ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 10, nº 5 (1993), p. 561-590

<http://www.numdam.org/item?id=AIHPC_1993__10_5_561_0>

Ann. Inst. Henri Poincaré,

Vol. 10, n° 5, 1993, p. 561-590. *Analyse non linéaire*

Looking for the Bernoulli shift

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ABSTRACT. — We prove a result on the topological entropy of a large class of Hamiltonian systems. This result is obtained variationally by constructing "multibump" homoclinic solutions.

Key words : Hamiltonian systems, convexity, dual variational methods, concentration-compactness, homoclinic orbits, Bernoulli shift, topological entropy, chaos.

RÉSUMÉ. — On démontre un résultat sur l'entropie topologique d'une grande classe de systèmes hamiltoniens. Ce résultat est obtenu par une méthode variationnelle qui permet de construire des solutions homoclines « multi-bosses ».

A.M.S. Classification: 58 E 05, 58 E 30, 58 F 05, 58 F 13.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 10/93/05/\$4.00/© Gauthier-Villars

I. INTRODUCTION

1. Some history

Homoclinic orbits were first introduced by H. Poincaré (*see* [M] for a modern exposition). Considering a hyperbolic fixed point p of a diffeomorphism φ in $\mathbb{R}^{2N}$, we say that a point $r \neq p$ is homoclinic if it belongs to the intersection of the unstable and stable manifolds W^u, W^s associated to (p, φ); the orbit of r is called a homoclinic orbit. Assuming that W^u, W^s intersect transversally at r, and that φ is symplectic, Poincaré proved that there are infinitely many homoclinic orbits, geometrically distinct in the following sense:

(the orbits of r, r' are geometrically distinct) $\Leftrightarrow$ $(\forall n \in \mathbb{Z} : \varphi^n(r) \neq r')$.

Birkhoff, Smale and other authors also studied homoclinic orbits, and their relation with Bernoulli shifts. We state here a result of Smale on homoclinics (*see* [M]): if $r \neq p$ is a point of transverse intersection of W^u, W^s, then there are $l \in \mathbb{N}^*$ and a homeomorphism $\tau : \{0, 1\}^{\mathbb{Z}} \to I$, where I is an invariant set for φ^l, such that $\varphi^l \circ \tau = \tau \circ \sigma$. Here, $\sigma((a_n)) = (b_n)$ with $b_n = a_{n+1}$ and $\{0, 1\}^{\mathbb{Z}}$ is endowed with the standard metric

$$d(a, b) = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{|b_n - a_n|}{2^{|n|}}.$$

This structure is called a Bernoulli shift.

Bernoulli shifts are an important tool in the study of chaotic behavior. For instance, Smale's result given above implies that the topological entropy of φ, $h_{\text{top}}(\varphi)$, is greater than $\frac{\operatorname{Ln} 2}{l}$. This is a direct consequence of the following definition (*see* [O], p. 182-183):

$$h_{\text{top}}(\varphi) = \sup_{R>0} \lim_{e \to 0} \left(\limsup_{n \to \infty} \frac{\operatorname{Log} s(n, e, R)}{n} \right),$$

where

$$s(n, e, R) = \max \{ \operatorname{Card}(E) : E \subset B(0, R), \\ (\forall x \neq y \in E)(\exists k \in [\![0, n]\!]) : |\varphi^k(x) - \varphi^k(y)| \geq e \}.$$

2. Variational approach

The results described in the preceding section were proved by dynamical systems methods, with a transversality assumption on W^u, W^s. The question examined in this paper is the following one:

We assume that φ is the time-one map of a Hamiltonian system $x' = J \nabla_x H(t, x)$, H being one-periodic in time. Is it possible to say something about Bernoulli shifts and topological entropy, using a variational method? We will see that this approach has several advantages:

- The existence of a homoclinic point r is not an assumption any more, but follows from global hypotheses on H that we call (hA), (hR).
- The classical transversality hypothesis can be replaced by a weaker condition, denoted $(\mathscr{H})$.

3. Main results

We work with the same Hamiltonian system as in the paper [CZ-E-S]:

$$x' = \mathrm{J A}\, x + \mathrm{J}\, \nabla_x \mathrm{R}(t, x), \qquad x \in \mathbb{R}^{2\mathrm{N}}, \quad t \in \mathbb{R}, \quad \mathrm{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

We are looking for non-zero solutions satisfying $x(\pm\infty) = 0$, *i.e.* solutions homoclinic to 0.

We make the following assumptions on A, R:

$$\left.\begin{array}{c} \mathrm{A}^* = \mathrm{A}, \text{ and } \mathrm{JA} = \mathrm{E} \text{ is a constant matrix,} \\ \text{all eigenvalues of which have a non-zero real part.} \end{array}\right\} \tag{hA}$$

- $\mathrm{R}(.+1, .) = \mathrm{R}(., .)$, and R is C^2.
- $(\forall\, t \in \mathbb{R})$, $\mathrm{R}(t, .)$ is strictly convex.
- for some $\alpha > 2$, $0 < k_1 < k_2 < +\infty$, we have

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2\mathrm{N}}, \quad \mathrm{R}(t, x) \leqq \frac{1}{\alpha} (\nabla_x \mathrm{R}, x), \tag{hR}$$

$$k_1 |x|^\alpha \leqq \mathrm{R}(t, x) \leqq k_2 |x|^\alpha.$$

In [CZ-E-S], it was proved under these assuptions that there are at least two homoclinic orbits x, y, geometrically distinct, *i.e.* such that $\forall n \in \mathbb{Z} : n * x \neq y$, where $n * x(t) = x(t-n)$. One of them was obtained by a mountain-pass argument on a dual action functional. This paper has motivated some related work.

Concerning the existence of at least one homoclinic solution, the convexity assumption was relaxed in [H-W] and [T], by two different methods.

Concerning multiplicity, a novel variational argument was introduced in [S], and the following result was proved:

THEOREM I. — *Assume* (hA), (hR) *are true. Then there are infinitely many orbits homoclinic to* 0, *geometrically distinct in the sense*

$$x_1 \neq x_2 \quad \Leftrightarrow \quad (\forall n : n * x_1 \neq x_2).$$

The idea in [S] was to look for solutions near $(-n) * x + n * x$, where x is the homoclinic orbit found in [CZ-E-S] by mountain-pass, and n is large enough. We call them "solutions with two bumps distant of $2n$".

The existence of such solutions is a well-known fact of classical dynamical systems theory, in many particular situations. Let describe briefly one of them (*see* [W]):

Consider the autonomous system associated to the Hamiltonian

$$\mathrm{H}(p, q)=p^2-q^2+p^4+q^4, \qquad (p, q)\in\mathbb{R}^2.$$

It is integrable, and does not have any solution with two (or more) bumps. But in the autonomous case, we have a continuum of solutions which are the translates of one of them in time, and Theorem I is not contradicted.

By Melnikov's theory, it is possible to find small non-autonomous perturbations $\mathrm{H}(p, q)+\varepsilon\,\mathrm{K}(t, p, q)$ of the Hamiltonian such that W^u, W^s intersect transversally. Then, using the implicit function theorem, multi-bump homoclinic solutions can be constructed.

To give more detailed comments on Theorem I, we need some notations:

f is the dual action functional introduced in [CZ-E-S]. It is defined on the space $\mathrm{L}^\beta(\mathbb{R}, \mathbb{R}^{2\mathrm{N}})$, with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ (the exact form of f will be given in section II). $f^a=\{x/f(x)\leqq a\}$, $\mathscr{C}$ is the set of non-zero critical points, and $\mathbb{Z}$ acts by integer translations in time.

L: $\mathrm{L}^\beta\to\mathrm{W}^{1,\beta}$ is an isomorphism such that, if $u\in\mathscr{C}$, then $\mathrm{L}u$ is a homoclinic orbit (*see* §II).

c is the mountain-pass level, let us define it precisely:

0 is a strict local minimum for f, and $f(0)=0$. Moreover, f is not bounded from below (*see* [CZ-E-S]. So we consider

$$\Gamma=\{\gamma\in\mathrm{C}^0([0, 1], \mathrm{L}^\beta)/\gamma(0)=0,\ f\circ\gamma(1)<0\}.$$

Γ is non-empty, and we choose $c=\inf_{\gamma\in\Gamma}(\max f\circ\gamma)>0$ as mountain-pass level.

In [S], the variational gluing of two bumps was possible under the following assumption:

(*): There is some $c'>c$ such that $(\mathscr{C}\cap f^{c'})/\mathbb{Z}$ is finite.

The following result, which is a more precise version of Theorem I, is an immediate consequence of the arguments given in [S]:

THEOREM I'. — *Assume that* (hA), (hR) *and* (*) *are true. Then there are two critical points u, v such that for any r, $h>0$ and $n\geqq\mathrm{N}(r, h)$, exists a critical point u_n, with*

$$\|u_n-[(-n)*u+n*v]\|_{\mathrm{L}^\beta}<r \qquad \textit{and} \qquad f(u_n)\in[2c-h, 2c+h].$$

u, v, possibly equal, satisfy $f(u)=f(v)=c$. The homoclinic orbit $y_n=\mathrm{L}u_n$ is called a solution with two bumps distant of $2n$. It satisfies

$$\|y_n-[(-n)*\mathrm{L}u+n*\mathrm{L}v]\|_{\mathrm{W}^{1,\beta}}<\|\mathrm{L}\|.r.$$

Theorem I is trivial when (*) is not satisfied ("degenerate" situation), and Theorem I′ implies Theorem I when (*) is satisfied ("non-degenerate" situation).

In the later work [CZ-R][1], Coti Zelati and Rabinowitz apply the ideas of [S] to the case of second order systems, and construct, under assumption (*), solutions with m bumps, *i.e.* located in a ball of center $p^1 * x_1 + \ldots + p^m * x_m$ and radius ε, for the norm of the functional space $E = W^{1,2}(\mathbb{R}, \mathbb{R}^N)$. The x_i are in a fixed finite set of critical points of the action functional $\int \frac{x^2}{2} - V$ defined on E. They are found thanks to a mountain-pass. Moreover, for any i, $(p^{i+1} - p^i) \geqq K(\varepsilon, m)$. In the construction of [CZ-R][1], the minimal distance K between bumps goes to infinity as m goes to infinity, for ε fixed.

Other applications, in the domain of partial differential equations, are given in [CZ-R][2], [LI][1], [LI][2].

In the paper [C-L] of Chang and Liu, the assumption (*) is replaced by (**) : $\mathscr{C} \cap f^{c'}$ contains only isolated points.

In the present work, (**) is replaced by the weaker assumption

$(\mathscr{H}) : \mathscr{C} \cap f^{c'}$ is at most countable.

Moreover, multibump solutions are constructed for a minimal distance K between bumps independent of m. This last point, whose proof requires many modifications in the arguments of [S], [CZ-R][1], allows to study the topological entropy of the Hamiltonian system. The main theorem that we will prove can be stated as follows:

THEOREM II. — *Assume* (hA), (hR) *and* $(\mathscr{H})$ *are true. Then there exists a homoclinic orbit* x *such that, for any* $\varepsilon > 0$, *and any finite sequence of integers* $\bar{p} = (p^1, \ldots, p^m)$, satisfying

$$(\forall i): \quad (p^{i+1} - p^i) \geqq K(\varepsilon),$$

there is a homoclinic orbit $y_{\bar{p}}$, *with*

$$(\forall t \in \mathbb{R}): \quad \left| y_{\bar{p}}(t) - \sum_{i=1}^{m} x(t - p^i) \right| \leqq \varepsilon.$$

Here, K *is a constant independent of m.*

Remark 1. — The assumption $(\mathscr{H})$ cannot be satisfied in the autonomous situation, where the translates of x in time form a continuum. Now, if W^u, W^s intersect transversally, then their intersection is at most countable, and so is the set of homoclinic solutions; but the converse is false.

