Lord-Shulman thermoelasticity with microtemperatures

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Abstract In this paper we consider the Lord-Shulman thermoelastic theory
with porosity and microtemperatures. The new aspect we propose here is to
introduce a relaxation parameter in the microtemperatures. Then we obtain
an existence theorem for the solutions. In the case that a certain symmetry is
satisfied by the constitutive tensors, we prove that the semigroup is dissipative.
In fact, an exponential decay of solutions can be shown for the one-dimensional
case. In the last section, we restrict our attention to the case where we have
an isotropic and homogeneous material without porosity effects and assuming
that two of the constitutive parameters have the same sign. We see that the
semigroup is dissipative.

Keywords Thermoelasticity · Microtemperatures · Lord-Shulman · Semi-
groups · Stability · Existence

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1 Introduction

The Fourier law is the most known constitutive equation to model the heat conduction. However, this law proposes a relevant drawback. That is, when one adjoins it with the usual heat equation

$$c\dot{\theta} = q_{i,i}$$

the thermal waves propagate instantaneously. Here θ is the temperature and q_i is the heat flux vector. This fact contradicts the *causality principle*. For this reason many scientists have been interested in the formulation of alternative constitutive relations in order to overcome this paradox. Perhaps the most known and studied choice is the one proposed by Cattaneo and Maxwell [5], where a relaxation parameter is introduced in the Fourier law to obtain the constitutive equation:

$$\tau\dot{q}_i + q_i = k\theta_{,i},$$

where τ is assumed small, but strictly positive. Such proposition has be extended among other considerations to the thermoelasticity to obtain the Lord-Shulman theory [27]. This theory recovers the proposition of Maxwell and Cattaneo and combines it with the system describing the elastic vibrations of a material. Lord-Shulman thermoelasticity has received much attention over the years and the quantity of contributions to understand this theory is huge. It proposes the study of a system of four *hyperbolic* equations with thermal dissipation. In this case, the heat equation is also hyperbolic in contrast with the one obtained for the Fourier law which is parabolic.

On the other hand an increasing interest has been developed to consider materials with microstructure along the last and current centuries [1, 16–18, 41, 42, 47]. In particular a class of materials is considered in the case that the microstructure [11] is determined by microtemperatures [14]. Because of the wide applicability of this kind of structures they are currently in fashion and many people are interested in the study of elastic materials with microtemperatures [2, 3, 6–8, 20, 21, 31, 32, 38–40, 43]. When one looks at the constitutive equations proposed for the classical version of these materials one finds a system of parabolic equations and therefore one obtains again the drawback pointed out previously. That is, the microthermal waves also propagate instantaneously. Hence, it is also natural to introduce a relaxation parameter in the constitutive equations proposing the microtemperatures to overcome the new paradox.

On the other side, the study of elastic materials with voids was proposed by Cowin and Nunziato [9, 10, 35]. These materials has deserved big attention in the last forty years [4, 12, 13, 24, 25, 28, 29, 33, 34, 36, 37]. There are interesting physical applications of the thermoelasticity with voids, so we can recall the study of solids with small distributed porous, in particular we mention, for instance, rocks, soils, woods, ceramics or even biological materials as bones. In fact, in recent years multiporosity structure has been considered [19, 22, 23, 44–46].

Our paper is addressed in these two directions. We want to consider a thermoelastic theory of *porous materials* with *microtemperatures* based on the Cattaneo-Maxwell law. That is, we propose the same relaxation parameter for the temperature and the microtemperatures. Existence of solutions can be obtained. The stability of the solutions is provided assuming a certain symmetry among the constitutive tensors. Later we study the exponential decay for *some* one-dimensional problems. In the last section, we consider the case when the porosity is not present and the material is isotropic and homogeneous. A stability result is obtained in the case that two of the constitutive parameters have the same sign.

As it is usual, we use the bold notation for vector and/or tensors. A subindex after a colon denotes partial derivative with respect to this component. Also the summation over repeated indices is assumed.

2 Basic equations

We consider a three-dimensional solid determined by a bounded domain B with boundary so smooth to apply the divergence theorem.

In order to clarify the meaning of the microtemperatures, we recall that, in the case of materials with microstructure, it is assumed the existence of microelements which can be considered as a solid with deformations and temperatures. In the case that we assume the existence of microtemperatures, we have that, when $\mathbf{x}$ is the center of mass of a microelement in the reference configuration and $\tilde{\theta}$ is the absolute temperature, we can consider the approximation

$$\tilde{\theta}(\mathbf{x}', t) = \tilde{\theta}(\mathbf{x}, t) + T_j(x'_j - x_j) + O(d^2),$$

where $O(d^2)$ represents a term of order two in a diameter of the microelement. These functions T_j are called the microtemperatures and represent the variation of the temperature inside the microelement.

The basic equations for the thermo-porous-elasticity with microtemperatures in the case of centrosymmetric[1] materials can be obtained from [15] in the case that we do not take into account the microrotation effects by means of the evolution equations:

$$\rho \ddot{u}_i = t_{ij,j} + \rho f_i, \tag{1}$$

$$J\ddot{\phi} = h_{k,k} + g + \rho l, \tag{2}$$

$$\rho T_0 \dot{\eta} = q_{j,j} + \rho s, \tag{3}$$

$$\rho \dot{\epsilon}_i = q_{ji,j} + q_i - Q_i + \rho G_i, \tag{4}$$

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[1] In the general case the materials responses depend on the loading directions. In the case that the response is invariant under inversion we say that the material is centrosymmetic. It implies that the constitutive tensors of odd order vanish.

and the constitutive equations:

$$t_{ij} = A_{ijrs}e_{rs} + D_{ij}\phi - a_{ij}\theta, \tag{5}$$
$$h_i = A_{ij}\phi_{,j} - N_{ij}T_j, \tag{6}$$
$$g = -D_{ij}e_{ij} - \xi\phi + F\theta, \tag{7}$$
$$\rho\eta = a_{ij}e_{ij} + F\phi + a\theta, \tag{8}$$
$$\rho\epsilon_i = -N_{ji}\phi_{,i} - B_{ij}T_j, \tag{9}$$
$$q_i = k_{ij}\theta_{,j} + H_{ij}T_j, \tag{10}$$
$$q_{ij} = -P_{ijrs}T_{r,s}, \tag{11}$$
$$Q_i = (k_{ij} - K_{ij})\theta_{,j} + (H_{ij} - \Lambda_{ij})T_j, \tag{12}$$

where the constitutive tensors satisfy the dissipation inequality:

$$k_{ij}\theta_{,i}\theta_{,j} + (H_{ji} + T_0K_{ij})T_i\theta_{,j} + T_0\Lambda_{ij}T_iT_j + T_0P_{ijrs}T_{i,j}T_{r,s} \geq 0.$$

In this system of equations we have that ρ is the mass density, u_i is the displacement vector, t_{ij} is the stress tensor, h_i is the equilibrated stress tensor, g is the equilibrated body force, η is the entropy, q_i is the heat flux vector, J is the equilibrated inertia, T_0 is the reference temperature at the equilibrium state (assumed uniform), ϵ_i is the first moment of energy vector, Q_i is the microheat flux average, q_{ij} is the first moment heat flux tensor, e_{ij} is the strain tensor which is related to the displacement by the relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

ϕ is the volume fraction, θ is the temperature, T_i are the microtemperatures, f_i, l, s and G_i are supply terms and $A_{ijrs}, D_{ij}, a_{ij}, A_{ij}, N_{ij}, \xi, F, a, \beta_{ij}, k_{ij}, H_{ij}, P_{ijrs}, K_{ij}, B_{ij}$ and Λ_{ij} are tensors of different order.

