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Loop percolation on discrete half-plane

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Abstract

We consider the random walk loop soup on the discrete half-plane $\mathbb{Z} \times \mathbb{N}^*$ and study the percolation problem, i.e. the existence of an infinite cluster of loops. We show that the critical value of the intensity is equal to $\frac{1}{2}$. The absence of percolation at intensity $\frac{1}{2}$ was shown in a previous work. We also show that in the supercritical regime, one can keep only the loops up to some large enough upper bound on the diameter and still have percolation.

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1 Introduction

We will consider discrete (rooted) loops on $\mathbb{Z}^2$, that is to say finite paths to the nearest neighbours on $\mathbb{Z}^2$ that return to the origin and visit at least two vertices. The rooted random walk loop measure $\mu_{\mathbb{Z}^2}$ gives to each rooted loop of lengths $2n$ the mass $(2n)^{-1}4^{-2n}$. It was introduced in [5]. In [3] are considered loops parametrised by continuous time rather than discrete time. $\mu_{\mathbb{Z}^2}$ has a continuous analogue, the measure $\mu_{\mathbb{C}}$ on the Brownian loops on $\mathbb{C}$. Let $\mathbb{P}^t_{z,z'}(\cdot)$ be the standard Brownian bridge probability measure from z to z' of length t. $\mu_{\mathbb{C}}$ is a measure on continuous time-parametrised loops on $\mathbb{C}$ defined as

$$\mu_{\mathbb{C}}(\cdot) := \int_{\mathbb{C}} \int_{t>0} \mathbb{P}^t_{z,z}(\cdot) \frac{dt}{2\pi t^2} \frac{d\bar{z} \wedge dz}{2i},$$

where $\frac{d\bar{z}\wedge dz}{2i}$ is the standard volume form on $\mathbb{C}$. The measure $\mu_{\mathbb{C}}$ was introduced in [6].

Given $\alpha > 0$ we will denote by $\mathcal{L}^{\mathbb{Z}^2}_\alpha$ respectively $\mathcal{L}^{\mathbb{C}}_\alpha$ the Poisson ensemble of intensity $\alpha\mu_{\mathbb{Z}^2}$ respectively $\alpha\mu_{\mathbb{C}}$, called random walk respectively Brownian loop soup. In [5] it was shown that one can approximate $\mathcal{L}^{\mathbb{C}}_\alpha$ by a rescaled version of $\mathcal{L}^{\mathbb{Z}^2}_\alpha$. If A is a subset of $\mathbb{Z}^2$ we will denote by $\mathcal{L}^A_\alpha$ the subset of $\mathcal{L}^{\mathbb{Z}^2}_\alpha$ made of loops contained in A. If U is an open subset of $\mathbb{C}$ we will denote by $\mathcal{L}^U_\alpha$ the subset of $\mathcal{L}^{\mathbb{C}}_\alpha$ made of loops contained in U. For $\delta > 0$ we will denote by $\mathcal{L}^{A,\geq\delta}_\alpha$ respectively $\mathcal{L}^{U,\geq\delta}_\alpha$ the subset of random walk loops $\mathcal{L}^A_\alpha$ respectively Brownian loops $\mathcal{L}^U_\alpha$ made of loops of diameter greater or equal to δ. Similarly we will use the notation $\mathcal{L}^{A,\leq\delta}_\alpha$ for the loops of diameter smaller or equal to δ.

We will consider clusters of loops. Two loops γ and γ' in a Poisson ensemble of discrete or continuous loops belong to the same cluster if there is a chain of loops

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$\gamma_0, \gamma_1, \dots, \gamma_n$ in this Poisson ensemble such that $\gamma_0 = \gamma$, $\gamma_n = \gamma'$ and γ_i and γ_{i-1} visit a common point. For all $\alpha > 0$, loops in $\mathcal{L}_\alpha^{\mathbb{Z}^2}$ as well as in $\mathcal{L}_\alpha^{\mathbb{C}}$ form a single cluster. Thus we will consider loops on discrete half-plane $\mathsf{H} = \mathbb{Z} \times \mathbb{N}^*$ and on continuous half-plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$, mainly from the angle of existence of an unbounded cluster.

The percolation problem for Brownian loops was studied in [10]. It was shown that there is a critical intensity $\alpha_*^{\mathbb{H}} \in (0, +\infty)$ such that for $\alpha \in (0, \alpha_*^{\mathbb{H}}]$, $\mathcal{L}_\alpha^{\mathbb{H}}$ has only bounded clusters, and for $\alpha > \alpha_*^{\mathbb{H}}$ the loops in $\mathcal{L}_{\alpha_*^{\mathbb{H}}}^{\mathbb{H}}$ form one single cluster. The critical intensity was identified to be equal to 1. But actually $\alpha_*^{\mathbb{H}} = \frac{1}{2}$. In [10] the outer boundaries of outermost clusters in a sub-critical Brownian loop soup were identified to be a Conformal Loop Ensemble CLE_κ with the following relation between α and κ.

