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LORENZ DOMINANCE AND THE VARIANCE OF LOGARITHMS

BY JAMES E. FOSTER AND EFE A. OK¹

1. INTRODUCTION

THE VARIANCE OF LOGARITHMS V_L is a widely used measure of dispersion, owing in part to its natural link with wage determination models, its useful decomposition formula, and its special relationship with the lognormal distribution.² At the same time there is a longstanding view that V_L is not a proper measure of inequality due to its clear violation of Dalton's (1920) "principle of transfers" and its consequent disregard for the Lorenz criterion—a ranking generally accepted as the unambiguous arbiter of inequality comparisons.³ This criticism of V_L is often dismissed on the grounds that problems can only occur at the extreme upper tail of the distribution. Cowell (1977), for example, has proved that a marginal transfer from poor to rich can decrease V_L (hence violating the transfer principle) only if the higher income is greater than e times the geometric mean of the distribution in question. This is commonly viewed as suggesting that when V_L violates the Lorenz criterion, the Lorenz curves can differ only slightly from one another in a very limited range of the "richest" population.

In this note we offer a strikingly different picture of the potential conflict between the variance of logarithms and the Lorenz criterion. In particular, we find that disagreements between V_L and the Lorenz ranking may involve broad-based changes in incomes up and down the distribution (not just in the extreme upper tail). Moreover, they can involve comparisons in which the Lorenz curves are widely divergent from one another. Our main result shows that the following "worst-case scenario" can occur: One distribution, whose Lorenz curve approximates the 45° line of equality, Lorenz dominates a second distribution, whose Lorenz curve approximates the L -shaped curve of complete inequality, and yet V_L concludes that the second is more equal than the first. Consequently, the variance of logarithms is capable of making very serious errors.

2. DEFINITIONS

An income distribution is a positive real vector $\mathbf{x} = (x_1, \dots, x_n)$, where n is its population size and x_i is the income of person i . Associated with \mathbf{x} is a *distribution*

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² See Foster and Sen (1997). V_L figures prominently in the recent literature on U.S. wage inequality (e.g., Karoly (1992), Levy and Murnane (1992), Juhn, Murphy, and Pierce (1993), and Card and Lemieux (1994)), discussions of regional (σ -)convergence (e.g., Sala-i-Martin (1996)), and self-selection models of inequality (e.g., Heckman and Honore (1990)).

³ See, for example, Atkinson (1970), Cowell (1977), and Sen (1997).

function F_x , which, for each s , indicates the share $F_x(s)$ of the population receiving an income of s or below. The *Lorenz curve* L_x gives the cumulative share of income $L_x(p)$ held by the poorest cumulative share p of the population. Noting that $F_x^{-1}(p)$ is the income of the person whose position in the distribution is p , we see that

$$L_x(p) \equiv \frac{1}{\mu_x} \int_0^p F_x^{-1}(t) dt, \quad 0 \leq p \leq 1,$$

defines the Lorenz curve of the distribution \mathbf{x} , where μ_x is the mean of the distribution.⁴ We say that \mathbf{x} *Lorenz dominates* \mathbf{y} , denoted by $\mathbf{x}Ly$, whenever $L_x(p) \geq L_y(p)$ holds for all $p \in [0, 1]$, with strict inequality for some p .⁵

An *inequality measure* I is a mapping from the set of all income distributions of arbitrary population size to the reals, where $I(\mathbf{x})$ denotes the level of inequality associated with the distribution \mathbf{x} . We say that I is *Lorenz consistent* if $\mathbf{x}Ly$ implies that $I(\mathbf{x}) < I(\mathbf{y})$ for any two distributions \mathbf{x} and \mathbf{y} . Two commonly used inequality measures, the *Gini coefficient* and *Theil's second measure*, are defined as $G(\mathbf{x}) \equiv (1/(2n^2\mu_x)) \sum_{k=1}^n \sum_{l=1}^n |x_k - x_l|$ and $T_2(\mathbf{x}) \equiv \ln \mu_x - \ln g_x$, where $g_x \equiv (\prod_{k=1}^n x_k)^{1/n}$ is the geometric mean of \mathbf{x} . It is readily observed that the inequality measures G and T_2 are Lorenz consistent.

