the general distribution $d\eta$ of the radii, when $0 < r < +\infty$. When a distribution $d\eta$ is truncated above $r_{\max}$, a new renormalized distribution $d\gamma$ is defined in such a way that $d\gamma = d\eta \left[\int_0^{r_{\max}} d\eta\right]^{-1}$. $r_{\max}$ verifies the following normalization equation

$$\int_0^{r_{\max}} \pi r^2 d\gamma = \frac{\phi}{n} \int_0^{r_{\max}} d\gamma, \tag{7.1}$$

where ϕ and n are the concentration of the holes and the number of inclusions, respectively.

The most important distribution is the lognormal distribution for which

$$d\gamma(r) = \frac{\alpha}{r} \exp\left[-\frac{(\ln r - \mu)^2}{2s^2}\right] dr,$$

where α, μ and s are given non-negative parameters. Then (7.1) becomes

$$\frac{\phi}{\pi n} = \frac{\int_0^{r_{\max}} r \exp\left[-\frac{(\ln r-\mu)^2}{2s^2}\right] dr}{\int_0^{r_{\max}} r^{-1} \exp\left[-\frac{(\ln r-\mu)^2}{2s^2}\right] dr}. \tag{7.2}$$

If $0 < r_{\max} << 1$, we propose an asymptotic formula for $r_{\max}$.

We shall use the standard error function

$$erf\ (u) := \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt$$

which satisfies the properties $erf\ (0) = 0$, $erf\ (+\infty) = 1$. Put $p := \frac{\phi}{\pi n}$, $x_0 := \frac{X+\mu}{\sqrt{2}s}$, $X := -\ln r_{\max}$. The integrals from (7.2) are transformed by substituting $t = -\ln r$ as follows

$$\int_0^{r_{\max}} r^{-1} \exp\left[-\frac{(\ln r - \mu)^2}{2s^2}\right] dr = \sqrt{2\pi} s \left[1 - erf\ (x_0)\right],$$

$$\int_0^{r_{\max}} r \exp\left[-\frac{(\ln r - \mu)^2}{2s^2}\right] dr = e^{-2(\mu+s^2)} \sqrt{2\pi} s \left[1 - erf\ \left(x_0 + \sqrt{2}s\right)\right].$$

Substitution of these expressions in (7.2) yields the following number equation for x_0

$$pe^{2(\mu+s^2)} = \frac{1 - erf\ \left(x_0 + \sqrt{2}s\right)}{1 - erf\ (x_0)}. \tag{7.3}$$

One can solve this equation numerically. However, when $r_{\max}$ is small, we can deduce an approximate analytical formula for x_0. Using the asymptotic relation

$$erf\ (x) \backsim 1 - \frac{1}{\sqrt{\pi} x} e^{-x^2}, \text{ as } x \to +\infty$$

since $erf\ (x) - 1$ is odd, we obtain the following equation

$$pe^{2(\mu+s^2)} = \frac{x_0}{x_0 - \sqrt{2}s}$$

and the desired formula for the maximal radius

$$r_{\max} = \exp\left[\mu + \frac{pe^{2(\mu+s^2)}}{1 - pe^{2(\mu+s^2)}}\right]. \tag{7.4}$$

As we saw in Section 4, the basic functions of the statistical representation (6.5) contain $R(k,m)$. We now try to estimate $R(k,m)$ defined by (6.3) for a lognormal distribution

$$R(k,m) = \alpha \int_0^{r_{\max}} \frac{r^{2k-1}}{\ln^m r} \exp\left[-\frac{(\ln r - \mu)^2}{2s^2}\right] dr. \tag{7.5}$$

