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LORENTZIAN SYMMETRIC SPACES[1]

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Communicated by Armand Borel, August 25, 1969

0. Introduction. M. Berger has classified the pseudo-riemannian symmetric spaces which are isotropy irreducible [1]. The notion of isotropy irreducibility is too restrictive and the proper building blocks are the indecomposable symmetric spaces (cf. §1). In other words, every pseudo-riemannian symmetric space is locally a product of indecomposable ones. In this paper we announce a complete classification of Lorentzian symmetric spaces; we emphasize the solvable case since the solvable and semisimple case separate (see Theorem 3). It has been conjectured that there were no pseudo-riemannian symmetric spaces with nilpotent isometry group: a class of examples is presented in §4. §4 also contains examples of Lorentzian symmetric tori, analogous to flat tori.

Our interest in this problem originated with the study of Huyghens' principle in general relativity [2]; the 4 dimensional solvable Lorentzian symmetric spaces satisfy Huyghens' principle.

We would like to thank J. Tits for the term indecomposable, C. C. Moore for pointing out that the Heisenberg algebra was related to our algebra $\mathfrak{g}_A$ and A. Taub and G. Walker for stimulating discussions.

1. Pseudo-riemannian symmetric quadruples. By a pseudo-riemannian symmetric space we mean a connected, C^∞ manifold, $(M, \langle\ ,\ \rangle)$

AMS Subject Classifications. Primary 5373, 5378.

Key Words and Phrases. Pseudoriemannian symmetric spaces, indecomposable, symmetric quadruple, semisimple Lie algebra, solvable Lie algebra, Riemann tensor.

[1] This work was supported in part by Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant No. 903-67 and National Aeronautics and Space Administration under NASA Grant No. NGL 44-004-001.

with a geodesically complete C^∞ pseudo-riemannian structure $\langle\ ,\ \rangle$, such that the Riemann curvature tensor of the Levi-Civita connection is covariantly constant. Let $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ where $\mathfrak{g}$ is a finite dimensional real Lie algebra, $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{p}$ is a nonzero, $\mathfrak{k}$-invariant, complementary vector subspace to $\mathfrak{k}$ in $\mathfrak{g}$, B is a nondegenerate, $\mathfrak{k}$-invariant, symmetric bilinear form on $\mathfrak{p}$. We call q a symmetric quadruple (abbreviated s.q.) of dimension n if:

(i) $[\mathfrak{p}, \mathfrak{p}]=\mathfrak{k}$.

(ii) $\mathfrak{k}$ contains no nonzero ideals of $\mathfrak{g}$.

(iii) $\dim \mathfrak{p}=n$.

Let $q_i=(\mathfrak{g}_i, \mathfrak{k}_i, \mathfrak{p}_i, B_i)$ be a s.q. $(i=1, 2)$; q_1 and q_2 are said to be isomorphic if there exists a Lie algebra isomorphism $A:\mathfrak{g}_1\rightarrow\mathfrak{g}_2$ such that:

(i) $A\mathfrak{k}_1=\mathfrak{k}_2$.

(ii) $A\mathfrak{p}_1=\mathfrak{p}_2$.

(iii) $B_2(Ax, Ay)=B_1(x, y)$, for any $x, y\in\mathfrak{p}_1$.

Let $(M, \langle\ ,\ \rangle)$ be a simply connected pseudo-riemannian symmetric space. $G(M)$, the group generated by the transvections of M (that is the extension to isometry of parallel translation along geodesics) is a Lie group acting transitively on M. Let $p_0\in M$ and s_{p_0} be the symmetry at $p_0\cdot\mathfrak{g}$ will denote the Lie algebra of $G(M)$, $\sigma:\mathfrak{g}\rightarrow\mathfrak{g}$ the involutive automorphism of $\mathfrak{g}$ corresponding to inner conjugation by s_{p_0}. We set $\mathfrak{k}=\{x\in\mathfrak{g}\mid\sigma x=x\}$, $\mathfrak{p}=\{x\in\mathfrak{g}\mid\sigma x=-x\}$; $\mathfrak{k}$ is the Lie algebra of the isotropy group of p_0 in $G(M)$. We "pull-back" $\langle\ ,\ \rangle$ to $\mathfrak{p}$ denoting the pull-back B. It is easy to see that $q(M)=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ is a s.q., which up to isomorphism is independent of p_0.

THEOREM 1. (a) *Let M_1 and M_2 be simply connected pseudo-riemannian symmetric spaces. Then M_1 and M_2 are isometric if and only if $q(M_1)$ is isomorphic with $q(M_2)$.*

(b) *Let q be a s.q. Then, there exists M, a simply connected pseudo-riemannian symmetric space such that q is isomorphic with $q(M)$.*

Part (b) is canonical. Indeed if $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ is a s.q., and if G is the connected Lie group with Lie algebra $\mathfrak{g}$, then the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$ is closed in G.

