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Lorenz rankings of rules for the adjudication of conflicting claims

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## Abstract

For the problem of adjudicating conflicting claims, we offer simple criteria to compare rules on the basis of the Lorenz order. These criteria pertain to three families of rules. The first family contains the constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules (Thomson, 2007a). The second family, which also contains the constrained equal awards and constrained equal losses rules, is obtained from the first one by exchanging, for each problem, how well agents with relatively larger claims are treated as compared to agents with relatively smaller claims. The third family consists of consistent rules (Young, 1987). We also address the issue whether certain operators on the space of rules preserve the Lorenz order.

JEL classification number: C79; D63; D74

Key-words: claims problems; constrained equal awards rule; constrained equal losses rule; Talmud rule; minimal overlap rule; Piniles' rule; constrained egalitarian rule; ICI rules; CIC rules; consistent rules; Lorenz domination; operators.

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# 1 Introduction

When several agents have claims on a resource adding up to more than is available, what should be done with what is available of the resource? For example, when a firm goes bankrupt, how should its liquidation value be divided among its creditors? A “division rule” is a function that associates with each such problem a recommendation for it. Alternatively, think of a tax authority having to raise a certain amount of money from taxpayers whose incomes differ, in order to cover the cost of some public project. How much should each taxpayer be assessed? This problem is mathematically identical to the problem of claims adjudication, but we will use language that fits that second interpretation, calling the data defining a problem “claims” and “endowment”, and calling the amount assigned to an agent his “award”.<sup>1</sup>

Our goal is to develop criteria to perform Lorenz comparisons of rules. An important issue when evaluating a rule is how differentially it treats agents with relatively large claims as compared to claimants with relatively small claims. The Lorenz order is commonly used to evaluate income distributions and it is just as natural in the context of claims adjudication. Because rules are complex objects, one should not expect to be able to make Lorenz comparisons of any two rules very generally, independently of the data of the problem. The comparisons may depend on the specific values claims take, the endowment, the number of claimants, and even their identities. In fact, for each specific problem, the awards vectors chosen by two rules may not be Lorenz comparable. Nevertheless, one may hope that within structured classes of rules, such comparisons would be feasible. The main results of this paper fulfill this hope, as they concern three families of rules that are sufficiently wide as to contain all of the rules that have been most often discussed in the literature.

A great variety of rules have indeed been proposed, motivated by a broad range of considerations, but several unifying principles can be invoked to organize them into families. We start with a family of rules (Thomson, 2000) whose definition is quite simple and intuitive—and in fact it expresses a fairly widely held view of how this type of problems should be solved—yet the family is rich enough to contain a number of important rules, rules that at first sight do not all appear to be related. They are the “constrained equal awards rule” and the “constrained equal losses rule”, both already

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<sup>1</sup>For a survey of the literature on this subject, see Thomson (2003, 2007c).

familiar to Maimonides (12th Century); the “Talmud rule” (Aumann and Maschler, 1985), proposed to rationalize certain numerical examples found in the Talmud (O’Neill, 1982); and the “minimal overlap rule” (O’Neill, 1982), which is an extension<sup>2</sup> of an incompletely specified medieval rule (Rabad, 12th Century). The second family is a counterpart of the first family in which relatively small claims and relatively large claims are treated in a reverse way to how they are treated in the first family. It contains the constrained equal awards and constrained equal losses rules. (These two rules are the only ones the two families have in common.) (Section 2)

The main ingredient in the definitions of the two families is the basic idea of equality, applied recursively either to the amounts claimants receive or to the losses they incur. Fix the claims vector and let the endowment grow. For the first family, initially all claimants are present to partake in the division; as the endowment increases, they drop out in the order of increasing claims until one claimant is left and he is the only one present for a while; they then return for more in the reverse order (Subsection 3.2). For the second family, they arrive to receive a share in the order of decreasing claims, each claimant staying until he is fully compensated; they drop out in the order of increasing claims (Subsection 3.3). For both families, all agents who are present at a given time partake equally in the division of each increment. The richness of the families comes from the freedom in choosing where claimants are dropped off and invited back in (or conversely, for the second family), and in particular, from the fact that these drop-off and pick-up points are allowed to depend on the claims vector.

The third family is defined by means of a property its members satisfy, called “consistency”.<sup>3</sup> Let a rule be given. Consider a problem, and apply the rule to solve it. Then imagine that some claimants leave with their awards, and reassess the situation. This situation can be seen as a “reduced problem” with fewer claimants and a smaller endowment. The requirement is that for this problem, the rule should assign the same amount to each remaining claimant as it did initially. Several of the aforementioned examples, such as the constrained equal awards, constrained equal losses, and Talmud rules, are consistent. So are Piniles’ rule (Piniles, 1861), which was an early, only partially successful, attempt at producing the numbers in the Talmud, the

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<sup>2</sup>Another extension of Ibn Ezra’s suggestion is proposed by Bergantiños and Méndez-Naya (2001) and Alcalde, Marco, and Silva (2005).

<sup>3</sup>For a survey of the applications of the “consistency principle”, see Thomson (2007b).

constrained egalitarian rule (Chun, Schummer, and Thomson, 2001), which offers another implementation of the idea of equality subject to constraints, and of course the proportional rule. The minimal overlap rule is not however.

For each of the three families, we formulate a general and simple criterion that allows Lorenz comparisons of their members. In addition to recovering a number of elementary rankings, we obtain several non trivial ones. A special case of our result concerning the first family is that the Talmud rule Lorenz dominates the minimal overlap rule. Another is a complete Lorenz ranking of the members of a one-parameter family of rules, a subfamily of the ICI family, proposed by Moreno-Tertero and Villar (2006b) as a generalization of the constrained equal awards, constrained equal losses, and Talmud rules (Section 4). Such a ranking is derived by these authors (Moreno-Tertero and Villar, 2006a) by direct calculation. We also derive a complete Lorenz ranking of the members of a one-parameter subfamily of the second family defined in parallel manner. (In both cases, the parameter is a point in the unit interval.)

Our results pertaining to consistency involve proving that, using Hokari and Thomson’s (2000) terminology, Lorenz domination is “lifted” by consistency from the two-claimant case to the case of arbitrarily many claimants for all rules satisfying a pair of elementary properties: this means that given two consistent rules satisfying these properties, if one of them Lorenz dominates the other in the two-claimant case, its domination extends to arbitrarily many claimants. Then, to establish Lorenz domination in general, it suffices to check the two-claimant case, which is trivial. Applications of this result are numerous because many rules are consistent. In particular we obtain rankings of several important rules. We also recover the Lorenz ranking of the members of the Moreno-Tertero–Villar family (2006b) and the ranking of the reverse family, both rankings already derived from their membership in the ICI and CIC families (Section 5).

