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Lorentz violation with an antisymmetric tensor

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Field theories with spontaneous Lorentz violation involving an antisymmetric 2-tensor are studied. A general action including nonminimal gravitational couplings is constructed, and features of the Nambu-Goldstone and massive modes are discussed. Minimal models in Minkowski spacetime exhibit dualities with Lorentz-violating vector and scalar theories. The post-Newtonian expansion for nonminimal models in Riemann spacetime involves qualitatively new features, including the absence of an isotropic limit. Certain interactions producing stable Lorentz-violating theories in Minkowski spacetime solve the renormalization-group equations in the tadpole approximation.

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I. INTRODUCTION

Among the simpler field theories in Minkowski spacetime are ones built from p-forms. These include electrodynamics, which is the Abelian gauge theory of a 1-form, and field theories constructed with antisymmetric p-tensors. The predominant examples of the latter include models with a gauge-invariant kinetic term for an antisymmetric 2-tensor, sometimes called the notoph [1] or the Kalb-Ramond field [2]. These theories have some elegant properties, including dualities to other p-form theories [1–3].

In this work, we consider Lorentz-violating field theories with an antisymmetric 2-tensor, including models coupled to gravity. In a generic Lagrange density, terms can be constructed that explicitly violate Lorentz symmetry by forming observer invariants from tensor operators and c-number coefficients. However, explicit breaking is generically incompatible with the Bianchi identities in Riemann geometry and hence is problematic for theories with gravity [4]. A viable alternative is spontaneous Lorentz violation, in which a potential term drives the development of a nonzero vacuum value for a tensor field [5]. In theories of this type, the Lagrange density is Lorentz invariant, but the presence of the tensor vacuum value means the physics can display Lorentz breaking. Here, our focus is on theories with spontaneous Lorentz violation triggered by a potential for an antisymmetric 2-tensor field.

Spontaneous Lorentz violation triggered by a potential for an arbitrary tensor field is accompanied by certain generic features. Massless Nambu-Goldstone (NG) modes [6] emerge that are associated with field fluctuations along the broken Lorentz generators [7]. When a smooth potential drives the Lorentz breaking, massive modes can also appear [8]. The role of the NG and massive modes is central to the physical content of a field theory with spontaneous Lorentz violation. Some of their properties are generic to any field theory, while others depend on the specific field content and structure of the action. One goal here is to apply this work to theories based on an antisymmetric 2-tensor field, establishing some basic properties of the corresponding NG and massive modes.

Another motivation for this work stems from the possibility of novel experimental signals for Lorentz violation involving gravitational couplings. Recent years have seen extensive tests of Lorentz symmetry in Minkowski spacetime [9], but the scope of searches involving gravitational couplings remains comparatively limited. Dominant curvature couplings involving Lorentz violation are controlled by three coefficient fields, conventionally denoted as $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$, and u [4]. Constraints on some $s^{\mu\nu}$ coefficients have been attained using lunar laser ranging [10] and atom interferometry [11], and numerous other experimental and observational signals from these couplings can arise at the post-Newtonian level [12]. However, to date no gravitational field theory has been constructed to yield nonzero $t^{\kappa\lambda\mu\nu}$ coefficients. We show here that a comparatively simple coupling of this type can appear in theories involving gravitational couplings to an antisymmetric 2-tensor. The post-Newtonian expansion is affected in a purely anisotropic way, offering a qualitatively distinct source of signals for Lorentz violation.

In Minkowski spacetime, spontaneous Lorentz violation arises whenever the potential for the interactions has a nontrivial stable extremum. An interesting issue is the behavior of the Lorentz-violating interactions under quantum corrections. In certain vector models with spontaneous Lorentz violation, nontrivial potentials solve the renormalization-group (RG) equations in the tadpole approximation [13]. Part of the present work revisits this possibility in the context of a minimal theory with an antisymmetric 2-tensor. We investigate the RG flow in the tadpole approximation and obtain analytical solutions for the potential. All the nontrivial stable potentials that result describe theories with spontaneous Lorentz violation.

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The organization of this work is as follows. We begin in Sec. II with the basic construction of the field theory, including the gravitational couplings. Some general features of the potential and consequences of the spontaneous Lorentz breaking are discussed. In Sec. III, properties of minimal models in Minkowski spacetime are established, including correspondences to dual theories. Section IV focuses on gravitational couplings and their consequences for post-Newtonian physics. The field equations are obtained, linearization is performed in a Minkowski background, and the post-Newtonian metric is obtained at third order. In Sec. V, we return to the minimal model in Minkowski spacetime and implement the tadpole approximation for the RG equations, obtaining solutions for the potential. A summary is provided in Sec. VI. Throughout this work, we adopt the conventions of Ref. [4].

II. FIELD THEORY

In this section, the action for an antisymmetric 2-tensor in four-dimensional Riemann spacetime is considered. Some definitions are introduced, and basic properties of the field theory are summarized. We construct the nonminimal gravitational couplings and discuss some general features associated with the potential driving spontaneous Lorentz violation.

A. Setup

The fundamental field of interest in this work is an antisymmetric 2-tensor, denoted $B_{\mu\nu} = -B_{\nu\mu}$. It is convenient to introduce the dual tensor $\mathfrak{B}_{\mu\nu}$, defined by

$$\mathfrak{B}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\kappa\lambda}B^{\kappa\lambda}, \tag{1}$$

where $\epsilon_{\kappa\lambda\mu\nu}$ is the totally antisymmetric Levi-Civita tensor. The covariant derivative of $B_{\mu\nu}$ is denoted $D_\lambda B_{\mu\nu}$. This work considers Riemann spacetimes, and the covariant derivative is constructed with the Levi-Civita connection. The generalization to the Cartan connection and the corresponding Riemann-Cartan spacetimes with torsion [14] is of potential interest, although current experimental constraints [15] suggest nonzero torsion components are likely to have at most a limited phenomenological impact.

A useful combination of derivatives is the totally antisymmetric field-strength tensor $H_{\lambda\mu\nu}$, given by

$$H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} \tag{2}$$

and its dual

$$\mathfrak{H}_\kappa \equiv \tfrac{1}{6}\epsilon_{\kappa\lambda\mu\nu}H^{\lambda\mu\nu}. \tag{3}$$

The field strength $H_{\lambda\mu\nu}$ can be viewed as the components of an exact 3-form field H constructed via the exterior derivative from the 2-form B associated with $B_{\mu\nu}$. Covariant derivatives can also be used in Eq. (2) because the connection coefficients cancel in Riemann spacetime. The field strength $H_{\lambda\mu\nu}$ satisfies the identity

$$\partial_\kappa H_{\lambda\mu\nu} - \partial_\lambda H_{\mu\nu\kappa} + \partial_\mu H_{\nu\kappa\lambda} - \partial_\nu H_{\kappa\lambda\mu} = 0, \tag{4}$$

which follows because an exact 3-form is closed. Again, covariant derivatives can be used in this expression instead.

The field strength $H_{\lambda\mu\nu}$ is invariant under a gauge transformation of $B_{\mu\nu}$ given by

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \tag{5}$$

which represents a shift of B by an exact 2-form. The gauge parameter Λ_μ has four components, but the transformation involves only three independent effects because the shift

$$\Lambda_\mu \rightarrow \Lambda_\mu + \partial_\mu\Sigma \tag{6}$$

leaves Eq. (5) unchanged. This latter shift represents a subsidiary gauge transformation involving an exact 1-form.

The action for the theory including gravitational and matter sectors can be written as

$$S = \int d^4x e(\mathcal{L}_g + \mathcal{L}_M + \mathcal{L}_B + \mathcal{L}_V), \tag{7}$$

where e is the metric determinant and the Lagrange density $e\mathcal{L}$ is split into four pieces, corresponding to the pure-gravity sector $e\mathcal{L}_g$, the matter sector $e\mathcal{L}_M$, the $B_{\mu\nu}$ kinetic term $e\mathcal{L}_B$, and the potential term $e\mathcal{L}_V$. For our purposes, it suffices for $e\mathcal{L}_g$ to adopt the usual Einstein-Hilbert action of general relativity with cosmological constant Λ,

$$e\mathcal{L}_g = \frac{e}{2\kappa}(R - 2\Lambda), \tag{8}$$

where $\kappa = 8\pi G_N$ with G_N the Newton gravitational constant. Also, the specific content of the matter-sector Lagrange density $e\mathcal{L}_M$ is secondary here, and in much of the analysis to follow it suffices to assume vanishing matter couplings to $B_{\mu\nu}$. When useful, a matter coupling to $B_{\mu\nu}$ can be introduced in the form

$$e\mathcal{L}_M \supset -\tfrac{1}{2}eB_{\mu\nu}j_B^{\mu\nu}, \tag{9}$$

where $j_B^{\mu\nu}$ is the corresponding current. This coupling is analogous to that of the Kalb-Ramond current in string theory [2]. For certain actions of the form (7), including ones in Minkowski spacetime that are invariant under the gauge transformation (5), the current $j_B^{\mu\nu}$ is conserved. Note also that a nonzero vacuum value for $B_{\mu\nu}$ can lead to terms in the effective action of the type found in the minimal standard-model extension (SME) [16]. For example, a current $j_B^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$ generates an SME coefficient of the $H_{\mu\nu}$ type.

B. Kinetic term

By definition, the kinetic term $e\mathcal{L}_B$ in the action (7) determines the dynamics of $B_{\mu\nu}$, including its nonminimal couplings to gravity. In this work, we restrict attention to kinetic terms of second order in derivatives of $B_{\mu\nu}$. For some of the analysis, it is useful to allow also nonminimal nonderivative gravitational couplings that are linear in the curvature tensor. Higher-order derivative couplings associated with Lorentz violation could in principle be incorporated, at least at the level of effective field theory [17]. A classification of all derivative operators might be achieved following the methodology adopted for Lorentz-violating electrodynamics [18], but this lies beyond our present scope.

In the present section, we provide the general Lagrange density $\mathcal{L}_B$ containing all independent quadratic kinetic terms for the antisymmetric tensor $B_{\mu\nu}$, along with all independent nonminimal nonderivative couplings to gravity that are linear in the curvature. It is convenient to split the Lagrange density $\mathcal{L}_B$ into two parts,

$$e\mathcal{L}_B = e\mathcal{L}_{BB} + e\mathcal{L}_{B\mathfrak{B}}, \tag{10}$$

where the parity-even term $\mathcal{L}_{BB}$ involves quadratic expressions in $B_{\mu\nu}$ and the parity-odd term $\mathcal{L}_{B\mathfrak{B}}$ involves the product of $B_{\mu\nu}$ and $\mathfrak{B}_{\mu\nu}$.

The general form of the parity-even term can be written as

$$\begin{aligned} e\mathcal{L}_{BB} = {}& \tau_1 e H_{\lambda\mu\nu}H^{\lambda\mu\nu} + \tau_2 e(D_\lambda B^{\lambda\nu})(D_\mu B^\mu{}_\nu) \\ & + \tau_3 e B^{\kappa\lambda}B^{\mu\nu}R_{\kappa\lambda\mu\nu} + \tau_4 e B^{\lambda\nu}B^\mu{}_\nu R_{\lambda\mu} \\ & + \tau_5 e B^{\mu\nu}B_{\mu\nu}R, \end{aligned} \tag{11}$$

where τ_1, τ_2, τ_3, τ_4, and τ_5 are arbitrary constants. Note that only the first term in this expression is invariant under the gauge transformation (5). In constructing Eq. (11), we can omit the two scalars $(D_\lambda B_{\mu\nu})(D^\mu B^{\nu\lambda})$ and $(D_\lambda B^{\mu\nu}) \times (D^\lambda B_{\mu\nu})$ because they are equivalent to other terms via the identity

$$H_{\lambda\mu\nu}H^{\lambda\mu\nu} = 3(D_\lambda B^{\mu\nu})(D^\lambda B_{\mu\nu}) + 6(D_\lambda B_{\mu\nu})(D^\mu B^{\nu\lambda}) \tag{12}$$

and the integral relation

$$\begin{aligned} \int d^4x e\bigg[& (D_\lambda B_{\mu\nu})(D^\mu B^{\nu\lambda}) + (D_\lambda B^{\lambda\nu})(D_\mu B^\mu{}_\nu) \\ & - B^{\lambda\nu}B^\mu{}_\nu R_{\lambda\mu} + \frac{1}{2}B^{\kappa\lambda}B^{\mu\nu}R_{\kappa\lambda\mu\nu}\bigg] = 0. \end{aligned} \tag{13}$$

This last relation holds up to surface terms, which leave the equations of motion unaffected.