The substitution $-t = \ln r$ transforms (7.5) into

$$R(k,m) = -\int_X^{+\infty} e(t) t^{-m} dt, \tag{7.6}$$

where $e(t) := \alpha \exp\left[-\frac{(t+\mu)^2}{2s^2} - 2kt\right]$, $X := -\ln r_{\max}$. The integral (7.6) is correctly defined as an improper one, since $X > 0$ and it decreases exponentially at infinity. Integrate $R(k,m)$ by parts

$$R(k,m) = \frac{1}{1-m} X^{1-m} e(X) + \frac{1}{1-m} \int_X^{+\infty} e^{/}(t) t^{1-m} dt,$$

where $m > 1$, $e^{/}(t) = -\left(\frac{(t+\mu)^2}{2s^2} + 2kt\right) e(t)$. We have

$$R(k,m) = \frac{1}{1-m} X^{1-m} e(X) + \frac{1}{(1-m)s^2} R(k,m-2) + \frac{\mu s^{-2} + 2k}{(1-m)s^2} R(k,m-1). \tag{7.7}$$

Hence, we obtain the recursive formula (7.7) for $R(k,m)$. Therefore, it is sufficient to calculate $R(k,0)$ and $R(k,1)$ in order to determine $R(k,m)$ for an arbitrary m by (7.7).

The integral $R(1,0)$ has been considered in relation with equation (7.2). Along similar lines, we obtain

$$R(k,0) = \alpha \sqrt{\frac{\pi}{2}} s \exp\left[2k\mu + 2s^2k^2\right] \left[1 - erf\ \left(\frac{X+\mu}{\sqrt{2}s} + \sqrt{2}ks\right)\right]. \tag{7.8}$$

The following integral requires a more complicated investigation

$$R(k,1) = -\int_X^{+\infty} e(t)t^{-1}dt.$$

As for $m > 1$, integrate by parts

$$R(k,1) = e(X)\ln X + \int_X^{+\infty} \ln t\, e^{/}(t)dt.$$

Using the identity (see, for instance, the program MATHEMATICA)

$$\int_0^{+\infty} \ln t\, e^{/}(t)dt = 0 \tag{7.9}$$

we obtain

$$R(k,1) = e(X)\ln X - \int_0^X \ln t\, e^{/}(t)dt.$$

It follows from (7.9) that the latter integral tends to zero as $X \to +\infty$ (or $r_{\max} \to 0$). If $X \to 0$(or $r_{\max} \to 1$), this integral also tends to zero. Its extremum is attached at the point $X = 1$. Investigate $R(k,1)$ as a function of X. We have $R^{/}(k,1) = X^{-1}e(X) > 0$ for $X > 0$ and $R^{//}(k,1) < 0$ for $X > 0$. Hence, the function $R(k,1)$ as a function of X increases from $-\infty$ for $X = 0$ to zero for $X = +\infty$, and it is convex. Near $X = 0$ the function $R(k,1)$ is similar to $\ln X$. We are interested by the behavior of $R(k,1)$ for large X. Application of de l'Hospital's rule yields the relation

$$\lim_{X \to +\infty} \frac{R(k,1)}{-\frac{s^2}{X^2}e(X)} = 1.$$

Therefore, the following asymptotic formula is valid

$$R(k,1) \backsim -\frac{\alpha s^2}{\ln^2 r_{\max}} \exp\left[-\frac{(\ln r_{\max} - \mu)^2}{2s^2} - 2k\ln r_{\max}\right] \tag{7.10}$$

for small positive $r_{\max}$.

Thus, in order to calculate the basic functions (6.4) from (6.2) for the lognormal distribution we put $R(k,0)$ and $R(k,1)$ from (7.8) and (7.10), after calculation of arbitrary $R(k,m)$ by the recurrent formula (7.7). Moreover, in these calculations, we assume that $r_{\max}$ has the form (7.4).

8 Numerical results and conclusion

In the present paper, we were mainly concerned with the theoretical investigation of the longitudinal permeability of unidirectional circular cylinders. An iterative convergent algorithm based on the method of functional equations was derived. We showed that the coefficient k_c^* has the general form (4.5). In particular we deduced the discrete lower-order formulae (5.9), (5.14) and the statistical average (6.5). We discussed in detail the lognormal distribution of the radii; namely, formula (7.4) for the maximal radius is deduced and a simple recurrent algorithm (7.7), (7.8), (7.10) is given for the basic functions (4.5).

Let us provide some examples demonstrating the possibilities of this method. The computations were performed by A.E. Malevich.