As usual, $A_{ijrs}, A_{ij}, B_{ij}, D_{ij}, a_{ij}, k_{ij}, \Lambda_{ij}$ and P_{ijrs} are assumed symmetric. That is,

$$A_{ijrs} = A_{rsij}, \ A_{ij} = A_{ji}, \ B_{ij} = B_{ji}, \ D_{ij} = D_{ji}, \ a_{ij} = a_{ji}, \ k_{ij} = k_{ji},$$
$$\Lambda_{ij} = \Lambda_{ji}, \ P_{ijrs} = P_{rsij}.$$

Moreover, in the case that we impose the Onsager Postulate, where it is assumed the existence of a C^2 function which is the potential for the dissipation (see also the book of Eringen [11], section 2.4), we have that $K_{ij} = T_0^{-1}H_{ji}$.

It is worth recalling that in the isotropic and homogeneous case we have the constitutive equations:

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \mu_0\phi\delta_{ij} - \beta_0\theta\delta_{ij}, \tag{13}$$

$$h_i = a\phi_{,i} - \mu_2 T_i, \tag{14}$$

$$g = -\mu_0 e_{ii} - \xi\phi + \beta_1\theta, \tag{15}$$

$$\rho\eta = \beta_0 e_{ii} + \beta_1\phi + a\theta, \tag{16}$$

$$\rho\epsilon_i = -\mu_2\phi_{,i} - bT_i, \tag{17}$$

$$q_i = k\theta_{,i} + k_1 T_i, \tag{18}$$

$$q_{ij} = -k_4 T_{r,r}\delta_{ij} - k_5 T_{j,i} - k_6 T_{i,j}, \tag{19}$$

$$Q_i = (k - k_3)\theta_{,i} + (k_1 - k_2)T_i. \tag{20}$$

Here, λ and μ are the usual Lamé parameters, β_0 determines the coupling between the displacement and the temperature, μ_0 is the coupling between the displacement and the porosity, k is the thermal conductivity and a the thermal capacity. The remaining parameters are usual in the study of the present theory.

In this case the dissipation inequality implies that

$$k \geq 0, \ \ 3k_4 + k_5 + k_6 \geq 0, \ \ k_6 + k_5 \geq 0, \ \ k_6 - k_5 \geq 0, \ \ (k_1 + T_0 k_3)^2 \leq 4T_0 k k_2.$$

Moreover, in the case that the Onsager Postulate is assumed we have that $k_3 = T_0^{-1}k_1$.

In this paper we consider the natural counterpart for the microtemperatures of the Lord-Shulman theory. In this situation we need to change the constitutive equations for the tensors q_i, q_{ij}, and Q_i in the following form[2]

$$\tau\dot{q}_i + q_i = k_{ij}\theta_{,j} + H_{ij}T_j, \tag{21}$$

$$\tau\dot{q}_{ij} + q_{ij} = -P_{ijrs}T_{r,s}, \tag{22}$$

$$\tau\dot{Q}_i + Q_i = (k_{ij} - K_{ij})\theta_{,j} + (H_{ij} - \Lambda_{ij})T_j, \tag{23}$$

where $\tau > 0$ is the relaxation parameter. In the isotropic and homogeneous case our equations become

$$\tau\dot{q}_i + q_i = k\theta_{,i} + k_1 T_i, \tag{24}$$

$$\tau\dot{q}_{ij} + q_{ij} = -k_4 T_{r,r}\delta_{ij} - k_5 T_{j,i} - k_6 T_{i,j}, \tag{25}$$

$$\tau\dot{Q}_i + Q_i = (k - k_3)\theta_{,i} + (k_1 - k_2)T_i. \tag{26}$$

Therefore, the system of equations that we will study are as follows,

$$\rho\ddot{u}_i = (A_{ijrs}e_{rs} + D_{ij}\phi - a_{ij}\theta)_{,j} + \rho f_i,$$

$$J\ddot{\phi} = (A_{ij}\phi_{,j} - N_{ij}T_j)_{,i} - D_{ij}e_{ij} - \xi\phi + F\theta + \rho l,$$

$$\tau a\ddot{\theta} + a\dot{\theta} = -\tau a_{ij}\ddot{u}_{i,j} - a_{ij}\dot{u}_{i,j} - \tau F\ddot{\phi} - F\dot{\phi} + (k_{ij}\theta_{,j} + H_{ij}T_j)_{,i} + \rho(\tau\dot{s} + s),$$

$$\tau B_{ij}\ddot{T}_j + B_{ij}\dot{T}_j = -\tau N_{ji}\ddot{\phi}_{,j} - N_{ji}\dot{\phi}_{,j} - K_{ij}\theta_{,j} - \Lambda_{ij}T_j + (P_{ijrs}T_{r,s})_{,j} - \rho(\tau\dot{G}_i + G_i),$$

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[2] It is clear that we could consider a general case when the relaxation parameter for the microstructure is different from the one for the macrostructure. But in this paper we want to restrict our attention to the easier case where both parameters agree.

where, to simplify the calculations, we have assumed that $T_0 = 1$. It is worth noting that the first four equations in this system came from a direct substitution of the first two evolution equations and the first three constitutive equations. To obtain the last four equations is a little more complicated. What we have to do is to consider the relations:

$$\rho T_0(\dot{\eta} + \tau \ddot{\eta}) = q_{j,j} + \tau \dot{q}_{j,j} + \rho(s + \tau \dot{s}), \tag{27}$$

$$\rho(\dot{\epsilon}_i + \tau \ddot{\epsilon}_i) = (q_{ji,j} + \tau \dot{q}_{ji,j}) + q_i - Q_i + \tau(\dot{q}_i - \dot{Q}_i) + \rho(G_i + \tau \dot{G}_i), \tag{28}$$

and make use of the constitutive equations for the entropy, the first heat flux moment tensor and the new constitutive equations for the heat flux vector, microheat flux average and the first heat flux moment tensor newly proposed to be compatible with the Cattaneo-Maxwell theory.

