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Lorentz covariant treatment of the Kerr-Schild geometry*

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It is shown that a Lorentz covariant coordinate system can be chosen in the case of the Kerr-Schild geometry which leads to the vanishing of the pseudo energy-momentum tensor and hence to the linearity of the Einstein equations. The retarded time and the retarded distance are introduced and the Liénard-Wiechert potentials are generalized to gravitation in the case of world-line singularities to derive solutions of the type of Bonnor and Vaidya. An accelerated version of the de Sitter metric is also obtained. Because of the linearity, complex translations can be performed on these solutions, resulting in a special relativistic version of the Trautman-Newman technique and Lorentz covariant solutions for spinning systems can be derived, including a new anisotropic interior metric that matches to the Kerr metric on an oblate spheroid.

1. INTRODUCTION

In general relativity, the field equations are often simplified when we deal with algebraically special or degenerate metrics. The degeneracy of the metrics is linked with the multiplicity of the Debever–Penrose directions.[1,2] One of the important examples for the algebraically special metrics is the Kerr–Schild[2] metric which is given as

$$g_{\mu\nu}=\eta_{\mu\nu}-2V\lambda_\mu\lambda_\nu \qquad (1.1)$$

where $\eta_{\mu\nu}=(1,-1,-1,-1)$ is the Minkowski metric, V is a scalar function, and λ_μ is a light like vector both with respect to $g_{\mu\nu}$ and $\eta_{\mu\nu}$:

$$g_{\mu\nu}\lambda^\mu\lambda^\nu=\eta_{\mu\nu}\lambda^\mu\lambda^\nu=0. \qquad (1.2)$$

This null vector is also geodesic both with respect to $g_{\mu\nu}$ and $\eta_{\mu\nu}$, that is,

$$\lambda^\mu\partial_\mu\lambda_\nu=\lambda^\mu\nabla_\mu\lambda_\nu=0, \qquad (1.3)$$

where ∂_μ and ∇_μ are covariant derivatives with respect to $\eta_{\mu\nu}$ and $g_{\mu\nu}$, respectively. These two properties of λ_μ, Eqs. (1.2), (1.3), make it a shear free double Debever–Penrose vector. If the scalar function V is a constant, λ_μ becomes a Killing vector.

The Kerr–Schild metric has been studied by several authors by using either the tetrad formalism[2] or the direct procedure[3] in solving the field equations. We use the second method in a special relativistic covariant way and find the energy–momentum tensor (e.m.t.) of the matter and the field other than the gravitational field. The mixed form of the e.m.t. is linear in the function V and also it is divergence free in the ordinary sense, that is,

$$\partial_\mu T^\mu{}_\nu=0. \qquad (1.4)$$

Therefore, the pseudo-energy–momentum tensor (p.e.m.t.) of the gravitational field must also be conserved. We find that it vanishes in this coordinate system. Vanishing of the p.e.m.t. makes the field equations linear. Because of this fact, the gravitational field is not its own source in this coordinate system. If a metric can be thrown into the Kerr–Schild form by a coordinate transformation, the gravitational energy and momentum are cancelled by this coordinate transformation which represents some kind of acceleration according to the equivalence principle.

A method, which leads to a new metric from an old one, is based on making a complex translation along one of the coordinates without changing the physical character of the source. Such a complex translation is allowed in classical electrodynamics and in linearized general relativity because of their linearity[4] of the equations. In exact general relativity complex translation was used several years ago by Newman and Janis[5] to obtain the Kerr metric from the Schwarzschild metric and by Newman *et al.*[6] to obtain the charged Kerr metric from the Reissner–Nordstrom metric. Recently, Adler *et al.*[3] used complex translation and reobtained the Kerr metric in the Kerr–Schild coordinate system without drawing attention to the linearity of the field equations. Now it becomes clear that complex translation is allowed in general relativity whenever we can find a coordinate system in which the p.e.m.t. vanishes or the Einstein equations are linear in this coordinate system. This is of course not true for an arbitrary metric. It happens to be true in the algebraically special Kerr–Schild geometry.

Another advantage of the Kerr–Schild metric is the following. When we take λ_μ as the gradient of the retarded time and V as a function of the retarded distance for an accelerated system (particles, charges, etc.) we get simply the result of Bonnor and Vaidya,[7] generalizing the Liénard–Wiechert potential in electromagnetism to the retarded gravitational potential. In addition to their result we also find the accelerated version of the de Sitter metric.

In Sec. 2, we find the Einstein tensor of the Kerr–Schild metric and show the linearity of the field equations. We also prove that λ_μ is a double Debever–Penrose vector for any e.m.t. In Sec. 3 we find the field of accelerated systems, especially of the charged particle in a de Sitter universe. In Sec. 4 we complexify the solutions discussed in the previous section for non-accelerated systems. We find the e.m.t. for this case. The Kerr[8] and charged Kerr metrics[6] are special cases of this e.m.t. For the interior metric, this tensor is shown to correspond to the e.m.t. of an anisotropic perfect fluid and to match the Kerr metric on an oblate spheroid. In Sec. 5, we show the resemblence between the linearized field equations obtained from an approximation procedure and the field equations obtained for the Kerr–Schild metric.