Remark 2. — The estimate on $y_{\bar{p}} - \sum_{i=1}^{m} x(t - p^i)$ is given in L^∞ norm. In [S] and [CZ-R][1], it was given in global $W^{1,q}(\mathbb{R})$ norm. Without this change, it seems impossible, or at least very difficult, to choose K independently of m.

Since K does not depend on m, we can study the limit $m \to \infty$, and get solutions with infinitely many bumps (those are not homoclinic orbits any more). We have

COROLLARY II.1. – *With the hypotheses and notations of Theorem II, for any interval* $I \subset \mathbb{Z}$, *finite or infinite, and any sequence of integers* $\bar{p} = (p^i)_{i \in I}$ *such that* $(\forall i) : (p^{i+1} - p^i) \geqq K(\varepsilon)$, *there is a solution* $y_{\bar{p}}$ *of* (1) *satisfying*

$$(\forall t \in \mathbb{R}): \quad \left| y_{\bar{p}}(t) - \sum_{i \in I} x(t - p^i) \right| \leqq \varepsilon.$$

If I *is infinite, we say that y has infinitely many bumps.*

As a consequence, we have an "approximate" Bernoulli shift structure:

COROLLARY II.2. – *Under the hypotheses of Theorem II, there is* $x_0 \in \mathbb{R}^{2N} \setminus \{0\}$ *such that, for any* $\varepsilon > 0$, *exist* $K = K(\varepsilon) > 0$ *and*

$$\tilde{\tau} = \tilde{\tau}(\varepsilon) : (\{0, 1\}^{\mathbb{Z}}, d) \to (\mathbb{R}^{2N}, |\,.\,|),$$

with:

- $\tilde{\tau}$ *is injective, and* $\tilde{\tau}^{-1}$ *is uniformly continuous.*
- $(\forall n \in \mathbb{Z}) \| \tilde{\tau} \circ \sigma^n - \varphi^{Kn} \circ \tilde{\tau} \|_{\infty} < 2\varepsilon.$
- $\begin{cases} s_0 = 1 \Rightarrow |\tilde{\tau}(s) - x_0| < \varepsilon \\ s_0 = 0 \Rightarrow |\tilde{\tau}(s)| < \varepsilon. \end{cases}$

Here, φ *is the time-one flow of* (1), *and* $\sigma(s)_n = s_{n+1}$. *Note that we cannot say that* $\tilde{\tau}$ *is continuous. We call* $(\tilde{\tau}(\{0, 1\}^{\mathbb{Z}}), \varphi^K)$ *an approximate Bernoulli shift structure.*

Corollary II.2 will be proved in section VI.

Now, we are in a position to state the result on topological entropy. Choose $\varepsilon \leqq \frac{|x_0|}{3}$. If two sequences s, s' are such that $s_k \neq s'_k$ for some k, then

$$|\Phi^{K(\varepsilon)k} \circ \tau(s) - \Phi^{K(\varepsilon)k} \circ \tau(s')| \geqq \frac{|x_0|}{3}.$$

So, for $e < \frac{|x_0|}{3}$ and $R > |x_0| + \varepsilon$, we get $s(Kn.\, e, R) \geqq 2^n$, and $h_{\text{top}}(\varphi) \geqq \frac{\text{Ln}\, 2}{K(\varepsilon)}$. So Corollary II.2 implies

COROLLARY II.3. – *With the hypotheses of Theorem I, the flow of* (1) *has a positive topological entropy.*

Note: Independently of the present paper, Bessi in [B] constructs variationally an approximate Bernoulli shift for the one-dimensional pendulum, by a method inspired of [S]. He replaces assumption (*) by a weakening of the classical Melnikov condition, and his result is given for small perturbations of an autonomous system.

II. VARIATIONAL FRAMEWORK AND SKETCH OF PROOF OF THEOREM II

We use a variational formulation based on Clarke's dual action principle (*see* [CZ-E-S], [E]). Define $G(t, y)=\max\{(z.y)-R(t, z)/z\in\mathbb{R}^{2N}\}$. G is 1-periodic in time, strictly convex in y, and satisfies, for $\frac{1}{\alpha}+\frac{1}{\beta}=1$:

$$0\leqq\frac{1}{\beta}(\nabla_y G, y)\leqq G(t, y)\leqq(\nabla_y G, y),$$
$$(\exists c_1, c_2>0)(\forall(y, t))\quad c_1|y|^\beta\leqq G(t, y)\leqq c_2|y|^\beta,$$
$$|\nabla_y G(t.y)|\leqq c_2|y|^{\beta-1}.$$

We define

$$\begin{aligned}D:\quad & W^{1,\beta}(\mathbb{R}, \mathbb{R}^{2N})\to L^\beta(\mathbb{R}, \mathbb{R}^{2N})\\ & z\mapsto\left(-J\frac{d}{dt}-A\right)z,\\ & L=D^{-1}.\end{aligned}$$

We call $\mathscr{C}$ the set of non-zero critical points of the following functional f:

$$f(u)=\int G(t, u)\,dt-\frac{1}{2}\int(u, Lu)\,dt,\qquad u\in L^\beta(\mathbb{R}, \mathbb{R}^{2N}).$$

We have (*see* [CZ-E-S])

LEMMA 1. — *If* $u\in\mathscr{C}$, *then* $x=Lu$ *is a non-zero solution of* (1) *such that* $x(\pm\infty)=0$, i.e. *an orbit homoclinic to* 0.

Our task will be to find a large class of elements of $\mathscr{C}$.

For this purpose, we need some compactness properties of f. Unfortunately, f does not satisfy the Palais-Smale (PS) condition, because it is invariant for the action of the non-compact group $\mathbb{Z}: n*u=u(.-n)$. To deal with this problem, we use the concentration-compactness theory of P. L. Lions (*see* [LS].

We have (*see* [CZ-E-S])

LEMMA 2. — *Suppose* (hA), (hR) *are true. Then f satisfies the following compactness property:*

Let $(u_n)_{n\geqq 0}$ *be a sequence such that*

$$f(u_n)\to a>0,\qquad f'(u_n)\to 0.$$

Then there exist $m>0$, *a subsequence* $(n_p)_{p\geq 0}$, *and* $u^1, \ldots, u^m$ *in* $\mathscr{C}$, *not necessarily distinct, such that*

$$\left\| u_{n_p} - \sum_{i=1}^{m} k_p^i * u^i \right\| \underset{p \to \infty}{\to} 0,$$

where $k_p^i \in \mathbb{Z}$, $(k_p^j - k_p^i) \to +\infty$ *as* $p \to +\infty$ *if* $i<j$.

To simplify notations, we will write

$$\bar{k}_p = (k_p^1 \ldots k_p^m) \in \mathbb{Z}^m, \qquad \bar{u} = (u^1 \ldots u^m) \in \mathscr{C}^m,$$

$$\bar{k}_p * \bar{u} = \sum_{i=1}^{m} k_p^i * u^i. \qquad \text{Moreover,} \qquad (\lim_{k \to \infty} (k_p^j - k_p^i) = +\infty \text{ if } i<j)$$

will be summarized by

$$(\bar{k}_p \to \Omega \text{ as } p \to +\infty).$$

Now, what is special here is that the splittings $\bar{k} * \bar{u}$ do not vary continuously when $\bar{k}$ varies. This leads to introduce a new compactness condition (*see* [CZ-E-S], [S]).

CONDITION $\overline{\text{PS}}$ (a). – *Let* (u_n) *be a sequence such that* $f(u_n) \leq a \in \mathbb{R}$, $f'(u_n) \to 0$, $(u_{n+1} - u_n) \to 0$. *Then* (u_n) *is convergent.*

We have:

LEMMA 3. – *Assume* (hA), (hR) *and* $(\mathscr{H})$ *are true. Then* $\overline{\text{PS}}(c')$ *holds.*

Lemma 3 will be proved in section III, and will be used in the proof of Lemma 7, section IV.

The interest of $\overline{\text{PS}}$ is that, if f is bounded on a pseudo-gradient line, then one can find a $\overline{\text{PS}}$ sequence on this line. So $\overline{\text{PS}}$ can give the same kind of deformation lemmas as the Palais-Smale condition. If $\overline{\text{PS}}$ is satisfied under level c', by deforming a particular curve in Γ, one finds at least one critical point u between levels c and c'. When (*) holds, one can impose $f(u)=c$. When only $(\mathscr{H})$ holds, the best that can be done is to take u with $(f(u)-c)$ arbitrarily small.

In [S], under assumption (*), a "product min-max" is constructed at level $2c$, for the "split" functional $\tilde{f}(x) = f(x\chi_{\mathbb{R}_-}) + f(x\chi_{\mathbb{R}_+})$, where χ_I is the caracteristic function of I. Theorems I and I′ are then proved by contradiction, thanks to a deformation argument. This argument works because the differentials f' and $\tilde{f}'$ "look the same" near $(-n)*u+n*v$, where u, v are critical points associated to the mountain-pass, possibly equal.

The proof of Theorem II is based on the same ideas, but contains several technical improvements.

We first construct, for any r, $h>0$, a non-trivial homology class in $H_1(f^{\bar{c}+h}, f^{\bar{c}})$, containing a chain included in $B(u, r)$, thanks to assumption

($\mathscr{H}$). Here, $\bar{c}=f(u)\in[c, c')$, and $u\in\mathscr{C}$, found thanks to the mountain-pass, is independent of r, h (*see* § IV).

Then, roughly speaking, we consider a product of m "copies" of this homology class, and find a "product min-max" in a neighborhood of $\sum_{i=1}^{m} p^i * u$. This is done in section IV thanks to Künneth's formula,

$$H_*(X\times Y, (Z\times Y)\cup(X\times T))=H_*(X, Z)\otimes H_*(Y, T).$$

Note that in [S], [CZ-R][1], a more elementary procedure (without homology) is used to construct the product min-max. It would be possible to use this procedure in the proof of Theorem II. But the method involving homology seems easier to generalize to situations where the min-max is not of mountain-pass type.

Finally, we find a critical point $u_{\bar{p}}$ in a neighborhood of $\sum_{i=1}^{m} p^i * u$, provided $(p^{i+1}-p^i)\geqq K$, K depending only on r, not on m. To do this, we assume that $u_{\bar{p}}$ does not exist, construct a more precise version of the deformation used in [S], and apply it to the "product min-max" to obtain a contradiction (*see* § V).

In the proof of Theorem II, a crucial point is to make a suitable choice of the neighborhood of $\sum_{i=1}^{m} p^i * u$ in which we want to find $u_{\bar{p}}$: this choice allows to control K as m increases. The correct neighborhood will be defined in the statement of Theorem III (*see* the end of section V), after the introduction of some technical notations. Theorem II will be a direct consequence of Theorem III.

III. COMPACTNESS PROPERTIES OF f

We first prove the following result:

LEMMA 4. — *Suppose* (hA), (hR) *and* ($\mathscr{H}$) *are true. Then there is an at most countable compact set* D *such that:*

If $(u_n)_{n\geqq 0}$ *satisfies* $f(u_n)\leqq c'$, $f'(u_n)\to 0$, *then*

$$(\forall r>0)\quad(\exists N>0),\qquad [p>q>N\Rightarrow \|u_p-u_q\|\in B(D, r)].$$

Here, $B(D, r)=\{x\in[0, +\infty)/d(x, D)<r\}$.