For isotropic and homogeneous materials the system becomes

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \mu_0 \phi_{,i} - \beta_0 \theta_{,i} + \rho f_i,$$

$$J\ddot{\phi} = a_0 \phi_{,jj} - \mu_2 T_{,i} - \mu_0 u_{i,i} - \xi\phi + \beta_1 \theta + \rho l,$$

$$\tau a \ddot{\theta} + a\dot{\theta} = -\tau\beta_0 \ddot{u}_{i,i} - \beta_0 \dot{u}_{i,i} - \tau\beta_1 \ddot{\phi} - \beta_1 \dot{\phi} + k\theta_{,jj} + k_1 T_{i,i} + \rho(\tau\dot{s} + s),$$

$$\tau b \ddot{T}_i + b\dot{T}_i = -\tau\mu_2 \ddot{\phi}_{,i} - \mu_2 \dot{\phi}_{,i} + k_6 T_{i,jj} + (k_4 + k_5)T_{j,ji} - k_2 T_i - k_3 \theta_{,i} - \rho(\tau\dot{G}_i + G_i).$$

If we consider the notation $\hat{f} = f + \tau\dot{f}$ our system of equations takes the form

$$\rho \ddot{\hat{u}}_i = (A_{ijrs}\hat{e}_{rs} + D_{ij}\hat{\phi} - \tau a_{ij}\dot{\theta} - a_{ij}\theta)_{,j} + \rho\hat{f}_i,$$

$$J\ddot{\hat{\phi}} = (A_{ij}\hat{\phi}_{,j} - \tau N_{ij}\dot{T}_j - N_{ij}T_j)_{,i} - D_{ij}\hat{e}_{ij} - \xi\hat{\phi} + \tau F\dot{\theta} + F\theta + \rho\hat{l},$$

$$\tau a\ddot{\theta} + a\dot{\theta} = -a_{ij}\dot{\hat{u}}_{i,j} - F\dot{\hat{\phi}} + (k_{ij}\theta_{,j} + H_{ij}T_j)_{,i} + \rho\hat{s},$$

$$\tau B_{ij}\ddot{T}_j + B_{ij}\dot{T}_j = -N_{ji}\dot{\hat{\phi}}_{,j} - K_{ij}\theta_{,j} - \Lambda_{ij}T_j + (P_{ijrs}T_{r,s})_{,j} - \rho\hat{G}_i.$$

It is clear that from the solutions to this system we obtain the solutions to the primitive system. Therefore, for the sake of simplicity in the notation, we drop the hat. For instance, in the isotropic and homogeneous case we obtain:

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \mu_0 \phi_{,i} - \beta_0(\tau\dot{\theta}_{,i} + \theta_{,i}) + \rho f_i,$$

$$J\ddot{\phi} = a_0 \phi_{,jj} - \mu_2(\tau\dot{T}_{i,i} + T_{,i}) - \mu_0 u_{i,i} - \xi\phi + \beta_1(\tau\dot{\theta} + \theta) + \rho l,$$

$$\tau a\ddot{\theta} + a\dot{\theta} = -\beta_0 \dot{u}_{i,i} - \beta_1 \dot{\phi} + k\theta_{,jj} + k_1 T_{i,i} + \rho s,$$

$$\tau b\ddot{T}_i + b\dot{T}_i = -\mu_2 \dot{\phi}_{,i} + k_6 T_{i,jj} + (k_4 + k_5)T_{j,ji} - k_2 T_i - k_3\theta_{,i} - \rho G_i.$$

To study a well posed problem we should impose initial and boundary conditions. That is, we assume that

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \quad \phi(\mathbf{x}, 0) = \phi^0(\mathbf{x}), \quad \dot{\phi}(\mathbf{x}, 0) = \varphi^0(\mathbf{x}),$$

$$\theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad \dot{\theta}(\mathbf{x}, 0) = \vartheta^0(\mathbf{x}), \quad T_i(\mathbf{x}, 0) = T_i^0(\mathbf{x}), \quad \dot{T}_i(\mathbf{x}, 0) = S_i^0(\mathbf{x}).$$

In case we assume homogeneous Dirichlet boundary conditions we impose

$$u_i(\mathbf{x}, t) = \phi(\mathbf{x}, t) = \theta(\mathbf{x}, t) = T_i(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial B.$$

3 Well posedness

The aim of this section is to obtain an existence and uniqueness result for the solutions to the general problem with the initial and boundary conditions proposed previously.

We first impose the conditions to satisfy our constitutive tensors. First we assume that all the constitutive tensors are bounded above.

(i) There exist three positive constants ρ_0, J_0, a_0 and such that

$$\rho(\mathbf{x}) \geq \rho_0, J(\mathbf{x}) \geq J_0, a(\mathbf{x}) \geq a_0.$$

(ii) There exists a positive constant B_0 such that

$$B_{ij}\xi_i\xi_j \geq B_0\xi_i\xi_i,$$

for every vector ξ_i.

(iii) There exists a positive constant C^* such that

$$k_{ij}\xi_i\xi_j + (H_{ji} + K_{ij})\eta_i\xi_j + \Lambda_{ij}\eta_i\eta_j + P_{ijrs}\xi_{ij}\xi_{ij} \geq C^*(\xi_i\xi_i + \eta_i\eta_i + \xi_{ij}\xi_{ij}).$$

for every vector ξ_i, η_i and tensor ξ_{ij}.

(iv) There exists a positive constant A^* such that

$$\int_B (A_{ijkl}u_{i,j}u_{k,l} + 2D_{ij}u_{i,j}\phi + \xi\phi^2)dv \geq A^* \int_B (u_{i,j}u_{i,j} + \phi^2)dv,$$

for every vector (u_i) and functions ϕ vanishing at the boundary of the domain.

(v) There exists a positive constant A_0 such that

$$A_{ij}\xi_i\xi_j \geq A_0\xi_i\xi_i,$$

for every vector ξ_i.

The meaning of conditions (i) and (ii) is clear. Condition (iii) is a consequence of the second law in the case of thermoelastic materials with microtemperatures. In particular, this condition implies that k_{ij}, Λ_{ij} and P_{ijrs} are positive definite. Conditions (iv) and (v) can be understood by means of the elastic stability.