2. THE KERR-SCHILD GEOMETRY

The light like character of the four vector λ_μ greatly simplifies the calculations. Because of this property, it can be raised and lowered with both $\eta_{\mu\nu}$ and $g_{\mu\nu}$, and we also have

$$\sqrt{-g}=1, \tag{2.1}$$

so that we have

$$g^{\mu\nu}=\eta^{\mu\nu}+2V\lambda^\mu\lambda^\nu. \tag{2.2}$$

The Riemann–Christoffel symbols and the curvature tensor, for this metric, are

$$\Gamma^\alpha{}_{\mu\nu}=-[(l^\alpha l_\nu)_{,\mu}+(l^\alpha l_\mu)_{,\nu}-\eta^{\gamma\alpha}(l_\mu l_\nu)_{,\gamma}+4Al^\alpha l_\mu l_\nu], \tag{2.3}$$

$$\begin{aligned}R^\gamma{}_{\mu\nu\alpha}&=\partial_\nu\Gamma^\gamma{}_{\mu\alpha}-\partial_\alpha\Gamma^\gamma{}_{\mu\nu}+\Gamma^\gamma{}_{\beta\nu}\Gamma^\beta{}_{\mu\alpha}-\Gamma^\gamma{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu}\\&=l^\gamma l_{[\alpha}\psi_{\nu]\mu}-\eta^{\gamma\tau}l_\mu l_{[\nu}\psi_{\alpha]\tau}+2Al_\mu l_{[\alpha}\Theta_{\nu]}{}^\gamma\\&\quad-2Al^\gamma l_{[\nu}\Theta'_{\alpha]\mu}-l^\gamma\Phi_{\nu\mu\alpha}+l^\gamma\Phi_{\alpha\mu\nu}\\&\quad-l_\nu\Phi_\mu{}^\gamma{}_\alpha+l_\mu\Phi_\alpha{}^\gamma{}_\nu-l_\mu\Phi_\nu{}^\gamma{}_\alpha+\eta^{\gamma\tau}l_\nu\Phi_{\tau\mu\alpha}\\&\quad-\eta^{\gamma\tau}l_\alpha\Phi_{\tau\mu\nu}-\eta^{\beta\sigma}(l^\gamma l_\nu)_{,\beta}(l_\mu l_\alpha)_{,\sigma}+\eta^{\beta\sigma}(l^\gamma l_\alpha)_{,\beta}(l_\mu l_\nu)_{,\sigma}\\&\quad-(l_\alpha l^\gamma)_{,\mu\nu}+(l_\nu l^\gamma)_{,\mu\alpha}+\eta^{\gamma\beta}(l_\mu l_\alpha)_{,\beta\nu}-\eta^{\gamma\beta}(l_\mu l_\nu)_{,\beta\alpha}\\&\quad-4(Al^\gamma l_\mu l_\alpha)_{,\nu}+4(Al^\gamma l_\mu l_\nu)_{,\alpha},\end{aligned} \tag{2.4}$$

where

$$l_\mu=\sqrt{V}\lambda_\mu,$$
$$\psi_{\mu\nu}=l_{\beta,\nu}l^\beta{}_{,\mu},$$
$$\Theta^\gamma{}_\mu=l^\gamma{}_{,\mu}-l_{\mu,}{}^\gamma,$$
$$\Theta'_{\mu\nu}=l_{\mu,\nu}+l_{\nu,\mu},$$
$$\Phi_{\mu\nu\alpha}=l^\beta{}_{,\mu}(l_\nu l_\alpha)_{,\beta},$$

and

$$l_{[\alpha}\psi_{\beta]\mu}=l_\alpha\psi_{\beta\mu}-l_\beta\psi_{\alpha\mu},$$
$$l_{[\alpha}\Theta^\gamma{}_{\nu]}=l_\alpha\Theta^\gamma{}_\nu-l_\nu\Theta^\gamma{}_\alpha.$$