Proof. — Consider the set

$$D=\left\{x\in[0,+\infty)/x=\sum_{i=1}^{m}\|u_i-v_i\|,\ m\geqq 1,\ u_i,\ v_i\in\mathscr{C}\cup\{0\},\ \sum_{i=1}^{m}f(u_i)\leqq c',\ \sum_{i=1}^{m}f(v_i)\leqq c'\right\}.$$

From ($\mathscr{H}$), D is at most countable.

Let us prove that D is compact. We know (*see* [CZ-E-S]) that there is $\Lambda>0$ such that

$$(\forall u\in\mathscr{C})\quad f(u)\geqq\Lambda.$$

Consider a sequence (d^n) in D, with

$$d^n=\sum_{i=1}^{M_n}\|u_i^n-v_i^n\|,\quad u_i^n, v_i^n\in\mathscr{C}\cup\{0\},\qquad \sum_{i=1}^{M_n}f(u_i^n)\leqq c',$$
$$\sum_{i=1}^{M_n}f(v_i^n)\leqq c',\quad (u_i^n=0\Rightarrow v_i^n\neq 0).$$

We have $M_n\leqq 2c'/\Lambda$.

So, after extraction, we may assume that $M_n=M$ is constant and, by Lemma 2, that, $\forall i\in[\![1, M]\!]$:

$$\|u_i^n-\bar{k}_i^n * \bar{U}_i\|\to 0,\qquad \bar{U}_i\in\mathscr{C}^{m(i)},\quad \bar{k}_i^n \underset{n\to\infty}{\to} \Omega,$$
$$\|v_i^n-\bar{l}_i^n * \bar{V}_i\|\to 0,\qquad \bar{V}_i\in\mathscr{C}^{m'(i)},\quad \bar{l}_i^n \underset{n\to\infty}{\to} \Omega.$$

One easily sees that

$$d_n\to\sum_{k=1}^{m''}\|\mathscr{U}_k-\mathscr{V}_k\|=d_\infty$$

where $\mathscr{U}_k$, resp. $\mathscr{V}_k$, if non-zero, are of the form $n*\bar{U}_i^j$, resp. $n*\bar{V}_i^j$, and $d_\infty\in D$.

We have thus proved that D is compact. The last step is to study (u_n) such that

$$f(u_n)\leqq c',\qquad f'(u_n)\to 0.$$

Assume there are two subsequences $(u_{p_m})_{m\geqq 0}$, $(u_{q_m})_{m\leqq 0}$ satisfying $\|u_{p_m}-u_{q_m}\|\notin B(D,\rho)$ for some $\rho>0$. After extraction, we may impose

$$\|u_{p_m}-\bar{\kappa}_m*\bar{\mu}\|\to 0,\qquad \bar{\mu}=(\mu^1,\ldots,\mu^r)\in\mathscr{C}^r,$$
$$\kappa_m\to\Omega,\qquad \sum f(\mu^i)\leqq c'$$
$$\|u_{q_m}-\bar{\lambda}_m*\bar{\nu}\|\to 0,\qquad \bar{\nu}=(\nu^1,\ldots,\nu^s)\in\mathscr{C}^s,$$
$$\bar{\lambda}_m\to\Omega,\qquad \sum f(\nu^i)\leqq c'.$$

After a new extraction, each sequence $(\kappa^i_m - \lambda^j_m)$ has a limit $l_{i,j}$ in $\mathbb{Z} \cup \{-\infty, +\infty\}$. Moreover, for each i, $\mathrm{Card}(\{j / |l_{i,j}| < +\infty\}) \leqq 1$.

Hence

$$\| u_{p_m} - u_{q_m} \| \to \sum_{k=1}^{t} \| l_k * w_k - w'_k \|,$$

where $(w_k)_{1 \leqq k \leqq t}$ is a reindexing of

$$(\mu^1, \ldots, \mu^r, \underbrace{0, \ldots, 0}_{(t-r)\text{ terms}}),$$

$(w'_k)_{1 \leqq k \leqq t}$ is a reindexing of

$$(\nu^1, \ldots, \nu^s, \underbrace{0, \ldots, 0}_{(t-s)\text{ terms}}),$$

and $l_k \in \mathbb{Z}$.

Clearly, $\sum f(w_k) = \sum f(\mu^i) \leqq c'$, $\sum f(w'_k) = \sum f(\nu^j) \leqq c'$. So $\sum_{k=1}^{t} \| w_k - w'_k \| \in \mathrm{D}$, which contradicts the assumption $\|u_{p_m} - u_{q_m}\| \notin \mathrm{B}(\mathrm{D}, \rho)$. The last assertion of Lemma 4 is thus proved by contradiction. □

We now give another lemma, that will be used in section V.

LEMMA 5. — *Suppose that f satisfies* (hA), (hR) *and* ($\mathscr{H}$). *Then the set*

$$\mathrm{F} = \left\{ x = \sum_{k=1}^{m} f(u_k)/m \geqq 1, (u_1, \ldots, u_m) \in \mathscr{C}^m, (\forall k), f(u_k) \leqq c' \right\}$$

is closed and a most countable.

The proof of Lemma 5 is analogous to that of Lemma 4, so we won't give it. Now, we prove Lemma 3 as a consequence of Lemma 4.

Proof. — Consider a sequence (u_n) such that

$$f(u_n) \leqq c', \qquad f'(u_n) \to 0, \quad (u_{n+1} - u_n) \to 0.$$

we want to prove by contradiction that (u_n) is a Cauchy sequence.

Assume the contrary, *i. e.* $\| u_{q_n} - u_{p_n} \| \to \delta > 0$, $p_n < q_n < p_{n+1}$.

The open set $]0, \delta[\setminus \mathrm{D}$ contains an interval $[d_1 - d_2, d_1 + d_2]$. And there is P such that

$$\left(p > \mathrm{P} \Rightarrow \| u_{p+1} - u_p \| \leqq \frac{d_2}{2} \right).$$

So, if $p_n > \mathrm{P}$,

$$\| u_{r_n} - u_{p_n} \| \in \left[d_1 - \frac{d_2}{2}, d_1 + \frac{d_2}{2} \right] \text{ for some } r_n \in [\![p_n, q_n]\!].$$

But this implies $\| u_{r_n} - u_{p_n} \| \notin \mathrm{B}(\mathrm{D}, d_2/2)$, which is impossible by Lemma 4.

So (u_n) is Cauchy, hence convergent. Lemma 3 is thus proved. □

We now study the local compactness of $\mathscr{C}$. We prove

LEMMA 6. – *Assume* (hA) *and* (hR) *are true. There is* $r_0>0$ *such that, if a sequence* (u_n) *satisfies*

$$\begin{cases} f'(u_n)\to 0 \\ (\exists\, \mathrm{R}>0),\ (\forall p, q), \quad \|(u_p-u_q)\chi_{\mathbb{R}\setminus[-\mathrm{R},\mathrm{R}]}\|\leqq 2r_0 \end{cases}$$

then (u_n) *is precompact.*

Proof. – We remark (*see* [CZ-E-S]) that there is $r_0>0$ such that

$$\frac{3r_0}{2}<\|u\| \qquad (\forall u\in\mathscr{C})$$

We now apply Lemma 2 to the sequence (u_n). If $m\geqq 2$ or if $(m=1)$ and $\lim_{p\to\infty}(|k_p^1|=+\infty)$, then for any $\mathrm{P}>0$, there are $p>q>\mathrm{P}$ such that

$$\|(\bar{k}_p*\bar{u}-\bar{k}_q*\bar{u})\chi_{\mathbb{R}\setminus[-\mathrm{R},\mathrm{R}]}\|\geqq 3r_0.$$

This contradicts $\|(u_p-u_q)\chi_{\mathbb{R}\setminus[-\mathrm{R},\mathrm{R}]}\|\leqq 2r_0$, for P large enough.

So $m=1$, and we may extract a subsequence $u_{n_{\varphi(p)}}$ such that $k^1_{\varphi(p)}=k$ is constant, and $u_{n_{\varphi(p)}}\underset{p\to\infty}{\to} k*u^1\in\mathscr{C}$. Lemma 6 is thus proved. □

Lemma 6 will be used in the proof of Lemma 12, section V.

IV. THE PRODUCT MIN-MAX

We want to find a min-max at each level kc, $k\geqq 2$. This will be done thanks to singular homology over $\mathbb{Z}$. We first need to "localize" the min-max

$$\inf_{\gamma\in\Gamma}(\max f\circ\gamma)=c.$$

This will be done thanks to $(\mathscr{H})$.

We recall some notations:

$$f^l=\{x/f(x)\leqq l\},\qquad f^{<l}=\{x/f(x)<l\},$$
$$f_l=(-f)^{-l},\qquad f_a^b=f_a\cap f^b,$$
$$\mathrm{B}(x,\rho)=\{y/\|y-x\|<\rho\},\qquad \mathrm{S}(x,\rho)=\{y/\|y-x\|=\rho\}.$$

We have

LEMMA 7. – *Assume* (hA), (hR) *and* $(\mathscr{H})$ *are true. Choose* $r\in\mathbb{R}_+^*\setminus\mathrm{D}$, *with the notation of Lemma* 4.

Then for any $h>0$, *exist* $p=p(h, r)\in\mathbb{N}^*$, $(u^1, \ldots, u^p)\in(\mathscr{C}\cap \bar{f}_c^{c+h})^p$, *and* $\gamma\in\Gamma$, *with:*

(i) $$\operatorname{Im}(\gamma)\cap f_c\subset \bigcup_{i=1}^{p} \mathrm{B}(u^i, r)$$

(ii) $$\operatorname{Im}(\gamma)\cap f_{c+h}=\varnothing$$

(iii) $$\operatorname{Im}(\gamma)\cap f_c\cap\left[\bigcup_{i=1}^{p} \mathrm{S}(u^i, r)\right]=\varnothing$$

Proof. – Given $r>0$, we just have to prove the result for h small enough. We take $\gamma^h\in\Gamma$ such that $f\circ\gamma^h<c+h$.