The general form of the parity-odd term involving both $B_{\mu\nu}$ and its dual $\mathfrak{B}_{\mu\nu}$ can be written as

$$\begin{aligned} e\mathcal{L}_{B\mathfrak{B}} = {}& \sigma_1 e(D_\lambda B_{\mu\nu})(D^\lambda \mathfrak{B}^{\mu\nu}) + \sigma_2 e(D_\lambda B^\lambda{}_\nu)(D_\mu \mathfrak{B}^{\mu\nu}) \\ & + \sigma_3 e B^{\kappa\lambda}\mathfrak{B}^{\mu\nu}R_{\kappa\lambda\mu\nu} + \sigma_4 e B_{\mu\nu}\mathfrak{B}^{\mu\nu}R, \end{aligned} \tag{14}$$

where σ_1, σ_2, σ_3, and σ_4 are arbitrary constants. Note that the nonminimal curvature-coupling term $B^{\mu\lambda}\mathfrak{B}^\nu{}_\lambda R_{\mu\nu}$ is proportional to the σ_4 term in Eq. (14) via the identity

$$B_\mu{}^\lambda \mathfrak{B}_{\nu\lambda} = \tfrac{1}{4}g_{\mu\nu}(B_{\alpha\beta}\mathfrak{B}^{\alpha\beta}). \tag{15}$$

Also, the scalar $(D_\lambda B_{\mu\nu})(D^\mu \mathfrak{B}^{\nu\lambda})$ is equivalent to terms in Eq. (14) via the integral relation

$$\begin{aligned} \int d^4x e\bigg[& (D_\lambda B_{\mu\nu})(D^\mu \mathfrak{B}^{\nu\lambda}) + (D_\lambda B^\lambda{}_\nu)(D_\mu \mathfrak{B}^{\mu\nu}) \\ & - \frac{1}{4}B_{\mu\nu}\mathfrak{B}^{\mu\nu}R + \frac{1}{2}B^{\kappa\lambda}\mathfrak{B}^{\mu\nu}R_{\kappa\lambda\mu\nu}\bigg] = 0. \end{aligned} \tag{16}$$

Terms quadratic in the dual tensor $\mathfrak{B}_{\mu\nu}$ can also be considered, including ones involving linear curvature couplings. However, all such terms are equivalent to combinations of ones in the parity-even Lagrange density (11). Some useful identities are

$$\begin{aligned} (D_\lambda \mathfrak{B}_{\mu\nu})(D^\mu \mathfrak{B}^{\nu\lambda}) &= \tfrac{1}{2}(D_\lambda B_{\mu\nu})(D^\lambda B^{\mu\nu}) \\ &\quad - (D_\lambda B^\lambda{}_\nu)(D_\mu B^{\mu\nu}), \\ \mathfrak{B}^{\kappa\lambda}\mathfrak{B}^{\mu\nu}R_{\kappa\lambda\mu\nu} &= -B^{\mu\nu}B^{\kappa\lambda}R_{\mu\nu\kappa\lambda} + 4B^{\lambda\nu}B^\mu{}_\nu R_{\lambda\mu} \\ &\quad - B^{\mu\nu}B_{\mu\nu}R, \\ \mathfrak{B}^{\mu\lambda}\mathfrak{B}^\nu{}_\lambda &= B^{\mu\lambda}B^\nu{}_\lambda - \tfrac{1}{2}g^{\mu\nu}B^{\alpha\beta}B_{\alpha\beta}. \end{aligned} \tag{17}$$

Using these identities, $\mathcal{L}_{BB}$ can be rewritten as an expression involving terms quadratic in $\mathfrak{B}_{\mu\nu}$.

One generalization of the above construction involves replacing the constants $\tau_1, \ldots, \sigma_4$ by arbitrary functions of $B_{\mu\nu}$. This idea has recently been used to identify an extension to the class of gravitationally coupled vector theories known as bumblebee models [19]. A similar extension of the models discussed in this work may also exist.

C. Potential term

The term $e\mathcal{L}_V$ in the action (7) incorporates the potential V triggering spontaneous symmetry breaking. We assume that V drives the formation of a nonzero vacuum value

$$\langle B_{\mu\nu}\rangle = b_{\mu\nu}, \tag{18}$$

which breaks local Lorentz and diffeomorphism symmetry. This implies a vacuum value for the dual field $\mathfrak{B}_{\mu\nu}$,

$$\langle \mathfrak{B}_{\mu\nu}\rangle = \mathfrak{b}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\kappa\lambda\mu\nu}b^{\kappa\lambda}. \tag{19}$$

In general, the potential V could include dependence on $B_{\mu\nu}$, on covariant derivatives of $B_{\mu\nu}$, on the Levi-Civita tensor $\epsilon_{\kappa\lambda\mu\nu}$, and on the metric $g_{\mu\nu}$. A pure-derivative potential has been investigated for the vector field in certain bumblebee models [20], and an analogous treatment could be considered here. However, for simplicity we disregard derivative couplings in V in this work.

Since the Lagrange density is an observer-scalar density, the dependence of the potential V on $B_{\mu\nu}$ can arise only through the invariants $B_{\mu\nu}B^{\mu\nu}$ and $B_{\mu\nu}\mathfrak{B}^{\mu\nu}$. Note that neither of these terms is invariant under the gauge transformation (5). Following the approach of Ref. [8], we introduce

$$X_1 \equiv B_{\mu\nu}B^{\mu\nu} - x_1, \qquad X_2 \equiv B_{\mu\nu}\mathfrak{B}^{\mu\nu} - x_2, \tag{20}$$

and we write the potential as

$$V = V(X_1, X_2), \tag{21}$$

where $\langle V\rangle = 0$ is assumed. In Eq. (20), x_1 and x_2 are two real numbers representing the vacuum values of the invariants,

$$\begin{aligned} x_1 &\equiv \langle B_{\mu\nu}B^{\mu\nu}\rangle = \langle g^{\kappa\mu}\rangle\langle g^{\lambda\nu}\rangle b_{\kappa\lambda}b_{\mu\nu}, \\ x_2 &\equiv \langle B_{\mu\nu}\mathfrak{B}^{\mu\nu}\rangle = \langle g^{\kappa\mu}\rangle\langle g^{\lambda\nu}\rangle b_{\kappa\lambda}\mathfrak{b}_{\mu\nu}, \end{aligned} \tag{22}$$

where $\langle g^{\mu\nu}\rangle$ is the vacuum value of the inverse metric.

For certain purposes, it is convenient to split $b_{\mu\nu}$ into the independent components b_{0j} and b_{jk} and to introduce spatial vectors $\vec{e}$ and $\vec{b}$ defined by

$$e^j = -b_{0j}, \qquad b^j = \tfrac{1}{2}\epsilon^{jkl}b_{kl}, \tag{23}$$

in analogy with the separation of the antisymmetric field strength into electric and magnetic vector fields in Maxwell electrodynamics. Under some circumstances, it is also convenient to perform observer rotation and boost transformations to attain a special observer frame in which $b_{\mu\nu}$ takes a simple block-diagonal form. This can be achieved in a local Lorentz frame in Riemann spacetime or everywhere in Minkowski spacetime. Provided at least one of x_1 and x_2 is nonzero, the special form can be chosen as

$$b_{\mu\nu} = \begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \tag{24}$$

where a and b are real numbers. In this special frame, $\vec{e} = (a, 0, 0)$ and $\vec{b} = (b, 0, 0)$, while $x_1 = -2(a^2 - b^2)$ and $x_2 = 4ab$. If both x_1 and x_2 vanish, then the replacements $b_{23} = -b_{32} \to 0$, $b_{13} = -b_{31} \to -a$ can be implemented in the above block-diagonal form instead. Note that $b_{\mu\nu}$ in the special frame is determined by no more than two nonzero real numbers, an improvement over the six real numbers required for the generic case. However, most of the analysis in this work makes no assumptions about the specific form of the vacuum value $b_{\mu\nu}$.

Adopting for the potential V the partial derivative notation

$$V_m = \frac{\partial V}{\partial X_m}, \qquad V_{mn} = \frac{\partial^2 V}{\partial X_m \partial X_n}, \quad \ldots \tag{25}$$

with $m, n, \ldots = 1, 2, \ldots$, the extremal conditions determining the vacuum are

$$V_m = 0 \qquad \text{(vacuum condition)}. \tag{26}$$

Since $X_m = 0$ in the vacuum, the potential can be expanded about the vacuum as the series

$$V(X_1, X_2) = \tfrac{1}{2}\lambda_{mn}X_mX_n + \tfrac{1}{6}\lambda_{mnp}X_mX_nX_p + \cdots, \tag{27}$$

where

$$\lambda_{mn} = V_{mn}(0,0), \qquad \lambda_{mnp} = V_{mnp}(0,0), \quad \ldots \tag{28}$$

are constants. A simple example of this type is provided by the smooth diagonal quadratic form with only λ_{11} and λ_{22} nonzero. Note that the values of the constants (28) are relevant to the issue of overall stability of a given vacuum, which in general is an involved question [21] and as yet remains only partially resolved even for comparatively simple vector-based models [22]. Another useful class of potentials involves linear or quadratic Lagrange multipliers [8]. However, for most of the analysis to follow, specifying the form of V is unnecessary.

In the theory (7), the field excitations of primary interest are the fluctuations in $g_{\mu\nu}$ and $B_{\mu\nu}$. The metric fluctuation $h_{\mu\nu}$ is given by

$$g_{\mu\nu} = \langle g_{\mu\nu}\rangle + h_{\mu\nu}, \tag{29}$$

where $\langle g_{\mu\nu}\rangle$ is the vacuum metric, while $B_{\mu\nu}$ can be expanded as

$$B_{\mu\nu} = b_{\mu\nu} + \tilde{\tilde{B}}_{\mu\nu}. \tag{30}$$

Note that the alternative expansion $B^{\mu\nu} = \langle B^{\mu\nu}\rangle + \tilde{B}^{\mu\nu}$ could in principle be adopted instead [8].

In Minkowski spacetime or in an asymptotically flat background, we can choose coordinates with

$$\langle g_{\mu\nu}\rangle = \eta_{\mu\nu} \qquad \text{(asymptotically flat)}. \tag{31}$$

For these cases, it is often convenient to introduce the simplifying assumption

$$\partial_\lambda b_{\mu\nu} = 0. \tag{32}$$

This preserves translation invariance and hence conservation of energy and momentum for the fluctuation fields $h_{\mu\nu}$ and $\tilde{\tilde{B}}_{\mu\nu}$. It also implies all solitonic solutions are disregarded. Note that imposing the conditions (31) and (32) removes most of the freedom associated with observer general coordinate transformations.

The excitations $h_{\mu\nu}$ and $\tilde{\tilde{B}}_{\mu\nu}$ contain a total of 16 modes. The explicit form of the action (7) is required to establish their complete nature and behavior, including whether they are NG, massive, gauge, or spectator modes, whether they propagate or are auxiliary, and whether the alternative Higgs mechanism occurs [8].

The NG modes can be identified as the field excitations that preserve the minimum of the potential. They are therefore solutions of the conditions

$$X_1 = X_2 = 0 \qquad \text{(NG modes).} \tag{33}$$

Assuming both conditions are imposed by the theory, these represent two independent constraints to be satisfied by the six possible virtual Lorentz excitations of $\tilde{\tilde{B}}_{\mu\nu}$. There could therefore in principle be as many as four Lorentz NG modes in the theory. Determining which ones propagate as physical massless excitations is of definite interest because such modes represent long-range forces and can therefore be expected to have phenomenological implications. Even if the spontaneous Lorentz breaking occurs at a large scale such as the Planck mass, resulting in suppressed massive modes at low energies, the propagating massless modes can be expected to play a significant role in the physics. In effect, the propagating Lorentz NG modes form the smallest unit of the field $B_{\mu\nu}$ carrying relevant dynamical meaning at all scales. We refer to them as "phon" modes, a terminology adapted from phoneme, which is the smallest unit of language capable of carrying meaning.

In a given model with spontaneous Lorentz breaking triggered by an antisymmetric 2-tensor field, determining the number and properties of phon modes is key to establishing the physical content and phenomenological implications of the theory. This parallels the situation for theories with spontaneous Lorentz violation triggered by a vector or a symmetric 2-tensor, where the NG modes can play a variety of phenomenologically important roles. For example, certain gravitationally coupled vector theories with spontaneous Lorentz violation known as bumblebee models reproduce the Einstein-Maxwell equations in a fixed gauge, with the NG modes identified as photon modes [7,23]. Similarly, in a suitable theory for a symmetric 2-tensor generating spontaneous Lorentz violation, the NG modes obey the nonlinear Einstein equations in a fixed gauge and can be identified as gravitons [24,25]. Composite gravitons have been proposed as NG modes of spontaneous Lorentz violation arising from self-couplings of vectors [26], fermions [27], or scalars [28], following related ideas for photons [29]. In some models, the NG modes can also be interpreted as a new spin-dependent interaction [30] or as various new spin-independent forces [31], while in others they can generate torsion masses via the Lorentz-Higgs effect [7].

In what follows, we show that certain theories with spontaneous Lorentz breaking triggered by an antisymmetric 2-tensor field contain a phon mode behaving like a scalar. Since the phon can have nonminimal gravitational couplings, one intriguing possibility is that it could play a cosmological role. Cosmologically varying scalars can produce Lorentz violation associated with varying couplings [32], and we can anticipate that phon modes could play the cosmological roles of the inflaton associated with inflation or the various scalar modes proposed to underlie dark energy. Details of these and other possible phenomenological roles for the phon modes are an interesting topic for future study.

In contrast to the NG modes, the massive modes are excitations increasing the value of the potential V above its minimum. It follows that there are two massive modes, which can be identified with X_1 and X_2 or with linear combinations of these quantities. The explicit form of X_1 and X_2 in terms of $h_{\mu\nu}$ and $\tilde{\tilde{B}}_{\mu\nu}$ can be found using Eq. (20), and their mass matrix is λ_{mn}. These modes can also play a phenomenological role. In gravitationally coupled bumblebee theories, the massive modes modify the Newton gravitational potential [8], and even modes with large masses are likely to affect cosmological dynamics in the very early Universe. Analogous possibilities can be expected to arise for the massive modes X_1 and X_2.

III. MINIMAL MODEL

This section discusses some aspects of models with a gauge-invariant kinetic term. The limit of Minkowski spacetime, with the conditions (31) and (32) satisfied, is considered first. For this purpose, we adopt the minimal Lagrange density,

$$\mathcal{L}^{\min}_{B,V} = -\tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} - V, \tag{34}$$

and examine its content for various choices of V. We then consider some simple extensions, including minimal current and curvature couplings.