We complete this study by exploiting the notion of an “operator” on the space of rules. This is a mapping that associates with each rule another one (Thomson and Yeh, 2001). Operators are very useful in structuring the space of rules. Given two rules that are Lorenz ordered, one should know if their images under an operator are ordered too. We consider three operators. Given a rule  $S$ , the “claims truncation operator” associates with  $S$  the rule obtained, for each problem, by applying  $S$  to the derived problem in which claims have been truncated at the endowment. The “attribution of minimal

rights operator” associates with  $S$  the rule that gives, for each problem, the awards vector in two steps: first, each agent receives the difference between the endowment and the sum of the claims of the others (or 0 if this difference is negative); second, he receives what  $S$  would give him in the division of the remainder, claims having been adjusted down by the first installments. The “duality operator” associates with  $S$  the rule that, for each problem, divides the endowment as  $S$  divides the deficit. We show that the first operator preserves the Lorenz order, the second one generally preserves it, and the third one reverses it. Finally, we provide a Lorenz domination result pertaining to each rule satisfying some basic properties and the rule obtained from it by subjecting it to any one of these operators (Section 6).

Bosmans and Lauwers (2007) address the same questions but the main thrust of their approach is to identify a certain rule as being Lorenz minimal (or Lorenz maximal) within a certain class of rules, defined by means of a list of properties they share. They offer several characterizations of this type. Some of the rankings they obtain as corollaries are rankings also derived in the present paper. We comment in greater detail on their contribution at the relevant points. Of course, some rules cannot be Lorenz ordered. Showing that, given a pair of rules, neither one Lorenz dominates the other is achieved by means of examples. A number of such examples can be found in Bosmans and Lauwers (2007), settling the issue of dominance in the negative for several important pairs.

There has been a growing interest in the distributional properties of division rules. Other papers on the subject are by Hougard and Østerdal (2005), and Ju and Moreno-Ternero (2006, 2007).

## 2 The model of claims resolution. Lorenz domination

A group  $N = \{1, \dots, n\}$  of agents have claims on a resource,  $c_i \in \mathbb{R}_+$  being the **claim** of agent  $i \in N$  and  $c \equiv (c_i)_{i \in N}$  the claims vector. Claims are ordered so that  $c_1 \leq \dots \leq c_n$ . There is an **endowment**  $E$  of the resource that is insufficient to honor all of the claims. Using the notation  $\mathbb{R}_+^N$  for the cross-product of  $n$  copies of  $\mathbb{R}_+$  indexed by the members of  $N$ , a claims problem, or simply a **problem**, is a pair  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$  such that  $\sum c_i \geq E$ . Let  $\mathcal{C}^N$  denote the domain of all problems.

A division rule, or simply a **rule**, is a function that associates with each problem  $(c, E) \in \mathcal{C}^N$  a vector  $x \in \mathbb{R}^N$  satisfying the non-negativity and claims boundedness inequalities  $0 \leq x \leq c$  and the efficiency equality  $\sum x_i = E$ . Such an  $x$  is an **awards vector for  $(c, E)$** . Let  $\mathbf{X}(c, E)$  be the set of these vectors.

An important way to compare rules is by means of how evenly distributed the awards they select are. Is a rule relatively favorable to claimants with large claims or to claimants with small claims? In the theory concerning the measurement of income inequality, a number of partial orders have been defined to perform comparisons of income distributions.<sup>4</sup> The central one is based on the successive sums of ordered incomes, and the obvious way to apply it here is to consider successive sums of ordered awards, as follows. Let  $x$  and  $y \in \mathbb{R}^N$  be such that  $x_1 \leq \dots \leq x_n$ ,  $y_1 \leq \dots \leq y_n$ , and  $\sum x_i = \sum y_i$ . Then,  **$x$  is greater than  $y$  in the Lorenz order**, which we write as  $x \succ_L y$ , if  $x_1 \geq y_1$ ,  $x_1 + x_2 \geq y_1 + y_2$ ,  $x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3$ , and so on, with at least one strict inequality. We also say then that  **$x$  Lorenz dominates  $y$** . If the successive partial sums are equal for the two vectors,  $x$  and  $y$  are **Lorenz equivalent**. If either  $x \succ_L y$  or these vectors are Lorenz equivalent, we write  $x \succeq_L y$ . **A rule  $S$  Lorenz dominates a rule  $S'$**  if for each  $(c, E) \in \mathcal{C}^N$ ,  $S(c, E) \succeq_L S'(c, E)$ , and there is  $(c, E) \in \mathcal{C}^N$  such that  $S(c, E) \succ_L S'(c, E)$ . We then write  $S \succ_L S'$ . Also,  **$S$  and  $S'$  are Lorenz-equivalent** if for each  $(c, E) \in \mathcal{C}^N$ ,  $S(c, E)$  and  $S'(c, E)$  are Lorenz-equivalent. Finally, if  $S \succ_L S'$  or  $S$  and  $S'$  are Lorenz equivalent, we write  $S \succeq_L S'$ .

We leave it to the reader's judgment how differences in claims should be reflected in awards vectors. Our goal here is simply to help in this evaluation, by developing general criteria to determine, within broad families, whether two rules are comparable in the Lorenz order.

### 3 Rules and families of rules

First, we describe several rules, and families of rules, that will be covered by our main theorems.

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<sup>4</sup>One is based on the gap between the smallest and largest awards; another on the variance of the awards. Comparisons of rules on the basis of either gap or variance are presented by Schummer and Thomson (1997) and Chun, Schummer, and Thomson (2001).



### 3.1 Four central rules

The following rules are central in the literature.

**Constrained equal awards rule, *CEA*:** For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$ , where  $\lambda$  is chosen so as to achieve efficiency.

**Constrained equal losses rule, *CEL*:** For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $CEL_i(c, E) \equiv \max\{0, c_i - \lambda\}$ , where  $\lambda$  is chosen so as to achieve efficiency.

**Talmud rule, *T*:** (Aumann and Maschler, 1985) For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,

$$T_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } \sum \frac{c_i}{2} \geq E, \\ c_i - \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case,  $\lambda$  is chosen so as to achieve efficiency.

**Minimal overlap rule, *MO*:** (O'Neill, 1982) Claims on specific parts of the endowment are arranged so that the part claimed by exactly one claimant (whoever that claimant is; different “subparts” may be claimed by different claimants) is maximized, and for each  $k = 2, \dots, n - 1$  successively, subject to the first  $k$  maximizations being solved, the part claimed by exactly  $k + 1$  claimants (whoever these claimants are; different subparts may be claimed by different groups of  $k + 1$  claimants) is maximized. Once claims are arranged in this way, for each part of the endowment, equal division prevails among all agents claiming it. Each claimant receives the sum of the partial compensations assigned to him for the various parts that he claimed.

Formulas for the rule are available (O'Neill, 1982; Chun and Thomson, 2005; Alcalde, Marco, and Silva, 2007).

### 3.2 The ICI family

The considerations underlying the definition of the minimal overlap rule seem to be far removed from those that underlie the constrained equal awards, constrained equal losses, and Talmud rules. However, by plotting for each claims vector the awards they recommend as a function of the endowment, one notes similarities. The award to each claimant increases initially, then it remains

constant for a while, then it increases again until he is fully compensated; given two claimants, the interval of constancy for the one with the smaller claim contains the interval of constancy for the one with the larger claim; finally, any two awards that are increasing at any given moment do so at the same rate. Let us consider all rules exhibiting these features. We designate the family they constitute by the name of “Increasing-Constant-Increasing” family, or **ICI family** for short (the interval in which each claimant’s award is increasing can be subdivided into subintervals in which the rate of increase is constant; for the one with the largest claim, the interval in which his award stays fixed is degenerate).