We are going to take γ as a deformation of γ^h. We choose $e>0$ such that $[r-2e, r+2e]\cap \mathrm{D}=\varnothing$. For $d\geqq 0$, we define

$$\begin{aligned}
\mathrm{U}^d &= \{x\in f_c^{c+h}/(\forall y\in\mathscr{C}\cap f_c^{c+h})\,\|x-y\|>r+d\}\\
\mathrm{V}^d &= \{x\in f_c^{c+h}/(\exists y\in\mathscr{C}\cap f_c^{c+h})\,\|x-y\|\in[r-d, r+d]\}\\
\mathrm{K}^d &= (\{x\in f_c^{c+h}/(\exists y\in\mathscr{C}\cap f_c^{c+h})\,\|x-y\|<r-d\}\\
&\quad \cup\{x\in f^{<c}/(\exists y\in\mathscr{C}\cap f^c)\,\|x-y\|<r-d\})\setminus \mathrm{V}^d
\end{aligned}$$

We assume $c+h<c'$. From Lemma 4, there is $\mu>0$, independent of h, and such that $\inf\{\|f'(x)\|/x\in\mathrm{V}^{2e}\}\geqq\mu$. We assume, moreover, that $h<\mu e/2$. We build a locally Lipschitz vector field V on f^{c+h}, such that:

(j) $$x\in\mathrm{K}^{2e}\cup f^{c-h}\ \Rightarrow\ \mathrm{V}(x)=0$$

(jj) $$(\forall x)\quad f'(x).\mathrm{V}(x)\leqq 0,\qquad |\mathrm{V}(x)|\leqq 2|f'(x)|^{-1}$$

(jjj) $$x\in\mathrm{U}^e\cup\mathrm{V}^e\ \Rightarrow\ f'(x).\mathrm{V}(x)\leqq -1$$

Consider the flow φ_t defined by

$$(\forall(t, x)\in\mathbb{R}_+\times f^{c+h})\quad \begin{cases}\varphi_0(x)=x\\ \dfrac{\partial}{\partial t}\varphi_t(x)=\mathrm{V}\circ\varphi_t(x).\end{cases}$$

Assume that for some $x\in f^{c+h}$, the maximal interval of definition of $t\mapsto\varphi_t(x)$ is $[0, \mathrm{L}[$, $\mathrm{L}<+\infty$. Then $\int_0^{\mathrm{L}}\|\mathrm{V}\circ\varphi_t(x)\|\,dt=+\infty$. So we can define a sequence (t_n) by

$$t_0=0$$

$$\int_{t_n}^{t_{n+1}}\|\mathrm{V}\circ\varphi_t(x)\|\,dt=\sqrt{\mathrm{L}-t_n}$$

So we get

(α) $\forall (u, v) \in [t_n, t_{n+1}]^2 : \| \varphi_u(x) - \varphi_v(x) \| \leqq \sqrt{L - t_n}$

(β) $\exists s_n \in [t_n, t_{n+1}] : \begin{cases} \| f' \circ \varphi_{s_n}(x) \| \leqq 2 \| V \circ \varphi_{s_n}(x) \|^{-1} \leqq 2 \sqrt{L - t_n} \\ \varphi_{s_n}(x) \in f^{c+h} \setminus K^{2e} \end{cases}$

(γ) $\int_0^l \| V \circ \varphi_t(x) \| \, dt = \sum_{n=0}^{+\infty} \sqrt{L - t_n}$, where $l = \lim_{n \infty} t_n$.

If $l < L$, the left term of (γ) is finite, and the right one infinite. So we have $l = L$, and

$$(\varphi_{s_{n+1}}(x) - \varphi_{s_n}(x)) \to 0, \qquad f' \circ \varphi_{s_n}(x) \to 0.$$

Since f satisfies property $\overline{\mathrm{PS}}(c')$, we get

$$u_\infty = \lim_{n \to \infty} \varphi_{s_n}(x) \in (f^{c+h} \setminus K^{2e}) \cap \mathscr{C}.$$

But this intersection is empty. So we have proved that φ_t is defined on $\mathbb{R}_+ \times f^{c+h}$.

Now, suppose that $f(x) < c + h$, and that $\varphi_h(x) \in U^0 \cup V^0$. Then three situations may occur:

- $$(\forall t \in [0, h]), \quad \varphi_t \in U^e \cup V^e$$

apply (jjj), and conclude $f \circ \varphi_h(x) < c$: contradiction.

- $$(\exists y \in \mathscr{C} \cap f_c^{c+h}) \quad (\exists [\alpha, \beta] \subset [0, h]),$$
$$\| \varphi_\alpha(x) - y \| = r - e, \qquad \| \varphi_\beta(x) - y \| = r,$$
$$(\forall t \in [\alpha, \beta]), \quad \| \varphi_t(x) - y \| \in [r - e, r].$$

- $$(\exists y \in \mathscr{C} \cap f_c^{c+h}) \quad (\exists [\alpha, \beta] \subset [0, h]),$$
$$\| \varphi_\alpha(x) - y \| = r + e, \qquad \| \varphi_\beta(x) - y \| = r,$$
$$(\forall t \in [\alpha, \beta]), \quad \| \varphi_t(x) - y \| \in [r, r + e].$$

In the second and third situations, we have $\| \varphi_\beta(x) - \varphi_\alpha(x) \| \geqq e$, and from (jj), (jjj), $f'_y . V_y \leqq -\frac{1}{2} \| f'_y \| . \| V_y \| \leqq -\frac{\mu}{2} \| V_y \|$ if $y \in \varphi_{[\alpha, \beta]}(x) \cap f_{c-h}$. Since $h < \mu e/2$, we also conclude $f \circ \varphi_h(x) < c$: contradiction.

So we have proved that if $f(x) < c + h$, then either $f \circ \varphi_h(x) < c$, or $\varphi_h(x) \in K^0$.

Finally, $\gamma = \varphi_h \circ \gamma^h$ is such that

$$\operatorname{Im} \gamma \cap [\bigcup_{y \in \mathscr{C} \cap f_c^{c+h}} S(y, r)] \cap f_c = \varnothing,$$

$$(\operatorname{Im} \gamma \cap f_c) \subset \bigcup_{y \in \mathscr{C} \cap f_c^{c+h}} B(y, r).$$

Since $\operatorname{Im}\gamma \cap f_c$ is compact, we can extract a finite subcovering:

$$(\operatorname{Im}\gamma \cap f_c) \subset \bigcup_{i=1}^{p} \mathrm{B}(u^i, r). \qquad u^i \in \mathscr{C} \cap f_c^{c+h}.$$

Lemma 7 is thus proved. □

Lemma 7 has a direct consequence:

COROLLARY 7.1. – *Assume* ($\mathscr{H}$) *is true. Choose* $r>0$, $h>0$. *Then there is* $u=u(r, h) \in \mathscr{C} \cap f_c^{c+h}$ *such that* $i_* \neq 0$, *where*

$$i_*: \ \mathrm{H}_1(f^{<(c+h)} \cap \mathrm{B}(u, r), f^{<c} \cap \mathrm{B}(u, r)) \to \mathrm{H}_1(f^{<(c+h)}, f^{<c})$$

is the morphism induced by the canonical injection

$$i: \ \mathrm{B}(u, r) \to \mathrm{L}^\beta.$$

Proof. – We juste have to prove the result when $r \in \mathbb{R}_+^* \setminus \mathrm{D}$: it will then be true for any $r' \geqq r$.

Let p_0 be the minimal value of p such that there are $(u^1, \ldots, u^p) \in \mathscr{C} \cap (f_c^{c+h})^p$ and $\gamma \in \Gamma$ satisfying the conclusion of Lemma 7. $\operatorname{Im}\gamma \cap \mathrm{B}(u^{p_0}, r)$ is the image of a 1-dimensional complex $\omega \in \mathrm{C}_1(f^{<(c+h)})$, with $\omega \in \bar{\omega}$, for some $\bar{\omega} \in \mathrm{H}_1(f^{<(c+h)} \cap \mathrm{B}(u^{p_0}, r), f^{<c} \cap \mathrm{B}(u^{p_0}, r))$.

If $i_* \bar{\omega} = 0$, then there is a singular 2-dimensional complex $\Omega \in \mathrm{C}_2(f^{<(c+h)})$ such that $\partial\Omega = \omega - \alpha$, with $\alpha \in \mathrm{C}_1(f^{<c})$. So, replacing the curves of ω by curves of α in γ, we get $\bar{\gamma}$ satisfying the conclusion of Lemma 7 with $u^1, \ldots, u^{p_0-1}$. This contradicts the minimality of p_0. So $i_* \bar{\omega} \neq 0$. Corollary 7.1 is thus proved, with $u = u^{p_0}$. □

Corollary 7.1 gives the existence of at least one critical point $u \neq 0$. The hypothesis ($\mathscr{H}$) seems too weak to get u independent of r, h, and we cannot say that $f(u)=c$. The fundamental reason for this is that the Palais-Smale condition is not satisfied. To overcome this difficulty, we shall make use of Lemma 6 which gives a local Palais-Smale condition.

We first choose $\rho^0 \in]0, r_0[$, $d^0 > 0$, such that $[\rho^0 - d^0, \rho^0 + d^0] \cap \mathrm{D} = \varnothing$, r_0 being defined in Lemma 6.

We define

$$\mu^0 = \frac{1}{2} \inf \{ \| f'(x) \| / x \in f^{c'}, (\exists y \in \mathscr{C} \cap f^{c'}) : \| x - y \| \in [\rho^0, \rho^0 + d^0] \}.$$

We take $0 < h < \min(\mu^0 d^0, c' - c)$. By Corollary 7.1, there are

$$u^0 \in \mathscr{C} \cap f_c^{c'}, \qquad \bar{\omega} \in \mathrm{H}_1(\mathrm{B}(u^0, p^0) \cap f^{<c+h}, \mathrm{B}(u^0, \rho^0) \cap f^{<c}),$$

such that $i_* \bar{\omega} \neq 0$, where

$$i_*: \ \mathrm{H}_1(f^{<c+h} \cap \mathrm{B}(u^0, \rho^0), f^{<c} \cap \mathrm{B}(u^0, \rho^0)) \to \mathrm{H}_1(f^{<c+h}, f^{<c})$$

is the morphism induced by the canonical injection

$$i: \ \mathrm{B}(u^0, \rho^0) \to \mathrm{L}^\beta.$$

We define

$$X=(f^{c+h}\cap B(u^0,\rho^0))$$
$$\cup\left\{x\in L^\beta/\|x-u^0\|\in[\rho^0,\rho^0+d^0[,f(x)<c+h\left(1-\frac{\|x-u^0\|-\rho^0}{d^0}\right)\right\},$$
$$Y=f^c\cap B(u^0,\rho^0+d^0).$$

We call

$$j_*:\quad H_1(f^{<c+h}\cap B(u^0,\rho^0),f^{<c}\cap B(u^0,\rho^0))\to H_1(X,Y)$$

the morphism induced by the canonical injections

$$j_+:\quad f^{<c+h}\cap B(u^0,\rho^0)\to X,$$
$$j_-:\quad f^{<c}\cap B(u^0,\rho^0)\to Y.$$

Clearly, we have $j_*\bar{\omega}\neq 0$.

We define $\bar{c}=\inf_{z\in j_*\bar{\omega}}(\max f(z))\in[c,c+h[$.

By arguments similar to those proving Lemma 7 and Corollary 7.1, we find, for any $n\in\mathbb{N}^*$, a critical point $u^n\in\mathscr{C}\cap f_{\bar{c}}^{\bar{c}+(1/n)}\cap B(u^0,\rho^0-d^0)$, such that $i_*^n\neq 0$, where

$$i_*^n:\quad H_1\left(f^{<\bar{c}+(1/n)}\cap B\left(u^n,\frac{d^0}{n}\right),f^{<\bar{c}}\cap B\left(u^n,\frac{d^0}{n}\right)\right)$$
$$\to H_1(f^{<(\bar{c}+(1/n)}\cap B(u^n,d^0),f^{<\bar{c}}\cap B(u^n,d^0))$$

is the morphism induced by the canonical injection

$$i_*^n:\quad B\left(u^n,\frac{d^0}{n}\right)\to B(u^n,d^0).$$

By Lemma 6, the sequence (u^n) is precompact (recall that $\rho^0<r_0$). Considering one of its limit points, and taking $r_1=d^0/2$, we get

LEMMA 8. – *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true.*

Then there are $u\in\mathscr{C}$ *with* $f(u)=\bar{c}\in[c,c')$ *and* $r_1>0$, *such that, for any* $r\in]0,r_1]$ *and* $h>0$, *we have* $i_*\neq 0$ *where*

$$i_*:\quad H_1(f^{<(\bar{c}+h)}\cap B(u,r),f^{<\bar{c}}\cap B(u,r))$$
$$\to H_1(f^{<(\bar{c}+h)}\cap B(u,r_1),f^{<\bar{c}}\cap B(u,r_1))$$

is the morphism induced by the canonical injection

$$i:\quad B(u,r)\to B(u,r_1).$$

The great difference with Corollary 7.1 is that u does not depend on r, h any more.