$$\alpha = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

The critical value of κ corresponds to CLE_4. Actually the right relation between α and κ is

$$\alpha = \frac{1}{2} \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

So the value of α that corresponds to $\kappa = 4$ is $\frac{1}{2}$ and not 1. The missing factor $\frac{1}{2}$ appears in the Lawler's work [7] (Proposition 2.1). The error in [10] comes from an error in the article [6] by Lawler and Werner. There the authors consider a Brownian loop soup in the half-plane and a continuous path cutting the half-plane, parametrised by the half-plane capacity. For such a path the half-plane capacity at time t equals $2t$. It discovers progressively new Brownian loops and the authors map these loops conformally to the origin. In the Theorem 1 they identify the processes of these conformally mapped Brownian loops to be a Poisson point process with intensity proportional to the Brownian bubble measure. In the identification of intensity there is a factor 2 missing. Actually in the article [6], the Theorem 1 is inconsistent with the Proposition 11.

The problem of percolation by random walk loops was studied in [4], [2], [9] and [1] in more general setting than dimension 2. We will focus on the percolation by loops in $\mathcal{L}_\alpha^{\mathsf{H}}$. The probability of existence of an infinite cluster of loops follows a $0 - 1$ law and there can be at most one infinite cluster ([9]). Moreover for $\alpha = \frac{1}{2}$ loops in $\mathcal{L}_{\frac{1}{2}}^{\mathsf{H}}$ do not percolate ([9]). This result was obtained through a coupling with the massless Gaussian free field. By considering just the loops that go back and forth between two neighbouring vertices we get a lower bound on clusters of loops by clusters of an i.i.d. Bernoulli percolation. In particular this implies that for α large enough loops in $\mathcal{L}_\alpha^{\mathsf{H}}$ percolate. Hence as the parameter α increases there is a phase transition and a critical value $\alpha_*^{\mathsf{H}} \in [\frac{1}{2}, +\infty)$ of the parameter. Using the results on the clusters of Brownian loops from [10] and the approximation result from [5] we will show in section 2 the following:

Theorem 1.1. *For all $\alpha > \frac{1}{2}$ there is an infinite cluster of loops in $\mathcal{L}_\alpha^{\mathsf{H}}$. In particular $\alpha_*^{\mathsf{H}} = \frac{1}{2}$. Moreover, given $\alpha > \frac{1}{2}$, there is $n \in \mathbb{N}^*$ large enough, such that $\mathcal{L}_\alpha^{\mathsf{H}, \leq n}$ percolates too.*

That is to say the critical intensity parameter for the two-dimensional Brownian loop soups and random walk loop soups is the same.

We will consider 1-dependent edge percolations on H, $(\omega(e))_{e \text{ edge}}$. By 1-dependent percolation we mean that if two disjoint subsets of edges E_1 and E_2 are at graph distance at least 1 then $(\omega(e))_{e \in E_1}$ and $(\omega(e))_{e \in E_2}$ are independent. According the results on locally dependent percolation by Liggett, Schonmann and Stacey in [8], for all 1-dependent edge percolations on H with p the probability of an edge to be open, there is an universal $\tilde{p}(p) \in [0, 1)$ such that the 1-dependent edge percolation contains an i.i.d. Bernoulli percolation with probability $\tilde{p}(p)$ of an edge to be open. Moreover the following constraint holds:

$$\lim_{p\to 1^-} \tilde{p}(p) = 1.$$

2 Critical intensity parameter

Let $\alpha, \delta > 0$. Given U an open subset of $\mathbb{H}$, we will denote by $\mathcal{L}_\alpha^{U,\geq\delta}$ respectively $\mathcal{L}_\alpha^{U\cap\mathsf{H},\geq\delta}$ the subset of $\mathcal{L}_\alpha^{\mathbb{H}}$ respectively $\mathcal{L}_\alpha^{\mathsf{H}}$ made of loops contained in U and with diameter greater or equal to δ. We will use the notations $\mathcal{L}_\alpha^{U}$ and $\mathcal{L}_\alpha^{U\cap\mathsf{H}}$ when there is a condition on the range but not on the diameter.

Let Q_{ext} and Q_{int} be the following rectangles:

$$Q_{ext} := (0,6)\times(0,3), \quad Q_{int} := (1,5)\times(1,2).$$

We consider the subset of Brownian loops $\mathcal{L}_\alpha^{Q_{ext},\geq\delta}$, which is a.s. finite. We introduce the events $C_1(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$, $C_2(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ and $C_3(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ depending on the loops in $\mathcal{L}_\alpha^{Q_{ext},\geq\delta}$. The event $C_1(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ will be satisfied if there is a cluster K_1 of loops in $\mathcal{L}_\alpha^{Q_{int},\geq\delta}$ such that in $\mathcal{L}_\alpha^{(0,6)\times(1,2),\geq\delta}$ there is a loop that intersects K_1 and $\{1\}\times(1,2)$ and a loop that intersects K_1 and $\{5\}\times(1,2)$. The two loops may be the same. $C_2(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ will be satisfied if there is a cluster K_2 in $\mathcal{L}_\alpha^{(1,2)^2,\geq\delta}$ such that in $\mathcal{L}_\alpha^{(1,2)\times(0,3),\geq\delta}$ there is a loop that intersects K_2 and $(1,2)\times\{1\}$ and a loop that intersects K_2 and $(1,2)\times\{2\}$. The event $C_3(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ is similar to the event $C_2(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ where the square $(1,2)^2$ is replaced by the square $(4,5)\times(1,2)$ and the rectangle $(1,2)\times(0,3)$ by the rectangle $(4,5)\times(0,3)$. Next figure illustrates the event $\bigcap_{i=1}^3 C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$.

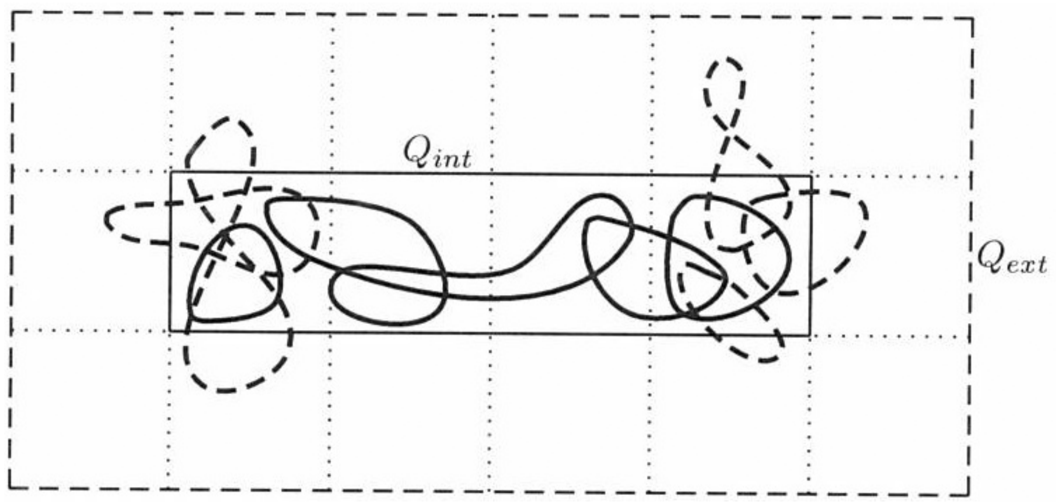

Figure 1: Illustration of the event $\bigcap_{i=1}^3 C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$. One should imagine that the smooth loops are actually Brownian. Only a set of loops that is sufficient for the event is represented. Full line loops stay inside Q_{int}. Dashed loops cross the boundary of Q_{int}.

We will call the event $\bigcap_{i=1}^3 C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ *special crossing event with exterior rectangle Q_{ext} and interior rectangle Q_{int}*. We will also consider translations, rotations and rescaling of Q_{ext} and Q_{int} and deal with *special crossing events* corresponding to the new rectangles. We are interested in the event $\bigcap_{i=1}^3 C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ because then the loops in $\mathcal{L}_\alpha^{Q_{ext},\geq\delta}$ achieve the three crossings drawn on the figure 2:

Next we show that if $\alpha > \frac{1}{2}$ and δ is small enough then the probability of the event $\bigcap_{i=1}^3 C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ is close to 1.

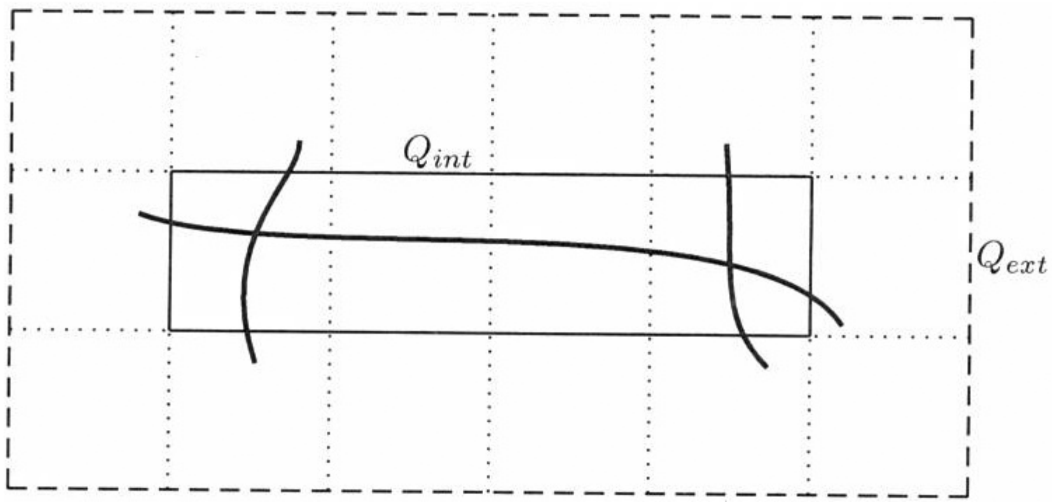

Figure 2: The three crossings we are interested in.