A. Minkowski spacetime

For the analysis, a first-order form of the Lagrange density (34) is useful. Introduce a vector field A_μ with field strength and its dual given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ \mathcal{F}_{\mu\nu} &\equiv \tfrac{1}{2}\epsilon_{\mu\nu\kappa\lambda} F^{\kappa\lambda}. \end{aligned} \tag{35}$$

Then, $\mathcal{L}^{\min}_{B,V}$ is equivalent to the first-order Lagrange density

$$\mathcal{L}^{\min}_{A,B,V} = \mathfrak{H}_\mu A^\mu - \tfrac{1}{2} A_\mu A^\mu - V \tag{36}$$

because the field A_μ is auxiliary and can be removed from the action, whereupon use of the identity $H_{\lambda\mu\nu} H^{\lambda\mu\nu} = -6 \mathfrak{H}_\mu \mathfrak{H}^\mu$ recovers $\mathcal{L}^{\min}_{B,V}$. Note that this procedure applies also to the path integral, so the equivalence holds at the quantum level. Partial integration on the first term shows that Eq. (36) can also be written as

$$\mathcal{L}^{\min}_{A,B,V} = \tfrac{1}{2}B_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}A_\mu A^\mu - V. \quad (37)$$

In this Lagrange density, which is also equivalent to the minimal theory (34), no derivatives act on the field $B_{\mu\nu}$.

Consider first the special case of the minimal model (34) with $V = 0$. The resulting Lagrange density $\mathcal{L}^{\min}_{B,0}$ is known to describe a free scalar field [1]. One way to see this is using the equivalent first-order form [33]. With $V = 0$ in Eq. (37), the field $B_{\mu\nu}$ acts as a Lagrange multiplier to enforce $\mathcal{F}_{\mu\nu} = 0$. In Minkowski spacetime, this implies the identity $A_\mu \equiv \partial_\mu \phi$. Substitution yields

$$\mathcal{L}^{\min}_{A,B,0} = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi, \quad (38)$$

which is the Lagrange density for a free scalar field.

Next, suppose a mass term is added for the field $B_{\mu\nu}$, so that $V = m^2 B_{\mu\nu}B^{\mu\nu}/4$. The resulting Lagrange density (34) is known to describe a massive vector field [1]. This can also be seen from the first-order form, which becomes

$$\mathcal{L}^{\min}_{A,B,V} = \tfrac{1}{2}B_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}A_\mu A^\mu - \tfrac{1}{4}m^2 B_{\mu\nu}B^{\mu\nu}. \quad (39)$$

The presence of the mass term means that $B_{\mu\nu}$ now plays the role of an auxiliary field rather than a Lagrange multiplier. Removing $B_{\mu\nu}$ from the action and using the identity $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$ gives

$$m^2\mathcal{L}^{\min}_{A,B,V} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}m^2 A_\mu A^\mu, \quad (40)$$

which is the Lagrange density for a massive vector field.

In the context of the present work, we are interested in the content of the theory (34) when the potential V takes a form that triggers spontaneous Lorentz breaking. For illustrative purposes, consider a potential $V = V(X_1)$ with nonzero quadratic coefficient $\lambda = \lambda_{11}$. This potential depends only on X_1, so the discussion in the previous section implies at most one massive mode can be expected.

Implementing the expansion (30), the Lagrange density becomes

$$\begin{aligned}\mathcal{L}^{\min}_{B,V} &= -\tfrac{1}{12}\tilde{H}_{\lambda\mu\nu}\tilde{H}^{\lambda\mu\nu} - V \\ &= -\tfrac{1}{12}\tilde{H}_{\lambda\mu\nu}\tilde{H}^{\lambda\mu\nu} - \tfrac{1}{2}\lambda X_1^2 - \cdots, \end{aligned}\quad (41)$$

which yields the equations of motion

$$\partial_\lambda \tilde{H}^{\lambda\mu\nu} \approx 4\lambda b^{\mu\nu}X_1. \quad (42)$$

The equivalent first-order Lagrange density is

$$\begin{aligned}\mathcal{L}^{\min}_{B,V} &\equiv \tfrac{1}{2}\tilde{B}_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}A_\mu A^\mu - V \\ &\approx \tfrac{1}{2}\tilde{B}_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}A_\mu A^\mu - 2\lambda(b_{\mu\nu}\tilde{B}^{\mu\nu})^2. \end{aligned}\quad (43)$$

In the last expression, only the leading-order term in $\tilde{B}_{\mu\nu}$ for the potential V is displayed. This is a mass term, involving the mass matrix

$$m_{\kappa\lambda\cdot\mu\nu} = 8\lambda b_{\kappa\lambda}b_{\mu\nu}. \quad (44)$$

However, only the single linear combination $b_{\mu\nu}\tilde{B}^{\mu\nu}$ of the six independent excitations in $\tilde{B}_{\mu\nu}$ is affected by this term. For example, in the special observer frame given by Eq. (24), the mass is associated with a linear combination of $\tilde{B}_{01}$ and $\tilde{B}_{23}$, so these field components determine the massive-mode content of the theory. The other modes remain massless. This shows that the situation with spontaneous Lorentz breaking is intermediate between the two Lorentz-invariant cases with zero mass and with a conventional mass term.

The presence of the vacuum value $b_{\mu\nu}$ defines an orientation in the theory that can be used for projection. Assuming $x_1 \neq 0$, we introduce for an antisymmetric 2-tensor $T_{\mu\nu}$ the orthogonal projections

$$\begin{aligned} T_{\|\mu\nu} &= \frac{1}{x_1}b_{\kappa\lambda}T^{\kappa\lambda}b_{\mu\nu}, \\ T_{\perp\mu\nu} &= T_{\mu\nu} - T_{\|\mu\nu}. \end{aligned}\quad (45)$$

With this notation, the Lagrange density (43) can be written as

$$\begin{aligned}\mathcal{L}^{\min}_{B,V} \approx{} & \tfrac{1}{2}\tilde{B}_{\perp\mu\nu}\mathcal{F}_\perp^{\mu\nu} + \tfrac{1}{2}\tilde{B}_{\|\mu\nu}\mathcal{F}_\|{}^{\mu\nu} \\ & - \tfrac{1}{2}A_\mu A^\mu - 2\lambda x_1 \tilde{B}_{\|\mu\nu}\tilde{B}_\|{}^{\mu\nu}. \end{aligned}\quad (46)$$

This form displays explicitly the intermediate nature of the minimal model with spontaneous Lorentz violation. In the expression (46), the projection $\tilde{B}_{\perp\mu\nu}$ is a Lagrange-multiplier field that acts to impose the condition

$$\mathcal{F}_{\perp\mu\nu} \approx 0 \quad (47)$$

in parallel with the situation when $V = 0$. However, the projection $\tilde{B}_{\|\mu\nu}$ is a massive auxiliary field obeying

$$\tilde{B}_{\|\mu\nu} \approx \frac{1}{8\lambda x_1}\mathcal{F}_{\|\mu\nu}, \quad (48)$$

in analogy with the case leading to Eq. (40). We see that the term proportional to $A_\mu A^\mu$ in the Lagrange density (46) plays a double role, with some combinations of the components of A_μ generating kinetic terms for massless NG modes while others form a mass term for the massive mode.

At leading order in $\tilde{B}_{\mu\nu}$, the solutions to Eq. (47) contain the massless NG modes in the theory, while the massive-mode content lies in the complement (48). However, examination of Eq. (42) reveals that no massive mode propagates at leading order. For example, taking a derivative of Eq. (42) gives $b^{\mu\nu}\partial_\mu X_1 \approx 0$, and working in the special frame (24) with nonzero x_1 and x_2 shows that X_1 is a constant. The result (48) implies $\mathcal{F}_{\parallel\mu\nu} \propto \tilde{\tilde{B}}_{\parallel\mu\nu} \propto X_1 b_{\mu\nu}$, so it follows that $\mathcal{F}_{\parallel\mu\nu}$ is a constant. Adopting natural boundary conditions with $X_1 = 0$, we obtain $\mathcal{F}_{\mu\nu} = 0$ and hence $A_\mu = \partial_\mu \phi$. At leading order in $\tilde{\tilde{B}}_{\mu\nu}$ and with these boundary conditions, the Lagrange density (46) therefore reduces to a theory of the form (38) describing a single free phon mode ϕ. In this limit, we see that the phon mode is the analog of the scalar associated with the massless notoph or Kalb-Ramond field [1,2].

Further insight can be obtained by performing a time-space decomposition on $\tilde{\tilde{B}}_{\mu\nu}$. Define

$$\tilde{\tilde{B}}_{0j} = -\Sigma^j, \qquad \tilde{\tilde{B}}_{jk} = \epsilon_{jkl}\Xi^l, \tag{49}$$

in analogy with the electrodynamic decomposition of the field strength into its electric and magnetic 3-vector fields. In terms of $\vec{\Sigma}$ and $\vec{\Xi}$, the Lagrange density (34) becomes

$$\mathcal{L}^{\min}_{B,V} = \tfrac{1}{2}(\dot{\vec{\Xi}} + \vec{\nabla} \times \vec{\Sigma})^2 - \tfrac{1}{2}(\vec{\nabla} \cdot \vec{\Xi})^2 - V(\vec{\Sigma}, \vec{\Xi}). \tag{50}$$

This form of the theory reveals that the only dynamical object is $\vec{\Xi}$, while $\vec{\Sigma}$ is auxiliary. It follows that at most three propagating modes can appear in the minimal model.

For $V = 0$, use of the Helmholtz decompositions $\vec{\Sigma} = \vec{\Sigma}_t + \vec{\Sigma}_l$ and $\vec{\Xi} = \vec{\Xi}_t + \vec{\Xi}_l$ into divergence-free transverse and curl-free longitudinal parts reveals the expected result that the curl-free single degree of freedom $\vec{\Xi}_l$ propagates a free scalar field, while the other fields are gauge or decouple. If instead the potential V is a conventional mass term, the three propagating modes are those of a massive vector. In contrast, for the case of interest here with V triggering spontaneous Lorentz violation, at most two of the six modes in $\tilde{\tilde{B}}_{\mu\nu}$ can be massive. For example, working in the special frame (24), the potential in the illustrative model (41) becomes

$$V(\vec{\Sigma}, \vec{\Xi}) \approx 8\lambda(a\Sigma^1 - b\Xi^1)^2 \tag{51}$$

at leading order in $\tilde{\tilde{B}}_{\mu\nu}$ and hence in $\vec{\Sigma}$ and $\vec{\Xi}$. This generates a mass matrix for the components Σ^1 and Ξ^1, with the linear combination $a\Sigma^1 - b\Xi^1$ representing the massive mode. The field Σ^1 is auxiliary. Although Ξ^1 could in principle be dynamical, it is nonpropagating at leading order in $\tilde{\tilde{B}}_{\mu\nu}$. Of the remaining 2 degrees of freedom in $\vec{\Xi}$, one is the free phon mode, while the other can be removed by the residual gauge freedom that leaves invariant the potential (51).

Analogous results are obtained for the minimal model (34) with more general potential $V = V(X_1, X_2)$. There can be up to two massive modes, with masses determined by the eigenvalues of the mass matrix for X_1 and X_2. In the special frame (24), X_1 and X_2 take the form

$$\begin{aligned} X_1 &= -4a\Sigma^1 + 4b\Xi^1 - 2\vec{\Sigma}^2 + 2\vec{\Xi}^2, \\ X_2 &= -4b\Sigma^1 - 4a\Xi^1 - 4\vec{\Sigma} \cdot \vec{\Xi}. \end{aligned} \tag{52}$$

Combinations of Σ^1 and Ξ^1 therefore represent the massive modes in the theory. The field $\vec{\Sigma}$ is auxiliary and can be eliminated from the Lagrange density, at least in principle, leaving only 1 massive degree of freedom. As before, this massive mode is nonpropagating at leading order. The issue of whether it propagates at higher orders is an interesting open question but lies beyond our present scope. This may most conveniently be addressed via the Hamiltonian formulation and the Dirac procedure for constraints [34].

B. Currents and curvature

Next, consider an extension of the theory (34) to include a coupling to a current $j_B^{\mu\nu}$, either specified externally or formed from fields other than $B_{\mu\nu}$. The relevant Lagrange density becomes

$$\mathcal{L}^{\min}_{B,V,j} = -\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} - V - \tfrac{1}{2}B_{\mu\nu}j_B^{\mu\nu}. \tag{53}$$

A gauge transformation of the form (5) changes $\mathcal{L}^{\min}_{B,V,j}$ by an amount

$$\delta\mathcal{L}^{\min}_{B,V,j} = \Lambda_\nu \partial_\mu (j_V^{\mu\nu} + j_B^{\mu\nu}), \tag{54}$$

where the potential current $j_V^{\mu\nu}$ is defined as

$$j_V^{\mu\nu} = 4V_1 B^{\mu\nu} + 4V_2 \mathfrak{B}^{\mu\nu}. \tag{55}$$

The result (54) represents the obstruction to gauge invariance in the theory. Off-shell invariance is achieved whenever the sum of the massive mode and the matter currents is conserved off shell. This occurs, for example, if the potential V vanishes and the current $j_B^{\mu\nu}$ is independently conserved.

In the present context with spontaneous breaking of Lorentz symmetry, the potential V is nonvanishing so gauge invariance is generically lost. However, the NG modes in the theory satisfy the conditions (26), so $j_V^{\mu\nu}$ vanishes in this sector. The current $j_V^{\mu\nu}$ is therefore associated with the massive modes. Moreover, in parallel with the case of classical electrodynamics, it is reasonable to take the current $j_B^{\mu\nu}$ to be independently conserved in this sector,

$$\partial_\mu j_B^{\mu\nu} = 0 \qquad \text{(NG sector)}, \tag{56}$$

when the massive modes are constrained to zero. It follows from Eq. (54) that the NG sector is off shell gauge invariant under the residual gauge transformations satisfying the conditions (33). Also, if $j_B^{\mu\nu}$ is specified externally, then it is conserved even in the presence of massive modes. However, if $j_B^{\mu\nu}$ is constructed from other fields, then it may be affected by the excitation of massive modes, whereupon conservation may fail.

