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LORENTZ COVARIANCE AND THE PHYSICAL STATES IN DUAL RESONANCE MODELS

P. Goddard, C. Rebbi and C.B. Thorn
CERN -- Geneva

A B S T R A C T

The properties of physical states in dual resonance models under rotations in the little group of the resonance momentum are studied. It is shown how at a critical number of space-time dimensions this group can be represented just on the space of "transverse" states constructed by Del Giudice, Di Vecchia and Fubini (because these are then essentially the only states which couple). Further, the action of rotations in this group on the "photon" operators "A_n^i" is used to produce, for any number of dimensions, enough operators to create all the physical states both in the conventional model and the model of Neveu and Schwarz.

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1. - INTRODUCTION

During the past few years a lot of progress has been made in the understanding of the structure of the physical states in dual resonance models (DRM) [1)-7)]. Recently it has finally been demonstrated that there are no ghosts either in the conventional model [6),7)] or in the Neveu-Schwarz (NS) [8)] model [7)], provided that the dimension of space-time, d, does not exceed a critical value.

An essential feature of these proofs is that for a critical number of dimensions of space-time ($d = 26$ for the conventional model, $d = 10$ for NS), the set of "transverse states" generated by the photon operators, A_n^i, constructed by Del Giudice, Di Vecchia and Fubini (DDF) [4)] is complete in the sense that any on-mass shell physical state can then be expressed as a sum of one of these transverse states plus a decoupled null state. We regard this fact as extremely important ; the transverse states are the only ones that admit a definite infinite momentum limit. Thus it seems that for this critical dimension the DRM can be reformulated starting from states with a definite infinite momentum limit, or from states quantized on the light cone [*),9)]. It is also probable that the theory is compatible with unitarity only for the critical dimension (since then the dual Pomeron becomes a pole) [6),7),10)].

Now the definition of a transverse state is not Lorentz covariant : given a physical state of momentum π_μ , a Lorentz transformation in the little group of π_μ can mix transverse and longitudinal components.

In fact, if E_{Li} is the generator of the little group which rotates in the i longitudinal plane, we shall find that [Eq. (3.11)]

$$[E_{Li}, A_m^j] = \delta^{ij} \hat{A}_m + m A_m^{ij}$$

where

$$A_m^{ij} = \sum_{n=1}^{\infty} \frac{1}{n} \left(A_{-n}^i A_{n+m}^j - A_{m-n}^j A_n^i \right)$$

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*) We are grateful to P. Di Vecchia for a discussion on this point.

and $\hat{A}_m$ *), defined by this equation, is an operator which commutes with the gauges but generates states independent of the "transverse" states. It is a consequence of the above equation that the subspace of states generated by $\hat{A}_n$, A^i_n is invariant under the little group. All of the algebraic properties of the $\hat{A}_m$, A^i_n follow from this defining equation. It can be shown that $\hat{A}_m$, A^i_n generate all the physical states, so we have the important result that one can obtain by commuting the "photon" operators with generators of the little group of the momentum π_μ enough additional operators to generate all the physical states. When we commute the "photon" operator in the Neveu-Schwarz model with generators of the little group of π_μ, we obtain not only the $\hat{A}_n$ but also B^i_r, which are transverse Fermion operators. Commuting once more we obtain $\hat{B}_r$; the operators A^i_n, B^i_r, $\hat{A}_n$, $\hat{B}_r$ then generate all the physical states in that model.

Of course, since at the critical dimension the transverse states are complete, we must be able to represent the little group on this subspace. As we shall show, this can be done since at that dimension, as Brower has pointed out [6], the algebra of the $\hat{A}_n$, A^i_m is isomorphic to that of the $\mathscr{L}_n$, A^i_m where $\mathscr{L}_n$ is the conformal generator constructed out of the A^i_m. This isomorphism allows one to substitute $\hat{A}_n \to \mathscr{L}_n$ to obtain the representation of the little group on the transverse subspace. At the critical dimension of the Neveu-Schwarz model there is a similar isomorphism between the algebras of $\hat{A}_n$, A^i_n, $\hat{B}_r$, B^i_r and $\mathscr{L}_n$, A^i_n, $\mathscr{G}_r$, B^i_r which again allows the realization of the little group on the "transverse" subspace.

We have organized our paper as follows. In Section 2 we give a new derivation of the "transverse" physical states by an iterative construction which is particularly suited for studying the infinite momentum limit. In this limit we show how to construct the representations of the little group of π_μ valid for the critical dimension. Although the considerations of this section motivated the work described subsequently, especially Eq. (3.3), Sections 3 and 4 are logically self-contained, and the reader already familiar with the DDF construction may wish to skip Section 2 on a first reading. Section 3 is devoted to a study of the covariance of the physical states in the conventional model. In Section 4 we apply the methods of Section 3 to obtain all the physical states of the Neveu-Schwarz model from the "photon" operators as a consequence of Lorentz covariance.

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*) These operators are what Brower has called $A^{(+)}_n$ and were first constructed by him [6].