The *variance of logarithms*, written as V_L , is defined as the variance applied to the distribution of log incomes. In other words, where $\ln \mathbf{x} \equiv (\ln x_1, \dots, \ln x_n)$, we have $V_L(\mathbf{x}) \equiv (1/n) \sum_{k=1}^n (\ln x_k - \mu_{\ln \mathbf{x}})^2$. Since $\mu_{\ln \mathbf{x}} = \ln g_x$, letting $f(s, g) \equiv (\ln s - \ln g)^2$, we may write

$$(1) \quad V_L(\mathbf{x}) = \frac{1}{n} \sum_{k=1}^n f(x_k, g_x).$$

Unlike the previous two measures, V_L^* is *not* Lorenz consistent.

3. THE EXTENT OF LORENZ INCONSISTENCY

We are interested in identifying situations where V_L violates Lorenz consistency, that is, $\mathbf{x}Ly$ and yet $V_L(\mathbf{x}) > V_L(\mathbf{y})$. Our starting point is the *single transfer* case. Let $\mathbf{y}(\delta)$ be the distribution obtained from \mathbf{y} by a transfer of size $\delta > 0$ from a richer individual (say person h) to a poorer individual (say person l). A violation of Lorenz consistency for $\mathbf{x} = \mathbf{y}(\delta)$ and \mathbf{y} then corresponds to a violation of the transfer principle—a topic that has been studied by Cowell (1977, p. 163) for a *marginal transfer* in which δ is arbitrarily small. Indeed,

$$(2) \quad \left. \frac{dV_L(\mathbf{y}(\delta))}{d\delta} \right|_{\delta=0} = \frac{1}{n} (f_1(y_l, g_y) - f_1(y_h, g_y))$$

describes the effect of a marginal progressive transfer on V_L . It is easily verified that $f(\cdot, g_y)$ is decreasing on $(0, g_y]$ and increasing on $[g_y, \infty)$, while it is convex on $(0, eg_y]$ and

⁴ Formally, $F_x^{-1}(p) \equiv \inf_{s \geq 0} \{F_x(s) \geq p\}$ is the *generalized inverse* of F_x (Gastwirth (1971)) and the Lorenz curve of \mathbf{x} is the *graph* of the function L_x .

⁵ It is well known that, where \mathbf{x} and \mathbf{y} have identical population sizes and means, $\mathbf{x}Ly$ holds if and only if \mathbf{x} can be obtained from \mathbf{y} by a finite number of *progressive transfers* (from richer to poorer individuals); see, for instance, Dasgupta, et al. (1973) and Foster (1985). This is one of the reasons why Lorenz dominance is generally accepted as an *unambiguous* method of making ordinal inequality comparisons.

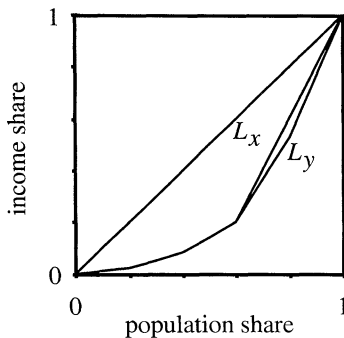


FIGURE 1.—Lorenz inconsistent example.

concave on $[eg_y, \infty)$. Consequently, $y_i \geq eg_y$ is a sufficient condition for a violation to occur, while $y_i > g_y$ and $y_h \geq eg_y$ are necessary. This observation leaves one with the impression that V_L will go against the transfer principle only when the associated transfer takes place in the extreme upper tail of the distribution, and hence that the extent of the disagreement of V_L with the Lorenz criterion is likely to be insignificant.

As an illustration, consider the distribution $\mathbf{y} = (2, 5, 10, 28, 40)$. Since $g_y = 10.229 > 10$ and $eg_y = 27.721 < 28$, the only possible violation of the transfer principle due to a marginal transfer involves the final two incomes. Indeed, $f_1(y_4, g_y) = 0.071 > 0.068 = f_1(y_5, g_y)$, and thus a marginal transfer from person 5 to person 4 increases V_L . Now to what extent can the Lorenz curve be altered by a progressive transfer that simultaneously raises V_L ? The maximal upward shift is obtained by choosing $\mathbf{x} = \mathbf{y}(6) = (2, 5, 10, 34, 34)$ in which the incomes of persons 4 and 5 are equalized. In this case, V_L rises slightly (from 1.2100 to 1.2124) while, as depicted in Figure 1, the resulting upward shift in the Lorenz curve is modest with only a slight change of 0.028 in the Gini coefficient.