In the first example, the unit cell contains 64 holes as displayed in Figure 2a. The position of each disc is obtained by a random deviation of the regular array displayed in Figure 2b. A formula for k_c^* for the regular array is given by Mityushev&Adler (2000). For the cell displayed in Figure 2, we have

$$k_c^* = -0.015625 \ln \phi - 0.0171142 + 0.36682\phi + \tag{8.1}$$

$$0.00446609\phi^2 - 0.0879461\phi^3 + 0.135427\phi^4 - 0.0617604\phi^5,$$

where ϕ is the area fraction of holes. Formula (8.1) is calculated with the accuracy divisor $r^{10} \ln^0 r$ (see Section 4) and it is illustrated in Figure 3.

In the second example, we consider a set of 8 unit cells each of them containing 16 holes (see Figure 4). The first cell contains a regular array of cylinders for which

$$k_{reg} = -0.0625 \ln \phi - 0.0922513 + 0.125\phi - 0.03125\phi^2 - 0.00318571\phi^4. \tag{8.2}$$

Formula (8.2) is taken from Mityushev&Adler (2000) dividing by 16 the corresponding formula for a regular array of cylinders. The formulae for the permeabilities of eight k_c^* which are denoted as $k_1, ..., k_8$ have the form

$$k_1 = -0.0625 \ \ln \phi - 0.0625551 + 0.0242069\,\phi + 0.178599\,\phi^2 - \tag{8.3}$$

$$0.325235\,\phi^3 + 1.12016\,\phi^4 - 1.25773\,\phi^5 +$$

$$\frac{0.171063 - 0.397395\,\phi + 0.469457\,\phi^2 - 0.911992\,\phi^3 + 1.77234\,\phi^4}{-3.91732 + \ln \phi},$$

$$k_2 = -0.0625 \ \ln \phi - 0.045064 + 0.470861\,\phi - 1.02981\,\phi^2 +$$

$$1.88933\,\phi^3 - 2.44339\,\phi^4 + 2.18095\,\phi^5 +$$

$$\frac{-0.79148 + 1.46906\phi - 2.44339\,\phi^2 + 4.90246\phi^3 - 8.02338\,\phi^4}{-3.91732 + \ln\phi},$$

$$k_3 = -0.0625\ \ln\phi - 0.0655803 + 0.275642\,\phi - 0.400689\,\phi^2 +$$

$$0.49855\,\phi^3 - 0.474074\,\phi^4 + 0.758116\,\phi^5 +$$

$$\frac{-0.357906 + 0.672457\,\phi - 0.647307\,\phi^2 + 0.00195946\,\phi^3 + 2.34301\,\phi^4}{-3.91732 + \ln\phi},$$

$$k_4 = -0.0625\ \ln\phi - 0.0514621 + 0.245081\,\phi - 0.250573\,\phi^2 -$$

$$0.113657\,\phi^3 + 0.810208\,\phi^4 - 0.437414\,\phi^5 +$$

$$\frac{-0.025441 + 0.157152\,\phi - 0.107024\,\phi^2 + 0.10414\,\phi^3 - 0.109507\,\phi^4}{-3.91732 + \ln\phi},$$

$$k_5 = -0.0625\ \ln\phi - 0.0532152 + 0.353858\,\phi - 0.626954\,\phi^2 +$$

$$0.378568\,\phi^3 - 0.486462\,\phi^4 + 2.74329\,\phi^5 +$$

$$\frac{-0.928426 + 1.32836\,\phi - 1.45804\,\phi^2 + 1.55225\,\phi^3 - 4.92257\,\phi^4}{-3.91732 + \ln\phi},$$

$$k_6 = -0.0625\ \ln\phi - 0.0482648 + 0.225533\,\phi - 0.698149\,\phi^2 +$$

$$1.43239\,\phi^3 - 1.94397\,\phi^4 + 2.07924\,\phi^5 +$$

$$\frac{-0.100747 + 0.231596\,\phi - 1.73211\,\phi^2 + 4.76384\,\phi^3 - 9.622\,\phi^4}{-3.91732 + \ln\phi},$$