2. - GENERATORS OF THE LITTLE GROUP OF p_0 FOR THE CRITICAL DIMENSION

Consider a physical state of momentum $p_0 = \sqrt{2}\pi$, $|\psi,\pi>$, in the conventional model. This state therefore satisfies

$$L_n|\psi,\pi\rangle \equiv (\bar{L}_n - \sqrt{2}\,\pi\cdot\alpha_n)|\psi,\pi\rangle = 0. \tag{2.1}$$

where

$$\alpha_n^\mu = i\sqrt{n}\,a_n^\mu,\quad \alpha_{-n}^\mu = \alpha_n^{\mu\dagger} \qquad n = 1,2,\dots,\dots$$

$$\alpha_0^\mu = -p_0$$

so that

$$[\alpha_m^\mu, \alpha_n^\nu] = m g^{\mu\nu}\delta_{m,-n}$$

with

$$-g^{00} = g^{11} = g^{22} = \dots = g^{d-1,d-1} = 1$$

and

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{-m}\cdot\alpha_{n+m}:$$

If $|\psi,\pi>$ represents an on-mass-shell state, then

$$(L_0 - 1)|\psi,\pi\rangle \equiv (\bar{L}_0 + \pi^2 - 1)|\psi,\pi\rangle = 0. \tag{2.2}$$

It is convenient to choose a frame in which π_μ has only one space component, say π_z, all others being zero. For every Lorentz vector u_μ we define $u_\pm = (u_t \pm u_z)/\sqrt{2}$ and denote by u_i the residual space components, then $\pi_\mu \equiv (\pi_+, \pi_-, \vec{0})$.

Notice that for ∞ boosts in the z direction $\pi_- \to +\infty$, $\pi_+ \to 0$, so that the dominant term in the gauge equations is $\pi_-\,\alpha_{+,n}$.

The equations

$$\alpha_{+,n}|\psi_0\rangle = 0 \qquad n = 1,2,\ldots,\ldots \tag{2.3}$$

admit as solutions all the vectors belonging to the space S_∞ spanned by the transverse and + modes alone (notice that $[\alpha_{+,n}, \alpha^+_{+n}] = 0$). S_∞ contains as subspace the Hilbert space $\mathcal{H}_n$ generated by the transverse creation operators $\alpha^+_{i,n}$.

We would like to construct the solutions of Eq. (2.1) in such a way that when $\pi_- \to +\infty$ they reduce to vectors belonging to S_∞. For this purpose we exploit the relation $2\pi_+\pi_- = M^2$, M being the mass of the state, and rewrite Eq. (2.1) as

$$\left(\sqrt{2}\,\pi_-\alpha_{+,n} + L_n + \frac{M^2}{\sqrt{2}\,\pi_-}\alpha_{-,n}\right)|\psi,\pi\rangle = 0. \tag{2.4}$$

We then look for solutions in the form of an expansion in decreasing powers of π_- :

$$|\psi,\pi\rangle = \sum_{j=0}^{\infty} \pi_-^{-j}\,|\psi_j\rangle \tag{2.5}$$

By inserting Eq. (2.5) into Eq. (2.4), we obtain the conditions

$$\alpha_{+,n}|\psi_0\rangle = 0. \tag{2.6}$$

for the leading term, and

$$\sqrt{2}\,\alpha_{+,n}|\psi_j\rangle = -L_n|\psi_{j-1}\rangle - \frac{M^2}{\sqrt{2}}\alpha_{-,n}|\psi_{j-2}\rangle \equiv -|\Psi_{j,n}\rangle \tag{2.7}$$

with $|\psi_{-1}\rangle = 0$, for the next terms. Equation (2.6) clearly admits as solution any vector belonging to S_∞.

By using the algebra satisfied by the α and L operators, it can be shown that Eqs. (2.7) are compatible (see Appendix) ; still they are ambiguous, because we can add to $|\psi_j>$ any vector belonging to S_∞.

We fix the ambiguity and solve the equations at the same time by the following procedure. Let us multiply Eqs. (2.7) by $-(\alpha^+_{-,n}/n\sqrt{2})$ and sum over n. We obtain

$$-\sum_{n\neq 0} \frac{\alpha^+_{-,n}\alpha_{+,n}}{n} |\psi_j\rangle = N_{\alpha_-}|\psi_j\rangle = \sum_n \frac{\alpha^+_{-,n}}{n\sqrt{2}} |\Psi_{j,n}\rangle , \tag{2.8}$$

where N_{α_-} is the operator that counts the number of - modes.

Since the right-hand side of Eq. (2.8) contains terms with at least one - mode, Eqs. (2.8) and (2.7) can be solved by

$$|\psi_j\rangle = N^{-1}_{\alpha_-} \sum_n \frac{\alpha^+_{-,n}}{n\sqrt{2}} |\Psi_{j,n}\rangle \tag{2.9}$$

In an Appendix, we shall pròve the following statements.

a) The series in j defined by Eq. (2.9) always terminates ; it follows that its sum $|\psi,\pi>$ is a physical state which we associate with the leading term $|\psi_0>$.

b) Vectors $|\psi,\pi>$ associated with linearly independent leading term $|\psi_0>$ are themselves linearly independent. By counting arguments, it follows that the vectors $|\psi,\pi>$ span the whole space of physical states of momentum π_μ.

c) If $|\psi_0>$ is an eigenvector of $\bar{L}_0$, its associated vector $|\psi,\pi>$ is also an eigenvector of $\bar{L}_0$, with the same eigenvalue. It follows that the on-mass-shell physical states can be obtained starting from leading term solutions of $(\bar{L}_0 + \pi^2 - 1)|\psi_0> = 0$.

d) If the maximum number of + modes present in the leading term $|\psi_0>$ is ℓ, then the number of α^+_+ operators in the various terms of the expansion of $|\psi,\pi>$ never exceeds ℓ, and the norm of $|\psi,\pi>$ depends only on the first $2\ell+1$ terms of the expansion.

e) The physical vectors $|\psi_{tr}, \pi>$ associated with leading terms $|\psi_{o,tr}>$ belonging to $\mathcal{H}_{tr}$ satisfy the equations

$$\alpha_{-,n} |\psi_{tr}, \pi\rangle = 0. \tag{2.10}$$

they are therefore the transverse states introduced by DDF [4)], and characterized in Ref. 7) by the defining equation $k_+ \alpha_{-,n} |\psi_{tr}, \pi> = 0$. We observe that because of d) the norm of $|\psi_{tr}, \pi>$ equals the norm of its leading term.