We note that

$$l^\gamma\psi_{\gamma\beta}=l^\gamma\Phi_{\alpha\gamma\beta}=l^\gamma\Phi_{\alpha\beta\gamma}=0, \tag{2.5}$$

$$l_\gamma\Theta^\gamma{}_\mu=Al_\mu,\quad A=-\lambda^\gamma(V^{1/2})_{,\gamma} \tag{2.6}$$

$$l^\gamma\Phi_{\alpha\gamma\beta}=2Al_\gamma l_\beta, \tag{2.7}$$

$$\begin{aligned}l_\gamma R^\gamma{}_{\mu\nu\alpha}&=-l_\gamma(l_\alpha l^\gamma)_{,\mu\nu}+l_\gamma(l_\nu l^\gamma)_{,\mu\alpha}\\&\quad+l^\beta(l_\mu l_\alpha)_{,\beta\nu}-l^\beta(l_\mu l_\nu)_{,\beta\alpha},\end{aligned} \tag{2.8}$$

$$\begin{aligned}l_\gamma l^\nu R^\gamma{}_{\mu\nu\alpha}&=l^\beta l^\gamma(l_\mu l_\alpha)_{,\beta\gamma}\\&=[(V_{,\gamma}\lambda^\gamma)_{,\beta}\lambda^\beta V]\lambda_\mu\lambda_\alpha.\end{aligned} \tag{2.9}$$

We can find the Ricci tensor by using the identities (2.5)–(2.7), and letting $\gamma=\nu$ in Eq. (2.4). It reads

$$\begin{aligned}R_{\mu\alpha}&=-(l_\alpha l^\gamma)_{,\mu\gamma}+\eta^{\gamma\beta}(l_\mu l_\alpha)_{,\beta\gamma}-(l_\mu l^\gamma)_{,\gamma\alpha}\\&\quad+2[\eta^{\gamma\tau}l^\beta{}_{,\gamma}l_{\beta,\tau}-(K\lambda^\gamma)_{,\gamma}+(A^2-L^2)]l_\mu l_\alpha,\end{aligned} \tag{2.10}$$

where

$$L=-l^\alpha{}_{,\alpha}=-(V^{1/2}\lambda^\alpha)_{,\alpha} \tag{2.11a}$$

$$K=(A+L)V^{1/2}=-(V\lambda^\mu)_{,\mu}\,. \tag{2.11b}$$

The Ricci tensor with mixed components $R^\mu{}_\alpha$ and curvature scalar are,

$$R^\mu{}_\alpha=(K\lambda^\mu)_{,\alpha}+\eta^{\mu\gamma}(K\lambda_\alpha)_{,\gamma}+\Box(V\lambda^\mu\lambda_\alpha), \tag{2.12}$$
$$R=2(K\lambda^\mu)_{,\mu},\quad \Box=\eta^{\mu\nu}\partial_\mu\partial_\nu.$$

Hence the Einstein tensor reads

$$G^\mu{}_\alpha=(K\lambda^\mu)_{,\alpha}+\eta^{\mu\gamma}(K\lambda_\alpha)_{,\gamma}+\Box(V\lambda^\mu\lambda_\alpha)-\delta^\mu{}_\alpha(K\lambda^\gamma)_{,\gamma}, \tag{2.13}$$

with K given by (2.11b).

The algebraic classification of space–times is done by means of Weyl's conformal tensor which is defined by

$$\begin{aligned}C^\gamma{}_{\mu\nu\alpha}&=R^\gamma{}_{\mu\nu\alpha}+\tfrac12 g_{\mu\alpha}R^\gamma{}_\alpha-\tfrac12 g_{\mu\nu}R^\gamma_\alpha+\tfrac12 g_{\mu\alpha}\delta^\gamma{}_\nu\\&\quad-\tfrac12 R_{\mu\nu}\delta^\gamma{}_\alpha-\tfrac16(g_{\mu\alpha}\delta^\gamma{}_\nu-g_{\mu\nu}\delta^\gamma{}_\alpha)R.\end{aligned} \tag{2.14}$$

We can easily find that

$$\lambda_\gamma\lambda^\nu C^\gamma{}_{\mu\nu\alpha}=H\lambda_\mu\lambda_\alpha, \tag{2.15}$$

where

$$H=V(V_{,\mu}\lambda^\mu)_{,\nu}\lambda^\nu+A^2-L^2+\eta^{\gamma\beta}l^\tau{}_{,\beta}l_{\tau,\gamma}-\tfrac23(K\lambda^\mu)_{,\mu},$$

with A and L given by (2.6) and (2.11a), respectively. Equation (2.15) tells us that λ_μ is a double Debever–Penrose vector, thus space–time is algebraically degenerate.