We now transform our initial boundary value problem into a Cauchy problem in a suitable Hilbert space defined as follows,

$$\mathcal{H} = \mathbf{H}_0^1(B) \times \mathbf{L}^2(B) \times H_0^1(B) \times L^2(B) \times H_0^1(B) \times L^2(B) \times \mathbf{H}_0^1(B) \times \mathbf{L}^2(B).$$

If $U = (\mathbf{u}, \mathbf{v}, \phi, \varphi, \theta, \vartheta, \mathbf{T}, \mathbf{S})$ and $U^* = (\mathbf{u}^*, \mathbf{v}^*, \phi^*, \varphi^*, \theta^*, \vartheta^*, \mathbf{T}^*, \mathbf{S}^*)$ we define the inner product

$$\langle U, U^* \rangle = \frac{1}{2} \int_B \left(\rho v_i \bar{v}_i^* + J\varphi\bar{\varphi}^* + A_{ijkl}u_{i,j}\bar{u}_{k,l}^* + D_{ij}(u_{i,j}\bar{\phi}^* + \bar{u}_{i,j}^*\phi) \right.$$
$$\left. + \xi\phi\bar{\phi}^* + A_{ij}\phi_{,i}\bar{\phi}_{,j}^* \right)dv$$
$$+ \frac{1}{2} \int_B \left(a(\theta + \tau\vartheta)\overline{(\theta^* + \tau\vartheta^*)} + B_{ij}(T_i + \tau S_i)\overline{(T_j^* + \tau S_j^*)} \right) dv$$
$$+ \frac{\tau}{2} \int_B \left(k_{ij}\theta_{,i}\bar{\theta}_{,j}^* + \Lambda_{ij}T_i\bar{T}_j^* + P_{ijrs}T_{i,j}\bar{T}_{r,s}^* \right) dv$$

As usual, the bar over a variable represents the conjugated complex. In view of the assumption (iii) we conclude that k_{ij} and Λ_{ij} are positive definite and therefore our product is positive. It is worth noting that under the assumptions proposed previously the norm defined by this inner product is equivalent to the usual one in $\mathcal{H}$.

We define the following operators:

$$A_i(\mathbf{u}) = \rho^{-1}(A_{ijkl}u_{k,l})_{,j}, \quad \mathbf{A} = (A_i), \quad B_i(\phi) = \rho^{-1}(D_{ij}\phi)_{,j}, \quad \mathbf{B} = (B_i),$$
$$C_i(\theta) = -\rho^{-1}(a_{ij}\theta)_{,j}, \quad \mathbf{C} = (C_i), \quad D_i(\vartheta) = -\rho^{-1}(\tau a_{ij}\vartheta)_{,j}, \quad \mathbf{D} = (D_i),$$
$$E(\mathbf{u}) = -J^{-1}(D_{ij}u_{i,j}), \quad F^*(\phi) = J^{-1}((A_{ij}\phi_{,j})_{,i} - \xi\phi), \quad G(\theta) = J^{-1}F\theta, \quad H(\vartheta) = J^{-1}\tau F\vartheta,$$
$$J^*(\mathbf{T}) = -J^{-1}(N_{ij}T_j)_{,i}, \quad K(\mathbf{S}) = -\tau J^{-1}(N_{ij}S_j)_{,i} \quad L(\mathbf{v}) = -(\tau a)^{-1}a_{ij}v_{i,j},$$
$$M(\varphi) = -(\tau a)^{-1}F\varphi, \quad N(\theta) = (\tau a)^{-1}(k_{ij}\theta_{,j})_{,i}, \quad P(\vartheta) = -\tau^{-1}\vartheta,$$
$$Q(\mathbf{T}) = (\tau a)^{-1}(H_{ij}T_j)_{,i}, \quad R_k(\varphi) = -\tau^{-1}C_{ki}N_{ji}\varphi_{,j}, \quad \mathbf{R} = (R_k),$$
$$S_k^*(\theta) = -\tau^{-1}C_{ki}K_{ji}\theta_{,j}, \quad \mathbf{S}^* = (S_k^*), \quad U_k(\mathbf{T}) = \tau^{-1}C_{ki}(\Lambda_{ij}T_j + (P_{ijrs}T_{r,s})_{,j}),$$
$$\mathbf{U} = (U_k), \quad V_i(\mathbf{S}) = -\tau^{-1}S_i, \quad \mathbf{V} = (V_i),$$

where, C_{ij} is the inverse of B_{ij}, and the matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{A} & 0 & \mathbf{B} & 0 & \mathbf{C} & \mathbf{D} & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ E & 0 & F^* & 0 & G & H & J^* & K \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & L & 0 & M & N & P & Q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & \mathbf{R} & \mathbf{S}^* & 0 & \mathbf{U} & \mathbf{V} \end{pmatrix}. \tag{29}$$

Our initial boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U + \mathcal{F}(t), \quad U(0) = U^0, \quad \mathcal{F}(t) = (\mathbf{f}, \frac{\rho}{J}l, \frac{\rho}{\tau a}s, \frac{\rho}{\tau}C_{ki}G_i), \tag{30}$$

where

$$U^0 = (\mathbf{u}^0, \mathbf{v}^0, \phi^0, \varphi^0, \theta^0, \vartheta^0, \mathbf{T}^0, \mathbf{S}^0).$$

The domain of the operator $\mathcal{D}(\mathcal{A})$ is given by

$$\mathbf{H}^2 \cap \mathbf{H}_0^1 \times \mathbf{H}_0^1 \times H^2 \cap H_0^1 \times H_0^1 \times H^2 \cap H_0^1 \times H_0^1 \times \mathbf{H}^2 \cap \mathbf{H}_0^1 \times \mathbf{H}_0^1.$$

It is clear that $\mathcal{D}(\mathcal{A})$ is a dense subspace on the Hilbert space $\mathcal{H}$.

Lemma 1 *There exists a positive constant K^* such that, for every $U \in \mathcal{D}(\mathcal{A})$,*

$$\mathrm{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq K^*\|U\|_{\mathcal{H}}^2.$$

Proof We note that

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = \frac{\tau}{4}\int_B ((H_{ij}T_j)_{,i}\bar{\vartheta} + (H_{ij}\bar{T}_j)_{,i}\vartheta - K_{ij}\theta_{,j}\bar{S}_i - K_{ij}\bar{\theta}_{,j}S_i)dv$$
$$- \frac{1}{2}\int_B (k_{ij}\theta_{,i}\bar{\theta}_{,j} + \frac{1}{2}(H_{ji} + K_{ij})(T_i\bar{\theta}_{,j} + \bar{T}_i\theta_{,j}) + \Lambda_{ij}T_i\bar{T}_j + P_{ijrs}T_{i,j}\bar{T}_{r,s})dv.$$

Using the arithmetic-geometric mean inequality we find that

$$
\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq \epsilon(||\mathbf{T}||^2_{\mathbf{H}^1} + ||\theta||^2_{H^1}) + C(||\mathbf{S}||^2_{\mathbf{L}^2} + ||\vartheta||^2_{L^2})
$$
$$
- \frac{1}{2}\int_B (k_{ij}\theta_{,i}\bar{\theta}_{,j} + \frac{1}{2}(H_{ji} + K_{ij})(T_i\bar{\theta}_{,j} + \bar{T}_i\theta_{,j}) + \Lambda_{ij}T_i\bar{T}_j + P_{ijrs}T_{i,j}\bar{T}_{r,s})dv,
$$

where ϵ is a positive constant as small as we want and C is a calculable positive constant. In view of the assumptions imposed previously the lemma is proved.