Lemma 8 gives a min-max localized around u. To get our multiplicity result, we are going to make products of several "copies" of this min-max. At each product will be associated a new critical point. We first enounce:

COROLLARY 8.1. — *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true. Choose* $r \in]0, r_1[$, $h>0$.

Then there is $N = N(r, h)$ *such that*

$$(\forall (a, b) \in [N, +\infty]^2): \quad I_* \neq 0,$$

where

$$I_*: \quad H_1(f^{<(\bar{c}+h)} \cap B(u, r) \cap L^\beta_{(-a, b)}, f^{<\bar{c}} \cap B(u, r) \cap L^\beta_{(-a, b)}) \\ \to H_1(f^{<(\bar{c}+h)} \cap B(u, r_1) \cap L^\beta_{(-a, b)}, f^{<\bar{c}} \cap B(u, r_1) \cap L^\beta_{(-a, b)}$$

is the morphism induced by

$$I: \quad B(u, r) \cap L^\beta_{(-a, b)} \to B(u, r_1) \cap L^\beta_{(-a, b)}.$$

and

$$L^\beta_{(-a, b)} = \{ x \in L^\beta / \mathrm{supp}(x) \subset [-a, b] \}.$$

Proof. — We choose $\bar{\omega} \in H_1(f^{<(\bar{c}+h)} \cap B(u, r), f^{<\bar{c}} \cap B(u, r))$ such that

$$i_* \bar{\omega} \neq 0,$$

with the notations of Lemma 8.

The class $\bar{\omega}$ has an element of the form $\sum_{i=1}^{r} \lambda_i \sigma_i$, satisfying

(P) $[\lambda_i \in \mathbb{R}$, and $\sigma_i : S^1 \to L^\beta$ continuous or $\sigma_i : [0, 1] \to L^\beta$ continuous, with $\sigma_i(0), \sigma_i(1) \in f^{<\bar{c}}$, and $\mathrm{Im}(\sigma_i) \subset f^{<(\bar{c}+h)} \cap B(u, r)$ in both cases].

For $t_1, t_2 \in \bar{\mathbb{R}}$, we define

$$K_{t_1, t_2}: \quad L^\beta(\mathbb{R}, \mathbb{R}^{2N}) \to L^\beta(\mathbb{R}, \mathbb{R}^{2N}) \\ x(t) \mapsto \chi_{[t_1, t_2]}(t)\, x(t)$$

We note that $\bigcup_{i=1}^{r} \mathrm{Im}\, \sigma_i$ is compact, so that

$$\lim_{(t_1, t_2) \to (-\infty, +\infty)} \left(\sup \left\{ \| x - K_{t_1, t_2}(x) \| ; \ x \in \bigcup_{i=1}^{r} \mathrm{Im}\, \sigma_i \right\} \right) = 0.$$

Moreover, $f^{<(\bar{c}+h)} \cap B(u, r)$ and $f^{<\bar{c}} \cap B(u, r)$ are open.

So there is $N = N(r, e, h) \in \mathbb{N}$ such that, if $(a, b) \in [N, +\infty]^2$, then

$$\sum_{i=1}^{r} \lambda_i (K_{-a, b} \circ \sigma_i) \in \bar{\omega}.$$

As a consequence, there is

$$\tilde{\omega} \in H_1(f^{<(\bar{c}+h)} \cap B(u, r) \cap L^\beta_{(-a, b)}, f^{<\bar{c}} \cap B(u, r) \cap L^\beta_{(-a, b)})$$

such that $\sum \lambda_i (K_{-a, b} \circ \sigma_i) \in \tilde{\omega}$, and $i_*(\bar{\omega}) \neq 0$ implies $I_*(\tilde{\omega}) \neq 0$. So I_* cannot be zero.

Corollary 8.1 is thus proved. □

We now have to introduce some notations.

Take $x \in L^\beta$, $\bar{p}=(p^1, \ldots, p^m) \in \mathbb{Z}^m$, $m \geqq 1$, $p^i < p^{i+1}$. Denote

$$x_i = x\,\chi_{[(p^{i-1}+p^i)/2,\,(p^i+p^{i+1})/2]}, \qquad f_i(x)=f(x_i),$$

with χ_I the characteristic function of I, $p^0=-\infty$, $p^{m+1}=+\infty$.

We have $x=\sum_{i=1}^{m} x_i$, but $f \neq \sum_{i=1}^{m} f_i$.

Consider the sets

$$\mathscr{L}_+(h)=\bigcap_{i=1}^{m} (f_i)^{<(\bar{c}+h)}, \qquad \mathscr{L}_-(h)=\bigcup_{i=1}^{m} (f_i)^{<(\bar{c}-h)},$$

and the "product" ball

$$B^u_{\bar{p},\rho}=\{ x \in L^\beta / (\forall i) \, \| (x-p^i * u)_i \|_{L^\beta} < \rho \}$$

for $\rho>0$, $u \in \mathscr{C}$.

From Künneth's formula,

$$H_*(X \times Y, (Z \times Y) \cup (X \times T)) = H_*(X, Z) \otimes H_*(Y, T),$$

immediately follows

LEMMA 9. – *Assume that* (hA), (hR) *and* ($\mathscr{H}$) *are true.* u, r_1 *are the same as in Lemma* 8. *Choose* $r \in]0, r_1]$, $h>0$.

Then there is $N=N(r, h)$ *such that, if* $m \geqq 1$ *and* $\bar{p}=(p^1 \ldots p^m)$ *satisfy* $p^{i+1}-p^i \geqq N$ *for* $1 \leqq i \leqq m-1$, *then*

$$J_* \neq 0,$$

where

$$J_*: \quad H_m(\mathscr{L}_+(h) \cap B^u_{\bar{p},r},\ \mathscr{L}_-(0) \cap \mathscr{L}_+(h) \cap B^u_{\bar{p},r}) \to H_m(\mathscr{L}_+(h) \cap B^u_{\bar{p},r_1},\ \mathscr{L}_-(0) \cap \mathscr{L}_+(h) \cap B^u_{\bar{p},r_1})$$

is the morphism associated to the canonical injection

$$J: \quad B_{\bar{p},r} \to B_{\bar{p},r_1}.$$

Lemma 9 gives the desired product min-max.

V. A DEFORMATION ARGUMENT

In what follows, we assume once again that (hA), (hR) and ($\mathscr{H}$) are true. D, F are the same as in Lemmas 4, 5, r_0 is the same as in Lemma 6, u, $\bar{c}$, r_1 are the same as in Lemmas 8, 9.

5.1. Construction of a vector field

From (hA) (hR), we know that $(\exists\,\theta,\ C_1>0)\ (\forall\,(X,\ Y)\in(L^\beta)^2)$:

$$\left|\int(X,\ LY)\right|\leqq C_1\exp(-\theta\delta(X,\ Y))\,\|X\|_\beta\,\|Y\|_\beta,$$

for $\delta(X,\ Y)=\operatorname{dist}(\operatorname{supp}X,\ \operatorname{supp}Y)$.

From (hR), we know that

$$(\exists\,c_1>0)\quad(\forall\,(y,\ t)\in\mathbb{R}^{2N}\times\mathbb{R}),\qquad c_1|y|^\beta\leqq G(y,\ t)\leqq(\nabla G(y,\ t),\ y),$$
$$(\exists\,c_2>0)\quad(\forall\,(y,\ t)\in\mathbb{R}^{2N}\times\mathbb{R}),\qquad |\nabla G(y,\ t)|\leqq c_2|y|^{\beta-1}.$$

We choose $0<r_2<\min(1,\ r_1)$ such that

$$\frac{c_1}{2}(r_2)^\beta>6\,C_1(r_2)^2,\qquad\text{and}\qquad B(u,\ r_2)\subset f^{c'}.$$

We are going to use these technical conditions in the proof of the following Lemma:

LEMMA 10. – *Assume that* (hA), (hR) *and* $(\mathscr{H})$ *are true, and to* $0<r<\frac{r_2}{2}$, *associate* $e=e(r)$ *such that*

$$r+2e\leqq\frac{r_2}{2}\qquad\textit{and}\qquad[r-2e,\ r+2e]\cap D=\varnothing.$$

There are $\mu=\mu(r)>0$, $A=A(r)>0$ *such that:*
If $m\geqq 2$, *and if* $\bar{p}\in\mathbb{Z}^m$ *satisfies* $(\forall\,i):p^{i+1}-p^i>A$, *then:*
$(\forall\,x\in B^u_{\bar{p},\,r+e}\setminus B^u_{\bar{p},\,r-e})\ (\exists\,V_x\in B^0_{\bar{p},\,1})$:

1) $$f'(x)\,.\,V_x>\mu;$$
2) $$(\forall\,i):(f_i)'(x)\,.\,V_x\geqq 0;$$
3) $$\|y_i\|\geqq r-e\Rightarrow(f_i)'(x)\,.\,V_x>\mu,$$

with the notation $y_i=(x-p^i*u)_i$. □

Proof. – Define

$$\bar{\mu}=\frac{1}{2}\inf\{\|f'(x)\|_\alpha/x\in B(u,\ r+2e(r))\setminus B(u,\ r-e(r))\}.$$

$\bar{\mu}$ depends only on r, and $\bar{\mu}>0$ by Lemma 4. Let $x\in B^u_{\bar{p},\,r+e}\setminus B^u_{\bar{p},\,r-e}$, $i\in[\![1,\ m]\!]$, and $y_i=(x-p^i*u)_i$. Impose $A>64$.

We always have $\|x_i\|\leqq\|u\|+r_2$. So there is $\tau^i\in[2\sqrt{A},\ A/2-2\sqrt{A}]$ such that

$$\|x_i\chi_{\{\tau^i-\sqrt{A}\leqq|t-p^i|\leqq\tau^i+\sqrt{A}\}}\|_\beta\leqq\frac{C_2}{A^{1/2\beta}}.$$

Here, C_2 is a constant, but τ^i depends on x, i, A, $\bar{p}$.

Now, impose $\| u\chi_{\{|t|>\sqrt{A}\}}\| \leqq \frac{e}{3}$, and $\frac{C_2}{A^{1/2\beta}} \leqq \frac{e}{3}$, which is possible for $A \geqq A^0(e)$.