Now let us show that Einstein's tensor is divergence free in the ordinary sense, that is,

$$\partial_\mu G^\mu{}_\nu=0. \tag{2.16}$$

From the Bianchi identities we know that $G^\mu{}_\nu$ is conserved covariantly,

$$\nabla_\mu G^\mu{}_\alpha=\partial_\mu G^\mu{}_\alpha+\Gamma^\mu{}_{\mu\beta}G^\beta{}_\alpha-\Gamma^\beta{}_{\mu\alpha}G^\mu{}_\beta=0; \tag{2.17}$$

from Eq. (2.1), we have

$$\Gamma^\mu{}_{\mu\beta}=\partial_\beta\sqrt{-g}=0;$$

and it is also straightforward to show that

$$\Gamma^\beta{}_{\mu\alpha}G^\mu{}_\beta=0.$$

Thus, we obtain Eq. (2.16). In general we know that the conservation law for the total energy–momentum tensor is given as

$$\partial_\mu(T^\mu{}_\nu+t^\mu{}_\nu)=0$$

where $t^\mu{}_\nu$ is the p.e.m.t. of the gravitational field. This p.e.m.t. is given in different forms, i.e., the Einstein[9] and the Landau[10] forms. In our case these two forms are the same because of Eq. (2.1) and they both vanish. The total e.m.t. $T_{\mu\nu}+t_{\mu\nu}$ is given by

$$T^\mu{}_\nu+t^\mu{}_\nu=\frac{-1}{2G}\partial_\rho f^{\rho\mu}{}_\nu, \tag{2.18}$$

where G is the gravitational constant and $f^{\rho\mu}{}_\nu$ is defined as

$$f^{\mu\rho}{}_\nu=-f^{\rho\mu}{}_\nu=\mathcal{G}_{\nu\sigma}\partial_\lambda(\mathcal{G}^{\sigma\mu}\mathcal{G}^{\rho\lambda}-\mathcal{G}^{\sigma\rho}\mathcal{G}^{\mu\lambda}),$$

where

$$\mathcal{G}^{\sigma\mu}=\sqrt{-g}\,g^{\sigma\mu},$$
$$\mathcal{G}_{\sigma\mu}=\frac{1}{\sqrt{-g}}g_{\sigma\mu}.$$

For the Kerr–Schild metric $f^{\mu\rho}{}_\nu$ becomes

$$f^{\mu\rho}{}_{\nu}=2\partial_\beta[V\delta^\mu{}_\nu\lambda^\rho\lambda^\beta+V\eta^{\rho\beta}\lambda_\nu\lambda^\mu-V\eta^{\mu\beta}\lambda^\rho\lambda_\nu-V\delta^\rho{}_\nu\lambda^\mu\lambda^\beta]. \tag{2.19}$$

We find that

$$G(T^\mu{}_\nu+t^\mu{}_\nu)=G^\mu{}_\nu. \tag{2.20}$$

Hence, from the Einstein equations

$$t^\mu{}_\nu=0. \tag{2.21}$$

Using this fact and Gupta's equation[11] which reads

$$\partial_\alpha\partial_\beta(\eta^{\alpha\beta}\mathcal{G}^{\mu\nu}-\eta^{\mu\alpha}\mathcal{G}^{\nu\beta}-\eta^{\nu\alpha}\mathcal{G}^{\mu\beta}+\eta^{\mu\nu}\mathcal{G}^{\alpha\beta})$$
$$=2G\eta^{\mu\nu}(T^\nu{}_\lambda+t^\nu{}_\lambda) \tag{2.22}$$

and using the metric in Eq. (2.2) we recover Eq. (2.13). The absence of $t^\nu{}_\lambda$ in Eq. (2.22) makes the field equations linear, because the total energy—momentum tensor $(T^\nu{}_\lambda+t^\nu{}_\lambda)$ can only depend on the metric itself not on its derivatives. Thus in the Kerr—Schild geometry Einstein's equations take the linear form

$$\partial_\alpha\partial_\beta(\eta^{\alpha\beta}g^{\mu\nu}-\eta^{\mu\alpha}g^{\nu\beta}-\eta^{\nu\alpha}g^{\mu\beta}+\eta^{\mu\nu}g^{\alpha\beta})=2G\eta^{\mu\lambda}T^\nu{}_\lambda \tag{2.23}$$

where $T^\nu{}_\lambda$ is the energy—momentum tensor of matter and radiation excluding the gravitational field.

3. GRAVITATIONAL FIELD OF ACCELERATED SYSTEMS (NONSPINNING CASE)

Assume that any element of the system under consideration is on a geodesic Γ which is described by an affine parameter τ. Construct a light cone from the observation point x^μ, which intersects the geodesic Γ at any point $Z^\mu(\tau)$. The velocity of the element of the system is

$$\dot{Z}_\mu=\frac{dZ_\mu}{d\tau},$$

with

$$\eta^{\mu\nu}\dot{Z}_\mu\dot{Z}_\nu=\epsilon$$

where $\epsilon=1$ and $\epsilon=0$ correspond to the timelike and lightlike cases, respectively. We define a retarded distance R by

$$R=\dot{Z}^\mu(x_\mu-Z_\mu(\tau_0)),$$

for the value τ_0 of τ for which the distance between the point $Z^\mu(\tau_0)$ and the point x^μ is lightlike, that is,