The picture changes appreciably when we go beyond the single transfer case. Consider, for example, the distributions

$$\mathbf{x} = (1, 1, 1, 41, 41, 41, 41, 41, 41, 41) \quad \text{and} \quad \mathbf{y} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 281).$$

Clearly, \mathbf{x} is obtained from \mathbf{y} by means of six progressive transfers. Figure 2 shows that the Lorenz dominance of \mathbf{x} over \mathbf{y} is broad-based, as the curves depart for 70% of the

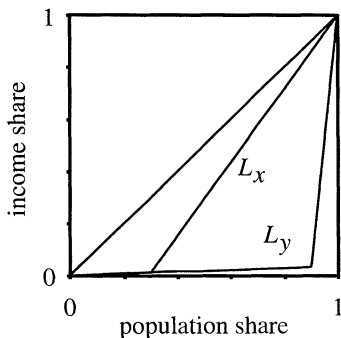


FIGURE 2.—Extreme Lorenz inconsistency.

population, and quite extensive since the area between them is very large. Moreover, the Gini coefficient and Theil's second measure strongly support the judgment of the Lorenz ranking.⁶ The Gini coefficient, in particular, rises by almost 200% indicating that the area between the line of perfect equality and L_y is three times larger than the area between the line of perfect equality and L_x . Yet, $V_L(\mathbf{x}) = 2.90 > 2.86 = V_L(\mathbf{y})$, that is, V_L declares \mathbf{x} to be *more unequal than y!* The Lorenz inconsistency of V_L can be very significant indeed.

The logic behind this example can be extended to obtain the following surprising result.

PROPOSITION: *For any $\epsilon > 0$, there exist two income distributions \mathbf{x} and \mathbf{y} satisfying $G(\mathbf{x}) < \epsilon$ and $G(\mathbf{y}) > 1 - \epsilon$ for which $\mathbf{x}L\mathbf{y}$ and $V_L(\mathbf{x}) > V_L(\mathbf{y})$.*

PROOF: For any integers $n \geq 3$ and $m \geq 1$ with $m < n$, define

$$\mathbf{x}^{m,n} \equiv \left(\underbrace{1, \dots, 1}_{m \text{ times}}, \underbrace{\theta, \dots, \theta}_{n-m \text{ times}} \right),$$

where $\theta \equiv (nB/(n - m)) + 1$ for an arbitrary $B > 0$. By direct computation, we find

$$(3) \quad G(\mathbf{x}^{m,n}) = \frac{B}{B+1} \left(\frac{m}{n} \right) \quad \text{and}$$

$$V_L(\mathbf{x}^{m,n}) = \frac{m}{n} \left(1 - \frac{m}{n} \right) \left(\ln \left(\frac{B}{1 - (m/n)} + 1 \right) \right)^2.$$

Define

$$\varphi(\alpha, B) \equiv \alpha(1 - \alpha) \left(\ln \left(\frac{B}{1 - \alpha} + 1 \right) \right)^2 \quad \text{for all } \alpha \in [0, 1) \text{ and } B > 0.$$

Four observations about the function $\varphi(\alpha, B)$ are in order. First, notice that φ can be continuously extended to $[0, 1]$ by using l'Hôpital's rule and setting $\varphi(1, B) = 0$ for all $B > 0$. Second, we have

$$(4) \quad \varphi(\alpha, B) < \varphi(1 - \alpha, B) \quad \text{for all } \alpha \in (0, 1/2) \text{ and } B > 0$$

as can be readily verified. Third,

$$\frac{\partial \varphi(\alpha, B)}{\partial \alpha} = \left((1 - 2\alpha) \ln \left(\frac{B}{1 - \alpha} + 1 \right) + \frac{2\alpha B}{B + (1 - \alpha)} \right) \ln \left(\frac{B}{1 - \alpha} + 1 \right)$$

so that φ is a strictly increasing function of α on $[0, 1/2]$, given any $B > 0$. Finally, it is clear that for any $\bar{\alpha} > 1/2$, there exists a $\bar{B} > 0$ large enough to guarantee that $(\partial \varphi(\alpha, B) / \partial \alpha) < 0$ for all $\alpha \geq \bar{\alpha}$ and $B \geq \bar{B}$.