We are ready now to study Lorentz covariance. Notice that the notion of a transverse state is not a covariant one. The transformations $\mathcal{M}$ of the little group of π may transform a transverse state into a non-transverse one, i.e., into a state having a leading term $|\psi_o> \notin \mathcal{H}_{tr}$. However, for a critical number of space-time dimensions on the mass-shell the transverse states form a complete set of physical states, in the sense that any physical state can be rewritten as a transverse state plus a decoupled null state. Then, whatever be $\mathcal{M}$, it is possible to perform the following decomposition

$$\mathcal{M} |\psi_{tr}, \pi\rangle = |\psi'_{tr}, \pi\rangle + |\psi^n, \pi\rangle \tag{2.11}$$

where $|\psi^n, \pi>$ is a decoupled state of zero norm. It follows that the linear mapping

$$|\psi_{tr}, \pi\rangle \to |\psi'_{tr}, \pi\rangle \equiv \tilde{\mathcal{M}} |\psi_{tr}, \pi\rangle \tag{2.12}$$

defines a unitary representation $\tilde{\mathcal{M}}$ of the little group of π. If we project both sides of Eq. (2.12) onto their leading terms $|\psi_{o,tr}>$, we define a representation of the same group within the Hilbert space of the transverse modes $\mathcal{H}_{tr}$. The form of this representation is most easily deduced by considering infinitesimal transformations

$$\mathcal{M}_\epsilon \equiv 1 + i\epsilon E_{-i} = 1 + i\epsilon\sqrt{2} \sum_{n \neq 0} \frac{1}{n} (\pi_- \alpha_{+,-n} - \pi_+ \alpha_{-,-n}) \alpha_{i,n} \tag{2.13}$$

Given the expansion

$$|\Psi_{tr},\pi\rangle = |\Psi_{o,tr}\rangle + \frac{1}{\sqrt{2}\pi_-}\sum_n \frac{\alpha_{-,-n}}{n}\bar{L}_n|\Psi_{o,tr}\rangle + \ldots, \ldots \tag{2.14}$$

it is easily seen that the leading term of $\mathfrak{M}_\epsilon|\Psi_{tr},\pi\rangle$ is

$$|\Psi_{o,tr}\rangle + i\epsilon\sum_{n\neq 0}\frac{\alpha_{i,n}\bar{L}_n}{n}|\Psi_{o,tr}\rangle + |\Psi_o''\rangle \tag{2.15}$$

where

$$|\Psi_o''\rangle = i\epsilon\sum_{n=1}^{\infty}\left(\sqrt{2}\pi_-\alpha_{+,-n} - \bar{L}_{-n}\right)\frac{\alpha_{i,n}}{n}|\Psi_{o,tr}\rangle \tag{2.16}$$

is the leading term of a state, orthogonal to all transverse states, of null norm. Then we deduce that the algebra of the $E_{L,i}$ can be represented within $\mathcal{H}_{tr}$ by

$$\hat{E}_{L,i} = \sum_{n=1}^{\infty}\frac{1}{n}\left(L^T_{-n}\alpha^i_n - \alpha^i_{-n}L^T_n\right) \tag{2.17}$$

where only the transverse parts of the operators $\bar{L}_n$ are included in L^T_n, since they operate only on the transverse modes. For the critical number of dimension only (i.e., for 24 transverse components) the commutators of two $\hat{E}_{L,i}$ close to

$$[\hat{E}_{L,i},\hat{E}_{L,j}] = 2\left(L^T_0 - 1\right)E_{i,j} \tag{2.18}$$

with

$$E_{ij} = \sum_{n\neq 0} :\frac{\alpha_{i,-n}\alpha_{j,n}}{n}:$$

Since on the mass shell

$$(L_0^T - 1)\,|\psi_{0,tr}\rangle = M^2\,|\psi_{0,tr}\rangle \tag{2.19}$$

Eq. (2.18) agrees with the algebra of the $E_{L,i}$.

Finally, let us notice that all our considerations can be immediately extended to the model of Neveu and Schwarz. The L_n are modified by the addition of a term

$$\frac{1}{4} : \sum_{r=-\infty}^{\infty} b_{-r}\, b_{r+n} :$$

and the gauge equations are supplemented by

$$G_r\,|\psi,\pi\rangle \equiv (\bar{G}_r - \sqrt{2}\,\pi . b_r)\,|\psi,\pi\rangle = 0\,,\quad r = \tfrac{1}{2}, \tfrac{3}{2}, \dots, \dots \tag{2.20}$$