$$\eta_{\mu\nu}(x^\mu-Z^\mu(\tau_0))(x^\mu-Z^\mu(\tau_0))=0. \tag{3.1}$$

From now on we shall use $\dot{Z}^\mu$ to denote $\dot{Z}^\mu(\tau_0)$. Differentiation of Eq. (3.1) with respect to x^μ gives us

$$\partial_\mu\tau_0=[x_\mu-Z_\mu(\tau_0)]/R\ .$$

Now, we can define the lightlike 4-vector λ_μ as

$$\lambda_\mu=\partial_\mu\tau_0. \tag{3.2}$$

It is straightforward to show that λ_μ satisfies Eqs. (1.2), (1.3). In order to find the e.m.t. we need the following identities:

$$\partial_\alpha R=\dot{Z}_\alpha-\lambda_\alpha(\epsilon-R\ddot{Z}_\beta\lambda^\beta), \tag{3.3a}$$

$$\lambda^\alpha\partial_\alpha R=\lambda^\alpha\dot{Z}_\alpha=1, \tag{3.3b}$$

$$\lambda_{\mu,\nu}=\frac{1}{R}[\eta_{\mu\nu}-\lambda_\mu\dot{Z}_\nu-\lambda_\nu\dot{Z}_\mu+\lambda_\mu\lambda_\nu(\epsilon-R\ddot{Z}^\alpha\lambda_\alpha)], \tag{3.3c}$$

$$K=-\frac{1}{R}(RV'-2V), \tag{3.3d}$$

$$V'=\frac{dV}{dR}.$$

Here, we assume that the scalar function V is only a function of the retarded distance R. Using Eq. (3.3a)—(3.3d) and Eq. (2.13), we get the e.m.t. as[12]

$$T^\mu{}_\nu=\left(V''+\frac{2V'}{R}\right)\delta^\mu{}_\nu+\left(-V''+\frac{2V}{R^2}\right)(\dot{Z}^\mu\lambda_\nu+\dot{Z}_\nu\lambda^\mu)$$
$$+[\epsilon V''-2zV'+2(-\epsilon+2zR)(V/R^2)]\lambda^\mu\lambda_\nu \tag{3.4}$$

where

$$z=\ddot{Z}^\alpha\lambda_\alpha. \tag{3.5}$$

This e.m.t. in Eq. (3.4) has some simple forms for some special V's. When

$V=m/R-e^2/2R^2$ (e and m are constants),

we get the Bonner and Vaidya[7] solution. When

$V=(\rho_0/6)R^2$, ρ_0 is const,

we get a new solution corresponding to the gravitational field generated by a de Sitter space in accelerated motion, i.e., the interior solution corresponds to a finite matter free space—time region with a cosmological constant, so that

$$G^\mu{}_\nu-\rho_0\delta^\mu{}_\nu=0.$$

We verify that the only vacuum solution with zero cosmological constant is the Schwarzschild metric with $z=0$ for a nonspinning system.

4. COMPLEXIFICATION: FIELD OF UNACCELERATED SPINNING SYSTEMS

The gravitational field of an accelerated spinning system can be found either by solving the Einstein equations given in Eq. (2.13) or by complexifying the solutions found in the previous section. We choose the second method because of its simplicity and use special relativistic spinor representations of the four vectors.

In our method we simply make a complex translation along x^μ and find the real and imaginary parts of every four vectors and scalar functions. The new complex quantities are

$$x'^\mu=x^\mu+ia^\mu,$$
$$\tau'=\tau_1+i\tau_2,$$
$$Z'_\mu=Z_{1\mu}+iZ_{2\mu},$$
$$\dot{Z}'_\mu=v_{1\mu}+iv_{2\mu},$$

where a_μ is a constant spacelike 4-vector and[13]

$$v_{1\mu}=\frac{\partial Z_{1\mu}}{\partial\tau_1}=\frac{\partial Z_{2\mu}}{\partial\tau_2},$$

$$v_{2\mu}=-\frac{\partial Z_{1\mu}}{\partial\tau_2}=\frac{\partial Z_{2\mu}}{\partial\tau_1}.$$

Instead of Eq. (3.1) we have the following two equations:

$$(x_\mu-Z_{1\mu})(x^\mu-Z_1{}^\mu)-(a_\mu-Z_{2\mu})(a^\mu-Z_2{}^\mu)=0, \tag{4.1}$$

$$(x_\mu - Z_{1\mu})(a^\mu - Z_2{}^\mu) = 0. \tag{4.2}$$

We shall look for solutions that can be expressed in terms of the complex null vector λ'_μ

$$\lambda'_\mu = \partial_\mu \tau' = \mu_\mu + i\nu_\mu, \tag{4.3}$$

where

$$\mu_\mu = \partial_\mu \tau_1 = \frac{1}{r_1{}^2 + r_2{}^2}[r_1(x_\mu - Z_{1\mu}) + r_2(a_\mu - Z_{2\mu})], \tag{4.4}$$