Related results emerge on shell. The equation of motion for $\tilde{\tilde{B}}_{\mu\nu}$ is

$$\partial_\lambda H^{\lambda\mu\nu} = j_V^{\mu\nu} + j_B^{\mu\nu}. \tag{57}$$

This implies that the total current is conserved on shell,

$$\partial_\mu(j_V^{\mu\nu} + j_B^{\mu\nu}) = 0. \tag{58}$$

It follows that the variation $\delta\mathcal{L}_{B,V,j}^{\min}$ of the Lagrange density vanishes on shell, so the gauge-symmetry breaking is an off-shell effect.

The first-order form of the theory (53) can be written as the Lagrange density

$$\mathcal{L}_{A,B,V,j}^{\min} = \tfrac{1}{2}B_{\mu\nu}\mathcal{F}^{\mu\nu} - \tfrac{1}{2}A_\mu A^\mu - V - \tfrac{1}{2}B_{\mu\nu}j_B^{\mu\nu}, \tag{59}$$

from which the original theory (53) can be recovered by partial integration on the first term followed by elimination of the auxiliary field A_μ, as before. In what follows, it is convenient to perform a time-space decomposition for $j_B^{\mu\nu}$ paralleling the decomposition (49). We introduce vectors $\vec{J}$, $\vec{K}$ as

$$j_B^{0j} = J^j, \qquad j_B^{jk} = \epsilon^{jkl}K_l. \tag{60}$$

Current conservation (56) implies

$$\vec{\nabla}\cdot\vec{J} = 0, \qquad \vec{\nabla}\times\vec{K} - \dot{\vec{J}} = 0. \tag{61}$$

Note these equations are equivalent to the homogeneous Maxwell equations for the pair $(\vec{E}, \vec{B}) = (\vec{K}, -\vec{J})$.

Consider first the Lagrange density (59) with $V = 0$. Then, $B_{\mu\nu}$ remains a Lagrange-multiplier field as before, but the associated constraint becomes $\mathcal{F}_{\mu\nu} = j_{B\mu\nu}$. The solution is $A_\mu = \alpha_\mu + \partial_\mu\phi$, where α_μ is the 4-vector potential associated with the Maxwell electromagnetic fields $(\vec{E}, \vec{B}) = (\vec{K}, -\vec{J})$. Substitution yields

$$\mathcal{L}_{A,B,V,j}^{\min} = -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \phi j_\phi, \tag{62}$$

where $j_\phi = \partial_\mu\alpha^\mu$. A term proportional to $\alpha_\mu\alpha^\mu$ that is irrelevant for the dynamics of ϕ has been dropped. This theory describes a scalar field ϕ interacting with the current j_ϕ.

If instead the potential in the Lagrange density (59) is the mass term $V = m^2 B_{\mu\nu}B^{\mu\nu}/4$, then $B_{\mu\nu}$ is auxiliary. Removing it from the action yields

$$m^2\mathcal{L}_{A,B,V,j}^{\min} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}m^2 A_\mu A^\mu - A_\mu j_A^\mu, \tag{63}$$

where $j_A^\mu = \epsilon^{\alpha\beta\gamma\mu}\partial_\alpha j_{B\beta\gamma}/2$. This equation omits the quadratic current-coupling term $j_B^{\mu\nu}j_{B\mu\nu}/4$, which is irrelevant for the dynamics of A_μ. The Lagrange density (63) describes a massive vector field A_μ interacting with the current j_A^μ.

For the case of interest here with V spontaneously breaking Lorentz symmetry, we again find the theory contains a mixture of phon and massive modes. However, these modes interact with currents, and additional SME-type couplings can appear. Consider, for example, the illustrative model with $V = V(X_1)$ approximated by a quadratic term with coefficient λ. Projecting the perpendicular and parallel components of $B_{\mu\nu}$, $\mathcal{F}_{\mu\nu}$, and $j_B^{\mu\nu}$ according to Eq. (45) and substituting into Eq. (59) gives the Lagrange density

$$\begin{aligned}\mathcal{L}_{A,B,V,j}^{\min} \approx{}& \tfrac{1}{2}\tilde{\tilde{B}}_{\perp\mu\nu}(\mathcal{F}_\perp^{\mu\nu} - j_{B\perp}^{\mu\nu}) \\ &+ \tfrac{1}{2}\tilde{\tilde{B}}_{||\mu\nu}(\mathcal{F}_{||}^{\mu\nu} - j_{B||}^{\mu\nu}) - 2\lambda x_1\tilde{\tilde{B}}_{||\mu\nu}\tilde{\tilde{B}}_{||}{}^{\mu\nu} \\ &- \tfrac{1}{2}A_\mu A^\mu - \tfrac{1}{2}b_{\mu\nu}j_B^{\mu\nu}.\end{aligned} \tag{64}$$

This reveals that the projection $\tilde{\tilde{B}}_{\perp\mu\nu}$ is a Lagrange-multiplier field imposing the constraint

$$\mathcal{F}_{\perp\mu\nu} \approx j_{B\perp}^{\mu\nu}, \tag{65}$$

while the projection $\tilde{\tilde{B}}_{||\mu\nu}$ is an auxiliary field given by

$$\tilde{\tilde{B}}_{||\mu\nu} \approx \frac{1}{8\lambda x_1}(\mathcal{F}_{||}^{\mu\nu} - j_{B||}^{\mu\nu}). \tag{66}$$

As before, the NG modes are contained in the solutions to Eq. (65), while the massive-mode content is in the complement (66) and is constrained by current conservation. Adopting natural boundary conditions for the equations of motion again leads to $X_1 = 0$. The solution for A_μ can be written as $A_\mu = \alpha_\mu + \partial_\mu\phi$, where ϕ is the phon mode and α_μ is the 4-vector potential for the Maxwell fields $(\vec{E}, \vec{B}) = (\vec{K}, -\vec{J})$. At leading order, the only propagating mode is the phon. Removing the Lagrange-multiplier and auxiliary modes from the theory yields the Lagrange density

$$\mathcal{L}_{A,B,V,j}^{\min} \approx -\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \phi j_\phi - \tfrac{1}{2}b_{\mu\nu}j_B^{\mu\nu} - \tfrac{1}{2}\alpha_\mu\alpha^\mu, \tag{67}$$

where $j_\phi = \partial_\mu\alpha^\mu$. This describes an interacting phon along with an SME-type coupling to the current $j_B^{\mu\nu}$ and an induced current-current coupling.

Another interesting extension of the minimal theory (34) is obtained in passing from Minkowski to Riemann spacetime and adding the Einstein-Hilbert term (8). The relevant Lagrange density is

$$e\mathcal{L}^{\rm min}_{R,B,V,j} = \frac{e}{2\kappa}(R-2\Lambda) - \frac{1}{12}eH_{\lambda\mu\nu}H^{\lambda\mu\nu} - eV - \frac{1}{2}eB_{\mu\nu}j_B^{\mu\nu}. \qquad (68)$$

Much of the discussion in Minkowski spacetime remains valid, but some derivations face obstructions.

The introduction of the vector field A_μ and the construction of the equivalent first-order form

$$e\mathcal{L}^{\rm min}_{R,A,B,V,j} = \frac{e}{2\kappa}(R-2\Lambda) + \frac{1}{2}eB_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}eA_\mu A^\mu - eV - \frac{1}{2}eB_{\mu\nu}j_B^{\mu\nu} \qquad (69)$$

proceeds as before because the derivatives in $H_{\lambda\mu\nu}$ can be taken as covariant and hence the partial integration performed. Global statements obtained from the Lagrange density become local statements, including the equations of motion and the results for the current $j_B^{\mu\nu}$.

If $V = 0$ and the topology of the spacetime manifold M is trivial, the theory describes a scalar field in Riemann spacetime. However, the solution for A_μ that leads to this interpretation is valid only locally if the first cohomology class $H^1(M, \mathbb{R})$ is nonvanishing. This issue is absent if V is taken as the mass term $V = m^2 B_{\mu\nu}B^{\mu\nu}/4$, when the theory describes a massive vector in Riemann spacetime.

If instead V triggers spontaneous Lorentz breaking, the vacuum value $b_{\mu\nu}$ can vary with spacetime position and hence have a nontrivial derivative in the general case [4]. The field strength $H_{\lambda\mu\nu}$ can therefore acquire a nonzero contribution even when $\tilde{\tilde{B}}_{\mu\nu}$ vanishes. However, this has no effect on the first-order form (69). For example, performing the decomposition (45) for the illustrative model with $V = V(X_1)$ approximated by a quadratic term yields the Lagrange density

$$e\mathcal{L}^{\rm min}_{R,A,B,V,j} \approx \frac{e}{2\kappa}(R-2\Lambda) + \frac{1}{2}e\tilde{\tilde{B}}_{\perp\mu\nu}(\mathcal{F}^{\mu\nu}_\perp - j^{\mu\nu}_{B\perp}) + \frac{1}{2}e\tilde{\tilde{B}}_{\|\mu\nu}(\mathcal{F}^{\mu\nu}_\| - j^{\mu\nu}_{B\|}) - 2\lambda x_1 e\tilde{\tilde{B}}_{\|\mu\nu}\tilde{\tilde{B}}_\|{}^{\mu\nu} - \frac{1}{2}eA_\mu A^\mu - \frac{1}{2}eb_{\mu\nu}j_B^{\mu\nu}, \qquad (70)$$

where $b_{\mu\nu}$ may now vary with position. The constraint (65) still holds, but if $H^1(M, \mathbb{R})$ is nontrivial then the general solution for A_μ includes all independent closed nonexact 1-forms with suitable support on the spacetime. The latter can be viewed as additional topological modes in the theory, but since these modes are nonexact they cannot play the role of topological phon modes. There is still only one phon, which propagates in Riemann spacetime and interacts with a current that includes a contribution from the topological modes. In the special case of an asymptotically flat spacetime with $\Lambda = 0$ and trivial topology, the condition (32) holds and the topological modes are absent. At leading order in $h_{\mu\nu}$, the phon then propagates in a Minkowski background with weak-field coupling to the metric.

IV. NONMINIMAL MODEL

The effects of Lorentz violation on gravity can be characterized in a general way by constructing the effective field theory for the metric and curvature while allowing arbitrary Lorentz-violating couplings [4]. This procedure generates the gravity sector of the SME in Riemann spacetime. At leading order in the curvature, three basic types of Lorentz-violating couplings arise. Each involves a coefficient field for Lorentz violation that upon acquiring a vacuum value generates a Lorentz-violating coupling for gravity. The theory (7) for the antisymmetric 2-tensor $B_{\mu\nu}$ has the interesting feature of containing all three kinds of couplings, despite being comparatively simple.

In this section, we consider a particular restriction of the theory (7) that suffices to exhibit all three kinds of couplings. The theory includes some of the nonminimal curvature couplings obtained in Sec. II B. Following the specification of the Lagrange density, the equations of motion and energy-momentum conservation law are obtained. The results are linearized and some implications for the mode content are obtained. We then apply the formalism of Ref. [12] to extract the post-Newtonian metric.

A. Action

At leading order in the curvature, the three basic types of Lorentz-violating couplings include ones to the traceless Ricci tensor, the Weyl tensor, and the scalar curvature. The corresponding SME coefficient fields are conventionally denoted as $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$, and u. We adopt here an extension of the minimal model of the previous section that suffices to include all three. It is constructed by adding nonzero couplings of the τ_3, τ_4, and τ_5 types displayed in Sec. II B. For simplicity, we assume $\Lambda = 0$ and $j_B^{\mu\nu} = 0$ but include a matter Lagrange density $\mathcal{L}_M$ to act as a gravitational source. The potential $V(X_1, X_2)$ triggering spontaneous Lorentz violation is taken to satisfy the vacuum condition (26) and to have the expansion (27) involving the constants λ_{mn}.

The chosen Lagrange density can be written in the form

$$e\mathcal{L}^{\rm nonmin} = \frac{e}{2\kappa}R - \frac{1}{12}eH_{\lambda\mu\nu}H^{\lambda\mu\nu} - eV + e\mathcal{L}_M + \frac{e}{2\kappa}(\xi_1 B^{\kappa\lambda}B^{\mu\nu}R_{\kappa\lambda\mu\nu} + \xi_2 B^{\lambda\nu}B^\mu{}_\nu R_{\lambda\mu} + \xi_3 B^{\mu\nu}B_{\mu\nu}R). \qquad (71)$$

For convenience in the analysis to follow, we have extracted a factor of 2κ from the coupling constants τ_3, τ_4, and τ_5 and relabeled them as ξ_1, ξ_2, and ξ_3.

At the level of the action, the theory (71) implies an explicit correspondence between $B_{\mu\nu}$ and the three SME coefficient fields $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$, and u. We find

$$\begin{aligned}(s_B)^{\mu\nu} &= (2\xi_1+\xi_2)(B^\mu{}_\alpha B^{\nu\alpha} - \tfrac{1}{4}g^{\mu\nu}B^{\alpha\beta}B_{\alpha\beta}),\\ (t_B)^{\kappa\lambda\mu\nu} &= \tfrac{2}{3}\xi_1(B^{\kappa\lambda}B^{\mu\nu} + \tfrac{1}{2}B^{\kappa\mu}B^{\lambda\nu} - \tfrac{1}{2}B^{\kappa\nu}B^{\lambda\mu})\\ &\quad - \tfrac{1}{2}\xi_1(g^{\kappa\mu}B^\lambda{}_\alpha B^{\nu\alpha} - g^{\lambda\mu}B^\kappa{}_\alpha B^{\nu\alpha}\\ &\quad - g^{\kappa\nu}B^\lambda{}_\alpha B^{\mu\alpha} + g^{\lambda\nu}B^\kappa{}_\alpha B^{\mu\alpha})\\ &\quad + \tfrac{1}{6}\xi_1(g^{\kappa\mu}g^{\lambda\nu} - g^{\lambda\mu}g^{\kappa\nu})B^{\alpha\beta}B_{\alpha\beta},\\ u_B &= -(\tfrac{1}{6}\xi_1 + \tfrac{1}{4}\xi_2 + \xi_3)B^{\alpha\beta}B_{\alpha\beta}.\end{aligned} \tag{72}$$

The reader is cautioned that the vacuum values of the coefficient fields implied by these equations differ by scalings from those that appear in the final linearized effective Einstein equations [12]. This issue is revisited in Sec. IV C.