Then, three possibilities may occur:

First case:

$$\| x_i \chi_{\{|t-p^i| \geqq \tau^i+\sqrt{A}\}} \| \geqq \frac{e}{3}.$$

We take

$$V_{x,i} = x_i (h_- \chi_{]-\infty, p^i-\tau^i-\sqrt{A}]} + h_+ \chi_{[p^i+\tau^i+\sqrt{A}, +\infty[})$$

with

$$h_+ = 1 \quad \text{if } \| x_i \chi_{[p^i+\tau^i+\sqrt{A}, +\infty[} \| \geqq \frac{e}{6}, \qquad h_+ = 0 \quad \text{otherwise,}$$

$$h_- = 1 \quad \text{if } \| x_i \chi_{]-\infty, p^i-\tau^i-\sqrt{A}]} \| \geqq \frac{e}{6}, \qquad h_- = 0 \quad \text{otherwise.}$$

We have

$$\begin{aligned}
(f_i)'(x).V_{x,i} &\geqq c_1 \| V_{x,i} \|_\beta^\beta - C_1 \| V_{x,i} \|_\beta^2 \\
&\quad - C_1 \| x\chi_{\{\tau^i-\sqrt{A} \leqq |t-p^i| \leqq \tau^i+\sqrt{A}\}} \|_\beta \cdot \| V_{x,i} \|_\beta \\
&\quad - C_1 \| x\chi_{\{|t-p^i| \leqq \tau^i-\sqrt{A}\}} \|_\beta \cdot \| V_{x,i} \|_\beta \exp(-2\theta\sqrt{A}) \\
&\geqq \frac{3c_1}{4} \| V_{x,i} \|_\beta^\beta - C_1 \frac{e}{3} \| V_{x,i} \|_\beta \\
&\quad - C_1 (\| u \|_\beta + r_2) \| V_{x,i} \|_\beta \exp(-2\theta\sqrt{A}) \\
&\geqq \frac{3c_1}{4} \| V_{x,i} \|_\beta^\beta - C_1 e \| V_{x,i} \|_\beta \qquad \text{for} \quad A \geqq A^1(e) \\
&\geqq \frac{3c_1}{4} \| V_{x,i} \|_\beta^\beta - 6C_1 \| V_{x,i} \|_\beta^2 \\
&\geqq \frac{c_1}{4} \| V_{x,i} \|_\beta^\beta \geqq \frac{c_1}{4} \left(\frac{e}{6}\right)^\beta.
\end{aligned}$$

[We recall that $\frac{e}{6} \leqq \| V_{x,i} \|_\beta \leqq \| u\chi_{\{|t| \geqq \sqrt{A}\}} \| + (r+e) \leqq r_2 < 1$, and that $\frac{c_1}{2}(r_2)^\beta > 6C_1 (r_2)^2$.]

Second case: $\| x_i \chi_{\{|t-p^i| \geqq \tau^i+\sqrt{A}\}} \| < \frac{e}{3}$, and $\| y_i \| < r-e$. Then we take $V_{x,i} = 0$.

Third case: $\|x_i\chi_{\{|t-p^i|\geqq\tau^i+\sqrt{A}\}}\|<\frac{e}{3}$, and $\|y_i\|<r-e$. Then

$$\begin{aligned}\|x\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{A}\}}-p^i*u\|&\geqq\|y_i\|-\|x_i\chi_{\{\tau^i-\sqrt{A}\leqq|t-p^i|\leqq\tau^i+\sqrt{A}\}}\|\\&\quad-\|u\chi_{|t|\geqq\sqrt{A}}\|-\|x_i\chi_{\{|t-p^i|\geqq\tau^i+\sqrt{A}\}}\|\\&\geqq r-e-\frac{e}{3}-\frac{e}{3}-\frac{e}{3}=r-2e.\end{aligned}$$

Finally,

$$\begin{aligned}r-2e&\leqq\|x\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{A}\}}-p^i*u\|\\&\leqq\|y_i\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{A}\}}\|+\|u\chi_{|t|\geqq\sqrt{A}}\|\\&\leqq r+e+\frac{e}{3}\\&\leqq r+2e.\end{aligned}$$

So there is $W_{x,i}\in L^\beta$ such that $\|W_{x,i}\|\leqq 1$, and

$$f'(x\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{A}\}}).W_{x,i}>\bar{\mu}.$$

Now,

$$\begin{aligned}f'(x)&=f'(x_i\chi_{\{|t-p^i|\leqq\tau^i-\sqrt{A}\}}+f'(x_i\chi_{\{\tau^i-\sqrt{A}\leqq|t-p^i|\leqq\tau^i+\sqrt{A}\}})\\&\quad+f'(x_i\chi_{\{|t-p^i|\geqq\tau^i+\sqrt{A}\}})+\sum_{j\neq i}f'(x_j)\\&=f'(x^a)+f'(x^b)+f'(x^c)+\sum_{j\neq i}f'(x_j).\end{aligned}$$

But $\|x^b\|\leqq\frac{C_2}{A^{1/2\beta}}$, and $\max\{\|x^a\|,\|x^c\|,\|x_j\|(j\neq i)\}\leqq\|u\|+r_2$.

We choose $V_{x,i}=W_{x,i}\chi_{\{|t-p^i|\leqq\tau^i\}}$. Clearly, $\|V_i\|\leqq 1$. Moreover, we have:

$$\begin{aligned}f'(x).V_{x,i}&\geqq f'(x^a).W_{x,i}-|f'(x^a).(V_{x,i}-W_{x,i})|\\&\quad-|f'(x^b).V_{x,i}|-|f'(x^c).V_{x,i}|-\sum_{j\neq i}|f'(x_j).V_{x,i}|\\&\geqq\bar{\mu}-C_1(\|u\|+r_2)\exp(-\theta\sqrt{A})\\&\quad-c_2\left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1}-C_1\frac{C_2}{A^{1/2\beta}}-C_1(\|u\|+r_2)\exp(-\theta\sqrt{A})\\&\quad-\sum_{j\neq i}C_1(\|u\|+r_2)\exp(-\theta\sqrt{A})\exp[-\theta(|i-j|-1)A]\\&\geqq\bar{\mu}-c_2\left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1}-C_1\frac{C_2}{A^{1/2\beta}}\\&\quad-C_1(\|u\|+r_2).\left(2+\frac{2}{1-\exp(-\theta A)}\right)\exp(-\theta\sqrt{A})\\&\geqq\bar{\mu}/2\qquad\text{for}\quad A\geqq A^2(r).\end{aligned}$$

Identically,

$$
\begin{aligned}
(f_i)'(x).\mathrm{V}_{x,i} &= f'(x^a+x^b+x^c).\mathrm{V}_{x,i}\\
&\geqq \bar{\mu}-c_2\left(\frac{\mathrm{C}_2}{\mathrm{A}^{1/2\beta}}\right)^{\beta-1}-\mathrm{C}_1\frac{\mathrm{C}_2}{\mathrm{A}^{1/2\beta}}-2\mathrm{C}_1(\|u\|+r_2)\exp(-\theta\sqrt{\mathrm{A}})\\
&\geqq \bar{\mu}/2 \quad \text{for} \quad \mathrm{A}\geqq \mathrm{A}^2.
\end{aligned}
$$

Conclusion. – We now take $\mathrm{V}_x=\sum_i \mathrm{V}_{x,i}$. By construction, $\mathrm{V}_x\in \mathrm{B}^0_{\bar{p},1}$. Denote by I^1, I^2, I^3 the sets of indices i corresponding to Cases 1, 2, 3 respectively. We write

$$
\begin{aligned}
f'(x).\mathrm{V}_x &= \sum_{i\in \mathrm{I}^1} f'(x).\mathrm{V}_{x,i}+\sum_{i\in \mathrm{I}^3} f'(x).\mathrm{V}_{x,i}\\
&\geqq \sum_{i\in \mathrm{I}^1} f'(x).\mathrm{V}_{x,i}+\frac{\bar{\mu}}{2}\operatorname{card}(\mathrm{I}^3).
\end{aligned}
$$

Now, there is a family $\mathrm{J}^1\subset [\![0, m]\!]$ such that

$$
\sum_{i\in \mathrm{I}^1}\mathrm{V}_{x,i}=\sum_{j\in \mathrm{J}^1}\mathrm{X}^j,
$$

where

$$
\begin{aligned}
\mathrm{X}^j &= (\xi^j_+\,\chi_{[((p^j+p^{j+1})/2),\, p^{j+1}-\tau^{j+1}-\sqrt{\mathrm{A}}]}+\xi^j_-\,\chi_{[p^j+\tau^j+\sqrt{\mathrm{A}},\, ((p^j+p^{j+1})/2)]})\,x\\
&= \xi^j_+\,\mathrm{X}^j_+ + \xi^j_-\,\mathrm{X}^j_-
\end{aligned}
$$

with $\xi^j_\pm\in\{0, 1\}$, and

$$
(\forall s\in\{+,-\})\quad (\forall j\in [\![0, m]\!])
$$
$$
\left(\xi^j_s=1\Rightarrow \|\mathrm{X}^j_s\|\geqq \frac{e}{6},\ \xi^j_s=0\Rightarrow \|\mathrm{X}^j_s\|<\frac{e}{3}.\right)
$$

So there are three possible situations

$$
(\xi^j_-=\xi^j_+=1),\quad (\xi^j_-=0 \text{ and } \xi^j_+=1),\quad (\xi^j_-=1 \text{ and } \xi^j_+=0).
$$

First situation: $\xi^j_-=\xi^j_+=1$.

Denote

$$
\mathrm{Y}^j=x\,\chi_{[p^j+\tau^j-\sqrt{\mathrm{A}};\, p^j+\tau^j+\sqrt{\mathrm{A}}]\,\cup\,[p^{j+1}-\tau^{j+1}-\sqrt{\mathrm{A}};\, p^{j+1}-\tau^{j+1}+\sqrt{\mathrm{A}}]}
$$
$$
\mathrm{Z}^j=x_j+x_{j+1}-\mathrm{X}^j-\mathrm{Y}^j.
$$

We have

$$
\begin{aligned}
f'(x).\mathrm{X}^j &= f'(\mathrm{X}^j).\mathrm{X}^j+f'(\mathrm{Y}^j).\mathrm{X}^j+f'(\mathrm{Z}^j).\mathrm{X}^j\\
&\quad+\sum_{k\neq j,\, j+1} f'(x_k).\mathrm{X}^j\\
&\geqq \frac{3c_1}{4}\|\mathrm{X}^j\|^\beta-\mathrm{C}_1\frac{2e}{3}\|\mathrm{X}^j\|-2\mathrm{C}_1\|\mathrm{X}^j\|(\|u\|+r_2)\exp(-2\theta\sqrt{\mathrm{A}})\\
&\quad-2\|\mathrm{X}^j\|(\|u\|+r_2)\sum_{l\geqq 0}\exp(-2\theta\sqrt{\mathrm{A}})\exp(-\theta l\mathrm{A})
\end{aligned}
$$

$$\geqq \frac{3c_1}{4}\|X^j\|^\beta - C_1 e\|X^j\| \qquad \text{for} \quad A \geqq A^3(e)$$

$$\geqq \frac{3c_1}{4}\|X^j\|^\beta - 6C_1\|X^j\|^2$$

$$\geqq \frac{c_1}{4}\|X\|^j\|^\beta \geqq \frac{c_1}{4}\frac{e^\beta}{6^\beta}—$$

since $\frac{e}{6} \leqq \|X^j\| \leqq 2\|u\chi_{\{|t| \geqq \sqrt{A}\}}\| + 2(r+e) \leqq r_2$.