with

$$G_r = \sum_{n=-\infty}^{\infty} \alpha_n . b_{n-r}$$

The iterative construction of the state reads

$$|\psi_j\rangle = (N_{\alpha_-} + N_{b_-})^{-1} \Big\{ \sum_{n=1}^{\infty} \Big(\frac{\alpha_{-,-n}\,\bar{L}_n}{n}\,|\psi_{j-1}\rangle + \frac{\alpha_{-,-n},\alpha_{-,n}}{\sqrt{2}}\, M^2\,|\psi_{j-2}\rangle \Big)$$
$$+ \sum_{r=\frac{1}{2}}^{\infty} \Big(b_{-,-r}\,\bar{G}_r\,|\psi_{j-1}\rangle + b_{-,-r}\,b_{-,r}\,|\psi_{j-2}\rangle \Big) \tag{2.21}$$

and the representation of the little group is given by (for eight transverse dimensions)

$$\hat{E}_{L,i} = \sum_{n=1}^{\infty} \frac{1}{n} \big(L_{-n}^T \alpha_n^i - \alpha_{-n}^i L_n^T \big) + \sum_{r=\frac{1}{2}}^{\infty} \big(G_{-r}^T b_r^i - b_{-r}^i G_r^T \big) \tag{2.22}$$

In the next sections we shall discuss the covariance of states from a more general view point. We shall take advantage of Eqs. (2.17) and (2.22) but their properties will be discussed a second time so the argument does not depend logically on this section.

3. - COVARIANCE OF THE PHYSICAL STATES IN THE CONVENTIONAL MODEL

In the previous section, we saw how considerations of the completeness of the "transverse" physical states when the space-time dimensions $d = 26$ and the form of these states in the infinite momentum limit enabled us to extend the representation of the generators, E_{ij}, of the $(d-2)$ dimensional rotation group defined on these states into a representation of $O(d-1)$. Del Giudice, Di Vecchia and Fubini [4] constructed the transverse states by introducing operators A^i_n describing the coupling of the zero mass vector particle ("photon") :

$$A^i_n = \frac{1}{2\pi}\int_0^{2\pi} d\tau \; P^i(z)\, V^0(-nk,z) \tag{3.1}$$

where

$$P^r(z) = \sum_{n=-\infty}^{\infty} \alpha^r_n z^{-n}, \qquad z = e^{i\tau}$$

$$V^0(k',z) = \exp\left\{-\sqrt{2}k'\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}z^n\right\}\exp\left\{\sqrt{2}k'(p_0\log z + iq_0)\right\}\exp\left\{\sqrt{2}k'\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}z^{-n}\right\}$$

These operators create the transverse physical states when applied to a tachyon vacuum state of momentum p $(p^2 = 1,\ k^2 = 0,\ p\cdot k = -\frac{1}{2})$; they have the same commutation relations as the α^i_n :

$$[A^i_m, A^j_n] = m\,\delta^{ij}\,\delta_{m,-n}$$

and additionally

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\,\delta_{m,-n}$$

$$[L_m, A^j_n] = [K_m, A^j_n] = 0.$$

for $K_n = k\cdot\alpha_n$. Thus for $d = 26$,

$$E_{ij} \to \hat{E}_{ij} = \sum_{n=1}^{\infty}\frac{1}{n}\left(A^i_{-n}A^j_n - A^j_{-n}A^i_n\right) \tag{3.2}$$

$$E_{L,i} \to \hat{E}_{L,i} = \sum_{n=1}^{\infty}\frac{1}{n}\left(\mathcal{L}_{-n}A^i_n - A^i_{-n}\mathcal{L}_n\right) \tag{3.3}$$

where

$$\mathcal{L}_n = \frac{1}{2}:\sum_{m=-\infty}^{\infty}\sum_{i=2}^{d-1} A^i_{-m}A^i_{n+m}:$$

(and A^i_0 is understood as to be equal to 0) defines a representation of $O(d-1)$ in the space of states created by the A^i_n. But, in general,

$$[\hat{E}_{Li}, \hat{E}_{Lj}] = -2\left(\mathcal{L}_0 - \frac{1}{24}[d-2]\right)\hat{E}_{ij} + \left(\frac{1}{24}[d-2]-1\right)\sum_{n=1}^{\infty} n\left(A^j_{-n}A^i_n - A^i_{-n}A^j_n\right) \tag{3.4}$$

and so the algebra does not close for $d \neq 26$. This is a direct result of the d dependence of the c number in

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{1}{12}[d-2]\,m(m^2-1)\,\delta_{m,-n}$$

$$[\mathcal{L}_m, A^i_n] = -n\,A^i_{m+n} \tag{3.5}$$

Using (3.3) we can compute the action of $\hat{E}_{Li}$ on the DDF states

$$[\hat{E}_{L,i}, A^j_n] = \delta^{ij}\mathcal{L}_n + n\,A^{ij}_n \tag{3.6}$$

where

$$A_n^{ij} = \sum_{m=1}^{\infty} \frac{1}{m}\left(A_{-m}^{i} A_{m+n}^{j} - A_{n-m}^{j} A_{m}^{i} \right)$$

Notice that (3.6) expresses $\mathcal{L}_n$ in terms of the DDF states and the effective little group generators. Furthermore, one can easily see that the structure (3.6) together with the algebra (3.4) for $d = 26$ implies the algebra (3.5).

The operators $A_n^{(+)}$ which Brower has introduced obey the algebra (3.5) with the c number $2m(m^2-1)\,\delta_{m,-n}$ for any dimension. This suggests that they might play the role of the $\mathcal{L}_n$'s for a non-critical dimension.