It is an exercise to construct the full algebras of Killing vector fields, and affine vector fields on M, from the structure of $q(M)$.

A pseudo-riemannian manifold $(M, \langle\ ,\ \rangle)$ is said to be decomposable if it can be written locally as a product $(M_1, \langle\ ,\ \rangle_1)\times(M_2, \langle\ ,\ \rangle_2)$; it is indecomposable if it is not decomposable.

Note. An indecomposable pseudo-riemannian symmetric space is not necessarily irreducible in the sense of J. Wolf [3].

Let $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ be a s.q. We call q decomposable if p is the direct sum of two $\mathfrak{k}$-invariant, B-orthogonal, nonzero subspaces; q is indecomposable if it is not decomposable.

THEOREM 2. *Let M be a pseudo-riemannian symmetric space and let M be its universal covering space. Then $\tilde{M}$ is indecomposable if and only if $q(M)$ is indecomposable.*

Before proceeding to the fine structure of s.q.'s we note that the duality in the theory of Riemannian symmetric spaces admits a natural extension to s.q.'s. Indeed, if $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$, and $\mathfrak{g}^C$ is the complexification of $\mathfrak{g}$ then the dual of q is $q'=(\mathfrak{g}', \mathfrak{k}, \sqrt{-1}\mathfrak{p}, B')$ where $\mathfrak{g}'$ is the real subalgebra $\mathfrak{k}+\sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}^C$ and $B'(\sqrt{-1}x, \sqrt{-1}y)=B(x, y)$ $\forall x, y, \in \mathfrak{p}$.

2. **Lorentzian quadruples.** A Lorentzian manifold of dimension $(n+1)$ is a pseudo-riemannian manifold, with fundamental tensor of signature $(1, n)$. The corresponding notions of Lorentzian symmetric space and Lorentzian symmetric quadruples (abbreviated l.s.q.) are the obvious ones.

THEOREM 3. *Let $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ be an indecomposable l.s.q. Then $\mathfrak{g}$ is either semisimple or solvable.*

To prove this result one uses a Levi decomposition of $\mathfrak{g}$, stable under the involution, the existence of which is assured by the Mostow-Taft theorem [4]. The key to the proof is an analysis of the Riemann tensor of the associated symmetric space.

If $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ is an s.q. and if $\mathfrak{g}$ is semisimple we call q semisimple. The next result reduces the classification of indecomposable semisimple l.s.q. to Berger's classification of isotropy irreducible pseudo-riemannian symmetric spaces.

PROPOSITION 1.[2] *Let $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ be a semisimple indecomposable l.s.q. such that the action of $\mathfrak{k}$ on $\mathfrak{p}$ is not irreducible. Then up to isomorphism $\mathfrak{g}=\mathrm{sl}(2, R)$, $\mathfrak{k}$ is the space of all real diagonal matrices in $\mathfrak{g}$, $\mathfrak{p}$ is the subspace of $\mathfrak{g}$ consisting of all elements of the form*

$$\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}, \quad x, y \in R$$

and $B(u, v)=\mathrm{tr}\, u\, v$ for $u, v \in \mathfrak{p}$.

To prove Proposition 1, one notes that since $\mathfrak{k}$ is the fixed point set

[2] The authors thank the referee for pointing out the omission of this result in our first version of this paper. From the proof of Proposition 1, one sees that there are many examples of indecomposable semisimple s.q.'s that are not irreducible.

of an involutive automorphism of $\mathfrak{g}$, $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Thus, as a $\mathfrak{k}$-module, $\mathfrak{p}$ splits into a sum of irreducible $\mathfrak{k}$ modules. Let $\mathfrak{p}_1$ be one of the irreducible submodules of $\mathfrak{p}$. By indecomposability $B(\mathfrak{p}_1, \mathfrak{p}_1)=0$. Thus we may assume that there is an irreducible submodule $\mathfrak{p}_2$ so that $B(\mathfrak{p}_1, \mathfrak{p}_2)\neq 0$. Now by indecomposability $B(\mathfrak{p}_2, \mathfrak{p}_2)=0$. Thus (again by indecomposability) $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_2$. The signature of B is thus $(\dim \mathfrak{p}_1, \dim \mathfrak{p}_2)=(\dim \mathfrak{p}_1, \dim \mathfrak{p}_1)$. Thus $\dim \mathfrak{p}_1=\dim \mathfrak{p}_2=1$, $\dim \mathfrak{k}=1$. The result follows.