$$k_7 = -0.0625\ \ln\phi - 0.0354179 + 0.182893\,\phi - 0.567517\,\phi^2 +$$

$$2.05524\,\phi^3 - 3.83446\,\phi^4 + 1.54685\,\phi^5 +$$

$$\frac{-0.51629 + 1.04549\,\phi - 0.257221\,\phi^2 - 2.96365\,\phi^3 + 22.5804\,\phi^4}{-3.91732 + \ln\phi},$$

$$k_8 = -0.0625\ \ln\phi - 0.0532151 + 0.0706493\,\phi - 0.0320416\,\phi^2 +$$

$$0.864306\,\phi^3 + 0.938413\,\phi^4 + 3.25426\,\phi^5 +$$

$$\frac{-0.327548 + 0.0673839\,\phi - 0.47006\,\phi^2 + 6.23122\,\phi^3 - 3.37361\,\phi^4}{-3.91732 + \ln\phi}.$$

Formula (8.3) is calculated with the accuracy divisor $r^{10}\ln^1 r$. k_c^* is illustrated in Figure 5. It should be noted that in all our computations the permeability is minimal for the regular array.

The third example is the finite Sierpinski carpet displayed in Figure 6. Let us apply to it formula (4.5) in the zero-th approximation

$$k_c^* = -2\left(\sum_{k=1}^{n} \frac{1}{\ln r_k}\right) \tag{8.4}$$

The construction of the domain D is started from the disc $|z| < r$. Similitudes have a contraction ratio equal to $\frac{1}{3}$. The set of first generation is obtained as the union of the discs $|z - a_{1k}| < \frac{1}{3}r$ $(k = 1, 2, ..., 8)$, where a_{1k} are the centers of first generation. And so on up to the $P-$th generation consisting of 8^P discs of radius $3^{-P}r$. Then (8.4) becomes

$$k_c^* = k_c^*(P) = 2\left(\sum_{p=1}^{P} \frac{1}{p - \ln 3 - \ln r}\right)^{-1}. \tag{8.5}$$

Calculations with (8.5) and $r = \frac{1}{6}$, the maximal possible value of r, show that $a := k_c^*(P+1)/k_c^*(P)$ is closed to $\frac{1}{8}$ for sufficiently large P. A similar flow in the Sierpinski carpet with square holes has been discussed by Adler (1986) and summarized in Adler (1992). It follows from (5.283a) of Adler (1992) that $k_c^*(P) \sim a_1^P$, where $a_1 = 8/81 = 0.0988$. The value of a_1 should be independent of the shape of the inclusions. Hence, the difference between a and a_1 is due to the fact that (8.4) is a zeroth-order approximation.

9 Concluding remarks

The method of functional equations provides a very powerful tool in order to derive analytical expressions for the longitudinal permeability of spatially periodic bundles of parallel circular cylinders of arbitrary centers and radii.

A general formula could be obtained which was specialized to a number of particular cases.

This technique will be extended to the case of transversal flow in a separate paper.

10 Appendix. Calculation of integrals

In order to calculate the effective permeability we have to calculate some double integrals. The integral I_1 is expressed as

$$I_1 = I_1^0 - \sum_{k=1}^{n} I_1^k, \tag{10.1}$$

where

$$I_1^0 = \int_{Q_{(0,0)}} \left(S_2 x^2 + (2\pi - S_2)\, y^2\right) d\sigma_z = \frac{1}{12}\left[S_2\alpha^2 + (2\pi - S_2)\,\alpha^{-2}\right], \tag{10.2}$$

$$I_1^k = \int_{D_k} \left(S_2 x^2 + (2\pi - S_2)\, y^2\right) d\sigma_z = \left[S_2 x_k^2 + (2\pi - S_2)\, y_k^2\right] \pi r_k^2 + \frac{1}{2}\left(\pi r_k^2\right)^2,$$

where $a_k = x_k + iy_k$. The integral I_1^0 is calculated by an iterative integral, I_1^k by polar coordinates.