Proceeding in this way is easily justified by recursive application of the widely held view that for a small endowment, differences in claims should be judged irrelevant. Then equal shares are appealing as providing “consolation prizes”. Starting with equal division, as  $E$  increases, at some point—let  $F_1(c)$  denote the value of  $E$  at which this occurs—claimant 1’s award is perceived as being too large in relation to his claim, so he is dropped off. Differences in the claims of the others might still be judged irrelevant for a while, and we continue with equal division for them until—let  $F_2(c)$  denote the value of  $E$  at which this occurs—it is felt that claimant 2’s award is too large in relation to his claim. Then, he is dropped off too and we continue with equal division for the remaining  $n-2$  claimants, and so on. We proceed until agent  $n-1$  has been dropped off—let  $F_{n-1}(c)$  denote the value of  $E$  at which this occurs. To complete the description of the rule, it is convenient to start with  $E = \sum c_i$  and let  $E$  decrease (and the deficit, namely the difference between  $E$  and  $\sum c_i$ , grow). This time, we focus on the losses claimants incur. The deficit is initially divided equally among all agents until it is felt that claimant 1’s loss is too large in relation to his claim—let  $G_1(c)$  denote the value of  $E$  at which this occurs. The next increments in the deficit are divided equally among claimants 2, . . . ,  $n$  until agent 2 is dropped off—let  $G_2(c)$  denote the value of  $E$  at which this occurs; we proceed until agent  $n-1$  is dropped off—let  $G_{n-1}(c)$  denote the value of  $E$  at which this occurs. When  $E$  decreases from  $G_{n-1}(c)$  to  $F_{n-1}(c)$ , agent  $n$  is the only one to have to absorb incremental deficits. Equivalently, when  $E$  increases from  $F_{n-1}(c)$  to  $G_{n-1}(c)$ , agent  $n$  is the only recipient of each new unit.

The freedom in defining the rules comes from that of choosing the various points at which claimants are dropped off and picked up again, and the fact that these drop-off and pick-up points may depend on the claims vector. These parameters are not independent however. Indeed, claimant 1’s award

increases twice, at Step 1 by  $\frac{F_1(c)}{n}$ , and at Step  $2n - 1$  (the last step) by  $\frac{\sum c_i - G_1(c)}{n}$ , for a total of  $c_1$ . Thus,  $\frac{F_1(c)}{n} + \frac{\sum c_i - G_1(c)}{n} = c_1$ . Claimant 2's award increases along with claimant 1's award on both occasions and at the same rate, also for a total of  $c_1$ ; in addition, it increases at Step 2 by  $\frac{F_2(c) - F_1(c)}{n-1}$  and at Step  $2n - 2$  by  $\frac{G_1(c) - G_2(c)}{n-1}$ . Thus,  $c_1 + \frac{F_2(c) - F_1(c)}{n-1} + \frac{G_1(c) - G_2(c)}{n-1} = c_2$ . Similar statements can be made about the other claimants. For the last agent, we obtain  $c_{n-1} + G_{n-1}(c) - F_{n-1}(c) = c_n$ .

Summarizing, here is the general definition of the family. Its members are indexed by a list  $H \equiv (F_k, G_k)_{k=1}^{n-1}$  of pairs of functions from  $\mathbb{R}_+^N$  to  $\mathbb{R}$  such that for each  $c \in \mathbb{R}_+^N$ ,  $(F_k(c))_{k=1}^{n-1}$  is nowhere decreasing,  $(G_k(c))_{k=1}^{n-1}$  is nowhere increasing,  $G_1(c) \leq \sum c_i$ , and the following **ICI relations** hold. For convenience, we introduce  $c_0 \equiv 0$ ,  $F_0(c) \equiv 0$ ,  $G_0(c) \equiv \sum c_i$ , and  $F_n(c) = G_n(c)$ . Then, for each  $k = 1, \dots, n$ ,

$$c_{k-1} + \frac{F_k(c) - F_{k-1}(c)}{n-k+1} + \frac{G_{k-1}(c) - G_k(c)}{n-k+1} = c_k$$

The ICI relations are not independent; multiplying the first one through by  $n$ , the second one by  $n - 1, \dots$ , and the last one by 1, gives new relations whose sum is an identity. If several agents have equal claims, the ICI relations imply that successive  $F_k$ 's are equal and that so are the corresponding  $G_k$ 's. Then, agents with equal claims drop out and come back together. Let  $\mathcal{H}^N$  denote the family of lists  $H \equiv (F_k, G_k)_{k=1}^{n-1}$  of pairs of functions satisfying the ICI relations.

Here is the formal definition of the ICI family (Thomson, 2007a):

**ICI rule relative to  $H \equiv (F_k, G_k)_{k=1}^{k=n-1} \in \mathcal{H}^N, S^H$ :** For each  $c \in \mathbb{R}_+^N$  with  $c_1 \leq \dots \leq c_n$ , the awards vector chosen by  $S^H$  is calculated as follows. As  $E$  first increases from 0 to  $F_1(c)$ , equal division prevails; as  $E$  increases from  $F_1(c)$  to  $F_2(c)$ , claimant 1's award remains constant, and equal division of each increment prevails among the others. As  $E$  increases from  $F_2(c)$  to  $F_3(c)$ , claimants 1 and 2's awards remain constant, and equal division of each increment prevails among the others... This goes on until  $E$  reaches  $F_{n-1}(c)$ . As  $E$  increases from  $F_{n-1}(c)$  to  $G_{n-1}(c)$ , each increment goes entirely to claimant  $n$ . As  $E$  increases from  $G_{n-1}(c)$  to  $G_{n-2}(c)$ , equal division of each increment prevails between claimants  $n$  and  $n - 1$ ... This goes on until  $E$  reaches  $G_1(c)$ , after which each increment is divided equally among all claimants, until all are fully compensated.

The four rules defined in Section 2 are ICI rules and our first lemma

identifies functions in  $\mathcal{H}^N$  that rationalize them as members of the family.<sup>5</sup>

**Lemma 1** (Thomson, 2000) *The following four rules are ICI rules:*

(a) *CEA: for each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv (nc_1, c_1 + (n-1)c_2, \dots, c_1 + c_2 + \dots + c_{k-1} + (n-k+1)c_k, \dots, c_1 + c_2 + \dots + c_{n-2} + 2c_{n-1})$ .*

(b) *CEL: for each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv (0, \dots, 0)$ .*

(c) *T: for each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv (n\frac{c_1}{2}, \frac{c_1}{2} + (n-1)\frac{c_2}{2}, \dots, \frac{c_1}{2} + \frac{c_2}{2} + \dots + \frac{c_{k-1}}{2} + (n-k+1)\frac{c_k}{2}, \dots, \frac{c_1}{2} + \frac{c_2}{2} + \dots + \frac{c_{n-2}}{2} + 2\frac{c_{n-1}}{2})$ .*

(d) *MO: for each  $c \in \mathbb{R}_+^N$ , set  $F(c) = (c_1, c_2, \dots, c_{n-1})$ .*

Moreno-Tertero and Villar (2006a,b) propose and study a one-parameter family of rules under the name of “Tal family”. This family happens to be a subfamily of the ICI family defined by selecting  $\theta \in [0, 1]$  and choosing  $F(c) \equiv \theta(nc_1, c_1 + (n-1)c_2, \dots, c_1 + c_2 + \dots + c_{k-1} + (n-k+1)c_k, \dots, c_1 + c_2 + \dots + c_{n-2} + 2c_{n-1})$ . Let  $\mathbf{T}^\theta$  denote the rule associated with  $\theta$  in this manner. Note that  $T^{\frac{1}{2}} = T$ ,  $T^1 = CEA$ , and  $T^0 = CEL$ .