Lemma 2 *Zero belongs to the resolvent of $\mathcal{A}$.*

Proof Let $(\mathbf{F}_1, \mathbf{F}_2, F_3, F_4, F_5, F_6, .\mathbf{F}_7, \mathbf{F}_8) \in \mathcal{H}$. We need to solve the system

$$
\mathbf{v} = \mathbf{F}_1,
$$
$$
\mathbf{Au} + \mathbf{B}\phi + \mathbf{C}\theta + \mathbf{D}\vartheta = \mathbf{F}_2,
$$
$$
\varphi = F_3,
$$
$$
E\mathbf{u} + F^*\phi + G\theta + H\vartheta + J^*\mathbf{T} + K\mathbf{S} = F_4,
$$
$$
\vartheta = F_5,
$$
$$
L\mathbf{v} + M\varphi + N\theta + P\vartheta + Q\mathbf{T} = F_6,
$$
$$
\mathbf{S} = \mathbf{F}_7,
$$
$$
\mathbf{R}\varphi + \mathbf{S}^*\theta + \mathbf{UT} + \mathbf{VS} = \mathbf{F}_8.
$$

After substitution we obtain

$$
\mathbf{Au} + \mathbf{B}\phi + \mathbf{C}\theta = \mathbf{F}_2 - \mathbf{D}F_5,
$$
$$
E\mathbf{u} + F^*\phi + G\theta + J^*\mathbf{T} = F_4 - HF_5 - K\mathbf{F}_7,
$$
$$
N\theta + Q\mathbf{T} = F_6 - L\mathbf{F}_1 - MF_3 - PF_5,
$$
$$
\mathbf{S}^*\theta + \mathbf{UT} = \mathbf{F}_8 - \mathbf{R}F_3 - \mathbf{VF}_7.
$$

In view of the condition (iii) we can find the solutions to the last two equations $(\theta, \mathbf{T}) \in H^2 \cap H^1_0 \times \mathbf{H}^2 \cap \mathbf{H}^1_0$. Therefore, we can substitute them into the two firsts equations to obtain the solution $(\mathbf{u}, \phi) \in \mathbf{H}^2 \cap \mathbf{H}^1_0 \times H^2 \cap H^1_0$. Moreover, we have that the norm of the solution can be controlled by the norm of the vector $(\mathbf{F}_1, \mathbf{F}_2, F_3, F_4, F_5, F_6, \mathbf{F}_7, \mathbf{F}_8)$. We can conclude that the lemma is proved.

As the operator is closed the use of these two lemmata and the Lummer-Phillips corollary to the Hille-Yosida theorem lead to the next theorem.

Theorem 1 *The operator $\mathcal{A}$ generates a quasi-contractive semigroup in $\mathcal{H}$.*

Thus, we conclude the following existence and uniqueness result.

Theorem 2 *Assume that $U(0) \in \mathcal{D}(\mathcal{A})$ and $\mathcal{F} \in C^1([0,\infty); \mathcal{H}) \cap C^0([0,\infty); \mathcal{D}(\mathcal{A}))$. Then, there exists a unique solution $U \in C^1([0,\infty); \mathcal{H}) \cap C^0([0,\infty); \mathcal{D}(\mathcal{A}))$ to problem (30).*

Remark 1 Since $\mathcal{A}$ is the generator of a quasi-contractive semigroup, the following estimate for the solution

$$||U(t)|| \leq K_1 \exp(K_2 t) \left(||U(0)|| + \int_0^t ||\mathcal{F}(s)|| \right) ds.$$

is satisfied where K_1 and K_2 are calculable positive constants. As we have seen the existence, uniqueness and continuous dependence of the solutions on the initial data and the supply terms, we can conclude that our problem is well posed in the sense of Hadamard.

Remark 2 The case when $K_{ij} = H_{ji}$, which is satisfied assuming the Onsager Postulate, deserves a particular attention. In this case, we can define the inner product

$$\langle U, U^* \rangle = \frac{1}{2} \int_B \left(\rho v_i \bar{v}_i^* + J\varphi \bar{\varphi}^* + A_{ijkl} u_{i,j} \bar{u}_{k,l}^* + D_{ij}(u_{i,j}\bar{\phi}^* + \bar{u}_{i,j}^* \phi) \right.$$
$$\left. + \xi\phi\bar{\phi}^* + A_{ij}\phi_{,i}\bar{\phi}_{,j}^* \right) dv$$
$$+ \frac{1}{2} \int_B \left(a(\theta + \tau\vartheta)\overline{(\theta^* + \tau\vartheta^*)} + B_{ij}(T_i + \tau S_i)\overline{(T_j^* + \tau S_j^*)} \right) dv$$
$$+ \frac{\tau}{2} \int_B \left(k_{ij}\theta_{,i}\bar{\theta}_{,j}^* + K_{ij}(\theta_{,i}\bar{T}_j^* + \bar{\theta}_{,i}^* T_j) + \Lambda_{ij}T_i\bar{T}_j^* + P_{ijrs}T_{i,j}\bar{T}_{r,s}^* \right) dv.$$

In this situation we have that

$$\text{Re } \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\frac{1}{2} \int_B (k_{ij}\theta_{,i}\bar{\theta}_{,j} + K_{ij}(T_i\bar{\theta}_{,j} + \bar{T}_i\theta_{,j}) + \Lambda_{ij}T_i\bar{T}_j + P_{ijrs}T_{i,j}\bar{T}_{r,s})dv,$$

and therefore,

$$\text{Re } \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq 0.$$

Thus, in this case our semigroup is dissipative and we can conclude the stability of the solutions.

4 One-dimensional isotropic and homogeneous case

In this section we restrict our attention to the one-dimensional isotropic and homogeneous case when $k_1 = k_3$ in the absence of supply terms. We note that this equality holds at least in the case that we assume Onsager's Postulate.