The gravitational field equations follow from the Lagrange density (71) by varying with respect to $g_{\mu\nu}$, while holding $B_{\mu\nu}$ and any matter fields fixed. Explicitly, we find

$$\begin{aligned}G^{\mu\nu} &= \kappa(T_M)^{\mu\nu} + \kappa(T_B)^{\mu\nu}\\ &\quad + (T_{\xi_1})^{\mu\nu} + (T_{\xi_2})^{\mu\nu} + (T_{\xi_3})^{\mu\nu}.\end{aligned} \tag{73}$$

The first term on the right-hand side is the matter energy-momentum tensor. The second term is the contribution to the energy-momentum tensor arising from the kinetic and potential terms for $B_{\mu\nu}$. It is given by

$$\begin{aligned}(T_B)^{\mu\nu} &= \frac{1}{2}H^{\alpha\beta\mu}H^\nu{}_{\alpha\beta} - \tfrac{1}{12}g^{\mu\nu}H^{\alpha\beta\gamma}H_{\alpha\beta\gamma} - g^{\mu\nu}V\\ &\quad + 4B^{\alpha\mu}B_\alpha{}^\nu V_1 + g^{\mu\nu}\mathfrak{B}_{\alpha\beta}B^{\alpha\beta}V_2.\end{aligned} \tag{74}$$

The remaining three terms in Eq. (73) are due to the nonminimal gravitational couplings. For the ξ_1 coupling, we find

$$\begin{aligned}(T_{\xi_1})^{\mu\nu} &= \xi_1(\tfrac{1}{2}g^{\mu\nu}B^{\alpha\beta}B^{\gamma\delta}R_{\alpha\beta\gamma\delta} + \tfrac{3}{2}B^{\beta\gamma}B^{\alpha\mu}R^\nu{}_{\alpha\beta\gamma}\\ &\quad + \tfrac{3}{2}B^{\beta\gamma}B^{\alpha\mu}R^\nu{}_{\alpha\beta\gamma} + D_\alpha D_\beta B^{\alpha\mu}B^{\nu\beta}\\ &\quad + D_\alpha D_\beta B^{\alpha\nu}B^{\mu\beta}).\end{aligned} \tag{75}$$

The contribution from the ξ_2 coupling is

$$\begin{aligned}(T_{\xi_2})^{\mu\nu} &= \xi_2(\tfrac{1}{2}g^{\mu\nu}B^{\alpha\gamma}B^\beta{}_\gamma R_{\alpha\beta} - B^{\alpha\mu}B^{\beta\nu}R_{\alpha\beta}\\ &\quad - B^{\alpha\beta}B^\mu{}_\beta R^\nu{}_\alpha - B^{\alpha\beta}B^\nu{}_\beta R^\mu{}_\alpha\\ &\quad + \tfrac{1}{2}D_\alpha D^\mu B^\nu{}_\beta B^{\alpha\beta} + \tfrac{1}{2}D_\alpha D^\nu B^\mu{}_\beta B^{\alpha\beta}\\ &\quad - \tfrac{1}{2}D^2 B^{\alpha\mu}B_\alpha{}^\nu - \tfrac{1}{2}g^{\mu\nu}D_\alpha D_\beta B^{\alpha\gamma}B^\beta{}_\gamma).\end{aligned} \tag{76}$$

Finally, for the ξ_3 coupling we obtain

$$\begin{aligned}(T_{\xi_3})^{\mu\nu} &= \xi_3(D^\mu D^\nu B^{\alpha\beta}B_{\alpha\beta} - g^{\mu\nu}D^2 B^{\alpha\beta}B_{\alpha\beta}\\ &\quad - B^{\alpha\beta}B_{\alpha\beta}G^{\mu\nu} + 2B^{\alpha\mu}B^\nu{}_\alpha R).\end{aligned} \tag{77}$$

The equations of motion for the antisymmetric 2-tensor are obtained by varying the Lagrange density (71) with respect to $B_{\mu\nu}$, while holding the metric and any matter fields fixed. They can be written in the form

$$D_\alpha H^{\alpha\mu\nu} = j_V^{\mu\nu} + j_R^{\mu\nu}, \tag{78}$$

where the potential current $j_V^{\mu\nu}$ is given by Eq. (55) and the curvature current $j_R^{\mu\nu}$ is defined as

$$j_R^{\mu\nu} = -\frac{2\xi_1}{\kappa}B_{\alpha\beta}R^{\alpha\beta\mu\nu} + \frac{2\xi_2}{\kappa}B_\alpha{}^{[\mu}R^{\nu]\alpha} - \frac{2\xi_3}{\kappa}B^{\mu\nu}R. \tag{79}$$

The sum of the currents is covariantly conserved on shell,

$$D_\mu(j_V^{\mu\nu} + j_R^{\mu\nu}) = 0, \tag{80}$$

as a consequence of the minimal kinetic term for $B_{\mu\nu}$ chosen for the theory (71). This result is the nonminimal analog of Eq. (58). Since $j_V^{\mu\nu}$ involves the derivatives V_1 and V_2, which are nonzero when the massive modes are excited, Eq. (80) can serve as a constraint on the massive modes. However, when the massive modes vanish, it can be viewed instead as a constraint on the curvature. This issue is revisited as part of the discussion of the linearized limit in Sec. IV B.

For the matter described by the Lagrange density $\mathcal{L}_M$, the equations of motion follow by variation with respect to the matter fields. The matter energy-momentum tensor $(T_M)^{\mu\nu}$ is covariantly conserved,

$$D_\mu(T_M)^{\mu\nu} = 0. \tag{81}$$

This can be verified explicitly as follows. First, note that the components $(T_B)^{\mu\nu}$, $(T_{\xi_1})^{\mu\nu}$, $(T_{\xi_2})^{\mu\nu}$, and $(T_{\xi_3})^{\mu\nu}$ of the total energy-momentum tensor satisfy the relation

$$\kappa D_\mu(T_B)^{\mu\nu} = -D_\mu[(T_{\xi_1})^{\mu\nu} + (T_{\xi_2})^{\mu\nu} + (T_{\xi_3})^{\mu\nu}]. \tag{82}$$

This can be checked by evaluating the left-hand side using the field equations (78), the identity (4), the Bianchi identities for the curvature tensor, and the identity (15). Next, take the covariant divergence of the gravitational field equations (73) and impose the traced Bianchi identities $D_\mu G^{\mu\nu} = 0$. Substitution of Eq. (82) then yields the matter energy-momentum conservation law (81).

B. Linearization

This section explores the linearized version of the theory (71) in an asymptotically flat spacetime. We choose coordinates as in Eq. (31) and impose the condition (32). The weak-field limit is taken, so only the leading-order terms in the fluctuations $h_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$ are kept. The fluctuations are assumed to vanish in the asymptotic region, far from any matter sources. As usual, raising and lowering of indices on linear quantities is understood to involve the Minkowski metric.

In the minimum of the potential, $X_1 = X_2 = 0$ and the vacuum solution satisfies

$$\eta^{\kappa\mu}\eta^{\lambda\mu}b_{\kappa\lambda}b_{\mu\nu} = x_1, \qquad \eta^{\kappa\mu}\eta^{\lambda\mu}b_{\kappa\lambda}\mathfrak{b}_{\mu\nu} = x_2. \quad (83)$$

At linear order, X_1 and X_2 take the form

$$\begin{aligned} X_1 &\approx 2b_{\mu\nu}\tilde{\tilde{B}}^{\mu\nu} - 2b_{\mu\alpha}b_\nu{}^\alpha h^{\mu\nu}, \\ X_2 &\approx 2\mathfrak{b}_{\mu\nu}\tilde{\tilde{B}}^{\mu\nu} - \tfrac{1}{2}x_2 h^\alpha{}_\alpha. \end{aligned} \quad (84)$$

These combinations represent the massive modes in the theory at this order.

At leading order, the field equations for the metric retain the form (73), but all quantities are understood to be linearized. The linearization of the Einstein tensor on the left-hand side is standard. The first term on the right-hand side is the linearized energy-momentum tensor for ordinary matter. Explicit expressions for the remaining terms on the right-hand side are

$$\begin{aligned} (T_B)_{\mu\nu} &\approx 4(\lambda_{11}X_1 + \lambda_{12}X_2)b_{\mu\alpha}b_\nu{}^\alpha \\ &\quad + \eta_{\mu\nu}(\lambda_{22}X_2 + \lambda_{12}X_1)x_2, \\ (T_{\xi_1})_{\mu\nu} &\approx \xi_1[\tfrac{1}{2}\eta_{\mu\nu}b^{\alpha\beta}b^{\gamma\delta}R_{\alpha\beta\gamma\delta} + 4b^{\beta\gamma}b^\alpha{}_{(\mu}R_{\nu)\alpha\beta\gamma} \\ &\quad + 2b^\alpha{}_\mu b^\beta{}_\nu R_{\alpha\beta} + 4b^\alpha{}_{(\mu}\partial^\beta D_\alpha B_{\nu)\beta}], \\ (T_{\xi_2})_{\mu\nu} &\approx \xi_2[\eta_{\mu\nu}(b^{\alpha\gamma}b^\beta{}_\gamma R_{\alpha\beta} - \tfrac{1}{2}b^{\alpha\gamma}\partial^\beta H_{\alpha\beta\gamma} \\ &\quad - \tfrac{1}{4}b^{\alpha\beta}b^{\gamma\delta}R_{\alpha\beta\gamma\delta} - \tfrac{1}{2}b^{\alpha\beta}\partial^\gamma D_\gamma B_{\alpha\beta}) \\ &\quad - b^\alpha{}_\mu b^\beta{}_\nu R_{\alpha\beta} - 2b^{\alpha\beta}R_{\alpha(\mu}b_{\nu)\beta} \\ &\quad + b^{\alpha\gamma}b^\beta{}_\gamma R_{\alpha\mu\nu\beta} - \tfrac{1}{2}b^\alpha{}_{(\mu}R_{\nu)\alpha\beta\gamma}b^{\beta\gamma} \\ &\quad + b^{\beta\gamma}\partial_{(\mu}D_\beta B_{\nu)\gamma} + b^\alpha{}_{(\mu}\partial^\beta D_{\nu)}B_{\alpha\beta} \\ &\quad + b^\alpha{}_{(\mu}\partial^\beta D_\beta B_{\nu)\alpha}], \\ (T_{\xi_3})_{\mu\nu} &\approx \xi_3[(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^\alpha\partial_\alpha)X_1 \\ &\quad - 2b_\mu{}^\alpha b_{\nu\alpha}R - x_1 G_{\mu\nu}]. \end{aligned} \quad (85)$$

In these expressions, all covariant derivatives and curvatures are taken to linear order in $h_{\mu\nu}$ and $\tilde{\tilde{B}}_{\mu\nu}$.

The linearized field equations for the fluctuations $\tilde{\tilde{B}}_{\mu\nu}$ take the form

$$\partial_\alpha H^{\alpha\mu\nu} = j_V^{\mu\nu} + j_R^{\mu\nu}, \quad (86)$$

where $H_{\lambda\mu\nu}$ is constructed using $\tilde{\tilde{B}}_{\mu\nu}$. The linearized currents $j_V^{\mu\nu}$ and $j_R^{\mu\nu}$ are given by

$$j_V^{\mu\nu} = 4(\lambda_{11}X_1 + \lambda_{12}X_2)b^{\mu\nu} + 4(\lambda_{22}X_2 + \lambda_{12}X_1)\mathfrak{b}^{\mu\nu} \quad (87)$$

and

$$j_R^{\mu\nu} = -\frac{2\xi_1}{\kappa}b_{\alpha\beta}R^{\alpha\beta\mu\nu} + \frac{2\xi_2}{\kappa}b_\alpha{}^{[\mu}R^{\nu]\alpha} - \frac{2\xi_3}{\kappa}b^{\mu\nu}R. \quad (88)$$

The identity $\partial^\nu\partial^\mu H_{\mu\nu\lambda} = 0$ implies that the total current is conserved,

$$\partial_\mu(j_V^{\mu\nu} + j_R^{\mu\nu}) = 0. \quad (89)$$

This can be interpreted as a constraint on massive-mode excitations, which appear in $j_V^{\mu\nu}$, and it also implies conditions on the linearized curvatures.

To investigate further, it is convenient to introduce two combinations of the linearized massive modes X_1, X_2 and the linearized scalar curvature R given by

$$\begin{aligned} X_1' &= 4(\lambda_{11}X_1 + \lambda_{12}X_2) - \frac{1}{2\kappa}(\xi_2 + 4\xi_3)R, \\ X_2' &= 4(\lambda_{22}X_2 + \lambda_{12}X_1). \end{aligned} \quad (90)$$

In terms of these variables, the conservation law (89) takes the simple form

$$b_{\alpha\nu}\partial^\alpha X_1' + \mathfrak{b}_{\alpha\nu}\partial^\alpha X_2' - \frac{1}{\kappa}(4\xi_1 + \xi_2)b^{\alpha\beta}\partial_\alpha R_{\beta\nu} = 0. \quad (91)$$

By applying the differential operator $b^\nu{}_\gamma b^\gamma{}_\delta\partial^\delta$, which cancels the first two terms containing the massive modes, we obtain a condition on derivatives of the linearized Ricci tensor,

$$\begin{aligned} 0 &= (4\xi_1 + \xi_2)b^\nu{}_\gamma b^\gamma{}_\delta b^{\alpha\beta}\partial^\delta\partial_\alpha R_{\beta\nu} \\ &= (4\xi_1 + \xi_2)b^\nu{}_\gamma b^\gamma{}_\delta b^{\alpha\beta}\partial^\delta\partial_\alpha[(T_M)_{\beta\nu} - \tfrac{1}{2}\eta_{\beta\nu}(T_M)^\mu{}_\mu] \\ &\quad + (4\xi_1 + \xi_2) \times O(\xi). \end{aligned} \quad (92)$$

In the second equation above, we have substituted for the linearized Ricci tensor in terms of the linearized matter energy-momentum tensor using the gravitational field equations.