### 3.3 Reverse family: the CIC family

A reverse algorithm to the one underlying the definition of the ICI family suggests itself. Fix the claims vector. There, we began with equal division and dropped claimants in succession, starting with the agents with the smallest claims. Here, as the endowment increases from 0 to the sum of the claims, we start by giving everything to claimant  $n$  and progressively enlarge the set of recipients, introducing them in the order of decreasing claims. More precisely, claimant  $n$  is the only one present until  $E$  reaches a first critical value—let  $F_1(c)$  denote this value—at which point claimant  $n-1$  enters the scene. Then, claimants  $n$  and  $n-1$  share equally each increment until  $E$  reaches a second critical value—let  $F_2(c)$  denote this value. Then, claimants  $n$ ,  $n-1$ , and  $n-2$  share equally each increment, and so on. This process goes on until claimant 1 enters the scene—let  $F_{n-1}(c)$  denote the value of  $E$  at which this occurs—at which point all claimants share equally each increment. At some point—let  $G_{n-1}(c)$  denote the value of  $E$  at which this occurs—claimant 1 is fully compensated and is dropped off. Then, equal division of each increment prevails among claimants 2 through  $n$  until claimant 2 is fully compensated—let  $G_{n-2}(c)$  denote the value of  $E$  at which this occurs—and is dropped off and so on. During the last step, claimant  $n$  is the only one left.

<sup>5</sup>The alternative extension of Rabad’s rule mentioned in footnote 2 is not an ICI rule.

A member of the family defined in this way can be described in terms of a list of pairs of functions satisfying relations parallel to those imposed on the pairs of defining the ICI rules. Specifically, consider a pair of functions  $H \equiv (F_k, G_k)_{k=1}^{n-1}$ , from  $\mathbb{R}_+^N$  to  $\mathbb{R}_+$ , such that for each  $c \in \mathbb{R}_+^N$ ,  $(F_k(c))_{k=1}^{n-1}$  is nowhere decreasing,  $(G_k(c))_{k=1}^{n-1}$  is nowhere increasing,  $G_1(c) \leq \sum c_i$ , and the following relations, which we call the **CIC relations**, hold. The acronym CIC reflects the fact that each claimant's award is first Constant, then Increasing, then Constant. For convenience, we introduce  $c_0 \equiv 0$ ,  $F_0(c) \equiv 0$ ,  $G_0(c) \equiv \sum c_i$ ,  $F_n(c) = G_n(0)$ . Then, for each  $k = 1, \dots, n$

$$c_{k-1} + \frac{F_{n-k+1}(c) - F_{n-k}(c)}{n-k+1} + \frac{G_{n-k}(c) - G_{n-k+1}(c)}{n-k+1} = c_k$$

The CIC relations are imposed to guarantee that as the endowment reaches the sum of the claims, each claimant is fully compensated. As for the ICI family, the relations are not independent: multiplying the first one through by  $n$ , the second one by  $n-1$ ,  $\dots$ , and the last one by 1, gives new relations whose sum is an identity. Also, if several agents have equal claims, the CIC relations imply that successive  $F_k$ 's are equal and that so are the corresponding  $G_k$ 's. Then, agents with equal claims come in and drop out together.

Let  $\bar{\mathcal{H}}^N$  denote the family of pairs of functions satisfying the CIC relations. We have the following formal definition of the CIC family (Thomson, 2007a):

**CIC rule relative to  $H \equiv (F_k, G_k)_{k=1}^{k=n-1} \in \bar{\mathcal{H}}^N, R^H$ :** For each  $c \in \mathbb{R}_+^N$  with  $c_1 \leq \dots \leq c_n$ , the awards vector chosen by  $R^H$  is calculated as follows. As  $E$  first increases from 0 to  $F_1(c)$ , everything goes to claimant  $n$ . As  $E$  increases from  $F_1(c)$  to  $F_2(c)$ , equal division of each increment prevails between claimants  $n$  and  $n-1$ . As  $E$  increases from  $F_2(c)$  to  $F_3(c)$ , equal division of each increment prevails among claimants  $n, n-1$ , and  $n-2$ .  $\dots$  This goes on until claimant 1 enters the scene, when  $E = F_{n-1}(c)$ , at which point equal division of each increment prevails until, when  $E = G_{n-1}(c)$ , he is fully compensated and is dropped off. As  $E$  increases from  $G_{n-1}(c)$  to  $E = G_{n-2}(c)$ , equal division of each increment prevails among claimants 2 through  $n$ . Then, claimant 2 is fully compensated and is dropped off.  $\dots$  At the end of the process, claimant  $n$  is the only one left, and he receives each increment until he is fully compensated.

The next lemma identifies three rules as CIC rules. The constrained equal awards and constrained equal losses rules are two of them. The third one is defined like the Talmud rule, which can be seen as a “hybrid” of the constrained equal awards and constrained equal losses rules, by exchanging the order in which these component rules are applied, with the half-claims being used in the formulas instead of the claims themselves. For that reason, we name it the **reverse Talmud** rule.<sup>6</sup>

**Reverse Talmud rule,  $T^r$ :** For each  $(c, E) \in \mathcal{C}^N$ ,

$$T_i^r(c, E) \equiv \begin{cases} \max\{\frac{c_i}{2} - \lambda, 0\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \frac{c_i}{2} + \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case,  $\lambda \in \mathbb{R}_+$  is chosen so as to achieve efficiency.

**Lemma 2** (Thomson, 2000) *The following rules are CIC rules:*

- (a) *CEA:* for each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv (0, 0, \dots, 0)$ .
- (b) *CEL:* for each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv (c_n - c_{n-1}, c_n + c_{n-1} - 2c_{n-2}, \dots, c_n + c_{n-1} + \dots + c_2 - (n-1)c_1)$ .
- (c)  *$T^r$ :* for each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv (-\frac{c_{n-1}}{2} + \frac{c_n}{2}, -2\frac{c_{n-2}}{2} + \frac{c_{n-1}}{2} + \frac{c_n}{2}, \dots, -k\frac{c_{n-k}}{2} + \frac{c_{n-k+1}}{2} + \dots + \frac{c_n}{2}, \dots, -(n-1)\frac{c_1}{2} + \frac{c_2}{2} + \dots + \frac{c_n}{2})$ .