Lemma 2.1. *Let Q be a rectangle of form $Q = (-a, a) \times (0, b)$. Let $\alpha > 0$. Let $(B_t)_{t \geq 0}$ be the standard Brownian motion on $\mathbb{C}$ started from 0 and let $\mathcal{L}_\alpha^Q$ be a Poisson ensemble of loops independent from B. Then for all $\varepsilon > 0$ there is $t \in (0, \varepsilon)$ such that B at time t intersects a loop in $\mathcal{L}_\alpha^Q$.*

Proof. First we consider a loops soup in $\mathbb{H}$, $\mathcal{L}_\alpha^{\mathbb{H}}$, independent of B. Let

$$T := \inf\{t > 0 | B_t \text{ is in the range of a loop in } \mathcal{L}_\alpha^{\mathbb{H}}\}.$$

T is a.s. finite. Indeed a loop in $\mathcal{L}_\alpha^{\mathbb{H}}$ delimits a domain with non-empty interior. Since the Brownian motion on $\mathbb{C}$ is recurrent, B will visit this domain and thus intersect the loop. Let $\lambda > 0$. The Poisson ensemble of loops $\mathcal{L}_\alpha^{\mathbb{H}}$ is invariant in law under the Brownian scaling

$$(\gamma(t))_{0 \leq t \leq t_\gamma} \longmapsto \lambda^{-\frac{1}{2}}(\gamma(\lambda t))_{0 \leq t \leq \lambda^{-1} t_\gamma}.$$

So does the Brownian motion B. Thus λT has the same law as T. It follows that $T = 0$ a.s.

The set of loops $\mathcal{L}_\alpha^{\mathbb{H}} \setminus \mathcal{L}_\alpha^Q$ is at positive distance from 0 thus B cannot intersect it immediately. It follows that B intersects immediately $\mathcal{L}_\alpha^Q$. □

Lemma 2.2. *Let $a, \alpha > 0$. There is a.s. a loop in $\mathcal{L}_\alpha^{(-a,a)^2}$ that intersects the real line $\mathbb{R}$.*

Proof. Let $\mathcal{L}_\alpha^{(n)}$ be the subset of $\mathcal{L}_\alpha^{(-a,a)^2}$ made of loops γ of duration t_γ comprised between 2^{-n-1} and 2^{-n}. The family $(\mathcal{L}_\alpha^{(n)})_{n \geq 0}$ is independent. By Brownian scaling, the probability that a loop in $\mathcal{L}_\alpha^{(n)}$ intersects $\mathbb{R}$ is the same as a loop in $\mathcal{L}_\alpha^{(-a2^{n/2}, a2^{n/2})^2}$ of duration comprised between $\frac{1}{2}$ and 1 intersects $\mathbb{R}$. This is at least as big as the similar probability for $\mathcal{L}_\alpha^{(0)}$. Since the latter probability is non-zero, the intersection events occurs a.s. for infinitely many of $\mathcal{L}_\alpha^{(n)}$. □

Lemma 2.3. *Let $a, \alpha > 0$. There is a.s. a loop in $\mathcal{L}_\alpha^{(-a,a)^2}$ that intersects the real line $\mathbb{R}$ and a loop in $\mathcal{L}_\alpha^{(-a,a)\times(0,a)}$.*

Proof. Consider the subset of $\mathcal{L}_\alpha^{(-a,a)^2}$ made of loops intersecting $\mathbb{R}$. It is non empty according the lemma 2.2. Moreover it is independent of $\mathcal{L}_\alpha^{(-a,a)\times(0,a)}$. The law of a

Brownian loop that intersects $\mathbb{R}$ is locally, near the point of intersection, absolutely continuous with respect to the law of a Brownian motion started from there. Applying lemma 2.1, we get that it intersects a.s. a loop in $\mathcal{L}_\alpha^{(-a,a)\times(0,a)}$. □

Lemma 2.4. *Let* $\alpha > \frac{1}{2}$*. Then*

$$\lim_{\delta\to 0^+} \mathbb{P}\Big(\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})\Big) = 1.$$