Lemma 10.1. *Let us consider the domain G which is the rectangle $Q_{(0,0)}$ without the disc $|z| \le r$. Then*

$$\int_G \mathcal{P}(z) d\sigma_z = \pi - S_2.$$

Proof. We apply the following form of the Green's formula

$$\int_G F'(z) d\sigma_z = \frac{i}{2} \int_{\partial G} F(t) d\bar{t}$$

with $F'(z) = \mathcal{P}(z)$ and $F(z) = -\zeta(z)$. Then we have

$$\int_G \mathcal{P}(z) d\sigma_z = \frac{i}{2} \int_{\partial Q_{(0,0)}} \zeta(t) d\bar{t} - \frac{i}{2} \int_{|t|=r} \zeta(t) d\bar{t}.$$

We calculate the first integral

$$\frac{i}{2} \int_{\partial Q_{(0,0)}} \zeta(t) d\bar{t} = \frac{1}{2} \int_{\partial Q_{(0,0)}} [-Re\, \zeta(t) dy + Im\, \zeta(t) dx +$$

$$iRe\, \zeta(t) dx - iIm\, \zeta(t) dy].$$

Using Mityushev&Adler (2000) we calculate

$$\int_{\partial Q_{(0,0)}} Re\, \zeta(t) dy = S_2, \ \int_{\partial Q_{(0,0)}} Im\, \zeta(t) dx = 2\pi - S_2,$$

$$\int_{\partial Q_{(0,0)}} Re\, \zeta(t) dx = \int_{\partial Q_{(0,0)}} Im\, \zeta(t) dy = 0.$$

By residua we also have

$$\int_{|t|=r} \zeta(t) d\bar{t} = \int_{|t|=r} \zeta\left(\frac{r^2}{\bar{t}}\right) d\bar{t} = 0$$

Hence,

$$\int_G \mathcal{P}(z) d\sigma_z = \frac{1}{2}\left[-S_2 + (2\pi - S_2)\right] = \pi - S_2.$$

The lemma is proved.

Definition 10.2. $\int_{Q_{(0,0)}} \mathcal{P}(z)d\sigma_z := \lim_{r\to 0}\int_G \mathcal{P}(z)d\sigma_z = \pi - S_2.$

This definition agrees with the v. p. integral.

Corollary 10.3. $\int_{Q_{(0,0)}} \mathcal{P}(z-w)d\sigma_z = \pi - S_2$, *since* $\mathcal{P}(z)$ *is doubly periodic.*

Remark 10.4. For the square array ($\alpha = 1$), we have

$$\int_{Q_{(0,0)}} \mathcal{P}(z)d\sigma_z = 0.$$

We propose the following numerical method for the exact calculation of the improper integrals

$$\mathfrak{Y}_k := \int_{Q_{(0,0)}} \zeta(z-a_k)d\sigma_z,\ \mathfrak{J}_k := \int_{Q_{(0,0)}} \ln\sigma(z-a_k)d\sigma_z.$$

Using the jump relation for $\sigma(z)$ and Green's formula

$$\int_G \frac{\partial Q}{\partial x}d\sigma_z = \int_{\partial G} Qdy$$

we can conclude that $\mathfrak{Y}_k$ is a linear function in $Re\ a_k$ and $Im\ a_k$, and that J_k is a square polynomial on $Re\ a_k$ and $Im\ a_k$. Applying the method of least squares for the square array ($\alpha = 1$), we obtain the relations

$$\mathfrak{Y}_k = -\pi\overline{a_k},\ J_k = T(0) + \frac{\pi}{2}\left|a_k\right|^2, \tag{10.3}$$

where

$$T(0) := \int_{Q_{(0,0)}} \ln\sigma(z)d\sigma_z \approx -1.048576. \tag{10.4}$$

Then the integrals (4.2) are calculated by the formulae

$$J_k = \mathfrak{J}_k - \sum_{m\neq k}\pi r_m^2\sigma_{mk} + \frac{\pi r_k^2}{2} - \pi r_k^2\ln r_k,\ Y_k = \mathfrak{Y}_k - \sum_{m\neq k}\pi r_m^2\zeta_{mk}. \tag{10.5}$$

Here, we use the integral

$$\int_{D_k} Re\ \left[\ln\sigma(z-a_k)\right]d\sigma_z = -\frac{\pi r_k^2}{2} + \pi r_k^2\ln r_k,$$

which was calculated by Mityushev&Adler (2000) and the mean value theorem of the harmonic function theory. It is written as follows

$$\int_{|z-a|<r} F(z)d\sigma_z = \pi r^2 F(a), \tag{10.6}$$

where the function $F(z)$ is harmonic in $|z-a|<r$ and continuous in $|z-a|\leq r$.