A counterpart of family  $\{T^\theta\}_{\theta \in [0,1]}$  can be defined by reversing the order in which the ideas of equal gains and equal losses are applied. For each  $c \in \mathbb{R}_+^N$ , set  $F(c) \equiv \theta(c_n - c_{n-1}, c_n + c_{n-1} - 2c_{n-2}, \dots, c_n + c_{n-1} + \dots + c_2 - (n-1)c_1)$ . Let  $\{U^\theta\}_{\theta \in [0,1]}$  denote this family. Note that  $U^{\frac{1}{2}} = T^r$ ,  $U^0 = CEA$ , and  $U^1 = CEL$ .

## 4 Lorenz comparisons of ICI rules and of CIC rules

Our first theorem formulates a simple sufficiency criterion on pairs of ICI rules that tells us when they can be Lorenz ordered. It suffices to check that the  $F$ -sequence of one dominates the  $F$ -sequence of the other; equivalently, that the  $G$ -sequence of one dominates the  $G$ -sequence of the other.

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<sup>6</sup>It is discussed under that name by Chun, Schummer, and Thomson (2001) and Hokari and Thomson (2003).

**Theorem 1** *Let  $S$  and  $S'$  be two ICI rules, associated with  $H \equiv (F, G)$  and  $H' \equiv (F', G') \in \mathcal{H}^N$ . If, for each  $k \in \{1, \dots, n-1\}$ ,  $F'_k \geq F_k$ , then  $S'$  Lorenz dominates  $S$ . Equivalently, it suffices that for each  $k \in \{1, \dots, n-1\}$ ,  $G'_k \geq G_k$ .*

The proof essentially consists of two lemmas. Let  $x$  and  $x' \in \mathbb{R}_+^N$  be such  $\sum x_i = \sum x'_i$ . We will use the fact that  $x' \succeq_L x$  if and only if one passes from  $x$  to  $x'$  by means of a sequence of Lorenz improving transfers.

Let  $c \in \mathbb{R}_+^N$ . If all coordinates of  $c$  differ, the passage from  $H$  to  $H'$  can be described as the composition of increases, for some  $k \in \{1, \dots, n-1\}$ , of the  $k$ -th component of the function  $F(c)$ . If some of the coordinates of  $c$  are equal, successive components of  $(F_k(c))$  have to increase simultaneously.

**Lemma 3** *Let  $S$  and  $S'$  be two ICI rules, associated with  $H = (F, G)$  and  $H' = (F', G') \in \mathcal{H}^N$ . If there are  $k \in \{1, \dots, n\}$  and  $\bar{k} \in \{0, \dots, n-k-1\}$  such that  $F_k = \dots = F_{k+\bar{k}} < F'_k = \dots = F'_{k+\bar{k}}$  and for each  $\ell \notin \{k, \dots, k+\bar{k}\}$ ,  $F_\ell = F'_\ell$ , then  $S'$  Lorenz dominates  $S$ .*

**Proof:** For each  $k = 1, \dots, n-1$ , let  $\delta_k(c) \equiv G_k(c) - F_k(c)$ . First, we show that the sequence  $(\delta_k(c))_{k=1}^{n-1}$  is independent of which ICI rule is considered.<sup>7</sup> Indeed:

$$\begin{array}{rcccccccc} \delta_1(c) & = & -(n-1)c_1 & + & c_2 & + & c_3 & + & \dots & + & c_{n-1} & + & c_n \\ \delta_2(c) & = & & & -(n-2)c_2 & + & c_3 & + & \dots & + & c_{n-1} & + & c_n \\ \dots & = & & & & + & \dots & + & \dots & + & c_{n-1} & + & c_n \\ \delta_{n-2}(c) & = & & & & & & & -2c_{n-2} & + & c_{n-1} & + & c_n \\ \delta_{n-1}(c) & = & & & & & & & & & -c_{n-1} & + & c_n \end{array}$$

Thus, the ICI rule associated with each  $(F, G) \in \mathcal{H}^N$  is defined, for each  $c \in \mathbb{R}_+$ , by the location of nested intervals in  $[0, \sum c_i]$ ,  $I_1(c) \equiv [F_1(c), G_1(c)]$ ,  $I_2(c) \equiv [F_2(c), G_2(c)]$ ,  $\dots$ ,  $I_{n-1}(c) \equiv [F_{n-1}(c), G_{n-1}(c)]$ , of lengths  $\delta_1(c)$ ,  $\delta_2(c)$ ,  $\dots$ ,  $\delta_{n-1}(c)$ .

To prove the lemma, let us suppose that no two claims are equal, which means that  $\bar{k} = 0$  in its statement. Since the calculations are performed for each claims vector separately, we omit  $c$  as an argument of the  $F_k$ 's and  $G_k$ 's. Let  $k \in \{1, \dots, n-1\}$ , and consider the passage from  $F_k$  to  $F'_k < F_k$ . We show that for each  $E \in [0, \sum c_i]$ , one passes from  $x \equiv S(c, E)$  to  $x' \equiv S'(c, E)$  by means of Lorenz improving transfers.

<sup>7</sup>In light of this fact, we could just as well have stated the theorem in terms of  $G$  and  $G'$  instead of  $S$  and  $S'$ .

**Case 1:**  $E \in [0, F_k]$ . Then  $x' = x$  and the conclusion holds trivially.

**Case 2:**  $E \in [F_k, F'_k]$ . Let  $\Delta \equiv E - F_k$ . Under  $S$ , this difference is distributed equally among claimants  $k + 1, \dots, n$ . Under  $S'$ , it is distributed equally among claimants  $k, \dots, n$ , whereas claimants  $1, \dots, k - 1$ 's awards remain at the values they are given by both rules when  $E = F_k$ . The differences between the successive partial sums for  $S'$  and  $S$ ,  $\sum_{i=1, \dots, m} [S'_i(c, E) - S_i(c, E)]$ , as  $m$  runs from 1 to  $n$ , are the following: the first  $k - 1$  differences are 0; then letting  $\ell$  run from 0 to  $n - k$ , the  $(k + \ell)$ -th term is  $\frac{\Delta(n-k-\ell)}{(n-k)(n-k+1)}$ . It is important to note for Step 4 that the maximal values of these differences are obtained for  $\Delta^* \equiv F'_k - F_k$  (then  $E = F'_k$ ).

**Case 3:**  $E \in [F'_k, G_k]$ . As  $E$  increases in that interval, each increment is distributed in the same fashion by both rules. Thus, the differences between the successive partial sums remain what they are under Case 2 for  $E = F'_k$ , the greatest value of the endowment covered under that case.