$$\nu_\mu = \partial_\mu \tau_2 = \frac{1}{r_1{}^2 + r_2{}^2}[-r_2(x_\mu - Z_{1\mu}) + r_1(a_\mu - Z_{2\mu})], \tag{4.5}$$

and the complex retarded distance is

$$R' = r_1 + ir_2, \tag{4.6}$$

where

$$r_1 = v_1{}^\mu(x_\mu - Z_{1\mu}) - v_2{}^\mu(a_\mu - Z_{2\mu}),$$

$$r_2 = v_2{}^\mu(x_\mu - Z_{1\mu}) + v_1{}^\mu(a_\mu - Z_{2\mu}).$$

Instead of the identities in Eqs. (3.3a)–(3.3d), we have the following:

$$r_{1,\alpha} = v_{1\alpha} + k\mu_\alpha + l\nu_\alpha, \tag{4.7a}$$

$$r_{2,\alpha} = v_{2\alpha} - l\mu_\alpha + k\nu_\alpha, \tag{4.7b}$$

$$\mu_\alpha v_1{}^\alpha - v_2{}^\alpha \nu_\alpha = 1, \tag{4.7c}$$

$$v_{1\alpha}\nu^\alpha + v_{2\alpha}\mu^\alpha = 0, \tag{4.7d}$$

$$\mu_\alpha\mu^\alpha = \nu_\alpha\nu^\alpha, \tag{4.7e}$$

$$\mu_\alpha\nu^\alpha = 0, \tag{4.7f}$$

$$\begin{aligned}\mu_{\alpha,\beta} &= \frac{1}{r_1{}^2 + r_2{}^2}[r_1\eta_{\alpha\beta} + (r_2 l - r_1 k)\mu_\alpha\mu_\beta - (r_2 l - r_1 k)\nu_\alpha\nu_\beta \\ &\quad - (r_1 l + r_2 k)(\mu_\alpha\nu_\beta + \mu_\beta\nu_\alpha) - r_2(\mu_\alpha v_{2\beta} + \mu_\beta v_{2\alpha} \\ &\quad + \nu_\alpha v_{1\beta} + \nu_\beta v_{1\alpha}) + r_1(-\mu_\alpha v_{1\beta} - v_{1\alpha}\mu_\beta + \nu_\alpha v_{2\beta} \\ &\quad + \nu_\beta v_{2\alpha})],\end{aligned} \tag{4.7g}$$

$$\begin{aligned}\nu_{\alpha,\beta} &= \frac{1}{r_1{}^2 + r_2{}^2}[-r_2\eta_{\alpha\beta} + (r_1 l + r_2 k)\mu_\alpha\mu_\beta - (r_1 l + r_2 k)\nu_\alpha\nu_\beta \\ &\quad + (-r_1 k + r_2 l)(\mu_\alpha\nu_\beta + \mu_\beta\nu_\alpha) - r_1(\nu_\alpha v_{1\beta} + \nu_\beta v_{1\alpha} \\ &\quad + \mu_\alpha v_{2\beta} + \mu_\beta v_{2\alpha}) + r_2(\mu_\alpha v_{1\beta} + \mu_\beta v_{1\alpha} - v_{2\alpha}\nu_\beta - v_{2\beta}\nu_\alpha)],\end{aligned} \tag{4.7h}$$

where

$$k = -1 + r_1 a_1{}^\alpha\mu_\alpha - r_2\nu_\alpha a_1{}^\alpha - r_2 a_2{}^\alpha\mu_\alpha - r_1 a_2{}^\alpha\nu_\alpha, \tag{4.8}$$

$$l = -r_1 a_2{}^\alpha\mu_\alpha + r_2 a_2{}^\alpha\nu_\alpha - r_2 a_1{}^\alpha\mu_\alpha - r_1 a_1{}^\alpha\nu_\alpha, \tag{4.9}$$

$$a_{1\alpha} = \frac{\partial v_{1\alpha}}{\partial \tau_1} = \frac{\partial v_{2\alpha}}{\partial \tau_2}, \tag{4.10}$$

$$a_{2\alpha} = -\frac{\partial v_{1\alpha}}{\partial \tau_2} = \frac{\partial v_{2\alpha}}{\partial \tau_2}. \tag{4.11}$$