If we solve the field equations perturbatively in the couplings ξ_1, ξ_2, then at lowest order Eq. (92) generates a direct constraint on ordinary matter. Consider, for example, a static distribution of mass given by

$$(T_M)_{\mu\nu} = \rho\delta^0{}_\mu\delta^0{}_\nu, \quad (93)$$

and adopt an observer coordinate system with $b_{\mu\nu}$ lying in a configuration with the vectors (23) given by $\vec{e} = (a, 0, 0)$ and $\vec{b} = (b\cos\theta, b\sin\theta, 0)$. Then the constraint (92) becomes

$$(4\xi_1 + \xi_2)a^2 b\sin\theta\frac{\partial^2\rho}{\partial z\partial x} = (4\xi_1 + \xi_2) \times O(\xi), \quad (94)$$

which is generically inconsistent with small corrections to the general-relativistic behavior of matter. In this work, we are interested in post-Newtonian corrections to general relativity rather than in more radical proposals. To retain conventional properties of matter, we therefore limit attention in what follows to models satisfying the condition

$$4\xi_1 + \xi_2 = 0. \quad (95)$$

In these models, the constraint (92) is satisfied automatically. Note that conditions of this type also arise in the post-Newtonian limit of other theories with nonminimal gravitational couplings, such as vector-tensor models without a potential term [35].

Imposing the condition (95) eliminates the last term in the conservation law (91), which reduces to a constraint on the massive-mode combinations X_1', X_2'. Assuming at least one of x_1 and x_2 is nonzero, we can choose the special observer reference frame (24) in which $\vec{b} = c\vec{e}$ for some nonzero real number c. The conservation law (91) then implies

$$\begin{aligned}(1 + c^2)\vec{e} \times \vec{\nabla} X_1' &= \vec{e} \times \vec{\nabla} \chi, \\ X_2' &= \frac{1}{c}(X_1' - \chi),\end{aligned} \tag{96}$$

where χ is a purely static function obeying $\vec{e} \cdot \vec{\nabla}\chi = 0$ in this special frame. Imposing the boundary conditions $\chi = 0$ at $t = t_0$ and $X_1' = 0$ at spatial infinity then implies

$$X_1' = X_2' = 0 \tag{97}$$

everywhere in spacetime. This shows that the only propagating modes in the theory (71) subject to the consistency requirement (95) and to a plausible choice of boundary conditions are gravitational and phon modes.

Other boundary conditions can also be adopted, for which X_1' and X_2' could potentially act as extra sources for nonmassive modes in $h_{\mu\nu}$ and $\tilde{\tilde{B}}_{\mu\nu}$. A similar situation arises for the massive mode in bumblebee models, which under suitable boundary conditions yields a modified Einstein-Maxwell theory even in the weak static limit [8]. An investigation along related lines for the Lagrange density (71) or the general action (7) is of interest but lies beyond our present scope.

C. Post-Newtonian metric

In this section, we manipulate the equations of motion to extract a version of the linearized gravitational field equations that depends on the vacuum values $b_{\mu\nu}$ but is independent of $\tilde{\tilde{B}}_{\mu\nu}$. This is achieved at leading order in the nonminimal couplings. A match is then made to the general form of the linearized gravitational field equations obtained in Ref. [12], and the post-Newtonian metric extracted.

Consider first the linearized dynamics of the fluctuations $\tilde{\tilde{B}}_{\mu\nu}$. We adopt the requirement (95) for compatibility with conventional properties of matter and choose boundary conditions yielding the condition (97) on the massive modes. The field equations (86) then simplify to the form

$$\partial^\alpha H_{\alpha\mu\nu} = \frac{\xi_2}{2\kappa}(b^{\alpha\beta}R_{\alpha\beta\mu\nu} + 4b^\alpha{}_{[\mu}R_{\nu]\alpha} + b_{\mu\nu}R). \tag{98}$$

This result can be interpreted as an equation for the fluctations $\tilde{\tilde{B}}_{\mu\nu}$ subject to the constraints

$$\begin{aligned}b_{\mu\nu}\tilde{\tilde{B}}^{\mu\nu} &= b_{\mu\lambda}b_\nu{}^\lambda h^{\mu\nu} + a_1 R, \\ \mathfrak{b}_{\mu\nu}\tilde{\tilde{B}}^{\mu\nu} &= \tfrac{1}{2}x_2 h^\alpha{}_\alpha + a_2 R,\end{aligned} \tag{99}$$

where a_1 and a_2 are given by

$$\begin{aligned}a_1 &= \left(\frac{\lambda_{22}}{\lambda_{22}\lambda_{11} - \lambda_{12}^2}\right)\left(\frac{-\xi_1 + \xi_3}{4\kappa}\right), \\ a_2 &= \left(\frac{\lambda_{12}}{\lambda_{12}^2 - \lambda_{22}\lambda_{11}}\right)\left(\frac{-\xi_1 + \xi_3}{2\kappa}\right).\end{aligned} \tag{100}$$

In these expressions, the coupling constant ξ_2 has been eliminated in favor of ξ_1 using the condition (95).

The desired goal is to use the field equations (98) to eliminate all appearances of $\tilde{\tilde{B}}_{\mu\nu}$ in the linearized gravitational field equations (73), which corresponds to eliminating $\tilde{\tilde{B}}_{\mu\nu}$ from the partial energy-momentum tensors (85). We work here at leading order in the coupling constants ξ_1 and ξ_3. A useful first step is to choose boundary conditions on the dynamics ensuring that the projection $b^{\mu\nu}H_{\lambda\mu\nu}$ is first order in ξ_1. To achieve this, consider the cyclic identity

$$\begin{aligned}\partial^\alpha\partial_\alpha(b^{\mu\nu}H_{\lambda\mu\nu}) = b^{\mu\nu}(&\partial_\mu\partial^\alpha H_{\alpha\nu\lambda} + \partial_\lambda\partial^\alpha H_{\alpha\mu\nu} \\ &+ \partial_\nu\partial^\alpha H_{\alpha\lambda\mu}).\end{aligned} \tag{101}$$

Inserting the field equations (98) yields

$$\begin{aligned}\partial^\alpha\partial_\alpha(b^{\mu\nu}H_{\lambda\mu\nu}) = \frac{8\xi_1}{\kappa}\Big(&b^\alpha{}_\lambda b^{\beta\gamma}\partial_\beta G_{\gamma\alpha} \\ &- b^{\alpha\beta}b^\gamma{}_\beta\partial_\alpha G_{\gamma\lambda} + b^{\alpha\beta}b^\gamma{}_\beta\partial_\lambda G_{\alpha\gamma} \\ &- \frac{1}{2}b^{\alpha\beta}b_{\lambda\beta}\partial_\alpha R + \frac{1}{4}x_1\partial_\lambda R\Big).\end{aligned} \tag{102}$$

This is a hyperbolic equation for the projection $b^{\mu\nu}H_{\lambda\mu\nu}$ with source term of order $O(\xi_1/\kappa)$, where ξ_1/κ is taken as a small dimensionless parameter controlling the size of the nonminimal couplings. We can ensure that the solutions are also of order $O(\xi_1/\kappa)$,

$$b^{\mu\nu}H_{\mu\nu\lambda} \sim O(\xi_1/\kappa), \tag{103}$$

by choosing boundary conditions to eliminate the homogeneous solutions to Eq. (102). This choice implies that the projected covariant derivative $b^{\mu\nu}D_\mu B_{\nu\lambda}$ is of order $O(\xi/\kappa)$,

$$b^{\mu\nu}D_\mu B_{\nu\lambda} \approx \tfrac{1}{2}b^{\mu\nu}H_{\lambda\mu\nu} - \tfrac{1}{2}\partial_\lambda(a_1 R) \sim O(\xi/\kappa). \tag{104}$$

With these results in hand, we can tackle the elimination of $\tilde{\tilde{B}}_{\mu\nu}$ from the partial energy-momentum tensors (85). Inspection reveals that the terms in the latter involving the fluctuations $\tilde{\tilde{B}}_{\mu\nu}$ either are higher order in the nonminimal couplings ξ_1, ξ_3 or are expressible in terms of the metric fluctuations $h_{\mu\nu}$. Some manipulation then yields effective linearized field equations for the metric fluctuations $h_{\mu\nu}$ at leading order in ξ_1 and ξ_3. In terms of linearized curvature tensors, these equations can be expressed as

$$\begin{aligned} R_{\mu\nu} \approx\; & \kappa(S_M)_{\mu\nu} - 2\xi_1 b_\mu{}^\alpha b_{\nu\alpha} R + 6\xi_1 b^{\alpha\beta} b^\gamma{}_{(\mu} R_{\nu)\gamma\alpha\beta} \\ & + 6\xi_1 b_\mu{}^\alpha b_\nu{}^\beta R_{\alpha\beta} + 8\xi_1 b^\beta{}_{(\mu} R_{\nu)\alpha} b_\beta{}^\alpha \\ & - 5\eta_{\mu\nu}\xi_1 b^{\alpha\gamma} b^\beta{}_\gamma R_{\alpha\beta} + \frac{3}{2}\eta_{\mu\nu}\xi_1 b^{\alpha\beta} b^{\gamma\delta} R_{\alpha\beta\gamma\delta} \\ & + 4\xi_1 b^\alpha{}_\gamma b^{\beta\gamma} R_{\mu\alpha\nu\beta} - \xi_3 b^{\alpha\beta} b_{\alpha\beta} R_{\mu\nu} \\ & + \eta_{\mu\nu}\xi_1 b^{\alpha\beta} b_{\alpha\beta} R, \end{aligned} \tag{105}$$

where $(S_M)_{\mu\nu}$ is the trace-reversed energy-momentum tensor for the matter.

At this stage, the expression (105) for the linearized gravitational field equations can be matched to the general form

$$R_{\mu\nu} = \kappa S_{\mu\nu} + (\Phi^{\bar{s}})_{\mu\nu} + (\Phi^{\bar{t}})_{\mu\nu} + (\Phi^{\bar{u}})_{\mu\nu}, \tag{106}$$

obtained in Ref. [12], where the quantities on the right-hand side are defined as

$$\begin{aligned} \Phi^{\bar{s}}_{\mu\nu} &= \tfrac{1}{2}\eta_{\mu\nu}(\bar{s}_B)^{\alpha\beta} R_{\alpha\beta} - 2(\bar{s}_B)^\alpha{}_{(\mu} R_{\nu)\alpha} + \tfrac{1}{2}(\bar{s}_B)_{\mu\nu} R \\ &\quad + (\bar{s}_B)^{\alpha\beta} R_{\alpha\mu\nu\beta}, \\ \Phi^{\bar{t}}_{\mu\nu} &= 2(\bar{t}_B)^{\alpha\beta\gamma}{}_{(\mu} R_{\nu)\gamma\alpha\beta} + 2(\bar{t}_B)_\mu{}^\alpha{}_\nu{}^\beta R_{\alpha\beta} \\ &\quad + \tfrac{1}{2}\eta_{\mu\nu}(\bar{t}_B)^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \\ &= 0, \\ \Phi^{\bar{u}}_{\mu\nu} &= \bar{u}_B R_{\mu\nu}. \end{aligned} \tag{107}$$

Note that the net contribution to $(\Phi^{\bar{t}})_{\mu\nu}$ vanishes, as a consequence of an identity satisfied by the coefficients $(\bar{t}_B)^{\kappa\lambda\mu\nu}$ [12]. In the expressions (107), the coefficients for Lorentz violation $(\bar{s}_B)^{\mu\nu}$, $(\bar{t}_B)^{\kappa\lambda\mu\nu}$, and $\bar{u}_B$ can be expressed explicitly in terms of the vacuum value $b_{\mu\nu}$ as

$$\begin{aligned} (\bar{s}_B)^{\mu\nu} &= 2\xi_1(b^\mu{}_\alpha b^{\nu\alpha} - \tfrac{1}{4}\eta^{\mu\nu} b^{\alpha\beta} b_{\alpha\beta}), \\ (\bar{t}_B)^{\kappa\lambda\mu\nu} &= 2\xi_1(b^{\kappa\lambda} b^{\mu\nu} + \tfrac{1}{2} b^{\kappa\mu} b^{\lambda\nu} - \tfrac{1}{2} b^{\kappa\nu} b^{\lambda\mu}) \\ &\quad - \tfrac{3}{2}\xi_1(\eta^{\kappa\mu} b^\lambda{}_\alpha b^{\nu\alpha} - \eta^{\lambda\mu} b^\kappa{}_\alpha b^{\nu\alpha} \\ &\quad - \eta^{\kappa\nu} b^\lambda{}_\alpha b^{\mu\alpha} + \eta^{\lambda\nu} b^\kappa{}_\alpha b^{\mu\alpha}) \\ &\quad + \tfrac{1}{2}\xi_1(\eta^{\kappa\mu}\eta^{\lambda\nu} - \eta^{\lambda\mu}\eta^{\kappa\nu}) b^{\alpha\beta} b_{\alpha\beta}, \\ \bar{u}_B &= (\tfrac{3}{2}\xi_1 - \xi_3) b^{\alpha\beta} b_{\alpha\beta}. \end{aligned} \tag{108}$$

Comparison of these vacuum-value coefficients with the results (72) for the coefficient fields appearing in the Lagrange density (71) reveals a rescaling of the latter of the type described in Ref. [12].