Equation (3.6) suggests that the sensible thing to do to obtain all the physical states for any number of dimensions is to rotate the A_n^i about the total momentum, p_o, into the plane of p and k. One can then extract new operators which commute with the gauges from the right-hand side. To this end we consider the commutator of A_n^i with the generators of the little group of the resonance momentum p_o in the whole Hilbert space

$$E_{i,j} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^{i} \alpha_{n}^{j} - \alpha_{-n}^{j} \alpha_{n}^{i} \right) \tag{3.7}$$

$$E_{L,i} = \sum_{n=1}^{\infty} \frac{1}{n}\left(p_L\cdot\alpha_{-n}\, \alpha_{n}^{i} - \alpha_{-n}^{i}\, p_L\cdot\alpha_{n} \right) \tag{3.8}$$

where p_L is orthogonal to p_o in the plane of p and k ; $p_L = 2\sqrt{2}(p+k) - p_o$ [so that $p_L\cdot p_o = 0$ for $p_o = \sqrt{2}\,\pi = \sqrt{2}(p+\beta k)$]. E_{ij} and E_{Li} satisfy the algebra of generators of $O(d-1)$:

$$[E_{ij}, E_{k,l}] = \delta^{jk} E_{i,l} - \delta^{il} E_{j,k} - \delta^{jl} E_{i,k} + \delta^{ik} E_{j,l}$$

$$[E_{ij}, E_{L,k}] = \delta^{jk} E_{L,i} - \delta^{ik} E_{L,j}$$

$$[E_{L,i}, E_{L,j}] = p_o^2\, E_{ij} \tag{3.9}$$

where we have made the substitution $-p_L^2 \to p_o^2$ valid on states of momentum $\pi = p + \beta k$. Now for $i \neq j$ it is straightforward to calculate the commutator $[E_{Li}, A_n^j]$ explicitly, using (3.1) and (3.8) except one has to use the formula

$$2\pi\,\delta(\tau-\tau') = \sqrt{2}\,k.P(z) \sum_{m=-\infty}^{\infty} V^o(mk,z)\,V^o(-mk,z) \tag{3.10}$$

(which is valid for matrix elements between states of momenta $p + \beta k$). This yields nA_n^{ij}. Thus, from tensorial properties in the transverse dimensions it follows that we may write

$$[E_{L,i}, A_n^j] = \delta^{ij}\hat{A}_n + nA_n^{ij} \tag{3.11}$$

and this defines $\hat{A}_n$. From (3.11) we may directly deduce all the algebraic properties of $\hat{A}_n$ from (3.11) and the algebra of E_{Li}, E_{ij} and A_n^i (with $\hat{A}_o \equiv 1 + \frac{1}{2}p_o^2$) :

$$[\hat{A}_m, \hat{A}_n] = (m-n)\,\hat{A}_{m+n} + 2\,\delta_{n,-m}\,m(m^2-1)$$

$$[\hat{A}_m, A_n^i] = -n A_{m+n}^i$$

$$[E_{ij}\,\hat{A}_m] = 0$$

$$[E_{L,i}, \hat{A}_m] = -A_m^i\left(\hat{A}_0 + \tfrac{3}{2}m^2 - \tfrac{1}{2}m - 2\right) + m\sum_{n=1}^{\infty}\tfrac{1}{n}\left(A_{-n}^i\hat{A}_{m+n} - \hat{A}_{m-n}A_n^i\right)$$

Because this algebra now closes, we have succeeded in restoring Lorentz convariance by the addition of the $\hat{A}_n$ to the "photon" operators A_n^i. It is now clear that the substitution $\mathcal{L}_n \to \hat{A}_n$ in (3.3) gives a realization of E_{Li}, E_{ij} which coincides with (3.7) and (3.8) on states generated by $\hat{A}_n$ and A_n^i *). Thus we have seen how the operators $\hat{A}_n$, first constructed by Brower (his $A_n^{(+)}$) [6], and their algebraic properties may be easily deduced from simple considerations of rotational invariance. In fact it can be shown that the A_n^i, $\hat{A}_n$ generate all the physical states. In the next section we show how these considerations of rotational properties will enable us to construct operators which create all the physical states in the Neveu-Schwarz model.

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*) In fact this coincidence occurs for all states with the appropriate collinear momentum.

Finally we would like to remark that (3.11) can be deduced on more abstract grounds. For $i \neq j$ $[E_{Li}, A^j_m]$ commutes with K_m as well as L_m and thus should be expressible in terms of the A^k_n. Further as E_{Li} and A^j_n are linear in α^i and α^j it follows that $[E_{Li}, A^j_m]$ must be of the form $\sum_r \alpha^r_m A^i_{-r} A^j_{r+m}$. The Jacobi identities applied to $[A^K_\ell [E_{Li}\, A^j_m]]$ and $[E_{Lk}[E_{Li}\, A^j_m]]$ are sufficient to determine the α^r_m, and so (3.11) deduced.