Now in order to find the real null 4-vector λ_μ we use the spinor representation of the 4-vectors. If A_μ is a 4-vector its spinor equivalent is given as

$$A = \sigma_\mu A^\mu \quad (A^\mu = \sigma^\mu{}_{\alpha\mathring{\beta}} A^{\alpha\mathring{\beta}};\ \alpha, \mathring{\beta} = 1, 2),$$

where

$$\sigma_\mu = (\sigma_0, \boldsymbol{\sigma}).$$

σ_0 is two-dimensional identity matrix and $\boldsymbol{\sigma}$'s are the Pauli spin matrices. They satisfy the following anti-commutation relation,

$$\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\eta_{\mu\nu}. \tag{4.12}$$

where

$$\bar{\sigma}_\mu = \sigma_2(\sigma_\mu)^T\sigma_2 = (\sigma_0, -\boldsymbol{\sigma}),$$

and $\sigma_\mu{}^T$ denotes the transpose of σ_μ. Using the spinor representations of the complex vectors λ'_μ and $\dot{Z}'_\mu$ we get the following identities:

$$\lambda'\bar{\dot{Z}}' + \dot{Z}'\bar{\lambda}' = 2, \tag{4.13a}$$

$$\lambda'\bar{\lambda}' = 0, \tag{4.13b}$$

$$\dot{Z}'\bar{\dot{Z}}' = 1. \tag{4.13c}$$

Then we define the spinor representation of the real null vector λ_μ as

$$\lambda = \lambda'\bar{v}_1\lambda'^\dagger[\mathrm{Tr}(\lambda')/\mathrm{Tr}(\lambda'\bar{v}_1\lambda'^\dagger)], \tag{4.14}$$

where $v_{1\mu}$ is the real part of Z'_μ. Note that when λ'_μ is real ($a_\mu = 0$), Eq. (4.14) becomes an identity. Since, in this section, we are only interested in the fields of the systems with uniform velocity, the procedure outlined above becomes simpler. The scalar functions k and l in Eqs. (4.8) and (4.9) become -1 and 0, respectively, and

$$r_{1,\alpha} = n_\alpha - \mu_\alpha,$$

$$r_{2,\alpha} = -\nu_\alpha,$$

where

$$\dot{Z}'_\mu = v_{1\mu} = n_\mu, \quad v_{2\mu} = 0.$$

Then, the null vector λ_μ can be found from Eq. (4.14) as

$$\lambda_\alpha = \frac{\mu^\beta\mu_\beta n_\alpha - \mu_\alpha + \epsilon_{\alpha\gamma\tau\beta}n^\gamma\mu^\tau\nu^\beta}{\mu^\beta\mu_\beta - 1}. \tag{4.15}$$

The derivative of λ_μ with respect to the coordinates x^μ is found as[14]

$$\begin{aligned}\lambda_{\mu,\nu} &= \frac{1}{r_1{}^2 + r_2{}^2}[r_1(\eta_{\mu\nu} + \lambda_{\mu\nu} - n_\mu\lambda_\nu - n_\nu\lambda_\mu) \\ &\quad + r_2\epsilon_{\mu\nu\alpha\beta}n^\alpha\lambda^\beta]\end{aligned} \tag{4.16}$$

Here, we notice that the velocity vector n_μ is a Killing vector, because it satisfies the equations

$$n^\alpha\lambda_\alpha = 1,$$

$$n^\alpha\lambda_{\beta,\alpha} = 0,$$

and since it is a timelike vector, it is always possible to bring it to its rest frame $\delta_\mu{}^0$ by a Lorentz transformation. Thus

$$n^\mu\partial_\mu g_{\alpha\beta} = \partial_0 g_{\alpha\beta} = 0.$$

To find the e.m.t. we make a further choice for the form of V by taking

$$V = f(r_1)/(r_1{}^2 + r_2{}^2). \tag{4.17}$$

With this assumption, the Kerr–Schild metric can be transformed into the Boyer–Lindquist[15] coordinate system, which reads

$$ds^2 = \left(1 - \frac{2f}{\Sigma}\right)d\bar{t}^2 - \frac{\Sigma}{\Delta}dr_1{}^2 - \Sigma d\Theta^2 - \frac{B}{\Sigma}\sin^2\Theta\, d\bar{\phi}^2 + \frac{4af}{\Sigma}\sin^2\Theta\, d\bar{\phi}\, dt,$$

where

$$\Sigma = r_1{}^2 + a^2\cos^2\Theta,$$

$$\Delta = r_1{}^2 + a^2 - 2f,$$

$$B = (r_1{}^2 + a^2)^2 - a^2\Delta\sin^2\Theta.$$

Transformations from Kerr–Schild coordinates into the Boyer–Lindquist coordinates are

$$(r_1 + ia)e^{i\phi}\sin\Theta = x + iy,$$

$$r_1\cos\Theta = z,$$

$$dt = d\bar{t} + \frac{2f}{\Delta}dr_1,$$

$$d\phi = d\bar{\phi} + \frac{a}{\Delta}dr_1.$$

In the new coordinate system λ_μ and the e.m.t. $T^\mu{}_\nu$ take the forms

$$\lambda_\mu = \left(1, \frac{\Sigma}{\Delta}, 0, -a\sin^2\Theta\right),$$

$$T_{\mu\nu} = (D + 4h)u_\mu u_\nu - (D + 4h)\frac{\Sigma}{\Delta}m_\mu m_\nu - (D + 2h)g_{\mu\nu},$$