3. Solvable Lorentzian quadruples. Let $q=(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ be a symmetric quadruple. If $\mathfrak{g}$ is solvable we call q solvable. After a standard notational convenience we describe a class of $(n+2)$ dimensional Lorentzian solvable s.q. Let W be a n-dimensional real vector space with a negative definite inner product $\langle\, ,\rangle$. Let W^* be the real dual of W. We denote by $w\mapsto w^*$ and $\lambda\mapsto\lambda^*$ the canonical maps $W\to W^*$ and $W^*\to W$ associated with $\langle\, ,\rangle$. To each symmetric bilinear form A on W we associate a l.s.q. $q_A=(\mathfrak{g}_A, \mathfrak{k}_A, \mathfrak{p}_A, B_A)$ where $\mathfrak{g}_A=W^*\oplus\overline{R}\oplus W\oplus R$, $\overline{R}$ and R being two copies of the reals and the multiplication table is:

$$\begin{aligned}
[\mathfrak{g}_A, R] &= 0,\\
[W^*, W^*] &= [W, W] = 0,\\
[1, w] &= w^* \quad \text{for} \quad w\in W \text{ and 1 is the identity of } \overline{R},\\
[\lambda, w] &= A(\lambda^*, w)1 \quad \text{for} \quad \lambda\in W^*, w\in W,\\
[\lambda, 1] &= \underline{A}\lambda^* \quad \text{for} \quad \lambda\in W^*
\end{aligned}$$

and $\langle \underline{A}w_1, w_2\rangle=-A(w_1, w_2)$ for $w_i\in W$.
$\mathfrak{k}_A=W^*$; $\mathfrak{p}_A=\overline{R}\oplus W\oplus R$ and B_A is defined by:

$$\begin{aligned}
B_{A|W} &= <,>,\\
B_A(\overline{R}\oplus R, W) &= B_A(\overline{R}, \overline{R}) = B_A(R, R) = 0,\\
B_A(1,1) &= 1.
\end{aligned}$$

LEMMA 1. (i) $\mathfrak{g}_A$ *is a solvable Lie algebra, and* q_A *is a l.s.q.*

(ii) q_A *is indecomposable if and only if* A *is nondegenerate.*

(iii) *If* A, B *are two nondegenerate symmetric bilinear forms on* W; *if* q_A *and* q_B *are the corresponding l.s.q., then* q_A *is isomorphic with* q_B *if and only if there exists an orthogonal transformation* $U: W\to W$ *and a positive scalar* c *such that* $\underline{B}=cU\underline{A}U^{-1}$.

THEOREM 4. *If* q *is an indecomposable, solvable* $(n+2)$ *dimensional l.s.q., there exists a nondegenerate symmetric bilinear form* A, *on* W *such that* q *is isomorphic to* q_A.

From Lemma 1 (iii) and Theorem 4, one sees that the isomorphism classes of indecomposable solvable l.s.q. are parametrized by points of $(S^{n-1}-X)/s_n$ where S^{n-1} is the unit sphere of R^n, $R^n \supset X = \{(x_1, \cdots, x_n) | x_1 \cdots x_n = 0\}$ and s_n is the symmetric group in n-letters acting on R^n in the standard fashion. The duality is given by the antipodal map.

PROPOSITION 2. *Let q_A be as above, M_A be the associated Lorentzian symmetric space. The full Lie algebra $\bar{\mathfrak{g}}$ of Killing vector fields on M_A is the canonical semidirect product of $\mathfrak{g}_A$ with the algebra of skew-symmetric linear maps of W which commute with A. The full algebra, $\tilde{\mathfrak{g}}$, of affine vector fields on M_A is a 2-dimensional nilpotent extension of $\mathfrak{g}$.*

It should be noted that $\bar{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ are solvable if and only if the eigenvalues of A have multiplicity <3.

THEOREM 5. (a) *M_A is isometric to the space $R\times R\times W$ with Lorentzian structure given by*

$$\langle (s, t, w), (s', t', w')\rangle_{(s_0, t_0, w_0)} = st' + s't + ss'A(w_0, w_0) + \langle w, w'\rangle$$

where $(s_0, t_0, w_0) \in R\times R\times W$ and $\langle w, w'\rangle$ denotes the previously defined scalar product on W. We use the usual identification of a vector space with its tangent space at each point.