4. - COVARIANCE OF THE PHYSICAL STATES IN THE NEVEU-SCHWARZ MODEL

In this section we apply the techniques developed in the last section to construct a closed algebra of operators which generates all the physical states in the Neveu-Schwarz model. Here the operators describing the coupling of the "photon", given by a construction analogous to that of DDF are

$$A^i_n = \frac{1}{2\pi}\int_0^{2\pi} d\tau \left\{ P^i(z) + \sqrt{2}\, n\, k.H(z)\, H^i(z) \right\} V^0(-nk, z)$$

where

$$H^r(z) = \sum_{r=-\infty}^{\infty} b^r_{-r} z^r$$

Then if $H_n = k \cdot b_n$ we have

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{d}{8} m(m^2-1)\, \delta_{m,-n}$$

$$\{G_r, G_s\} = 2 L_{r+s} + \frac{d}{2}\left(r^2 - \frac{1}{4}\right) \delta_{r,-s}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{r+m}$$

$$[L_m, A^i_n] = [G_r, A^i_n] = [K_m, A^i_n] = [H_r, A^i_n] = 0 .$$

The generators of the little group of p_o are now :

$$E_{i,j} = \sum_{m=1}^{\infty} \frac{1}{m}\left(\alpha_{-m}^{i}\alpha_{m}^{j} - \alpha_{-m}^{j}\alpha_{m}^{i}\right) + \sum_{r=\frac{1}{2}}^{\infty}\left(b_{-r}^{i}b_{r}^{j} - b_{-r}^{j}b_{r}^{i}\right)$$

$$E_{L,i} = \sum_{m=1}^{\infty} \frac{1}{m}\left(p_L\cdot\alpha_{-m}\alpha_{m}^{i} - \alpha_{-m}^{i}\,p_L\cdot\alpha_{m}\right)$$
$$+ \sum_{r=\frac{1}{2}}^{\infty}\left(p_L\cdot b_{-r}\,b_{r}^{i} - b_{-r}^{i}\,p_L\cdot b_{r}\right)$$

They obey the same algebra as before, of course. Then following Section 3 we compute for $i \neq j$

$$\left[E_{L,i}, A_{m}^{j}\right] = m A_{m}^{ij} + m B_{m}^{ij} \equiv m T_{m}^{ij}$$

where A_m^{ij} is still defined by (3.6) and explicitly

$$B_{m}^{ij} = \frac{1}{2\pi}\int_{0}^{2\pi} d\tau \left\{H^{i}(z) - \frac{k\cdot H(z)P^{i}(z)}{k\cdot P(z)}\right\}\left\{H^{j}(z) - \frac{k\cdot H(z)P^{j}(z)}{k\cdot P(z)}\right\}$$
$$\times\left\{1 + \frac{i}{2}\,\frac{k\cdot\dot{H}(z)\,k\cdot H(z)}{\{k\cdot P(z)\}^{2}}\right\}^{2} V^{0}(-nk,z)$$

and using (3.10) we can write

$$B_{m}^{ij} = \sum_{m=\frac{1}{2}}^{\infty}\left(B_{-r}^{i}B_{r+m}^{i} - B_{m-r}^{j}B_{r}^{i}\right)$$

where

$$B_s^i = \frac{1}{2\pi}\int_0^{2\pi} d\tau \quad \{\sqrt{2}\, k.P(z)\}^{-\frac{1}{2}}\left\{H^i(z) - \frac{k.H(z)\,P^i(z)}{k.P(z)}\right\}$$

$$\times\left\{1 + \frac{i}{2}\,\frac{k.\dot{H}(z)\,k.H(z)}{\{k.P(z)\}^2}\right\} V^0(-sk,z)$$

Although B_m^{ij} is a physical operator (i.e., commutes with G_r) it does not follow that B_s^i is. However, one can prove that it is by direct computation,

$$\{G_r, B_s^i\} \propto \frac{-i}{2\pi}\int_0^{2\pi} d\tau\, \frac{d}{d\tau}\left\{V^0(-sk,z)\;\frac{k.\dot{H}(z)H^i(z)}{\{k.P(z)\}^{\frac{3}{2}}}\right\} = 0 .$$

because the integrand is single-valued on states with momentum $p_0 = p + \beta k$. [In spite of their somewhat bizarre appearance, operators like $\{k\cdot P(z)\}^{\frac{1}{2}}$, $\{k\cdot p(z)\}^{-1}$ are well defined on the occupation number states, with momentum $p + \beta k$, since only a finite number of terms in their expansion about one contribute to any matrix elements ; cf. Ref. 6).] Furthermore, it is straightforward to verify

$$\{B_r^i, B_s^j\} = \delta^{ij}\delta_{r,-s}$$

$$[B_r^i, A_m^j] = 0 .$$

We can now continue applying the method of Section 3 by defining the operator $\hat{A}_n$ by

$$[E_{i,i}\; A_m^j] = \delta^{ij}\hat{A}_m + m A_m^{ij} + m B_m^{ij} \tag{4.1}$$

Analogously we must consider similarly rotating B_r^j :

$$[E_{Li}, B_r^j] = \delta^{ij}\hat{B}_r + M_r^{ij} \tag{4.2}$$

where

$$M_r^{ij} = \sum_{n=-\infty}^{\infty}\left\{\left(\tfrac{1}{2}-\tfrac{r}{n}\right)B_{r-n}^j A_n^i - B_{r-n}^i A_n^j\right\}$$

which we regard as defining $\hat{B}_r$. B_r^i, $\hat{B}_s$, A_m^i, $\hat{A}_n$ generate all the physical states by counting arguments. Again, from (4.1) and (4.2), we may deduce, algebraically, the algebra (with $A_0 - \frac{1}{2} = \frac{1}{2}p_0^2$) :

$$[\hat{A}_m, \hat{A}_n] = (m-n)\hat{A}_{m+n} + \delta_{n,-m}\, m(m^2-1)$$

$$[\hat{A}_n, \hat{B}_r] = \left(\tfrac{n}{2}-r\right)\hat{B}_{n+r}$$

$$\{\hat{B}_r, \hat{B}_s\} = 2\hat{A}_{r+s} + 4\,\delta_{r,-s}\left(r^2-\tfrac{1}{4}\right)$$