The integral F_1^k is defined and expressed as follows

$$F_1^k := \int_D E_1(z-a_k)d\sigma_z = \int_D \zeta(z-a_k)d\sigma_z - S_2\int_D (z-a_k)d\sigma_z =$$

$$\int_{Q_{(0,0)}} \zeta(z-a_k)d\sigma_z - \sum_{m=1}^{n}\int_{D_m}\zeta(z-a_k)d\sigma_z - S_2\int_D (z-a_k)d\sigma_z.$$

We have $\int_{D_m}\zeta(z-a_k)d\sigma_z = -\pi r_m^2\zeta_{mk}$, where $\zeta_{mk} := \zeta(a_k-a_m)$ for $m\neq k$ and $\zeta_{kk} := 0$. We calculate by using (10.6)

$$\int_D (z-a_k)d\sigma_z = \int_{Q_{(0,0)}}(z-a_k)d\sigma_z - \sum_{m=1}^{n}\int_{D_m}(z-a_k)d\sigma_z = -a_k + \sum_{m=1}^{n}\pi r_m^2(a_k-a_m)$$

Then

$$F_1^k = 2\pi i Im\ a_k + \sum_{m=1}^{n}\pi r_m^2\left[\zeta_{mk}-(a_k-a_m)\right]. \tag{10.7}$$

Let us also calculate

$$F_2^k := \int_D E_2(z-a_k)d\sigma_z = \int_D \mathcal{P}(z-a_k)d\sigma_z + S_2\int_D d\sigma_z =$$

$$\int_{Q_{(0,0)}}\mathcal{P}(z-a_k)d\sigma_z - \sum_{m=1}^{n}\int_{D_m}\mathcal{P}(z-a_k)d\sigma_z + S_2\,|D| = \pi - \sum_{m=1}^{n}\pi r_m^2\rho_{mk} - S_2\sum_{m=1}^{n}\pi r_m^2. \tag{10.8}$$

For the square array, (10.6) becomes

$$F_2^k = \pi - \sum_{m=1}^{n}\pi r_m^2\left(\pi+\rho_{mk}\right). \tag{10.9}$$

Along similar lines, we have

$$F_l^k := \int_D E_l(z-a_k)d\sigma_z = \int_{Q_{(0,0)}\setminus D}E_l(z-a_k)d\sigma_z - \sum_{m\neq k}^{n}\int_{D_k}E_l(z-a_k)d\sigma_z,\ l=3,4,\dots\ .$$

We use the periodicity of E_l and the following formula from Mityushev&Adler (2000)

$$\int_{Q_{(0,0)}\setminus U}E_l(z)d\sigma_z = -\pi r^2 S_l,$$

where $U = \{z \in \mathbb{C} : \ |z| < r\}$. Then (10.6) yields

$$F_l^k = -\pi r_k^2 S_l - \sum_{m \neq k}^{n} (-1)^l E_{mk}^l \pi r_m^2. \tag{10.10}$$

where $E_{mk}^l := E_l(a_k - a_m)$.

(10.6) yields the following integral

$$\int_D y d\sigma_z = \sum_{m=1}^{n} \pi r_m^2 y_m. \tag{10.11}$$

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Figure 1: Unit cell of size $(\alpha, 1/\alpha)$

Figure 2: Random cell (a) and regular cell (b) with 64 discs

Figure 3: The effective permeability k_c^* as function of the solid concentration for the configurations displayed in Figure 2

Figure 4: Regular cell and eight random cells with 16 discs

Figure 5: The effective permeability k_c^* for the configurations displayed in Figure 4

Figure 6: Sierpinski carpet with circular discs