Our system can be written as

$$\rho\ddot{u} = \mu^* u_{xx} + \mu_0\phi_x - \beta_0(\tau\dot{\theta}_x + \theta_x),$$
$$J\ddot{\phi} = a_0\phi_{xx} - \mu_2(\tau\dot{T}_x + T_x) - \mu_0 u_x - \xi\phi + \beta_1(\tau\dot{\theta} + \theta),$$
$$\tau a\ddot{\theta} + a\dot{\theta} = -\beta_0\dot{u}_x - \beta_1\dot{\phi} + k\theta_{xx} + k_1 T_x,$$
$$\tau b\ddot{T} + b\dot{T} = -\mu_2\dot{\phi}_x + k_6^* T_{xx} - k_2 T - k_1\theta_x.$$

Here, $\mu^* = \lambda + 2\mu$ and $k_6^* = k_4 + k_5 + k_6$.

To make the analysis easier in this section we propose a small change in the boundary conditions; however, we point out that a similar result could be obtained in the case of Dirichlet boundary conditions for each variable. We assume that

$$u(0,t) = u(l,t) = \phi_x(0,t) = \phi_x(l,t) = \theta_x(0,t) = \theta_x(l,t) = T(0,t) = T(l,t) = 0,$$

where l is a strict positive constant. We assume the same initial conditions proposed before, but restricted to the one-dimensional case. In this section, we also assume that

$$\begin{aligned}
&\rho > 0, \quad J > 0, \quad a > 0, \quad b > 0, \quad \tau > 0, \quad \mu^* > 0, \quad \mu^*\xi > \mu_0^2, \\
&a_0 > 0, \quad k_6^* > 0, \quad k > 0, \quad kk_2 > k_1^2.
\end{aligned}$$

We also assume that β_0 and μ_2 are different from zero. Our conditions are the natural counterpart of the conditions proposed in the previous section in the case of isotropic and homogeneous materials. We also note that the condition $kk_2 > k_1^2$ is the natural consequence of the condition $(k_1 + k_3)^2 < 4kk_2$ in the case that $k_1 = k_3$.

We want to study the time decay of the solutions of the problem proposed. To avoid undamped solutions we need to assume that the initial conditions satisfy

$$\int_0^l \phi^0 dx = \int_0^l \varphi^0 dx = \int_0^l \theta^0 dx = \int_0^l \vartheta^0 dx = 0.$$

Our problem can be studied in the Hilbert space

$$\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2 \times H_0^1 \times L_*^2,$$

where

$$L_*^2 = \{f \in L^2, \int_0^l f dx = 0\} \text{ and } H_*^1 = H^1 \cap L^2.$$

So it can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0 = (u^0, v^0, \phi^0, \varphi^0, \theta^0, \vartheta^0, T^0, S^0), \tag{31}$$

where U^0 is given as in the previous section but in dimension one and $\mathcal{A}$ is the following matrix

$$
\begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\mu^* D^2}{\rho} & 0 & \frac{\mu_0 D}{\rho} & 0 & -\frac{\beta_0 D}{\rho} & -\frac{\tau\beta_0 D}{\rho} & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
-\frac{\mu_0 D}{J} & 0 & \frac{a_0 D^2 - \xi}{J} & 0 & \frac{\beta_1}{J} & \frac{\tau\beta_1}{J} & -\frac{\mu_2 D}{J} & -\frac{\tau\mu_2 D}{J} \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & -\frac{\beta_0 D}{\tau a} & 0 & -\frac{\beta_1}{\tau a} & \frac{kD^2}{\tau a} & -\tau^{-1} & \frac{k_1 D}{\tau a} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & -\frac{\mu_2 D}{\tau b} & -\frac{k_1 D}{\tau b} & 0 & \frac{k_6^* D^2 - k_2}{\tau b} & -\tau^{-1}
\end{pmatrix}. \tag{32}
$$

and $D = d/dx$.

We can define the inner product

$$
\begin{aligned}
\langle U, U^* \rangle = \frac{1}{2} \int_0^l & \left(\rho v \bar{v}^* + J \varphi \bar{\varphi}^* + \mu^* u_x \bar{u}_x^* + \mu_0 (u_x \bar{\phi}^* + \bar{u}_x^* \phi) + \xi \phi \bar{\phi}^* + a_0 \phi_x \bar{\phi}_x^* \right) dx \\
& + \frac{1}{2} \int_0^l \left(a(\theta + \tau \vartheta) \overline{(\theta^* + \tau \vartheta^*)} + b(T + \tau S) \overline{(T^* + \tau S^*)} + \tau k \theta_x \bar{\theta}_x^* + \tau k_1 (\theta_x \bar{T}^* + \bar{\theta}_x T) \right) dx, \\
& + \frac{1}{2} \int_0^l \left(\tau k_2 T \bar{T}^* + \tau k_6^* T_x \bar{T}_x^* \right) dx
\end{aligned}
$$

where $U = (u, v, \phi, \varphi, \theta, \vartheta, T, S)$ and $U^* = (u^*, v^*, \phi^*, \varphi^*, \theta^*, \vartheta^*, T^*, S^*)$. Clearly, it is equivalent to the usual one in the Hilbert space.

It is worth recalling that in this case the domain of the operator is

$$
H_0^1 \cap H^2 \times H_0^1 \times H_*^1 \cap H^2 \times H_*^1 \times H_*^1 \cap H^2 \times H_*^1 \times H_0^1 \cap H^2 \times H_0^1.
$$

Hence, we find that the domain of the operator is dense.

Lemma 3 *The operator $\mathcal{A}$ satisfies that for every $U \in \mathcal{D}(\mathcal{A})$,*

$$
\mathrm{Re} \, \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq 0.
$$

Proof In this case we have that

$$
\mathrm{Re} \, \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\frac{1}{2} \int_B (k|\theta_x|^2 + k_1(T \bar{\theta}_x + \bar{T} \theta_x) + k_2 T^2 + k_6^* |T_x|^2) dx,
$$

and therefore the lemma is proved in view of the assumptions.

At the same time, we point out that similar arguments to those used in the previous section allow us to conclude that zero belongs to the resolvent of the operator $\mathcal{A}$.

In view of these facts and recalling that the domain of the operator is dense, we can use the Lumer-Phillips corollary to the Hille-Yosida theorem to obtain the following result.

Theorem 3 *The operator given by matrix $\mathcal{A}$ generates a contractive C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t \geq 0}$ in $\mathcal{H}$.*

Now, we will show the exponential decay of the solutions for our problem.

To prove the exponential decay, we recall the characterization stated in the book of Liu and Zheng [26].

Theorem 4 *Let $S(t) = \{e^{\mathcal{A}t}\}_{t \geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R} \subset \rho(\mathcal{A})$,
(ii) $\overline{\lim\limits_{|\lambda| \to \infty}} \|(i\lambda \mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

First, we have the following.