Now pick any $\epsilon > 0$, and choose any $\alpha_1 \in (0, \min\{\epsilon, 1/2\})$. By using the last observation on φ noted above, we may find a $B_1 > 0$ such that

$$(5) \quad \varphi(\cdot, B) \text{ is strictly decreasing on } [1 - \alpha_1, 1] \text{ for all } B \geq B_1.$$

⁶ Here we have $G(\mathbf{x}) = .29$, $G(\mathbf{y}) = .87$, $T_2(\mathbf{x}) = .77$, and $T_2(\mathbf{y}) = 2.80$.

But then, using the intermediate value theorem, we conclude that there must exist an $\alpha_1^*(B) \in (1/2, 1)$ such that

$$\varphi(\alpha_1, B) = \varphi(\alpha_1^*(B), B) \quad \text{for all } B \geq B_1.$$

Moreover, by (4) and (5), we must have $1 - \alpha_1 < \alpha_1^*(B)$ for all $B \geq B_1$. Now take any $\gamma < \epsilon$ and define

$$\alpha_2(B) \equiv \max\left\{\alpha_1^*(B) + \frac{1}{2}(1 - \alpha_1^*(B)), (1 - \epsilon) + \gamma\right\} \quad \text{for all } B > 0.$$

Clearly, $1 > \alpha_2(B) > \alpha_1^*(B) > 1 - \alpha_1 > \alpha_1$ (for $B \geq B_1$) so that, by (5), we have

$$(6) \quad \varphi(\alpha_2(B), B) < \varphi(\alpha_1^*(B), B) \quad \text{for all } B \geq B_1.$$

In addition, by the definition of $\alpha_2(B)$, there must exist $B_2 > 0$ such that

$$(7) \quad \frac{B}{B+1} \alpha_2(B) \geq \frac{B}{B+1} ((1 - \epsilon) + \gamma) > 1 - \epsilon \quad \text{for all } B \geq B_2.$$

Now let $\hat{B} = \max\{B_1, B_2\}$ and notice that $\alpha_1 < \alpha_2(\hat{B})$. Consequently, by denseness of the rationals in the reals, we may choose integers n, m_1 , and m_2 such that $m_1 < m_2$, $m_1/n \approx \alpha_1$, and $m_2/n \approx \alpha_2(\hat{B})$. But then, letting $\mathbf{x} \equiv \mathbf{x}^{m_1, n}$ and $\mathbf{y} \equiv \mathbf{x}^{m_2, n}$, we obtain

$$G(\mathbf{x}) \approx \frac{\hat{B}}{\hat{B}+1} \alpha_1 < \epsilon \quad \text{and} \quad G(\mathbf{y}) \approx \frac{\hat{B}}{\hat{B}+1} \alpha_2(\hat{B}) > 1 - \epsilon$$

by (3), the definition of α_1 , and (7). Moreover, (3) and (6) yield that $V_L(\mathbf{y}) \approx \varphi(\alpha_2(\hat{B}), \hat{B}) < \varphi(\alpha_1^*(\hat{B}), \hat{B}) = \varphi(\alpha_1, \hat{B}) \approx V_L(\mathbf{x})$. Finally, $\mathbf{x} L \mathbf{y}$ follows from $m_1 < m_2$ by construction.

Q.E.D.

Thus, the variance of logarithms is not only Lorenz inconsistent, but it can disagree with the Lorenz criterion in such a radical way that it misranks two distributions that are deemed to be almost perfectly equal and almost perfectly unequal by the Gini index.⁷ We thus conclude that there is essentially no limit to the disagreement between V_L and the Lorenz ordering of income distributions.

Why does V_L have this unfortunate characteristic? In other words, what is the driving force behind the above proposition? To answer this, let us return to the example depicted in Figure 2. The geometric mean rises in going from \mathbf{y} to \mathbf{x} , which is only natural given the fact that the geometric mean is a Lorenz consistent equality measure for comparing distributions with fixed average income. Now observe that

$$\begin{aligned} V_L(\mathbf{x}) - V_L(\mathbf{y}) &= \frac{1}{10} \sum_{k=1}^{10} (f(x_k, g_y) - f(y_k, g_y)) \\ &\quad + \frac{1}{10} \sum_{k=1}^{10} (f(x_k, g_x) - f(x_k, g_y)) \\ &= (7.01 - 2.86) + (2.90 - 7.01). \end{aligned}$$

⁷ In other words, the area between the two Lorenz curves is arbitrarily close to its maximum displacement of 1/2. The use of the Gini index here is natural, but not essential. For instance, we can adapt the proof of the proposition to show the following result for Theil's second measure: For any $\epsilon > 0$, there exist two income distributions \mathbf{x} and \mathbf{y} satisfying $T_2(\mathbf{x}) < \epsilon$ and $T_2(\mathbf{y}) > 1/\epsilon$ for which $\mathbf{x} L \mathbf{y}$ and $V_L(\mathbf{x}) > V_L(\mathbf{y})$.