$$[\hat{A}_m, A_n^i] = -n\, A_{m+n}^i$$

$$[\hat{A}_n, B_r^i] = -\left(\tfrac{n}{2}+r\right)B_{n+r}^i$$

$$\{\hat{B}_r, B_s^i\} = A_{r+s}^i$$

$$[\hat{B}_r, A_m^i] = -m\, B_{m+r}^i$$

Now, reversing the deductive procedure in Section 3 we may replace the generators of the little group of p_0 in the space of physical states by

$$E_{ij} \longrightarrow \sum \frac{1}{n}\left(A^i_{-n}A^j_n - A^j_{-n}A^i_n\right) + \sum\left(B^i_{-r}B^j_r - B^j_{-r}B^i_r\right)$$

$$E_{L,i} \longrightarrow \sum \frac{1}{n}\left(\hat{A}_{-n}A^i_n - A^i_{-n}\hat{A}_n\right) + \sum\left(\hat{B}_{-r}B^i_r - B^i_{-r}\hat{B}_r\right) \qquad (4.3)$$

Further, for $d = 10$, there must be a representation of this little group in the space generated by the A^i_{-n} and B^j_{-r} only. This can be achieved using the isomorphism between the algebras $\{A^i_n, B^j_r, \hat{A}_m, \hat{B}_s\}$ and $\{A^i_n, B^j_r, \mathcal{L}_m, \mathcal{G}_s\}$ at $d = 10$, where

$$\mathcal{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{i=2}^{d-1} :A^i_{-n}A^i_{m+n}: + \frac{1}{4}\sum_{s=-\infty}^{\infty}\sum_{i=2}^{d-1} :B^i_{-s}B^i_{n+s}:(2s+n)$$

$$\mathcal{G}_s = \sum_{n=-\infty}^{\infty}\sum_{i=2}^{d-1} A^i_{-n}B^i_{n+s}$$

by replacing $\hat{A}_m$ by $\mathcal{L}_m$ and $\hat{B}_s$ by $\mathcal{G}_s$ in (4.3), to obtain $\hat{E}_{Li}$ say. If we similarly define $\hat{E}_{Li}$ where d is not necessarily 10 we find

$$[\hat{E}_{Li}, \hat{E}_{Lj}] = -2\left(\mathcal{L}_0 - \frac{1}{16}[d-2]\right)E_{ij}$$

$$+ \left(\frac{d-2}{8} - 1\right)\sum_{n=1}^{\infty} n\left(A^j_{-n}A^i_n - A^i_{-n}A^j_n\right) + \left(\frac{d-2}{2} - 4\right)\sum_{r=\frac{1}{2}}^{\infty} r^2\left(B^j_{-r}B^i_r - B^i_{-r}B^j_r\right)$$

Thus we can only extend the representation of $O(d-2)$ in the space of transverse states to one of $O(d-1)$ in this way when $d = 10$. We remark that, as in Section 3, we can construct (4.1) and (4.2) by more abstract arguments, which we will not elaborate further here.

The argument of the present section should serve to demonstrate the power of considering rotations in the little group of p_0 in determining all the physical states from those generated by the transverse "photon" operators.

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APPENDIX

In this Appendix we prove some theorems that we only quoted in Section 2.

Theorem

The equations (2.7)

$$\sqrt{2}\,\alpha_{+,n}|\psi_j\rangle = -L_n|\psi_{j-1}\rangle - \frac{M^2}{\sqrt{2}}\alpha_{-,n}|\psi_{j-2}\rangle$$
$$= -|\Phi_{j,n}\rangle$$

are compatible.

For the proof of this Theorem we need the following Lemma :

Lemma

A necessary and sufficient condition for the system of Eqs. (2.7) to admit a solution is

$$\alpha_{+,n}|\Phi_{j,m}\rangle = \alpha_{+,m}|\Phi_{j,n}\rangle \tag{A.1}$$

for all m and $n > 0$.

Proof of Lemma

The necessity of this condition follows immediately from $[\alpha_{+,m}, \alpha_{+,n}] = 0$. To show that it is sufficient, consider any vector $|\psi_j\rangle$ solution of

$$N_-|\psi_j\rangle = \sum_{n=1}^{\infty} \frac{\alpha^+_{-n}}{n}|\Phi_{j,n}\rangle \tag{A.2}$$

with

$$N_- = -\sum_n \frac{\alpha^+_{-,n}\alpha_{+,n}}{n} \tag{A.3}$$

[In Section 2 it has been shown that solutions to (A.2) always exist.] We prove that any $|\Psi_j>$ solution of Eq. (A.2) solves all Eqs. (2.7) when (A.1) is satisfied.

Indeed, if (A.1) is true, we have

$$\begin{aligned}
(N_-+1)\,\alpha_{+,m}|\Psi_j\rangle &= \alpha_{+,m} N_- |\Psi_j\rangle \\
&= \sum_{n=1}^{\infty} \alpha_{+,m} \frac{\alpha^{+}_{-,n}}{n} |\Phi_{j,n}\rangle \\
&= \sum_{n=1}^{\infty} \frac{\alpha^{+}_{-,n}}{n} \alpha_{+,m} |\Phi_{j,n}\rangle - |\Phi_{j,m}\rangle \\
&= \left(\sum_{n=1}^{\infty} \frac{\alpha^{+}_{-,n}\,\alpha_{+,n}}{n}\right)|\Phi_{j,m}\rangle - |\Phi_{j,m}\rangle \\
&= -(N_-+1)\,|\Phi_{j,m}\rangle \;;
\end{aligned} \tag{A.4}$$

hence

$$\alpha_{+,m}|\Psi_j\rangle = -|\Phi_{j,m}\rangle$$

We can now proceed to prove the Theorem.