Proof. It is enough to show that the probability of each of the $C_i(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ converges to 1 as δ tends to 0. Since the three cases are very similar, we will do the proof only for $C_1(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$. According to lemma 2.3 there is a loop γ in $\mathcal{L}_\alpha^{(0,6)\times(1,2)}$ that intersects $\{1\}\times(1,2)$ and a loop γ' in $\mathcal{L}_\alpha^{Q_{int}}$. Similarly there is a loop $\tilde{\gamma}$ in $\mathcal{L}_\alpha^{(0,6)\times(1,2)}$ that intersects $\{5\}\times(1,2)$ and a loop $\tilde{\gamma}'$ in $\mathcal{L}_\alpha^{Q_{int}}$. Since $\alpha > \frac{1}{2}$, γ' and $\tilde{\gamma}'$ belong to the same cluster in $\mathcal{L}_\alpha^{Q_{int}}$ ([10]). Thus there is a chain of loops $(\gamma_0,\dots,\gamma_n)$ in $\mathcal{L}_\alpha^{Q_{int}}$, with $\gamma_0 = \gamma'$ and $\gamma_n = \tilde{\gamma}'$, joining γ' and $\tilde{\gamma}'$. If δ is the minimum of diameters of $(\gamma_0,\dots,\gamma_n)$ and γ and $\tilde{\gamma}$ then $C_1(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ is satisfied. Let $\bar{\delta}$ be maximal value of δ such that $C_1(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})$ is satisfied. $\bar{\delta}$ is a well defined random variable with values in $(0,+\infty)$. Then

$$\lim_{\delta\to 0^+} \mathbb{P}(C_1(\mathcal{L}_\alpha^{Q_{ext},\geq\delta})) = \lim_{\delta\to 0^+} \mathbb{P}(\delta \leq \bar{\delta}) = 1.$$

□

Next we recall the result on approximation of Brownian loops by random walk loops from [5]. Let $N \in \mathbb{N}^*$. We consider the discrete loops γ on $\mathbb{Z}\times\mathbb{N}^*$. We define on these loops a map Φ_N to continuous loops on $\mathbb{H}$. Given γ a discrete loop and $(z_0,\dots,z_{n-1},z_0)$ the sequence of the vertices it visits, the continuous loop $\Phi_N\gamma$ satisfies:

- the duration of $\Phi_N\gamma$ is $\frac{n}{2N^2}$;
- for $j \in \{0,\dots,n-1\}$, $\Phi_N\gamma(\frac{j}{2N^2}) = \frac{z_j}{N}$;
- $\Phi_N\gamma(\frac{n}{2N^2}) = \Phi_N\gamma(0) = \frac{z_0}{N}$;
- between the times $\frac{j}{2N^2}$, $j \in \{0,\dots,n\}$, $\Phi_N\gamma$ interpolates linearly.

The number of jumps n of a discrete loop γ will be denoted s_γ. The life-time of a continuous loop $\tilde{\gamma}$ will be denoted by $t_{\tilde{\gamma}}$. Let $\theta \in (\frac{2}{3},2)$ and $r \geq 1$. There is a coupling between $\mathcal{L}_\alpha^{\mathsf{H}}$ and $\mathcal{L}_\alpha^{\mathbb{H}}$ such that except on an event of probability at most $cste\cdot(\alpha+1)r^2N^{2-3\theta}$ there is a one to one correspondence between the two sets

- $\{\gamma \in \mathcal{L}_\alpha^{\mathsf{H}} | s_\gamma > 2N^\theta, |\gamma(0)| < Nr\}$,
- $\{\tilde{\gamma} \in \mathcal{L}_\alpha^{\mathbb{H}} | t_{\tilde{\gamma}} > N^{\theta-2}, |\tilde{\gamma}(0)| < r\}$,

such that given a discrete loop γ and the continuous loop $\tilde{\gamma}$ corresponding to it,

$$\Big|\frac{s_\gamma}{2N^2} - t_{\tilde{\gamma}}\Big| \leq \frac{5}{8}N^{-2}, \qquad \sup_{0\leq u\leq 1}\Big|\Phi_N\gamma\Big(u\frac{s_\gamma}{2N^2}\Big) - \tilde{\gamma}(ut_{\tilde{\gamma}})\Big| \leq cste\cdot N^{-1}\log(N).$$

Next we state without proof a lemma that follows immediately from this approximation.

Lemma 2.5. *Let* $\alpha > 0$ *and* $\delta > 0$*. As* N *tends to* $+\infty$ *the random set of interpolating continuous loops*

$$\big\{\Phi_N\gamma | \gamma \in \mathcal{L}_\alpha^{NQ_{ext}\cap\mathsf{H},\geq N\delta}\big\}$$

converges in law to the set of Brownian loops $\mathcal{L}_\alpha^{Q_{ext},\geq\delta}$.

We need to show that the above convergence for the uniform norm also implies a convergence of the intersection relations, that is to say that

$$\{(\gamma,\gamma')|\gamma,\gamma'\in\mathcal{L}_\alpha^{NQ_{ext}\cap\mathsf{H},\geq N\delta},\gamma\text{ intersects }\gamma'\}$$

converges in law to

$$\{(\tilde\gamma,\tilde\gamma')|\tilde\gamma,\tilde\gamma'\in\mathcal{L}_\alpha^{Q_{ext},\geq\delta},\tilde\gamma\text{ intersects }\tilde\gamma'\}.$$