**Case 4:**  $E \in [G_k, G'_k]$ . (Here, claimant  $k$  reenters the distribution later for  $S'$  than for  $S$ .) Let  $\bar{\Delta} \equiv E - G'_k$ . Under  $S$ , this difference is distributed equally among claimants  $k, \dots, n$ . Under  $S'$ , it is distributed equally among claimants  $k + 1, \dots, n$ . Starting from an endowment of  $G_k$ , the contributions to the differences between the successive partial sums for  $S'$  and  $S$  of an increase of  $\bar{\Delta}$  of the endowment are the following: the first  $k - 1$  differences are 0; then letting  $\ell$  run from 0 to  $n - k$ , the  $(k + \ell)$ -th term is  $\frac{\Delta(\ell-n+k)}{(n-k)(n-k+1)}$ . The maximal values of these differences are obtained for  $\Delta^* \equiv G'_k - G_k$  (then  $E = G'_k$ ). At that point and since by the ICI relations,  $G'_k - G_k = F'_k - F_k$ , these differences are the negative of the differences of the values of the successive partial sums for  $S'$  and  $S$  reached for  $E = F'_k$ , so that the successive partial sums are equal for both rules.

**Case 5:**  $E \in [G'_k, \sum c_i]$ . As  $E$  increases in that interval, each increment is distributed in the same fashion by both rules. The successive partial sums remain equal until  $E$  reaches  $\sum c_i$ .

When several claims are equal, several adjacent parameters are equal for all ICI rules. Then, to pass from one to the other, these parameters have to be increased together. This requires a straightforward adaptation of the calculations just made. If exactly  $\bar{k}$  successive parameters are equal, starting with the  $k$ -th one,  $F_k$ , the transfer is from claimants  $k, \dots, k + \bar{k}$  to claimants  $k + \bar{k} + 1, \dots, n$  initially, and later on, from claimants  $k + \bar{k} + 1, \dots, n$  to claimants  $k, \dots, k + \bar{k}$ .  $\square$



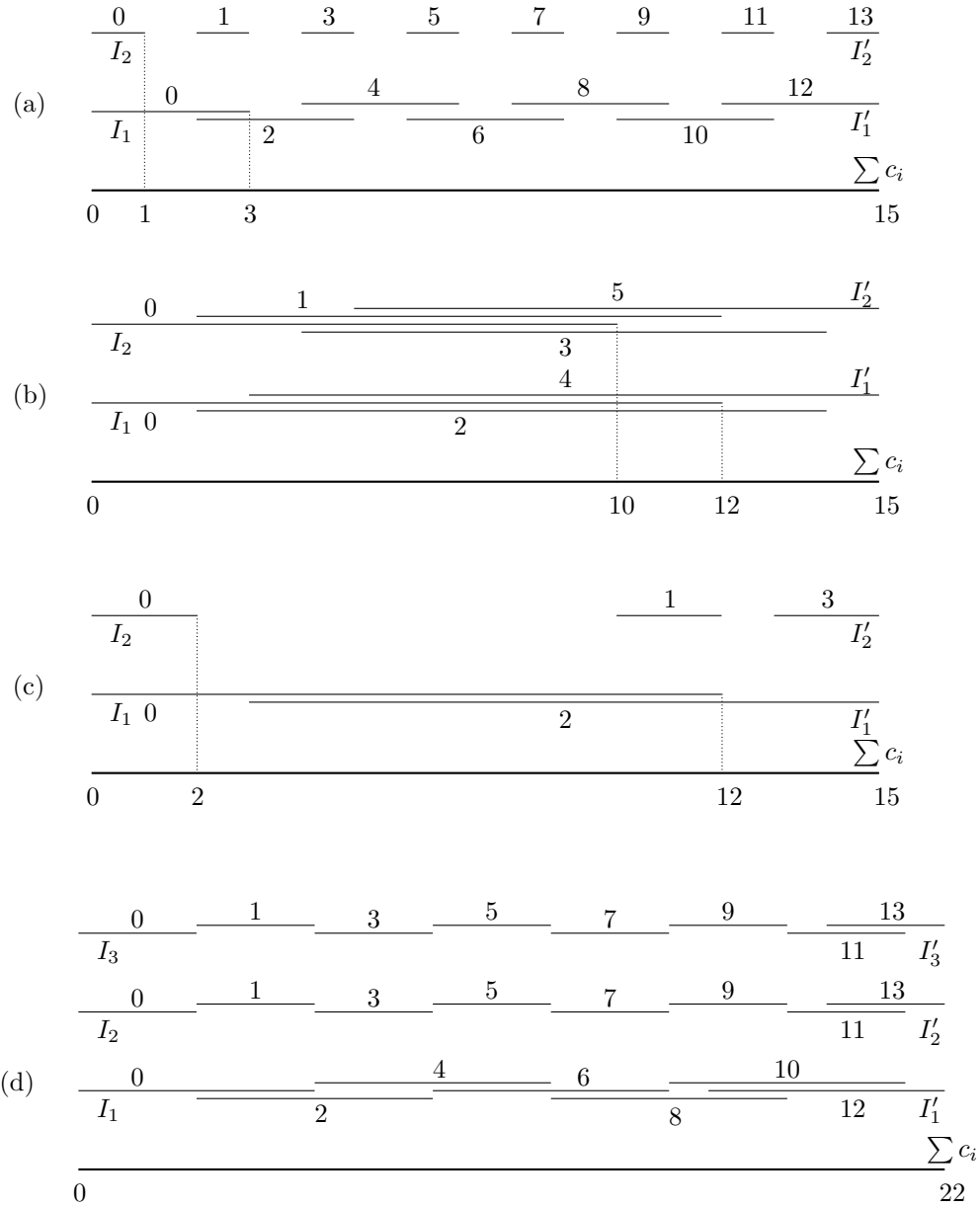
**Lemma 4** *One can pass from  $H$  to  $H'$  by a succession of adjustments as described in Lemma 3.*

The proof is simple but not standard and it is notationally cumbersome, so we will first illustrate it by means of examples, supposing initially that no two claims are equal. Again, we drop  $c$  as an argument of the  $F_k$ 's, the  $G_k$ 's, and the  $\delta_k$ 's. Given two ICI rules  $S$  and  $S'$ , associated with  $H = (F, G)$  and  $H' = (F', G') \in \mathcal{H}^N$ , if  $F \leq F'$ , then each interval  $I_k = [F_k, G_k]$  for  $S$  lies to the left of the corresponding interval  $I'_k = [F'_k, G'_k]$  for  $S'$ . (This is possible because the intervals have equal lengths, as we have seen). For the two-claimant case, there is a single interval, and there are no nesting constraints: one can pass from  $I_1$  to  $I'_1$  in one step.

If  $n \geq 3$ , more than one step may be needed. Suppose  $n = 3$ . Then there are two intervals,  $I_1$  and  $I_2$ . In each of the examples below,  $F_1 = F_2 = 0$  (thus,  $S = CEL$ ) and  $G'_1 = G'_2 = \sum c_i$  (thus,  $S' = CEA$ ). This is the case for which the required movements are the largest, and they give us an upper bound on the number of steps. To preserve nesting, we move  $I_1$  to the right, then  $I_2$ , then  $I_1$  again, and so on.