Second situation: $\xi_-^j = 0$, $\xi_+^j = 1$.
We now take

$$Y^j = x(\chi_{[p^j + \tau^j + \sqrt{A}, ((p_j + p_{j+1})/2)]} + \chi_{[p^{j+1} - \tau^{j+1} - \sqrt{A}, p^{j+1} - \tau^{j+1} + \sqrt{A}]}),$$

$$Z^j = x_j + x_{j+1} - X^j - Y^j.$$

We have $\|Y^j\| \leqq \frac{e}{3} + \frac{e}{3} = \frac{2e}{3}$, $\operatorname{dist}(\operatorname{supp} Z^j, \operatorname{Supp} X^j) \geqq \sqrt{A}$. As in the first situation, we get

$$f'(x).X^j \geqq \frac{3c_1}{4}\|X^j\|^\beta - C_1 e\|X^j\| \qquad \text{for} \quad A \geqq A^4(u, e)$$

$$\geqq \frac{3c_1}{4}\|X^j\|^\beta - 6C_1\|X^j\|^2$$

$$\geqq \frac{c_1}{4}\|X^j\|^\beta \geqq \frac{c_1}{4}\frac{e^\beta}{6^\beta}.$$

The third situation is identical to the second one. Since $I^1 \cup I^3$ is non-empty, we take

$$A(r) = \max(A^0, A^1, A^2, A^3, A^4) \qquad \text{and} \qquad \mu(r) = \min\left(\bar{\mu}, \frac{c_1}{4}\frac{e^\beta}{6^\beta}\right),$$

and Lemma 10 is proved. □

LEMMA 11. — *Suppose f satisfies* (hA), (hR) *and* ($\mathscr{H}$). *To $l < c'$, associate $\eta = \eta(l) > 0$ such that $l + 2\eta \leqq c'$, and $[l - 2\eta, l + 2\eta] \cap F = \varnothing$.*

Then there are $\mathscr{A} = \mathscr{A}(l)$ and $\nu = \nu(l)$ such that for any $m \geqq 2$, $\bar{p} \in \mathbb{Z}^m$, with $(\forall i)\, p^{i+1} - p^i > \mathscr{A}$, we have:

$$\left(\forall x \in B^u_{\bar{p}, (r_2/2)} \cap \bigcup_{i=1}^{m} (f_i)_{l-\eta}^{l+\eta}\right)(\exists \mathscr{V}_x \in B^0_{\bar{p}, 1}):$$

- $f'(x).\mathscr{V}_x > \nu$;
- $(\forall i \in [\![1, m]\!]) : (x \in (f_i)_{l-\eta}^{l+\eta} \Rightarrow (f_i)'(x).\mathscr{V}_x > \nu)$;
- $(\forall i) : (f_i)'(x).\mathscr{V}_x > 0$.

Proof. — We know that f is uniformly continuous on any bounded part of L^β. So there is $\mathscr{E}(\eta)>0$ such that, if X, $Y \in B(0, \|u\|+r_2)$, then

$$\|X-Y\| \leqq \mathscr{E} \Rightarrow |f(x)-f(y)| \leqq \eta.$$

Now, consider $\bar{\nu}=\frac{1}{2}\inf\{\|f'(x)\|;\ x \in f_{l-2\eta}^{l+2\eta}\}$. From Lemma 5, $\bar{\nu}>0$. The proof of Lemma 11 is similar to that of Lemma 10, replacing V by $\mathscr{V}$, $\bar{\mu}$ by $\bar{\nu}$, A by $\mathscr{A}$, e by $\mathscr{E}$. So we just sketch it. The three possibilities are:

First case: $\|x_i \chi_{\{|t-p^i| \geqq \tau^i+\sqrt{\mathscr{A}}\}}\| \geqq \frac{\mathscr{E}}{3}$, then

$$\mathscr{V}_{x,i}=x_i(h_- \chi_{]-\infty, p^i-\tau^i-\sqrt{\mathscr{A}}]}+h_+ \chi_{[p^i+\tau^i+\sqrt{\mathscr{A}}, +\infty[}),$$

$$(f_i)'(x) \cdot \mathscr{V}_{x,i} \geqq \frac{c_1}{2} \frac{\mathscr{E}^\beta}{6^\beta} \qquad \text{for} \quad \mathscr{A} \geqq \max(\mathscr{A}^0, \mathscr{A}^1).$$

Second case: $\|x_i \chi_{\{|t-p^i|>\tau^i+\sqrt{\mathscr{A}}\}}\|<\frac{\mathscr{E}}{3}$, *and* $f_i(x) \notin [l-\eta, l+\eta]$, then $\mathscr{V}_{x,i}=0$.

Third case: $\|x_i \chi_{\{|t-p^i|>\tau^i+\sqrt{\mathscr{A}}\}}\|<\frac{\mathscr{E}}{3}$, *and* $f_i(x) \in [l-\eta, l+\eta]$, then

$$f(x \chi_{\{|t-p^i| \leqq \tau^i-\sqrt{\mathscr{A}}\}}) \in [l-2\eta, l+2\eta] \qquad \text{for} \quad \mathscr{A} \geqq \mathscr{A}^0,$$

hence $f'(x \chi_{\{|t-p^i| \leqq \tau^i-\sqrt{\mathscr{A}}\}}) \cdot \mathscr{W}_{x,i}>\bar{\nu}$,

$$\|\mathscr{W}_{x,i}\| \leqq 1, \qquad \mathscr{V}_{x,i}=\mathscr{W}_{x,i}\chi_{\{|t-p^i| \leqq \tau^i\}},$$

$$f'(x) \cdot \mathscr{V}_{x,i} \geqq \bar{\nu}/2, \qquad (f_i)'(x) \cdot \mathscr{V}_{x,i} \geqq \bar{\nu}/2, \quad \text{for } \mathscr{A} \geqq \mathscr{A}^2.$$

The final study of $f'(x) \cdot \mathscr{V}_x$ is the same as in Lemma 10, and 11 is proved with $\mathscr{A}=\max(\mathscr{A}^0, \ldots, \mathscr{A}^4)$, $\nu=\min\left(\frac{\bar{\nu}}{2}, \frac{c_1}{2} \frac{\mathscr{E}^\beta}{6^\beta}\right)$. □

LEMMA 12. — *Suppose f satisfies* (hA), (hR) *and* ($\mathscr{H}$).

r, $e(r)$, $A(r)$, $\mu(r)$ *are the same as in Lemma* 10. *We impose, moreover,* $r<r_0$, *with the notation of Lemma* 6.

Choose $\lambda>0$ *such that* $\bar{c}+\lambda<c'$,

$$\text{and} \quad \begin{cases} \bar{c}+\lambda \notin F \\ \bar{c}-\lambda \notin F. \end{cases}$$

Suppose $m \geqq 2$, $\bar{p} \in \mathbb{Z}^m$,

$$(p^{i+1}-p^i) \geqq \max(A(r), \mathscr{A}(\bar{c}-\lambda), \mathscr{A}(\bar{c}+\lambda)) = \mathscr{B}(r, \lambda)$$

($\mathscr{A}$ *has been defined in Lemma* 11).

If $\mathscr{C} \cap B^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda) = \varnothing$, *then there are* $\xi = \xi(\bar{p}, r, \lambda) > 0$ *and a locally Lipschitz vector field* $V(x)$ *such that:*

(i) $(\forall x): V(x) \in B^0_{\bar{p}, 1}$, *and* $(x \notin B^u_{\bar{p}, (r_2/2)} \Rightarrow V(x) = 0)$;

(ii) $\forall x \in [B^u_{\bar{p}, r} \setminus B^u_{\bar{p}, (r-e)}]$, $\forall i \in [\![1, m]\!]$,

$$\left(\| y_i \| \in [r-e, r] \Rightarrow (f_i)'(x) . V(x) > \frac{\mu(r)}{3} \right).$$

(iii) $(\forall x \in B^u_{\bar{p}, r} \cap (\mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda)) : f'(x) . V(x) > \xi$.

(iv) $(\forall x \in B^u_{\bar{p}, (r_2/2)}) (\forall i \in [\![1, m]\!])$:

$$(f_i(x) \in \{ \bar{c} + \lambda, \bar{c} - \lambda \} \Rightarrow (f_i)'(x) . V(x) > 0).$$

Proof. – In Lemma 6, take $R = \max(|p^1|, |p^m|)$. Consider a sequence

$$(u_n) \in B^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda - \eta(\bar{c} + \lambda)) \setminus \mathscr{L}_-(\lambda - \eta(\bar{c} - \lambda)).$$

(u_n) satisfies

$$(\forall p, q), \quad \| (u_p - u_q) \chi_{\mathbb{R} \setminus [-R, R]} \| < 2 r_2 < 2 r_0.$$

So, if $\mathscr{C} \cap B^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda) = \varnothing$, we cannot have $f'(u_n) \to 0$, and there is $\alpha(\bar{p}, u, r, \lambda) > 0$ such that

$$\forall x \in B^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda - \eta(\bar{c} + \lambda)) \setminus \mathscr{L}_-(\lambda - \eta(\bar{c} - \lambda)) : \quad \| f'(x) \| \geqq 2 \alpha.$$

Now, if $x \in [B^u_{\bar{p}, (r+e)} \setminus B^u_{\bar{p}, (r-e)}]$, we find V_x satisfying the conclusion of Lemma 10, and we choose $V_x = 0$ otherwise.

For $s \in \{-, +\}$. if $x \in B^u_{\bar{p}, (r_2/2)} \cap \bigcup_i (f_i)^{\bar{c} + s\lambda + \eta(\bar{c} + s\lambda)}_{\bar{c} + s\lambda - \eta(\bar{c} + s\lambda)}$, we find $\mathscr{V}^s_x$ satisfying the conclusion of Lemma 11 with $l = c + s\lambda$, and we choose $\mathscr{V}^s_x = 0$ otherwise.

If $x \in B^u_{\bar{p}, r} \cap \mathscr{L}_+(\lambda) \setminus \mathscr{L}_-(\lambda)$ and if $V_x = \mathscr{V}^+_x = \mathscr{V}^-_x = 0$, we find $\bar{V}_x \in B^0_{\bar{p}, 1}$ such that $f'(x) . \bar{V}_x > \alpha$, and we choose $\bar{V}_x = \frac{1}{3}(V_x + \mathscr{V}^+_x + \mathscr{V}^-_x)$ otherwise.

We take $\xi = \min \left\{ \alpha, \frac{1}{3}(\mu(r) + \nu(\bar{c} + \lambda) + \nu(\bar{c} - \lambda)) \right\}$.

$\bar{V}_x$ satisfies:

(I) $(\forall x): \bar{V}_x \in B^0_{\bar{p}, 1}$, and $(x \notin B^u_{\bar{p}, (r_2/2)} \Rightarrow \bar{V}_x = 0)$.

$$\text{(II)} \quad \begin{cases} \forall x \in [B^u_{\bar{p}, r+e} \setminus B^u_{\bar{p}, (r-e)}], \ \forall i \in [\![1, m]\!], \\ \| y_i \| \in [r-e, r+e] \quad \Rightarrow \quad (f_i)'(x) . \bar{V}_x > \dfrac{\mu(r)}{3}. \end{cases}$$

$$\text{(III)}\quad \left\{\begin{array}{l}(\forall x \in \mathrm{B}^u_{\bar{p},\, r+e} \cap (\mathscr{L}_+ (\lambda+\eta)(\bar{c}+\lambda)) \setminus \mathscr{L}_- (\lambda+\eta(\bar{c}-\lambda)))\,: \\ \qquad\qquad f'(x).\bar{\mathrm{V}}_x > \xi.\end{array}\right.$$

$$\text{(IV)}\quad \left\{\begin{array}{l}(\forall x \in \mathrm{B}^u_{\bar{p},\, (r_2/2)})(\forall i \in [\![1, m]\!])\,: \\ (f_i(x) \in \{\bar{c}+\lambda,\ \bar{c}-\lambda\} \Rightarrow (f_i)'(x).\bar{\mathrm{V}}_x > 0).\end{array}\right.$$

But $\bar{\mathrm{V}}_x$ is not continuous. A classical pseudo-gradient construction ends the proof. □

5.2. The contradiction

We suppose (hA), (hR) and ($\mathscr{H}$) are true. r, $e(r)$, $\mu(r)$, λ are the same as in Lemma 12. On λ, we impose one more condition:

$$\lambda \leqq \frac{\mu(r)\, e(r)}{6}.$$

As in Lemma 12, we suppose that

$$\mathscr{C} \cap \mathrm{B}^u_{\bar{p},\, r} \cap (\mathscr{L}_+ \setminus \mathscr{L}_-)(\lambda) = \varnothing,$$

and we take $m \geqq 2$, $\bar{p} \in \mathbb{Z}^m$ with

$$(\forall i) \quad (p^{i+1} - p^i) \geqq \mathscr{B}(r, \lambda).$$

We define $\varphi(t, x)$ for $(t, x) \in \mathbb{R} \times \mathrm{L}^\beta$ by

$$\varphi(0, x) = x$$

$$\frac{\partial \varphi}{\partial t}(t, x) = -\mathrm{V} \circ \varphi(t, x),$$

where $\mathrm{V}(x)$ is the vector field of Lemma 12.