Lemma 4 *The operator $\mathcal{A}$ defined in (32) satisfies $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof We here follow the arguments given in the book of Liu and Zheng ([26], page 25). In the case that the intersection of the imaginary axis and the spectrum is non-empty, there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi \neq 0$, $|\lambda_n| < |\varpi|$ and a sequence of unit norm vectors $U_n = (u_n, v_n, \phi_n, \varphi_n, \theta_n, \vartheta_n, T_n, S_n)$ in the domain of the operator $\mathcal{A}$, such that

$$\|(i\lambda_n \mathcal{I} - \mathcal{A})U_n\| \to 0. \tag{33}$$

In our case, writing this condition term by term we get

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{34}$$

$$i\rho\lambda_n v_n - \mu^* D^2 u_n - \mu_0 D\phi_n + \beta_0 D\theta_n + \tau\beta_0 D\vartheta_n \to 0 \text{ in } L^2, \tag{35}$$

$$i\lambda_n \phi_n - \varphi_n \to 0 \text{ in } H^1, \tag{36}$$

$$iJ\lambda_n \varphi_n + \mu_0 Du_n - (a_0 D^2 - \xi)\phi_n - \beta_1 \theta_n - \tau\beta_1 \vartheta_n$$
$$+ \mu_2 DT_n + \tau\mu_2 DS_n \to 0 \text{ in } L^2, \tag{37}$$

$$i\lambda_n \theta_n - \vartheta_n \to 0 \text{ in } H^1, \tag{38}$$

$$i\tau a\lambda_n \vartheta_n + \beta_0 Dv_n + \beta_1 \varphi_n - kD^2 \theta_n + a\vartheta_n - k_1 DT_n \to 0 \text{ in } L^2, \tag{39}$$

$$i\lambda_n T_n - S_n \to 0 \text{ in } H^1, \tag{40}$$

$$i\tau b\lambda_n S_n + \mu_2 D\varphi_n - k_6^* D^2 T_n + k_2 T_n + k_1 D\theta_n + bS_n \to 0 \text{ in } L^2. \tag{41}$$

In view of the dissipative term for the operator, we see that

$$D\theta_n, \ DT_n \to 0 \text{ in } L^2, \text{ and then } \theta_n, \ T_n \to 0 \text{ in } L^2.$$

From (38) and (40) we also see that $D\vartheta_n, \ DS_n \to 0$ in L^2. and $\vartheta_n, \ S_n \to 0$ in L^2. Now, we multiply (41) by $\lambda_n^{-1} D\phi_n$ and in view 36 we obtain that

$$i\mu_2 \|D\phi_n\|^2 + k_6^* \langle DT_n, \frac{D^2\phi_n}{\lambda_n} \rangle \to 0.$$

From (37) we see that $\frac{D^2\phi_n}{\lambda_n}$ is bounded. As DT_n tends to zero we have that $D\phi_n$ tends to zero. In a similar way, we can also obtain that Du_n tends to zero. To be precise from 35 we see that $\frac{D^2 u_n}{\lambda_n}$ is bounded. If we multiply 39 by $\lambda_n^{-1} Du_n$ we see

$$i\beta_0 \|Du_n\|^2 + k\langle D\theta_n, \frac{D^2 u_n}{\lambda_n} \rangle \to 0.$$

Therefore we conclude that $Du_n \to 0$ in L^2. We also conclude that φ_n and v_n converge to zero. This contradicts the fact that the sequence has unit norm and the lemma is proved.

Secondly, we have the following property for operator $\mathcal{A}$.

Lemma 5 *The operator $\mathcal{A}$ satisfies*

$$\overline{\lim_{|\lambda| \to \infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof The proof also follows a contradiction argument; however, the analysis is a little more complex. Assume that the thesis is not true. Then, there exists a sequence of real numbers λ_n such that $|\lambda_n| \to \infty$ and a sequence of unit vectors in $\mathcal{D}(\mathcal{A})$ in such a way that (33) holds. Therefore, conditions (34)–(41) still hold. The same argument as in the previous lemma proves that $D\theta_n$, $DT_n \to 0$ in L^2. Therefore, $\lambda_n^{-1} D\vartheta_n$ and $\lambda_n^{-1} DS_n$ also tend to zero. Proceeding also as in the previous lemma we can conclude that $Du_n, D\phi_n$ tend to zero. After multiplication of (35) by u_n we see that v_n tends to zero in L^2. A similar argument with (37) shows that φ_n converges to zero in L^2. The convergence of S_n and ϑ_n can be obtained by a similar argument. Again, this is not possible because the sequence has unit norm and the lemma is proved.

The two previous lemmata give rise to the following result.

Theorem 5 *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t \geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)U\| \leq M\|U\|e^{-\alpha t}$.*

Proof The proof is a direct consequence of the two previous lemmata and Theorem 4.

We can ask ourselves for the regularity of the solutions to the problem. We point out that we cannot expect analyticity of the solutions in view that the dissipation is determined by linear hyperbolic equations and the coupling terms are weak.

5 Case without porosity

The aim of this final section is to point out that, in the case that we do not assume porosity effects, it is possible to obtain the stability of the solutions for the isotropic and homogeneous case whenever we assume that the signs of k_1 and k_3 agree. We note that in this case we do not have micromechanical deformations and then, the variables and tensors corresponding to ϕ, φ, h_i and g are not present.

We recall that, in this case, the system of equations is

$$\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} - \beta_0(\tau\dot{\theta}_{,i} + \theta_{,i}),$$
$$\tau a\ddot{\theta} + a\dot{\theta} = -\beta_0\dot{u}_{i,i} + k\theta_{,jj} + k_1 T_{i,i},$$
$$\tau b\ddot{T}_i + b\dot{T}_i = k_6 T_{i,jj} + (k_4 + k_5)T_{j,ji} - k_2 T_i - k_3\theta_{,i}.$$

In what follows, we will consider the case where k_1 and k_3 are positive but different from zero. The other case could be done in a similar way. Moreover, we note that in the case that $k_1 = k_3 = 0$ the analysis is even easier. We note that this is a technical requirement that we assume to guarantee the stability of the solutions. Of course, Onsager's Postulate implies this fact, but our aim is trying to weak this assumption, having stability yet. That is, we will see that we can obtain the stability of the solutions (and exponential stability for

the one dimensional case) even in case that k_1 and k_3 are different, but they have the same sign.