where

$$u_\mu = \frac{\Delta}{\Sigma}(1, 0, 0, -a\sin^2\Theta),$$

$$m_\mu = (0, -1, 0, 0),$$

$$D = -f_{r_1 r_1}/\Sigma \quad \left(f_{r_1} = \frac{df}{dr_1}\right)$$

$$h = \frac{r_1 f_{r_1} - f}{\Sigma^2}.$$

The Kerr and the charged Kerr metrics correspond to the vanishing of $f_{r_1 r_1}$. For an interior metric, the e.m.t. in Eq. (4.18) corresponds to the e.m.t. of an anisotropic perfect fluid distribution. Isotropy is destroyed in the radial direction. We note that the deviation from a perfect fluid distribution can also be regarded as arising from the contribution of a moving Nambu string.[16] Such anisotropic energy–momentum tensors have also been discussed recently by Bowers and Liang.[17] This interior metric matches to the Kerr metric on an oblate spheroid, $r_1 = r_0$ the equation of this surface being

$$r_0{}^4 - r_0{}^2(r^2 - a^2) - a^2z^2 = 0$$

where

$$r^2 = x^2 + y^2 + z^2$$

and the function $f(r_1)$ satisfies the following boundary conditions

$$f(r_0) = mr_0,$$

$$\left.\frac{d}{dr_1}f(r_1)\right|_{r_1=r_0} = m,$$

m being the total mass.

5. LINEARIZED GENERAL RELATIVITY AND TRAUTMAN'S COMPLEX TRANSLATION

The linearized theory of the gravitational field can be developed by regarding the actual Riemannian space—time as a first order perturbation of flat space—time. Here, in contrast to many authors we take $\mathcal{G}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ as the gravitational field and assume that[11,18]

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} + 2\epsilon\phi^{\mu\nu}, \tag{5.1}$$

where ϵ is a constant and $\phi^{\mu\nu}$ is a symmetric tensor. In the linearized theory, we neglect terms of all but the first order in ϵ. Hence

$$(-\det\mathcal{G}^{\mu\nu})^{1/2} = (-\det g_{\mu\nu})^{1/2} = (-g)^{1/2} = 1 + \epsilon\phi, \tag{5.2}$$

where

$$\phi = \eta^{\mu\nu}\phi_{\mu\nu}. \tag{5.3}$$

Field equations, in terms of $\phi^{\mu\nu}$, follow as

$$GT^\mu{}_\nu = +\epsilon(-\eta^{\mu\beta}\phi^\gamma{}_{\nu,\beta\gamma} - \eta^{\alpha\beta}\phi^\mu{}_{\alpha,\beta\nu} + \Box\phi^\mu{}_\nu - \delta^\mu{}_\nu\phi^{\alpha\beta}{}_{,\alpha\beta}). \tag{5.4}$$

Without any choice of gauge, it is easy to show that

$$\partial_\mu T^\mu{}_\nu = 0,$$

and, of course, the pseudo-energy–momentum tensor vanishes in this approximation.[11]

It is remarkable that the field equation (5.4) is exactly the same as the one (2.33) which was obtained for the Kerr–Schild metric. All the metrics which are in the Kerr—Schild class are also the solution of linearized field equations (5.4), but the reverse is not true in general.

Trautman[4] has developed a method of constructing classes of new solutions to linear special relativistic partial differential equations. In particular, he used the method to produce null curling solutions of Maxwell's equations and he stated that the same method can also be used in linearized Einstein's equations. Now it becomes very clear that Newman's complex translation is nothing but Trautman's complex translation.

6. CONCLUSION

To obtain linear gravitational field equations there are two possible methods. In the first one we use an approximation procedure which leads to linearized general relativity. In the second one we put some constraints on the symmetric tensor $\phi^{\mu\nu}$ in Eq. (5.1), in such a way that the pseudo-energy—momentum tensor vanishes. In this work we showed the existence of the second possibility. It is an open question whether the Kerr—Schild coordinate system is the only coordinate system in which Einstein's equations become linear for a special geometry.

We have further obtained the gravitational field of accelerated nonspinning particles and unaccelerated spinning particles. It is also possible to obtain the gravitational field of accelerated spinning particles. Work on the latter type solution is in progress.

If we take the null vector λ_μ as a constant null vector, the Kerr–Schild metric describes gravitational waves such that plane fronted waves are in this class of metrics with nonvanishing Weyl tensor. The corresponding space—time is of Petrov-type N.

As another possible application of our method the following remark is in order. Quantization of general relativity becomes simple for the linearized approximate theory. Since in the case of special geometries Einstein's theory becomes exactly linear in the Kerr–Schild coordinates, the same quantization procedure could also be applied in these special cases.

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