(b) *The group $G(M_A)$ is the space $R\times R\times V$, where $V = W\oplus W^*$, with the following multiplication table:*

$$(s, t, v)(s', t', v') = (s + s', t + t' + \tfrac{1}{2}B(e^{-s'\operatorname{ad}\bar{1}}v, v'), e^{-s'\operatorname{ad}\bar{1}}v + v')$$

where for any $v_1, v_2 \in V$, $B(v_1, v_2) = [v_1, v_2]$ as given in $\mathfrak{g}_A$ or explicitly:

$$B(w_1 + \overset{*}{\lambda}_1, w_2 + \overset{*}{\lambda}_2) = -A(\overset{*}{\lambda}_2, w_1) + A(\overset{*}{\lambda}_1, w_2).$$

(c) *The group action of $G(M_A)$ on M_A can be described by realizing M_A in $G(M_A)$ as $R\times R\times W$. The action is left multiplication followed by projection on M_A.*

Note. The full group of isometries of M_A is never connected. Indeed the symmetry at any point of M_A is not in the connected component of the isometry group as can be seen from Theorem 5 (b), (c).

THEOREM 6. *M_A can be realized as the algebraic pseudo-riemannian submanifold of $R^4\times W$ given by the equations:*

$$x_1^2 - 2x_3 = 0, \qquad x_4 - \tfrac{1}{2}A(w, w) = 0.$$

$R^4\times W$ has the flat pseudo-riemannian metric:

$$\langle (x, w), (x', w') \rangle = x_1 x'_2 + x_2 x'_1 - x_3 x'_4 - x_4 x'_3 + \langle w, w' \rangle.$$

4. Notes and examples. (a) *Some nilpotent quadruples.* Let V be a n-dimensional real vector space, and R a 4-linear form on V having all the algebraic properties of a Riemann tensor. A nilpotent quadruple $q_R = (\mathfrak{g}_R, \mathfrak{k}_R, \mathfrak{p}_R, B_R)$ is defined by:

$$\mathfrak{g}_R = V \wedge V \oplus V \oplus V^*$$

where V^* is the real dual of V. The multiplication is given by:

$$[\mathfrak{g}_R, V^*] = 0, \qquad [V \wedge V, V \wedge V] = 0,$$

$$[v_1, v_2] = v_1 \wedge v_2 \quad \text{for} \quad v_1, v_2 \in V, \qquad [v_1 \wedge v_2, v_3] = \underline{R}(v_1 \wedge v_2, v_3)$$

where $\underline{R}(v_1 \wedge v_2, v_3) \in V^*$ is given by:

$$\underline{R}(v_1 \wedge v_2, v_3) v_4 = R(v_1, v_2, v_3, v_4),$$

$\mathfrak{k}_R = V \wedge V$; $\mathfrak{p}_R = V \oplus V^*$; B_R is the canonical pairing of V with V^*.

LEMMA 2. (1) $\mathfrak{g}_R$ *is a nilpotent Lie algebra and* q_R *is a s.q. of dimension* $2n$ *and signature* (n, n).

(2) q_R *is indecomposable if and only if the symmetric bilinear form on* $V \wedge V$ *associated with* R *is nondegenerate.*

(3) q_R *and* $q_{R'}$ *are isomorphic if and only if there exists a linear isomorphism* $E: V \to V$ *such that*: $R(v_1, v_2, v_3, v_4) = R'(Ev_1, Ev_2, Ev_3, Ev_4)$.

(b) *Four dimensional s.q.'s.* The complete classification of these quadruples has been achieved. The solvable case consists of examples analogous to the ones described in §3 and §4(a). Examples in this case show that Theorem 3 does not generalize to general s.q.'s. If $q = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, B)$ is neither solvable nor semisimple and is indecomposable, then the semisimple part of $\mathfrak{g}$ is either sL(2, R) or sO(3). The duality has interesting implications in this case. Results of the (2, 2) case seem to generalize in the (2, n) case.

(c) *Examples of compact Lorentzian symmetric spaces.* We find Lorentzian symmetric tori associated with certain of the q_A described in §3. Let A be such that the eigenvalues of $\underline{A}$ are of the form $-4\pi^2 p^2/q^2$ where $p, q \in Z$ (where Z is the integers). Then on $\mathfrak{g}_A$ the map $t \to e^{t\,\mathrm{ad}\,T}$ has period q. Let L be a lattice of W (free subgroup of rank n) and set $\Gamma = (Zq, Z, L) \subset G(M_A)$. Then Γ is a discrete subgroup of $G(M_A)$ acting freely and properly discontinuously on M_A, and M_A/Γ is homeomorphic to T_{n+2}.

References

1. M. Berger, *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177. MR **21** #3516.

2. M. Cahen and R. McLenaghan, *Métriques des espaces lorentziens symétriques à quatre dimensions*, C. R. Acad. Sci. Paris Ser. A–B (1968), A1125–A1128. MR **37** #6881.

3. J. Wolf, *Spaces of constant curvature*, McGraw-Hill, New York, 1967. MR **36** #829.

4. E. J. Taft, *Orthogonal conjugacies in associative and Lie algebras*, Trans. Amer. Math. Soc. **113** (1964), 18–29. MR **29** #1229.

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