It is instructive to compare the present results for the antisymmetric 2-tensor to the equivalent ones for bumblebee theories. In these models, a potential for a vector field B_μ drives the formation of a vacuum value b_μ and thereby triggers spontaneous Lorentz violation. Possible nonminimal curvature couplings include Lorentz-violating couplings of the $s^{\mu\nu}$ and u types, but $t^{\kappa\lambda\mu\nu}$ couplings cannot appear [4]. In contrast, the theory (71) investigated here provides an explicit example of how nonzero $t^{\kappa\lambda\mu\nu}$ couplings can arise. Although the coefficients $(\bar{t}_B)^{\kappa\lambda\mu\nu}$ produce no leading-order contribution to the linearized gravitational field equations, they may generate nonzero contributions at higher orders. Moreover, the coefficients $(\bar{t}_B)^{\kappa\lambda\mu\nu}$ contain information about $b_{\mu\nu}$ that is absent in $(\bar{s}_B)^{\mu\nu}$, as can be verified by inspection of Eq. (108) in the special frame (24). Establishing the phenomenological role of the coefficients $(\bar{t}_B)^{\kappa\lambda\mu\nu}$ is an interesting open issue for future investigation.

Given the linearized gravitational field equations in the form (106) and the explicit expressions (108) for the coefficients for Lorentz violation, we can extract the post-Newtonian metric sourced by a given distribution of matter. For this purpose, we assume the matter is described as a conventional perfect fluid generating the gravitational potentials U, U^{jk}, V^j, W^j, X^{jkl}, and Y^{jkl} defined in Eq. (28) of Ref. [12]. We work at post-Newtonian order $O(3)$, and choose the post-Newtonian gauge at this order as

$$\partial_j g_{0j} = \tfrac{1}{2}\partial_0 g_{jj}, \qquad \partial_j g_{jk} = \tfrac{1}{2}\partial_k(g_{jj} - g_{00}). \tag{109}$$

Including terms to post-Newtonian order $O(3)$, we obtain

$$\begin{aligned} g_{00} &= -1 + 2U + 3(\bar{s}_B)^{00} U + (\bar{s}_B)^{jk} U^{jk} \\ &\quad - 4(\bar{s}_B)^{0j} V^j + O(4), \\ g_{0j} &= -(\bar{s}_B)^{0j} U - (\bar{s}_B)^{k0} U^{jk} \\ &\quad - \tfrac{7}{2}(1 + \tfrac{1}{28}(\bar{s}_B)^{00}) V^j - \tfrac{1}{2}(1 + \tfrac{15}{4}(\bar{s}_B)^{00}) W^j \\ &\quad + \tfrac{3}{4}(\bar{s}_B)^{jk} V^k + \tfrac{5}{4}(\bar{s}_B)^{jk} W^k + \tfrac{9}{4}(\bar{s}_B)^{kl} X^{klj} \\ &\quad - \tfrac{15}{8}(\bar{s}_B)^{kl} X^{jkl} - \tfrac{3}{8}(\bar{s}_B)^{kl} Y^{klj}, \\ g_{jk} &= \delta^{jk} + [(2 - (\bar{s}_B)^{00})\delta^{jk}] U \\ &\quad + [(\bar{s}_B)^{lm}\delta^{jk} - (\bar{s}_B)^{lj}\delta^{mk} \\ &\quad - (\bar{s}_B)^{lk}\delta^{mj} + 2(\bar{s}_B)^{00}\delta^{jl}\delta^{km}] U^{lm}. \end{aligned} \tag{110}$$

Note that the corresponding explicit post-Newtonian solutions for $\tilde{B}_{\mu\nu}$ can also be obtained from the equations of motion (98).

The above result for the post-Newtonian metric involves the vacuum coefficients (108). However, with the assumption of a conventional perfect fluid and the gauge choice (109), the result (110) retains the same form as the general expression for the pure-gravity sector of the minimal SME. As a consequence, the implications for experimental and observational tests derived in Ref. [12] apply directly to the theory (71) in the form considered here. For example, the constraints on the SME coefficients $\bar{s}^{\mu\nu}$ obtained via lunar laser ranging [10] and from atom interferometry [11] can be reinterpreted as limits on $(\bar{s}_B)^{\mu\nu}$. Other potential methods to measure these coefficients include laboratory experiments with torsion pendula or gravimeters, observations of the precession of orbiting gyroscopes, analyses of timing signals from binary pulsars, solar-system tests involving perihelion precessions, and time-delay and Doppler measurements [12,36].

We conclude this section with a brief discussion of the relation of the post-Newtonian metric (110) to the parametrized post-Newtonian (PPN) formalism [35,37] developed for testing gravitational physics. The PPN formalism assumes the existence of a special frame in which all unconventional effects are controlled by isotropic parameters, so any putative match to Eq. (110) requires identifying a frame in which the coefficients $(\bar{s}_B)^{\mu\nu}$ and $(\bar{t}_B)^{\kappa\lambda\mu\nu}$ are isotropic. For such a frame to exist, the coefficients $(\bar{s}_B)^{\mu\nu}$, $(\bar{t}_B)^{\kappa\lambda\mu\nu}$ must satisfy the isotropic constraints

$$\begin{aligned} (\bar{s}_B)^{0j} &= 0, \\ (\bar{s}_B)^{jk} &= \tfrac{1}{3}\delta^{jk}(\bar{s}_B)^{00}, \\ (\bar{t}_B)^{\kappa\lambda\mu\nu} &= 0. \end{aligned} \tag{111}$$

However, no such frame exists when $b_{\mu\nu}$ is nonzero. One way to see this is to use the separation (23) into two spatial vectors $\vec{e}$, $\vec{b}$ to write the isotropic constraints on $(\bar{s}_B)^{\mu\nu}$ in the form

$$\begin{aligned} \vec{e}\times\vec{b} &= 0, \\ (\vec{e}^2+\vec{b}^2)\delta^{jk} - 3e^je^k - 3b^jb^k &= 0. \end{aligned} \tag{112}$$

Some manipulation then reveals that only $\vec{e} = \vec{b} = 0$ can satisfy these constraints. The present theory for an antisymmetric 2-tensor $B_{\mu\nu}$ with nonzero vacuum value $b_{\mu\nu}$ therefore lacks an isotropic post-Newtonian limit, and hence it lies outside the PPN. This implies no experimental or observational limits on the theory can be placed from post-Newtonian tests analyzed via the PPN formalism.

V. TADPOLES IN THE MINIMAL MODEL

In this section, we return to the minimal theory (34) in Minkowski spacetime and investigate one aspect of its quantum behavior. While the renormalizability of various sectors of the SME viewed as an effective field theory has been studied at one loop [38], less is known about the issue of renormalizability and its relation to the potential V in theories with spontaneous Lorentz breaking. Here, we consider the effective action at linear order in the bare couplings and study the behavior of the resulting interactions under the renormalization group. The Wilson formulation of the RG [39–41] has been used to adduce evidence for relevant nonpolynomial interactions in scalar field theories [42,43], while exact RG methods [44] imply an essentially regularization-independent differential equation governing the RG flow for these interactions [45]. Similar methods can be applied to models with Lorentz violation [46], including bumblebee theories [13]. In what follows, we briefly summarize the scalar and vector cases and outline results for the minimal model (34) involving the antisymmetric 2-tensor field $B_{\mu\nu}$. Details of the methodology and a summary of possible issues can be found in Ref. [13].

A. Scalar and vector

Consider a theory with a single real scalar field, with Euclidean action in d dimensions given in terms of bare fields by

$$S_b = \int d^dx\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V_b\right). \tag{113}$$

In what follows, the interaction Lagrange density V_b is taken to be representable as a power series in ϕ^2, and a momentum cutoff Λ is used to regulate loop integrals.

Loop corrections generate the renormalized effective action S. Finding an exact expression for S requires determining the coefficient of every effective n-particle vertex, which requires all n-point correlation functions. This is a challenging task. One approach yielding an approximate expression is to limit attention to interactions that are at most linear in the bare couplings. All contributions to the effective n-particle amplitude then involve bare $(n+2k)$-point vertices attached via k tadpole loops. These diagrams can be summed [47]. Each loop contributes a factor of $\Delta_F(0)/2$. The factor of 2 is the symmetry factor for the loop, while $\Delta_F(x-y)$ is the Feynman propagator for a massless scalar, which differs from the negative inverse Laplacian only through its large-momentum regulation. In four dimensions, $\Delta_F(0) = \Lambda^2/16\pi^2$. Each diagram also acquires an additional symmetry factor of $k!$ corresponding to the interchange of k loops.

In terms of dimensionless effective coupling constants, the effective interaction action can be written as

$$S_b^{\text{int}} = \int d^dx\Lambda^dU_b, \tag{114}$$

where the dimensionless potential $U_b = U_b(\Lambda^{-(d-2)/2}\phi)$ depends upon Λ as a parameter as well as on ϕ. This dependence determines the nontrivial RG flow. Including all the first-order contributions, U_b must satisfy

$$\begin{aligned} \Lambda\frac{\partial U_b}{\partial\Lambda} + dU_b - \frac{1}{2}(d-2)\Lambda^{-(d-2)/2}\phi U_b' \\ = -\frac{1}{2}(d-2)C_bU_b''. \end{aligned} \tag{115}$$

The right-hand side of this equation contains the quantum corrections and arises entirely from tadpole contributions. The numerical value of the constant C_b depends on the regulator, but the result (115) is otherwise regulator independent in the tadpole approximation. Using a cutoff regulator yields $C_b = 1/16\pi^2$ in four dimensions.

The solutions U_b of the differential equation (115) with power-law dependences on Λ, $\Lambda\partial U_b/\partial\Lambda = -\lambda U_b$, are eigenmodes of the RG flow near the Gaussian fixed point. Solutions with positive anomalous dimension λ correspond to asymptotically free theories. They have stronger scale dependences than superficially renormalizable theories and involve relevant nonpolynomial interactions. Each value of λ gives only one functional form for the interaction, at least at the lowest nontrivial order in the coupling. Each model is therefore renormalizable at this level, being specified completely by the value of the coupling at a fixed energy and by the anomalous dimension λ, which controls the energy dependence of the cross section.

In the above, renormalizability is understood to be the statement that all divergences can be eliminated and all experimental properties determined by specifying only a finite number of observable quantities. It may seem counterintuitive that a nonpolynomial theory can be renormalizable in this sense because expanding in monomials produces an infinite number of coefficients that could be deemed adjustable. However, a sum of monomials is only one way to express a function. For example, although $g\exp(c\phi^2/\Lambda^2)$ can be expanded as an infinite number of monomial operators, the polynomial $\mu^2\phi^2 + \lambda\phi^4$ could also require an infinite sum to represent it in terms of other operators. The sine-Gordon theory in $1+1$ dimensions has a potential with an infinite number of monomial terms, but the theory is known to be renormalizable [48]. Furthermore, the RG relevance or irrelevance of nonpolynomial potentials for ϕ is distinct from the known irrelevance of all monomial potentials of degree greater than 4. The monomials fail to span the infinite-dimensional vector space of entire functions and hence form an incomplete basis for the space of allowed potentials, so the generic behavior of nonpolynomial theories cannot be inferred from the triviality of interacting polynomial theories. The widespread use of the incomplete basis of monomials in perturbative calculations originates in their special and convenient relationship to external states of known particle number, but this feature is inessential in the RG context.

Next, we summarize briefly the case of an action for a vector field B_μ with Maxwell kinetic term and potential V_b expressible as a power series in $B_\mu B^\mu$ [23]. Note that this bumblebee theory has no gauge invariance. The RG calculations in Euclidean space parallel those for a multiplet of four scalars except for minor changes arising from the structure of the kinetic term [13]. As in the scalar case, the eigenmodes of the RG flow with positive anomalous dimension correspond to asymptotically free theories.

All nontrival potentials of this type generate a vacuum value b_μ for the bumblebee field B_μ and trigger spontaneous Lorentz breaking. However, only a subset lead to stable theories in Minkowski spacetime, where $B_\mu B^\mu$ can be either positive or negative. Stable renormalizable theories arise when the anomalous dimension λ is less than 2 and b_μ is spacelike or when the anomalous dimension λ is greater than 8 and b_μ is timelike.

B. Antisymmetric 2-tensor

For the case of the antisymmetric 2-tensor $B_{\mu\nu}$, we consider the minimal theory with Lagrange density (34). Only two independent observer scalars can be constructed from $B_{\mu\nu}$, which we choose as

$$X = B_{\mu\nu}B^{\mu\nu}, \qquad Y = B_{\mu\nu}\mathfrak{B}^{\mu\nu}. \tag{116}$$

The scalars X_1, X_2 defined in Eq. (20) could also be adopted, but the above choice simplifies the presentation of the RG equation. The bare potential V_b is taken to be expressible as a power series in X and Y. As occurs for the bumblebee theory, the equations for the RG flow are equivalent up to numerical factors to those for a multicomponent scalar field. The differences are encoded in a fraction f, which is the number of propagating degrees of freedom divided by the total number of degrees of freedom appearing in the kinetic term. For the antisymmetric 2-tensor, $f = 1/2$ because 3 of the 6 degrees of freedom have propagators in the kinetic term.

Restricting attention to interactions that are at most linear in the bare couplings implies as before that the diagrams for the effective amplitudes involve tadpole loops attached to bare vertices. Each tadpole loop arises from a contraction of two factors of $B_{\mu\nu}$ closing an external line, with the specific contraction determining the resulting contribution from the diagram. In forming the loops, an external line can be closed in one of five ways. Connecting the two fields within a single X yields a factor of $12C\Lambda^2$, where $C = 2fC_b$. For two tensors from different X terms, there are four possibilities generating a net contribution of $8C\Lambda^2 X$. Contracting two fields in a single Y gives zero. The four ways to connect a field in X with one in Y yield $8C\Lambda^2 Y$, while the four possibilities using tensors from two different Y factors yield $8C\Lambda^2 X$. Note that there is no mixing of parity-odd and parity-even parts of the interaction.