Proof of Theorem

We proceed by induction. Suppose Eqs. (2.7) are satisfied up to $|\Psi_{j-1}>$: we demonstrate then that the vectors $|\Phi_{j,n}>$ satisfy the condition of the Lemma, and therefore that a solution for $|\Psi_j>$ exists. In fact, by the use of Eqs. (2.7), assumed up to $|\Psi_{j-1}>$, and of the algebra of the L's and α operators, we deduce

$$\sqrt{2}\,\alpha_{+,m}|\Phi_{j,n}\rangle - \sqrt{2}\,\alpha_{+,n}|\Phi_{j,m}\rangle$$

$$= \sqrt{2}\,\alpha_{+,m}\bar{L}_n|\Psi_{j-1}\rangle + M^2\alpha_{+,m}\,\alpha_{-,n}|\Psi_{j-2}\rangle - (m \leftrightarrow n)$$

$$= \sqrt{2}\left(\bar{L}_n\alpha_{+,m} + m\,\alpha_{+,m+n}\right)|\Psi_{j-1}\rangle + M^2\alpha_{+,m}\alpha_{-,n}|\Psi_{j-2}\rangle - (m \leftrightarrow n)$$

$$= -(\bar{L}_n\bar{L}_m + m\bar{L}_{m+n})|\psi_{j-2}\rangle - \frac{M^2}{\sqrt{2}}(\bar{L}_n\alpha_{-,m} + m\alpha_{-,m+n} + \alpha_{-,n}\bar{L}_n)|\psi_{j-3}\rangle$$
$$-\frac{M^2}{2}\alpha_{-,n}\alpha_{-,m}|\psi_{j-n}\rangle \quad + \quad (m \leftrightarrow n)$$

$$= \{[\bar{L}_m,\bar{L}_n] - (m-n)\bar{L}_{m+n}\}|\psi_{j-2}\rangle$$
$$+\frac{M^2}{\sqrt{2}}\{[\bar{L}_m,\alpha_{-,n}] - [\bar{L}_n,\alpha_{-,m}] - (m-n)\alpha_{-,m+n}\}|\psi_{j-3}\rangle = 0. \quad (A.5)$$

which completes the inductive proof.

We now demonstrate the statements a)-e) of Section 2. It is convenient to recall the explicit construction of the vectors $|\psi_j\rangle$. This is given by

$$|\psi_j\rangle = -N_-^{-1}\sum_n\left(\frac{\alpha_{-,n}^+\bar{L}_n}{n\sqrt{2}}|\psi_{j-1}\rangle + \frac{M^2\alpha_{-,n}^+\alpha_{-,n}}{2n}|\psi_{j-2}\rangle\right) \quad (A.6)$$

Notice that N_-^{-1}, $\alpha_{-,n}^+\bar{L}_n$ and $\alpha_{-,n}^+\alpha_{-,n}$ all commute with $\bar{L}_o$. Then, if $|\psi_o\rangle$ is an eigenvector of $\bar{L}_o$, so are all the terms $|\psi_j\rangle$. This proves statement c), once we have shown that the sum of the series is well defined.

To prove statement a), that the series in j terminates, it is useful to decompose the vectors $|\psi_j\rangle$ into sums of terms having definite numbers ℓ_+ and ℓ_- of + and - modes

$$|\psi_j\rangle = \sum_{\ell_+\ell_-}|\psi_j,\ell_+,\ell_-\rangle \quad (A.7)$$

We see that the first operator $(\alpha_{-,n}^+\bar{L}_n)$ in the right-hand side of Eq. (A.6) transforms a term $|\psi_{j-1}, \ell_+, \ell_-\rangle$ in a sum of terms of the form $|\psi_j, \ell_+, \ell_- +1\rangle$ or $|\psi_j, \ell_+ -1, \ell_-\rangle$, and that the second operator $(\alpha_{-,n}^+\alpha_{-,n})$ transform a term $(\psi_{j-2}, \ell_+, \ell_-\rangle$ into a term of the form $|\psi_j, \ell_+ -1, \ell_- +1\rangle$. It follows that the quantity $j + \ell_+ - \ell_-$ is conserved in the defining equation of the series, and that ℓ_+ is never increased. If ℓ is the maximum number of + modes present in ℓ_o, then $\ell_+ \leq \ell$, and we have the following bound

$$\ell_- \geq j - 1$$

Then it is clear that the series in j terminates [statement a)], because the number of $-$ modes can never exceed the maximum eigenvalue of $\bar{L}_0$ contained in the leading term.

Moreover, since

$$\langle \psi_j, \ell_+, \ell_- | \psi_{j'}, \ell'_+, \ell'_- \rangle = 0$$

unless $\ell_+ = \ell'_-$ $\ell_- = \ell'_+$, it is also obvious that only a finite number of $|\psi_j>$ $(j \leq 2\ell + 1)$ can contribute to the norm of

$$|\psi, \pi\rangle = \sum_j \pi^j |\psi_j\rangle$$

[statement d)].

Statement b) is proved by noting that $|\psi_0>$ is the only term of the series that has no $-$ modes. Statement e) follows from the equation $\ell_+ \leq \ell$ and the fact that $|\psi_{0,tr}>$ has $\ell = 0$.

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