Let $j\in\mathbb{N}$. Let γ be a continuous path on $\mathbb{C}$ (not necessarily a loop) of lifetime t_γ. For $r>0$ let

$$T_r(\gamma):=\inf\{s>0||\gamma(s)|\geq r\}\in(0,+\infty].$$

If $T_r(\gamma)<+\infty$ let

$$e^{i\omega_r}:=\frac{\gamma(T_r(\gamma))}{r}.$$

Let I_j be the real interval

$$I_j:=\Big(\frac{7}{12}2^{-j},\frac{9}{12}2^{-j}\Big).$$

For $0<r_1<r_2$ let $\mathcal{A}(r_1,r_2)$ be the annulus

$$\mathcal{A}(r_1,r_2):=\{z\in\mathbb{C}|r_1<|z|<r_2\}.$$

For $r>0$ let $HD(r)$ be the half-disc

$$HD(r):=B(0,r)\cap\{z\in\mathbb{C}|\Re(z)>0\}.$$

We will say that the path γ satisfies the condition $\mathcal{C}_j$ if

- $T_{\frac{11}{12}2^{-j}}(\gamma)<+\infty$,
- after time $T_{\frac{11}{12}2^{-j}}(\gamma)<+\infty$, γ hits $e^{i(\omega_{2^{-j-1}}+\frac{\pi}{2})}I_j$ at a time $\tilde t_j$ before hitting the circle $S(0,2^{-j})$,
- on the time interval $(T_{2^{-j-1}}(\gamma),\tilde t_j)$ γ stays in the half-disc $e^{i\omega_{2^{-j-1}}}HD(2^{-j})$,
- from time $\tilde t_j$ the path γ stays in the annulus $\mathcal{A}(\frac{7}{12}2^{-j},\frac{9}{12}2^{-j})$ until surrounding the disc $B(0,\frac{7}{12}2^{-j})$ once clockwise and hitting $e^{i(\omega_{2^{-j-1}}+\pi)}I_j$.

Figure 3 illustrates a path satisfying the condition $\mathcal{C}_j$. If this condition is satisfied than γ disconnects the disc $B(0,\frac{7}{12}2^{-j})$ from infinity. Moreover if one perturbs γ by any continuous function $f:[0,t_\gamma]\to\mathbb{C}$ such that $\|f\|_\infty\leq\frac{1}{12}2^{-j}$ then the path $(\gamma(s)+f(s))_{0\leq s\leq t_\gamma}$ disconnects the disc $B(0,2^{-j-1})$ from infinity. The disconnection is made inside the annulus $\mathcal{A}(2^{-j-1},2^{-j})$.

Lemma 2.6. *Let $(B_t)_{0\leq t\leq T}$ be a standard Brownian path on $\mathbb{C}$ starting from 0. Then almost surely it satisfies the condition $\mathcal{C}_j$ for infinitely many values of $j\in\mathbb{N}$.*

Proof. Let $\widetilde B$ be the Brownian path B continued for $t\in(0,+\infty)$. The events "$\widetilde B$ satisfies the condition $\mathcal{C}_j$" are i.i.d. Indeed such an event is rotation invariant and depends only on $\widetilde B$ on the time interval $(T_{2^{-j-1}}(\widetilde B),T_{2^{-j}}(\widetilde B))$. Moreover the probability of such an event is non-zero. Thus $\widetilde B$ satisfies the condition $\mathcal{C}_j$ for infinitely many values of $j\in\mathbb{N}$. Since

$$\lim_{j\to+\infty}T_{2^{-j}}(\widetilde B)=0,$$

so does B. □

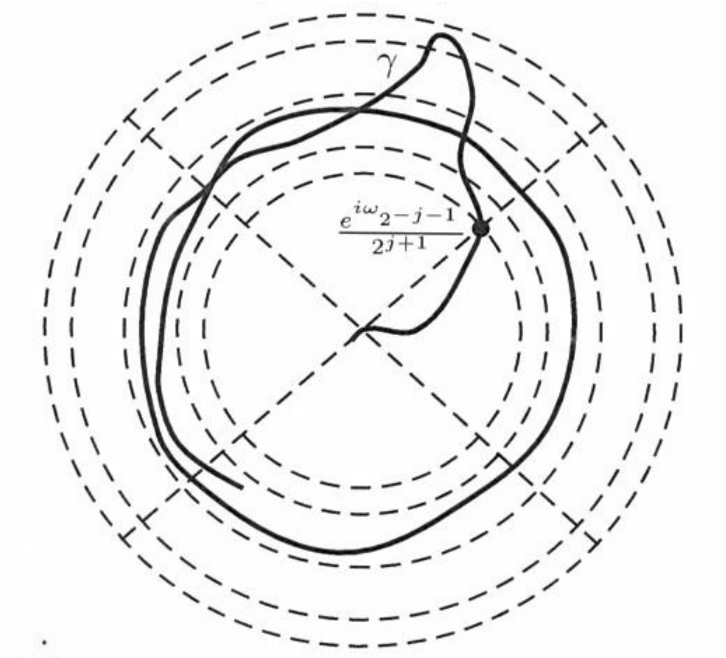

Figure 3: Representation of a path γ satisfying the condition $\mathcal{C}_j$.