This decomposition clearly shows that while the constituent transfers decrease the variance of log incomes evaluated at the original geometric mean, the resultant change in g_y raises V_L back above its original level. In other words, the *direct* effect the transfers have on the income levels of the individuals is here outweighed by their *indirect* effect via the geometric mean. Consequently, this example points to the variation of the geometric mean as a main culprit in the massive disagreement between V_L and the Lorenz criterion.⁸

4. CONCLUDING COMMENTS

The Lorenz criterion is generally recognized as the fundamental tool for making inequality comparisons. When Lorenz curves cross, the criterion has nothing to say, and Lorenz-consistent indices—including the Gini coefficient, the coefficient of variation, and the Theil measures—can disagree. When Lorenz curves are unambiguously ranked so that the criterion applies, all of these standard indices will unerringly follow its judgment. While it has long been recognized that the variation of logarithms may go against the Lorenz criterion, the specific cases where this occurs were thought to involve relatively minor disagreements, with Lorenz curves departing slightly at the upper incomes. Our main result shows that this view is incorrect—that the variance of logarithms can err in even the most extreme examples of Lorenz-comparable distributions. The notion of dispersion captured by variance of logarithms can dramatically depart from the notion of “inequality” as traditionally defined.

How likely are such errors to arise in practice? On this question there are as yet no definitive answers. Creedy (1977) is often cited as suggesting that problems from using V_L are highly unlikely.⁹ However, his analysis only considers the case of a single marginal transfer; and since Lorenz comparisons typically involve a series of discrete transfers (and a changing geometric mean), these results are not directly relevant. Moreover, even if one were interested in the likelihood that a marginal transfer between two incomes drawn at random will violate the transfer principle, the estimate given by Creedy is actually an order of magnitude too low. We have shown elsewhere (Foster and Ok (1997)) that when incomes are drawn directly from microdata or from standard statistical distributions, such as the lognormal, Singh-Maddala and beta-II, fitted to US data (cf. Singh and Maddala (1976) and McDonald (1984)) the likelihood of a violation is somewhere between 8% to 12%.¹⁰ The probability that a single marginal transfer will generate a Lorenz inconsistency for V_L is not easily dismissed as negligible.

⁸ Therefore, such large disagreements might be avoided if the variation in the geometric mean stays within certain bounds. Given the definition of Theil's second measure as $T_2(\mathbf{x}) = \ln \mu_x - \ln g_x$, one may thus benefit from using V_L in conjunction with T_2 in practice.

⁹ See, for instance, Heckman and Honore (1990, footnote 6).

¹⁰ The methodology behind these estimates can be briefly described as follows. Suppose that two incomes, s and t , are chosen at random from a probability distribution with a cdf F and a geometric mean g . Let π denote the probability of the event $\{\Delta V_L \geq 0 \text{ and } s \geq t\}$ conditional on a *small* transfer taking place from s to t . By (2) and the discussion that follows it, π must be the probability of the event $\{\min\{s, t\} > eg\} \cup \{s \in (g, eg), t > eg, f_1(s, g) > f_1(t, g)\} \cup \{t \in (g, eg), s > eg, f_1(s, g) < f_1(t, g)\}$. But then we have

$$\pi = (1 - F(eg))^2 + 2 \int_g^{eg} (1 - F(h(\omega))) dF(\omega)$$

where $h: (g, eg) \rightarrow (eg, \infty)$ is defined via $f_1(h(\omega), g) = f_1(\omega, g)$. Estimates are then obtained from this formula using a standard numerical approximation method.

Finally, we note that even when an outright conflict is not observed, there is significant scope for the phenomenon to affect the cardinal values of V_L , thereby altering perceptions and explanations of distributional changes. For example, a significant general compression of incomes may instead be interpreted as a minor compression if the variance of logarithms misreads the smoothing at the top as an increase in inequality. This built-in possibility for error would thus raise questions about analyses based exclusively on V_L . Future studies should provide evidence of the robustness of their conclusions by employing one or more of the standard, well-behaved inequality measures in addition to the variance of logarithms.

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