To investigate the RG flow, we write the effective potential in terms of dimensionless couplings $g_{j,k}$ as

$$V(X, Y) = \sum_{j,k=0}^{\infty} g_{j,k}\Lambda^4 \frac{X^j Y^k}{\Lambda^{2(j+k)}}. \tag{117}$$

Operating on V to obtain $\Lambda dV/d\Lambda$ yields two kinds of contributions, those derived from direct differentiation of Eq. (117) and those arising via $g_{j,k}$ from the differentiation of loop diagrams [40,43]. Keeping only quantum corrections linear in the field, which correspond to the tadpole diagrams, the result is

$$\Lambda \frac{dV}{d\Lambda} = \sum_{j,k=0}^{\infty} \bigg\{ \bigg[\Lambda \frac{dg_{j,k}}{d\Lambda} + 4g_{j,k} - 2(j+k)g_{j,k} \bigg] \Lambda^4 \frac{X^j Y^k}{\Lambda^{2(j+k)}} + g_{j,k}\Lambda^4 \frac{1}{\Lambda^{2(j+k)}} [12jX^{j-1}Y^k + 4j(j-1)X^{j-1}Y^k + 4jkX^{j-1}Y^k + 4k(k+1)X^{j+1}Y^{k-2}] C\Lambda^2 \bigg\}. \tag{118}$$

Some combinatorial factors appear in the quantum corrections, which are the terms involving $C\Lambda^2$.

The effective potential should be independent of the cutoff, $\Lambda dV/d\Lambda = 0$. Also, if the potential is an eigenmode of the RG flow near the Gaussian fixed point, then the couplings $g_{j,k}$ should have power-law scaling with Λ,

$$\Lambda \frac{dg_{j,k}}{d\Lambda} = -\lambda g_{j,k}. \tag{119}$$

Here, λ is the anomalous dimension of the potential. Inserting these conditions into Eq. (118) and equating powers of X and Y yields the recurrence relation for the couplings $g_{j,k}$ of an eigenmode as

$$\begin{aligned}[]
&[\lambda - 4 + 2(j+k)]g_{j,k} \\
&= C\{4(k+1)(k+2)g_{j-1,k+2} \\
&\quad + [12(j+1) + 4j(j+1) + 8k(j+1)]g_{j+1,k}\}.
\end{aligned} \tag{120}$$

By definition, $g_{-1,k} = 0$.

The recurrence relation (120) is more complicated than the equivalent expressions for the scalar and vector cases because three couplings are involved rather than two. The number of different interaction terms involving $2n$ powers of $B_{\mu\nu}$ is $n+1$. If all the couplings at order $(2n-2)$ are known, then the couplings at order $2n$ are constrained by n equations, one for each lower-order coupling. This means one coupling is undetermined at each order. For example, arbitrary values for the entire set $\{g_{0,k}\}$ can be chosen, whereupon all other couplings are fixed. The freedom to adjust infinitely many nonzero parameters is an indication of possible nonrenormalizability, since an infinite number of measurements is then required to specify the theory. However, renormalizability can be restored if at most finitely many parameters are nonzero. For example, if the effective potential depends only on X, so that all the couplings $g_{0,k}$ vanish, then a stable theory can be specified by the anomalous dimension λ and a single coupling g. This suffices for renormalizability, since only two measurements can fix λ and g.

The general case has nontrivial dependence on both X and Y. The key feature of the theory responsible for the possible nonrenormalizability is the existence of more than one independent observer scalar, as in Eq. (116). We therefore expect that other theories with general interactions involving tensors of higher rank also exhibit possible nonrenormalizability. Note, however, that nonrenormalizable interactions may nonetheless be relevant, since for a stable theory a positive anomalous dimension λ implies the effective potential grows at large scales, the free-field fixed point is ultraviolet stable, and the theory displays asymptotic freedom.

The recurrence relation (120) is equivalent to a partial differential equation for the effective potential. It is convenient to introduce the dimensionless independent variables x, y and dimensionless effective potential U by

$$x = \frac{X}{\Lambda^2}, \qquad y = \frac{Y}{\Lambda^2}, \qquad U(x,y) = \frac{V(X,Y)}{\Lambda^4}. \tag{121}$$

Then, the recurrence relation becomes

$$(\lambda - 4)U + 2xU_x + 2yU_y - 12CU_x - 4Cx(U_{xx} + U_{yy}) - 8CyU_{xy} = 0, \tag{122}$$

where partial derivatives of U are denoted by subscripts, U_x, U_y, etc.

We know of no exact solutions to Eq. (122) that are both absolutely stable and have a nontrivial dependence on y. An example of a solution with weak instability is the effective potential

$$U(x,y) = g\exp\bigg(\frac{x}{4C}\bigg)\cosh\bigg(\frac{y}{4C}\bigg), \tag{123}$$

which satisfies Eq. (122) with anomalous dimension $\lambda = 7$. This potential is strictly positive but tends to zero as $x \to -\infty$ for $|y| < |x|$, so there is no lowest-energy field configuration. The instability is weak because the energy approaches a limiting constant instead of diverging negatively. This implies tiny modifications of the potential suffice to restore stability. Adding a superficially renormalizable interaction such as $g'X^2$ suffices to obtain a physically meaningful model at the level of effective field theory with a finite cutoff, and it triggers a Lorentz-violating vacuum expectation value with $y = 0$ and large negative x. However, the RG flow suggests the extra term is irrelevant and fails to produce a stable continuum limit. It is also conceivable that stability could be restored at the nonlinear level.

The relationship between the effective potential and Lorentz violation is encoded in the recurrence relation (120). If $B_{\mu\nu}$ develops a nonzero vacuum value, the theory must either spontaneously break Lorentz symmetry or be unstable. A necessary condition for Lorentz invariance is the existence of a local minimum of the effective potential U at $B_{\mu\nu} = 0$ or, equivalently, at $x = y = 0$. This implies that

$$U_x(0,0) = 0, \qquad U_y(0,0) = 0,$$
$$U_{xx}(0,0) \geq 0, \qquad U_{yy}(0,0) \geq 0. \tag{124}$$

We next examine the implications of these conditions and the recurrence relation (120) at each order in $B_{\mu\nu}$.

Consider first the lowest-order couplings, corresponding to terms up to fourth order in $B_{\mu\nu}$. The conditions (124) imply

$$g_{1,0} = g_{0,1} = 0, \qquad g_{2,0} \geq 0, \qquad g_{0,2} \geq 0. \tag{125}$$

The recurrence relation (120) imposes two additional linear equations relating the quadratic and quartic couplings,

$$(\lambda - 2)g_{1,0} = 32Cg_{2,0} + 8Cg_{0,2},$$
$$(\lambda - 2)g_{0,1} = 20Cg_{1,1}. \tag{126}$$

The only way to satisfy all these conditions is to have $g_{2,0} = g_{1,1} = g_{0,2} = 0$. A similar argument holds at sixth order. Since $U(x, 0)$ is required to be stationary at $x = 0$, the coupling $g_{3,0}$ must vanish. Likewise, $g_{0,3} = 0$. The recurrence relation (120) then implies the remaining coefficients $g_{2,1}$, $g_{1,2}$ vanish at this order as well. These analytic arguments become more subtle at eighth order. Although Eq. (120) forces the condition $g_{3,1} = g_{1,3} = 0$, nonzero values of the other coefficients are allowed because the recurrence relation is satisfied for $g_{4,0} = g_{0,4}/16$, $g_{2,2} = -3g_{0,4}/4$. However, inspection of the graph of $(x^4/16) - (3x^2y^2/4) + y^4$ reveals that $x = y = 0$ is a saddle point instead of a local extremum, so all the eighth-order coefficients must vanish too. Elementary analytic arguments of this type suffice to show that all coefficients vanish up to the 14th order in $B_{\mu\nu}$. We expect this result to hold at all orders. Even if this conjecture is incorrect, the above arguments show that most effective potentials either trigger spontaneous Lorentz violation or are unstable.

Consider now the special case of potential $V = V(B_{\mu\nu})$ depending only on the parity-even observer scalar X or, equivalently, only on X_1 as defined in Eq. (20). This restriction implies a unique solution to the recurrence relation (120) up to an overall constant. We find

$$V(B_{\mu\nu}) = g\Lambda^4[M(\tfrac{1}{2}\lambda - 2, 3, z) - 1], \tag{127}$$

where the argument z is given by

$$z = \frac{X}{2C\Lambda^2} = \frac{8\pi^2}{\Lambda^2} B_{\mu\nu}B^{\mu\nu}. \tag{128}$$

The function $M(\alpha, \beta, z)$ is the confluent hypergeometric Kummer function, defined as [49]

$$M(\alpha, \beta, z) = 1 + \frac{\alpha}{\beta}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!} + \cdots. \tag{129}$$

Plots of the function $M(\alpha, 3, z)$ can be found in Ref. [50].

The effective potential (127) for $B_{\mu\nu}$ is closely related to that of the effective potential $V(B_\mu)$ in the bumblebee theory. The latter takes the form [13]

$$V(B_\mu) = g\Lambda^4[M(\tfrac{1}{2}\lambda - 2, 2, z) - 1], \tag{130}$$

with $z = -32\pi^2 B_\mu B^\mu/3\Lambda^2$. The functional properties of $V(B_{\mu\nu})$ and $V(B_\mu)$ are therefore similar. In both cases, Lorentz violation is ubiquitous. Stable theories exist for a range of positive values of the anomalous dimension λ, and all the corresponding potentials exhibit spontaneous Lorentz breaking.

The effective potential (127) allows stable theories with both positive and negative vacuum values x_1 for X. Decomposing $B_{\mu\nu}$ as $B_{0j} = -\Sigma^j$, $B_{jk} = \epsilon_{jkl}\Xi^l$ in analogy to Eq. (49) yields $X = \vec{\Xi}^2 - \vec{\Sigma}^2$, which can be either positive or negative. The argument z in the effective potential (127) can therefore acquire either sign in the local minimum. An analysis paralleling that in Ref. [13] reveals that stable theories exist for positive x_1 when the anomalous dimension lies between 0 and 2. For negative x_1, stability appears for λ greater than 10. In the latter case, metastable vacua also occur with larger vacuum values for X.

VI. SUMMARY

In this work, we have studied field theories with spontaneous Lorentz violation involving an antisymmetric 2-tensor $B_{\mu\nu}$. The theories are defined through a general class of actions of the form (7). The core of the action includes kinetic terms for $B_{\mu\nu}$ and a potential V driving spontaneous Lorentz violation. Other components include a gravity sector, a matter sector, and nonminimal gravitational couplings.

All nonminimal nonderivative gravitational couplings to $B_{\mu\nu}$ that are linear in the curvature are displayed in Sec. II B. Section II C discusses aspects of the potential, which can be taken as a function of the two observer-scalar field operators X_1 and X_2 defined in Eq. (20). The Lorentz-violating solutions to the equations of motion are classified by two vacuum values, x_1 and x_2. Generic features of these theories include the appearance of massless NG modes, which are solutions of Eq. (33), and the massive modes, which can be identified with X_1 and X_2. In some models, certain NG modes appear as physical modes, called phon modes, that propagate at long range.

A comparatively simple class of theories with some elegant features consists of Lagrange densities with gauge-invariant kinetic term for $B_{\mu\nu}$ and without nonminimal couplings. These minimal models are the subject of Sec. III. We show they are equivalent to certain field theories with spontaneous Lorentz violation involving a vector A_μ. In Minkowski spacetime and in the absence of Lorentz violation, these equivalences reduce to the known dualities between massless $B_{\mu\nu}$ and scalar fields and between massive $B_{\mu\nu}$ and vector fields [1]. The potential for Lorentz violation produces a hybrid duality in which phon mode and massive modes appear as different combinations of the components of the vector A_μ. Couplings to external currents and to gravity in Riemann spacetime leave unaffected this basic picture, as shown in Sec. III B.

Some features of nonminimal curvature couplings of $B_{\mu\nu}$ are considered in Sec. IV. In gravitational theories with spontaneous Lorentz breaking, the dominant curvature couplings generating Lorentz violation involve one or more of the three coefficient fields $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$, and u [4]. The action (7) for $B_{\mu\nu}$ incorporates all three types of couplings. We demonstrate this using the Lagrange density (71), which is a restriction of the theory (7) both simple enough for illustrative purposes and sufficiently general to exhibit nonzero coefficient fields $s^{\mu\nu}$, $t^{\kappa\lambda\mu\nu}$, and u. In Sec. IV B, this theory is linearized about an asymptotically flat background. Given suitable boundary conditions, the massive modes become frozen at this level, and only the phon and gravitational modes propagate. The post-Newtonian expansion for the theory is developed in Sec. IV C. This produces the nonzero vacuum values (108) for all three coefficient fields, a feature absent from other gravitationally coupled models with Lorentz violation discussed in the literature. The post-Newtonian metric is constructed as Eq. (110). It predicts a variety of signals in post-Newtonian tests of gravity. Many can be measured in existing or planned searches, while none are accessible to analyses using the PPN formalism.

In Sec. V, we return to the minimal model in Minkowski spacetime and study the quantum behavior of the Lorentz-violating potential. The RG flow in the tadpole approximation is determined by Eq. (118). An analytic solution for the special case with potential depending only on the parity-even observer scalar is obtained in Eq. (127). For potentials of this form, stable theories exist with anomalous dimensions lying between 0 and 2 or larger than 10. All potentials of this type exhibit spontaneous Lorentz breaking.

In conclusion, the spontaneous breaking of Lorentz symmetry via an antisymmetric 2-tensor offers some intriguing features. While these field theories display the properties expected from the broad existing treatment for general tensor fields [4,7,8], the structure of the NG and massive modes and of the gravitational couplings arising from the case of the antisymmetric 2-tensor implies distinctive physical content. The properties discussed in the present work suggest interesting possibilities for phenomenological applications, with definite signals that can be sought in present or forthcoming experimental and observational tests of Lorentz symmetry.

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