**Example 1** (*Figure 1a*): Let  $N \equiv \{1, 2, 3\}$  and  $c \equiv (4, 5, 6)$ . Then,  $\delta_1 = 3$  and  $\delta_2 = 1$ . The initial positions of the two intervals  $[F_1, G_1]$  and  $[F_2, G_2]$  are labelled 0. Since their left endpoints are equal at first, we can only move the upper one. The difference  $\delta_1 - \delta_2$  of their lengths is equal to 2. It is the maximal amount by which we can move the upper interval. The result of this first move is labelled 1. Once this is done, we can move the lower interval, by a maximal amount also equal to  $\delta_1 - \delta_2 = 2$ . The result of this second move is labelled 2. We now return to the higher interval. Once again, we can move it by the same amount. The result of this third move is labelled 3. We continue in this manner until the right endpoints of both intervals are  $\sum c_i$ . Altogether, for the example, we need 13 steps. (The successive values of the left endpoints of the two intervals are  $(0, 0)$ ,  $(0, 2)$ ,  $(2, 2)$ ,  $(2, 4)$ ,  $(4, 4)$ ,  $(4, 6)$ ,  $(6, 6)$ ,  $(6, 8)$ ,  $(8, 8)$ ,  $(8, 10)$ ,  $(10, 10)$ ,  $(10, 12)$ ,  $(12, 12)$ ,  $(12, 13)$ .)

**Example 2** (*Figure 1b*): Let  $N \equiv \{1, 2, 3\}$  and  $c \equiv (1, 2, 12)$ . Then,  $\delta_1 = 12$  and  $\delta_2 = 10$ . The sum of the claims is the same as in Example 1 but we need fewer steps to pass from the constrained equal losses rule to the constrained equal awards rule. The difference  $\delta_1 - \delta_2$  is still equal to 2, but because  $\delta_1$  is close to  $\sum c_i$ , the required move is more limited. (The 5 successive values of



**Figure 1: Illustrating Lemma 4.** (a) Example 1. (b) Example 2. (c) Example 3. The intervals  $I_1$  and  $I_2$  are represented as horizontal segments at two heights. Their successive positions are numbered, starting with “0”. At Step 1, we move the higher interval to the right, labelling its new position “1”. At Step 2, we move the lower interval to the right, labelling its new position “2”, and so on. (d) For Example 4, there are three intervals, at three different heights,  $I_1$ ,  $I_2$ , and  $I_3$ . Since  $c_2 = c_3$ ,  $I_2$  and  $I_3$  have equal lengths, and we have to move them together.

the left endpoints of the two intervals are  $(0, 0)$ ,  $(0, 2)$ ,  $(2, 2)$ ,  $(2, 4)$ ,  $(3, 4)$ ,  $(3, 5)$ .)

**Example 3** (Figure 1c): Let  $N \equiv \{1, 2, 3\}$  and  $c \equiv (1, 6, 8)$ . Then,  $\delta_1 = 12$  and  $\delta_2 = 2$ . The higher interval has to move by a large amount but because the difference  $\delta_1 - \delta_2$  is larger than for Example 2, at each step, the higher interval can move by a larger amount. (The 3 successive values of the left endpoints of the two intervals are  $(0, 0)$ ,  $(0, 10)$ ,  $(3, 10)$ ,  $(3, 13)$ .)

These examples show that when claims are all distinct, we can move each interval by the difference between the (positive) lengths of two successive intervals. Since the distance by which we have to move any interval is finite, we need a finite number of steps. The next example is one for which two claims are equal. Then, we have to move two successive intervals together.

**Example 4** (Figure 1d): Let  $N \equiv \{1, 2, 3, 4\}$  and  $c \equiv (4, 5, 5, 8)$ . Then,  $\delta_1 = 6$  and  $\delta_2 = \delta_3 = 3$ . Here, two intervals have equal lengths, so we have to move them together. (The 13 successive values of the left endpoints of the three intervals are  $(0, 0, 0)$ ,  $(0, 3, 3)$ ,  $(3, 3, 3)$ ,  $(3, 6, 6)$ ,  $(6, 6, 6)$ ,  $(6, 9, 9)$ ,  $(9, 9, 9)$ ,  $(9, 12, 12)$ ,  $(12, 12, 12)$ ,  $(12, 15, 15)$ ,  $(15, 15, 15)$ ,  $(15, 18, 18)$ ,  $(16, 18, 18)$ ,  $(16, 19, 19)$ .)

We can now generalize the process just described to prove Lemma 4.

**Proof:** (of Lemma 4) The proof is by induction on the number of claimants. The assertion is true for two claimants as there is only one interval then, and we can move it to its required position in one step. Suppose the assertion is true for  $n - 1$  claimants. We prove it for  $n$  claimants. We undertake a sequence of moves of intervals to the right. Suppose first that all claims are distinct. We move  $I_{n-1}$  and  $I_{n-2}$  in turn. Moving  $I_{n-2}$  by any amount may require moving  $I_{n-3}, \dots, I_1$  first but by the induction hypothesis, this requires finitely many steps. If at any step, one of the intervals  $I_k$  reaches the position  $I'_k$ , we stop for that interval, and the exercise splits into two parts: one of them involves all claimants  $\ell > k$  and the other involves all claimants  $\ell < k$ . We proceed with each group separately but for each of them, by the induction hypothesis, the number of required steps is finite.

The sum of the maximal movements of  $I_{n-1}$  and  $I_{n-2}$  during the first two steps is given by the difference  $\delta_{n-1} - \delta_{n-2} = 2(c_{n-1} - c_{n-2}) > 0$ . At each of the next steps, we can move either one or the other of these intervals by this

difference. As the distance by which we have to move each of them is finite, we need finitely many steps. Altogether, we can therefore move all intervals from their initial to their final positions in finitely many steps.

If some claims are equal, we have to move some intervals together, but this can only accelerate the process.

That  $S'$  Lorenz dominate  $S$  if for each  $k \in \{1, \dots, n-1\}$ ,  $G'_k \geq G_k$ , follows from the facts that Lorenz domination holds if for each  $k \in \{1, \dots, n-1\}$ ,  $F'_k \geq F_k$ , and that the sequence  $(\delta)_{k=1}^{k=n-1} \equiv (G_k - F_k)_{k=1}^{k=n-1}$  is the same for all members of the family, as established in Lemma 3.  $\square$

**Proof:** (of Theorem 1). It is a direct consequence of Lemmas 3 and 4  $\square$

The following Lorenz rankings of several rules can be obtained as easy corollaries of Theorem 1.

**Corollary 1** (a) *The constrained equal awards rule Lorenz dominates all ICI rules.*

(b) *All ICI rules Lorenz dominate the constrained equal losses rule.*

(c) *The Talmud rule Lorenz dominates the minimal overlap rule.*

(d) *For each pair  $\{\theta^1, \theta^2\}$  of elements of  $[0, 1]$ , if  $\theta^1 \leq \theta^2$ , then  $T^{\theta^2}$  Lorenz dominates  $T^{\theta^1}$ .*

**Proof:** Given an ICI rule  $S$ , let us use the notation  $H^S = (F^S, G^S) \in \mathcal{H}^N$  for the function associated with it. We show that in each case, the inequalities of Theorem 1 are met.

(a) For each ICI rule  $S$ , we have  $F^{CEA} = 0 \leq F^S$ .