Moreover, in this case we assume that

$$k > 0, \ \ 3k_4 + k_5 + k_6 > 0, \ \ k_6 + k_5 > 0, \ \ k_6 - k_5 > 0, \ \ (k_1 + k_3)^2 < 4kk_2.$$

To make the analysis easier we consider again the Dirichlet boundary conditions:

$$u_i(\mathbf{x}, t) = \theta(\mathbf{x}, t) = T_i(\mathbf{x}, t) = 0, \ \ \mathbf{x} \in \partial B,$$

and the initial conditions:

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \ \ \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}),$$
$$\theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \ \ \dot{\theta}(\mathbf{x}, 0) = \vartheta^0(\mathbf{x}), \ \ T_i(\mathbf{x}, 0) = T_i^0(\mathbf{x}), \ \ \dot{T}_i(\mathbf{x}, 0) = S_i^0(\mathbf{x}).$$

We consider the following Hilbert space:

$$\mathcal{H} = \mathbf{H}_0^1(B) \times \mathbf{L}^2(B) \times H_0^1(B) \times L^2(B) \times \mathbf{H}_0^1(B) \times \mathbf{L}^2(B),$$

and we define the corresponding inner product:

$$\langle U, U^* \rangle = \frac{1}{2} \int_B k_3 \left(\rho v_i \bar{v}_i^* + \mu u_{i,j} \bar{u}_{i,j}^* + (\lambda + \mu) u_{i,i} \bar{u}_{j,j}^* \right) dv$$
$$+ \frac{1}{2} \int_B \left(k_3 a(\theta + \tau\vartheta)\overline{(\theta^* + \tau\vartheta^*)} + k_1 b(T_i + \tau S_i)\overline{(T_i^* + \tau S_i^*)} \right) dv$$
$$+ \frac{\tau}{2} \int_B \left(kk_3 \theta_{,i} \bar{\theta}_{,i}^* + k_6 k_1 \bar{T}_{i,j}^* T_{i,j} + k_4 k_1 T_{i,i} \bar{T}_{j,j}^* + k_5 k_1 T_{i,j} \bar{T}_{j,i}^* + k_2 k_1 T_i \bar{T}_i^* \right.$$
$$\left. + k_1 k_3 (\theta_{,i} \bar{T}_i^* + \bar{\theta}^* T_i) \right) dv,$$

where $U = (u_i, v_i, \theta, \vartheta, T_i, S_i)$ and $U^* = (u_i^*, v_i^*, \theta^*, \vartheta^*, T_i^*, S_i^*)$. To guarantee that this is a inner product, we need to impose that $kk_2 > k_1 k_3$, but this condition is guaranteed whenever we assume that $4kk_2 > (k_1 + k_3)^2$. In view of the required assumptions, this product defines a norm which is equivalent to the usual one in the Hilbert space $\mathcal{H}$.

It is worth noting that we have introduced several weights in the definition of the inner product and the norm. We have multiplied the mechanical component of the energy by k_3 as well as the thermal component of the energy; however, we have multiplied the microtermal component of the energy by k_1. The introduction of these weights allow us to work with an inner product equivalent to the usual one, but in such a way that the dissipation cannot be negative.

Our problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \ \ U(0) = U^0, \tag{42}$$

where $U^0 = (\mathbf{u}^0, \mathbf{v}^0, \theta^0, \vartheta^0, \mathbf{T}^0, \mathbf{S}^0)$ is given as in Section 3 and $\mathcal{A}$ is the following matrix:

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \mathbf{A} & 0 & \mathbf{C} & \mathbf{D} & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & L & N & P & Q & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & \mathbf{S}^* & 0 & \mathbf{U} & \mathbf{V} \end{pmatrix}, \tag{43}$$

where

$$A_i(\mathbf{u}) = \rho^{-1}(\mu u_{i,jj} + (\lambda + \mu)u_{j,ji}), \quad \mathbf{A} = (A_i), \quad C_i(\theta) = -\rho^{-1}\beta_0\theta_{,i}, \quad \mathbf{C} = (C_i),$$
$$D_i(\vartheta) = -\rho^{-1}\tau\beta_0\vartheta_{,i}, \quad \mathbf{D} = (D_i), \quad L(\mathbf{v}) = -(\tau a)^{-1}\beta_0 v_{i,i}, \quad N(\theta) = (\tau a)^{-1}k\theta_{,jj},$$
$$P(\vartheta) = -\tau^{-1}\vartheta, \quad Q(\mathbf{T}) = (\tau a)^{-1}k_1 T_{i,i}, \quad S_i^*(\theta) = -\tau^{-1}b^{-1}k_3\theta_{,i}, \quad \mathbf{S}^* = (S_k^*),$$
$$U_i(\mathbf{T}) = \tau^{-1}b^{-1}(-k_2 T_i + (k_4 + k_5)T_{j,ji} + k_6 T_{i,jj}), \quad \mathbf{U} = (U_k), \quad V_i(\mathbf{S}) = -\tau^{-1}S_i, \quad \mathbf{V} = (V_i).$$

The domain of the operator $\mathcal{D}(\mathcal{A})$ is given by

$$\mathbf{H}^2 \cap \mathbf{H}_0^1 \times \mathbf{H}_0^1 \times H^2 \cap H_0^1 \times H_0^1 \times \mathbf{H}^2 \cap \mathbf{H}_0^1 \times \mathbf{H}_0^1.$$

Therefore, it is clear that $\mathcal{D}(\mathcal{A})$ is a dense subspace on the Hilbert space $\mathcal{H}$.

In this situation we have

$$\text{Re } \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\frac{1}{2} \int_B (kk_3\theta_{,i}\bar{\theta}_{,i} + k_1 k_3(T_i\bar{\theta}_{,i} + \bar{T}_i\theta_{,i})$$
$$+ k_2 k_1 T_i\bar{T}_i + k_5 k_1 T_{i,j}\bar{T}_{j,i} + k_6 k_1 \bar{T}_{i,j}T_{i,j} + k_4 k_1 T_{i,i}\bar{T}_{j,j})dv.$$

In view of the assumptions we see that

$$\text{Re } \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq 0.$$

At the same time, we point out that the analysis proposed in Section 3 can be adapted here to prove that zero belongs to the resolvent of the operator. Therefore, in this case our semigroup is dissipative and we can conclude the stability of the solutions.

We point out that the exponential stability for the one-dimensional problem can be also obtained, but we do not want to repeat the arguments proposed previously.

Theorem 6 *Assume that $U(0) \in \mathcal{D}(\mathcal{A})$ Then, there exists a unique solution $U \in C^1([0,\infty); \mathcal{H}) \cap C^0([0,\infty); \mathcal{D}(\mathcal{A}))$ to problem (42).*

Remark 3 Since $\mathcal{A}$ is the generator of a contractive semigroup, the following stability result for the solutions

$$||U(t)|| \leq ||U(0)||$$

is satisfied.

Acknowledgments

Conflict of Interest

The authors declare that they have no conflict of interest.

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