Lemma 2.7. *Let $z_1, z_2 \in \mathbb{C}$ and $t_1, t_2 > 0$. Let $(b_s^{(1)})_{0\le s\le t_1}$ and $(b_s^{(2)})_{0\le s\le t_2}$ be two independent standard Brownian bridges from z_1 to z_1 and z_2 to z_2 respectively. On the event that $b^{(1)}$ intersects $b^{(2)}$ there is a.s. $\varepsilon > 0$ such that for all continuous functions $f_1 : [0, t_1] \to \mathbb{C}$ and $f_2 : [0, t_2] \to \mathbb{C}$ of infinity norm $\|f_i\|_\infty \le \varepsilon$, $(b_s^{(1)} + f_1(s))_{0\le s\le t_1}$ intersects $(b_s^{(2)} + f_2(s))_{0\le s\le t_2}$.*

Proof. Let $T_2^{(1)}$ be the first time $b^{(1)}$ hits the range of $b^{(2)}$. If the two path do not intersect each other $T_2^{(1)} = +\infty$. On the event $T_2^{(1)} < +\infty$ the conditional law of $(b^{(1)}_{T_2^{(1)}+s} - b^{(1)}_{T_2^{(1)}})_{0\le s\le t_1 - T_2^{(1)} - \varepsilon}$ ($\varepsilon > 0$ a small constant) given the value $T_2^{(1)}$ is absolutely continuous with respect the law of a Brownian path starting from 0. From lemma 2.6 follows that the path $(b^{(1)}_{T_2^{(1)}+s} - b^{(1)}_{T_2^{(1)}})_{0\le s\le t_1 - T_2^{(1)}}$ satisfies the condition $\mathcal{C}_j$ for infinitely many values of $j \in \mathbb{N}$. Let

$$\tilde{j} := \max\Big\{ j \in \mathbb{N} | (b^{(1)}_{T_2^{(1)}+s} - b^{(1)}_{T_2^{(1)}})_{0\le s\le t_1 - T_2^{(1)}} \text{ satisfies the condition } \mathcal{C}_j \\ \text{and } \exists s \in [0, t_2], |b_s^{(2)} - b^{(2)}_{T_2^{(1)}}| \ge \frac{13}{12} 2^{-j} \Big\}.$$

$\tilde{j}$ is a r.v. defined on the event where $b^{(1)}$ and $b^{(2)}$ intersect. If f_1 and f_2 are such that $\|f_i\| \le \frac{1}{12} 2^{-\tilde{j}}$ then the path $b^{(1)} + f_1$ disconnects the disc $B(b^{(1)}_{T_2^{(1)}}, 2^{-\tilde{j}-1})$ from infinity inside the annulus $b^{(1)}_{T_2^{(1)}} + \mathcal{A}(2^{-\tilde{j}-1}, 2^{-\tilde{j}})$ and the path $b^{(2)} + f_2$ crosses from the circle $S(b^{(1)}_{T_2^{(1)}}, 2^{-\tilde{j}-1})$ to the circle $S(b^{(1)}_{T_2^{(1)}}, 2^{-\tilde{j}})$, so the two must intersect. □

Observe that two discrete loops γ and γ' intersect each other if and only if the continuous loops $\Phi_N \gamma$ and $\Phi_N \gamma'$ do. From lemmas 2.5 and 2.7 follows:

Corollary 2.8. *Let $\alpha > 0$ and $\delta > 0$. As N tends to $+\infty$ the random set of interpolating continuous loops*

$$\left\{\Phi_N \gamma | \gamma \in \mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}, \geq N\delta}\right\}$$

jointly with the intersection relations

$$\{(\gamma, \gamma') | \gamma, \gamma' \in \mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}, \geq N\delta}, \gamma \text{ intersects } \gamma'\}$$

converges in law to the set of Brownian loops $\mathcal{L}_\alpha^{Q_{ext}, \geq \delta}$ jointly with the intersection relations

$$\{(\tilde{\gamma}, \tilde{\gamma}') | \tilde{\gamma}, \tilde{\gamma}' \in \mathcal{L}_\alpha^{Q_{ext}, \geq \delta}, \tilde{\gamma} \text{ intersects } \tilde{\gamma}'\}.$$

We consider the scaled up rectangle NQ_{ext} and NQ_{int}. The next lemma deals with the probability that the discrete loops $\mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}}$ realise the *special crossing event with exterior rectangle NQ_{ext} and interior rectangle NQ_{int}*. See figures 1 and 2 and consider that Q_{ext} is replaced by NQ_{ext}, Q_{int} by NQ_{int} and $\mathcal{L}_\alpha^{Q_{ext}, \geq \delta}$ by $\mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}}$.