We have

LEMMA 13. – *With the notations and hypotheses above, there is* $\mathscr{T} = \mathscr{T}(r, \lambda, \bar{p})$ *such that*

$$\varphi(\mathscr{T}, .)[\mathrm{B}^u_{\bar{p},\, r-e} \cap \mathscr{L}_+(\lambda)] \subset \mathscr{L}_-(\lambda) \cap \mathscr{L}_+(\lambda).$$

Proof. – Take $x \in \mathrm{B}^u_{\bar{p},\, r-e} \cap \mathscr{L}_+(\lambda)$. Then

$$(\forall t \geqq 0), \quad \varphi(t, x) \in \mathrm{B}^u_{\bar{p},\, (r_2/2)} \cap \mathscr{L}_+(\lambda),$$

by (i) and (iv) of Lemma 12. Moreover, if $\varphi(t, x) \in \mathscr{L}_-(\lambda)$, then for any $t' \geqq t$, $\varphi(t', x) \in \mathscr{L}_-(\lambda)$, by (iv). Now, define

$$\mathrm{S} = \mathrm{S}(\bar{p}) = \sup\{|f(\mathrm{X}) - f(\mathrm{Y})|;\ (\mathrm{X}, \mathrm{Y}) \in (\mathrm{B}^u_{\bar{p},\, r_2})^2\}.$$

Define

$$\mathscr{T} = \frac{2\,\mathrm{S}(\bar{p})}{\xi(\bar{p}, r, \lambda)}.$$

By (iii) of Lemma 12, there is $t_x \in [0, \mathcal{T}]$ such that

$$\varphi(t_x, x) \notin \mathrm{B}^u_{\bar{p}, r} \cap (\mathcal{L}_+(\lambda) \setminus \mathcal{L}_-(\lambda)).$$

By (i), (ii) of Lemma 12, this implies $\varphi(\mathcal{T}, x) \in \mathcal{L}_-(\lambda)$ (we recall that $2\lambda \leqq \mu(r) e(r)/3$).

Lemma 13 is thus proved. □

Now, we impose

$$(\forall i) \quad (p^{i+1} - p^i) \geqq \mathrm{N}(r - e(r), \lambda),$$

with the notations of Lemma 9.

The conclusion of Lemma 13 clearly implies $\mathrm{J}_* = 0$, which contradicts the conclusion of Lemma 9.

Now, for any $h > 0$, we may choose $\lambda < h$ satisfying all the conditions above.

So, by contradiction, we have proved the following result:

THEOREM III. – *Assume that* (hA), (hR) *and* ($\mathcal{H}$) *are true.*

Then there is $u \in \mathcal{C}$, *with* $f(u) = \bar{c} \in [c, c')$, *and such that for any* $r, h > 0$, *for all* $m \geqq 1$ *and* $\bar{p} = (p^1, \ldots, p^m) \in \mathbb{Z}^m$:

$$[(\forall i): (p^{i+1} - p^i) \geqq \mathrm{M}(r, h)] \quad \Rightarrow \quad [\mathcal{C} \cap \mathrm{U}_{\bar{p}, r, h} \neq \varnothing].$$

$\mathrm{M}(r, h)$ *is a constant independent of* m, *and* $\mathrm{U}_{\bar{p}, r, h}$ *is a neighborhood of* $\sum_{i=1}^{m} p^i * u$ *defined as follows:*

$\mathrm{U}_{\bar{p}, r, h} = \mathrm{B}^u_{\bar{p}, r} \cap (\mathcal{L}_+(h) \setminus \mathcal{L}_-(h))$, *with the notations of Lemma* 9.

We now prove Theorem II:

We take a fixed value of h, and we write $\mathrm{M}(r)$ instead of $\mathrm{M}(r, h)$. We may choose $\mathrm{K} > \mathrm{M}(r)$ large enough to get $\| u \chi_{\{|t| \geqq \mathrm{K}/2\}} \| \leqq r$, which implies $\sum_{i=1}^{m} p^i * u \in \mathrm{B}^u_{\bar{p}, r}$ for any $m \geqq 2$, and $\bar{p} \in \mathbb{Z}^m$ such that $(\forall i)(p^{i+1} - p^i) \geqq \mathrm{K}$. So, from Theorem III, there is $u_{\bar{p}} \in \mathcal{C}$ such that

$$(\forall i \in \mathbb{Z}): \quad \left\| \left(u_{\bar{p}} - \sum_{i=1}^{m} p^i * u \right) \chi_{[((p^{i-1} + p^i)/2);\, ((p^i + p^{i+1})/2)]} \right\|_\beta \leqq 2r.$$

So, defining $y_{\bar{p}} = \mathrm{L} u_{\bar{p}}$:

$$\left\| y_{\bar{p}} - \sum_{i=1}^{m} p^i * x \right\|_\infty \leqq 3 \mathrm{C}_3 \sum_{n \geqq 0} 2r \exp[-2\theta' n \mathrm{M}(r)]$$

$$= \frac{6 \mathrm{C}_3 r}{1 - \exp(-2\theta' \mathrm{K})} \leqq \varepsilon,$$

for $\mathrm{K}(\varepsilon)$ large enough. So Theorem II is a direct consequence of Theorem III. □

We are now going to study the limit $(m \to +\infty)$.

VI. THE APPROXIMATE BERNOULLI SHIFT

Our first taks here is to prove Corollary II.1 of Theorem II. We consider a sequence $\bar{p}=(p^i)_{i\in I}$ of integers with $I\subset\mathbb{Z}$ a finite or infinite interval, and $p^{i+1}-p^i\geqq K(\varepsilon)$.

The case $0\leqq \mathrm{Card}(I)<\infty$ is clear. So we just consider the case of an infinite I. We may write $I=\bigcup_{k\geqq 0} I^k$, each I^k being finite. From Theorem II, we get an orbit y^k such that

$$\|y^k-\sum_{i\in I^k} p^i * x\|_\infty\leqq\varepsilon.$$

The y^k's being orbits, $\|y^k\|_\infty+\left\|\frac{d}{dt}y^k\right\|_\infty$ is a bounded sequence. So, after extraction, by Ascoli's theorem. y^k converges to some orbit $y_{\bar{p}}$ in the C^0_{loc} topology, and Corollary II.1 is proved.

Now, we take $s\in\{0, 1\}^{\mathbb{Z}}$ arbitrary (*i.e.* with possibly infinitely many 1's). There are an interval I of integers and a sequence $(q^i)_{i\in I}\subset\mathbb{Z}$, with $(\forall i)\, q^{i+1}>q^i$, and $s_n=\chi_{\{q^i,\, i\in\mathbb{Z}\}}(n)$.

We denote $p^i=K(\varepsilon)\,q^i$, and we define $\mathscr{T}(s)=y_{\bar{p}}$, using Corollary II.1.

We recall that $\{0, 1\}^{\mathbb{Z}}$ may be given the topology associated to the metric $d(s, s')=\frac{1}{3}\sum_{n\in\mathbb{Z}}\frac{|s_n-s'_n|}{2^{|n|}}$.

We define

$$\tilde{\tau}:\ \{0, 1\}^{\mathbb{Z}}\to\mathbb{R}^{2N}$$
$$s\mapsto\mathscr{T}(s)(0).$$

Since

$$\|\mathscr{T}(s)-\sum_n s_n(Kn * x)\|_\infty\leqq\varepsilon,$$

we have $\limsup_{d(s, s')\to 0}|\tilde{\tau}(s')-\tilde{\tau}(s)|\leqq 2\varepsilon$.

Now, we take $\delta>0$. There is $I(\delta)>0$ such that if $d(s, s')\geqq\delta$, then $s^I\neq(s')^I$.

So, taking $K(\varepsilon)$ large enough in Corollary II.1, there is $\rho>0$ independent of s, s', ε, with

$$\|(\sum_n s_n(Kn * x)-\sum_n s'_n(Kn * x))\,\chi_{[-2I, 2I]}\|_\infty\geqq 2\rho.$$

So

$$\|(\mathscr{T}(s)-\mathscr{T}(s'))\,\chi_{[-2I, 2I]}\|_\infty\geqq\rho$$

for $\varepsilon<\frac{\rho}{2}$.

Now, define

$$\mathcal{O}: \quad \mathbb{R}^{2N} \to C^0([-2I, 2I], \mathbb{R}^{2N})$$
$$x \mapsto \mathcal{O}(x)$$

where

$$\frac{d}{dt}\mathcal{O} - JA\,\mathcal{O} = J\nabla R(t, \mathcal{O})$$
$$\mathcal{O}(x)(0) = x.$$

By the classical continuity results on the Cauchy problem, $\mathcal{O}$ is uniformly continuous on any bounded part of $\mathbb{R}^{2N}$. So there is $\rho'(\delta) > 0$, independent of s, s', r, such that

$$\tilde{d}(s, s') \geqq \delta \quad \Rightarrow \quad \|\tilde{\tau}(s) - \tilde{\tau}(s')\| \geqq \rho'.$$

So $\tilde{\tau}$ is injective, and $\tilde{\tau}^{-1}$ is uniformly continuous. The other assertions of Corollary II.2 are easy to check, if we choose $x_0 = x(0)$. Corollary II.2 is thus proved. One would like $\tilde{\tau}$ to give a Bernoulli shift structure, *i.e.* $\tilde{\tau}$ homeomorphism, and $\tilde{\tau} \circ \sigma = \varphi^K \circ \tilde{\tau}$ (*see* [M], [W]). Unfortunately, this is not the case. We only have the estimate $\|\mathcal{T}(s) - \sum_n s_n (n * x)\|_\infty \leqq \varepsilon$. The points s such that $s_n = 0$ except for a finite number of n's correspond to homoclinic orbits passing through $\tilde{\tau}(s)$ at time 0: there are infinitely many of them.

ACKNOWLEDGEMENT

The author thanks I. Ekeland who suggested him to study Bernouilli shifts. He also thanks C. Viterbo and J.-C. Bourguignon for fruitful conversations. A preliminary version of this work was presented in a seminar at Bochum University in May 1990. The author thanks H. Hofer and his colleagues for their kind hospitality, and useful remarks. The final version of this work, including the link with topological entropy, has been written after a stay at E.T.H. Zürich in May 1991, where he was invited by E. Zehnder. The author thanks E. Zehnder and S. Golé for their interest in the subject, and useful suggestions.

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(*Manuscript received June* 3, 1992.)