Lemma 2.9. *Let $\alpha > \frac{1}{2}$. As N tends to $+\infty$, the probability that the loops $\mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}}$ realise a special crossing event with exterior rectangle NQ_{ext} and interior rectangle NQ_{int} converges to 1.*

Proof. Let $\delta > 0$. The probability that the loops $\mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}}$ realise the *special crossing event with exterior rectangle NQ_{ext} and interior rectangle NQ_{int}* is at least as large as the probability that the loops $\mathcal{L}_\alpha^{NQ_{ext} \cap \mathsf{H}, \geq N\delta}$ realise the *special crossing event* with the same interior and exterior rectangle. From the corollary 2.8 follows that the latter probability converges as $N \to +\infty$ to

$$\mathbb{P}\left(\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext}, \geq \delta})\right).$$

We conclude by applying the lemma 2.4. □

To conclude that for $\alpha > \frac{1}{2}$, $\mathcal{L}_\alpha^{\mathsf{H}}$ has an infinite cluster we will use a block percolation construction that will combine *special crossing events*.

Proof of the Theorem 1.1. From [9] we know already that $\alpha_*^{\mathsf{H}} \leq \frac{1}{2}$. We need to show that for $\alpha > \frac{1}{2}$, $\mathcal{L}_\alpha^{\mathsf{H}}$ has an infinite cluster.

Let $\alpha > \frac{1}{2}$ and $N \geq 1$. We consider a dependent edge percolation $(\omega^N(e))_{e \text{ edge of } \mathsf{H}}$ on the discrete half plane H. If e is an edge of form $\{(j,k),(j+1,k)\}$, $k \geq 1$, then $\omega^N(e) = 1$ (open edge) if $\mathcal{L}_\alpha^{(NQ_{int}+3Nj+i3Nk) \cap \mathsf{H}}$ achieves a *special crossing event with exterior rectangle $NQ_{ext} + 3Nj + i3Nk$ and interior rectangle $NQ_{int} + 3Nj + i3Nk$*. If e is an edge of form $\{(j,k),(j,k+1)\}$, $k \geq 1$, then $\omega^N(e) = 1$ if $\mathcal{L}_\alpha^{(iNQ_{int}+3Nj+i3Nk) \cap \mathsf{H}}$ achieves a *special crossing event with exterior rectangle $iNQ_{ext} + 3Nj + i3Nk$ and interior rectangle $iNQ_{int} + 3Nj + i3Nk$*, where the multiplication by i means rotation by $+\frac{\pi}{2}$. ω^N is a 1-dependent edge percolation: if two disjoint subsets of edges E_1 and E_2 are such that no edge is adjacent to both E_1 and E_2, then $(\omega^N(e))_{e \in E_1}$ and $(\omega^N(e))_{e \in E_2}$ are independent. This is due to the fact that the subsets of loops involved in the definition of *special crossing events* for edges in E_1 and and edges in E_2 are disjoint. To an open path in ω^N corresponds a cluster of $\mathcal{L}_\alpha^{\mathsf{H}}$ whose loops form crossings of related interior rectangles. Thus if ω^N has an unbounded cluster, then so does $\mathcal{L}_\alpha^{\mathsf{H}}$. See next picture.

The probability $\mathbb{P}(\omega^N(e) = 1)$ is uniform and we will denote it p_N. According to the lemma 2.9

$$\lim_{N \to +\infty} p_N = 1.$$

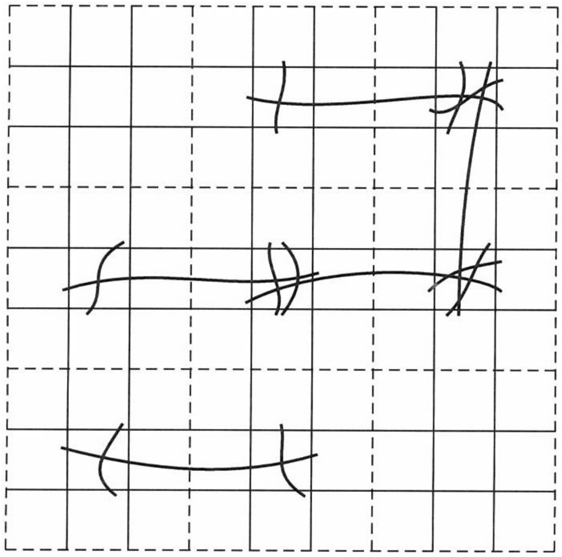

Figure 4: Crossings achieved by subsets of loops in $\mathcal{L}_{\alpha}^{\mathsf{H}}$, corresponding to five open edges in ω^N.

Thus for N large enough $\tilde{p}(p_N) > \frac{1}{2}$. $\frac{1}{2}$ is the critical probability for the i.i.d. Bernoulli edge percolation on H. So for N large enough ω^N contains a supercritical i.i.d. Bernoulli edge percolation and percolates itself. Thus $\mathcal{L}_{\alpha}^{\mathsf{H}}$ percolates too. Actually, since our construction only uses loops of diameter less or equal to $6N$, we have also percolation for $\mathcal{L}_{\alpha}^{\mathsf{H};\leq 6N}$. $\square$

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