(b) The sequences  $(F_k^T)_{k=1}^{k=n-1}$  and  $(F_k^{MO})_{k=1}^{k=n-1}$  are given in Lemma 1. We compare them term by term:

$$\begin{array}{rcl}
n \frac{c_1}{2} & & \geq c_1 \\
\frac{c_1}{2} + (n-1) \frac{c_2}{2} & & \geq c_2 \\
\dots & & \geq \dots \\
\frac{c_1}{2} + \frac{c_2}{2} + \dots + (n-k+1) \frac{c_k}{2} & & \geq c_k \\
\dots & & \geq \dots \\
\frac{c_1}{2} + \frac{c_2}{2} + \dots + \dots + 2 \frac{c_{n-1}}{2} & \geq & c_{n-1}
\end{array}$$

The first inequality is obviously satisfied. The remaining ones can be written, for each  $k = 2, \dots, n-1$ , as  $\sum_{\ell=1}^{k-1} c_\ell + (n-k+1)c_k \geq 0$ .

(c) Here, it is simpler to compare the  $G_k$ 's. For each ICI rule  $S$ , we have  $G^S \leq G^{CEL} = (\sum c_i, \dots, \sum c_i)$ .

(d) The functions  $F^{\theta^1}$  and  $F^{\theta^2}$  are non-negative and proportional. Thus, they trivially satisfy the hypotheses of Theorem 1.  $\square$

We need to introduce additional properties of rules to relate Corollary 1 to existing literature. **Order preservation of awards** (Aumann and Maschler, 1985) says that for each problem, given any two claimants, the one with the larger claim should be awarded at least as much as the one with the smaller claim, and **order preservation of losses** (Aumann and Maschler, 1985) says that the former should incur a loss that is at least as large as the loss incurred by the latter. **Order preservation** is the conjunction of the two requirements.

Parts (a) and (b) of Corollary 1 are not the interesting ones, as more general results are obviously true: the constrained equal awards rule Lorenz dominates all rules and the constrained equal losses rule is dominated by all rules satisfying *order preservation* (Moreno-Ternerero and Villar, 2006b, state the first of these results for rules satisfying *order preservation*, this property being incorporated in their definition of a rule, but (a) holds without it; Bosmans and Lauwers, 2007, state a related result). Bosmans and Lauwers (2007) obtain (c) as a corollary of a result stating that the minimal overlap rule is Lorenz minimal in a class of rules satisfying a certain list of properties, all of which are satisfied by the Talmud rule.<sup>8</sup> A direct proof of Part (d) is the object of Moreno-Ternerero and Villar (2006b).

A parallel conclusion to that of Theorem 1 holds for the CIC family. The proof is parallel too. The only difference is that at each step, the transfer is

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<sup>8</sup>Consider the family of rules satisfying the *reasonable lower bound* (which says that each agent's award should be at least as large as  $1/n$  times his claim truncated at the endowment (Moreno-Ternerero and Villar, 2004; Dominguez and Thomson, 2006), *order preservation*, *order preservation under claims variations* (which says that if some agent's claim increases, given two other agents, the one with the larger claim should incur a loss that is at least as large as the loss incurred by the one with the smaller claim; Thomson, 2007b), and *limited consistency* (which says that if an agent's claim is 0 and he leaves, the awards to the other claimants should not be affected; Section 5 is devoted to a study of *consistent* rules.) Bosmans and Lauwers show that the minimal overlap rule is Lorenz dominated by all rules satisfying these properties; moreover, it is the only rule to be so dominated. All ICI rules satisfy *order preservation*, but not all satisfy the *reasonable lower bound*, the constrained equal losses rule being an example of one that does not. Also, an ICI rule satisfies *order preservation under claims variations* only for some choices of the function specifying the breakpoints. Finally an ICI rule satisfies *limited consistency* only if its components are appropriately related.

from one claimant to several others, instead of from several claimants to one of them.

**Theorem 2** *Let  $S$  and  $S'$  be two CIC rules, associated with  $H \equiv (F, G)$  and  $H' \equiv (F', G') \in \bar{\mathcal{H}}^N$ . If, for each  $k \in \{1, \dots, n-1\}$ ,  $F'_k \leq F_k$ , then  $S' \succ_L S$ . Equivalently, it suffices that for each  $k \in \{1, \dots, n-1\}$ ,  $G'_k \leq G_k$ .*

**Proof:** We omit the details, as the proof is parallel to that of Theorem 1. It involves the observation that each CIC rule can be described by means of nested intervals whose lengths are independent of which member of the family is considered, as was the case for the ICI family. For each  $k = 1, \dots, n-1$ , let  $\epsilon_k(c) \equiv G_k(c) - F_k(c)$ . For each member of the CIC family, the sequence  $(\epsilon_k(c))_{k=1}^{n-1}$ , which we calculate from the CIC relations, is given by:<sup>9</sup>

$$\begin{aligned}
 \epsilon_1(c) &= c_1 + c_2 + \dots + c_{n-3} + c_{n-2} + 2c_{n-1} \\
 \epsilon_2(c) &= c_1 + c_2 + \dots + c_{n-3} + 3c_{n-2} \\
 \dots &= \dots + \dots + \dots + \dots \\
 \epsilon_{n-2}(c) &= c_1 + (n-1)c_2 \\
 \epsilon_{n-1}(c) &= nc_1
 \end{aligned}$$

□

## 5 Lorenz comparisons of consistent rules

Next, we turn to the family of rules satisfying the following requirement. Consider a rule and apply it to some problem. Then, imagine that a group of claimants take their awards and leave. Reassess the situation at this point. It can be seen as the problem of allocating among the remaining claimants the sum of the amounts initially awarded to them. Reapply the rule to solve this problem. The requirement is that the rule should assign to each of them the same amount as it did initially. It expresses the robustness of the rule “under partial implementation”.

For a formal statement, one has to generalize the model so as to allow variations in the population of claimants. Let there be an infinite set of “potential” claimants, indexed by the natural numbers,  $\mathbb{N}$ . In each given

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<sup>9</sup>We have  $\delta_1 + \epsilon_{n-1} = \delta_2 + \epsilon_{n-2} = \dots = \delta_{n-1} + \epsilon_1 = \sum c_i$ .

problem, however, only a finite number of them are present. Let  $\mathcal{N}$  be the class of finite subsets of  $\mathbb{N}$ . A (claims) **problem** is a pair  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ , where  $N \in \mathcal{N}$ , such that  $\sum_N c_i \geq E$ . A **rule** is a function defined over  $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$  consisting of all problems involving some population in  $\mathcal{N}$ , which associates with each  $N \in \mathcal{N}$  and each  $(c, E) \in \mathcal{C}^N$  a vector in  $X(c, E)$ . Given  $N \in \mathcal{N}$ , we refer to the restriction of a rule to the subdomain  $\mathcal{C}^N$  as its “ $N$ -component”.

**Consistency:** For each  $N \in \mathcal{N}$ , each  $(c, E) \in \mathcal{C}^N$ , and each  $N' \subset N$ , if  $x \equiv S(c, E)$ , then  $x_{N'} = S(c_{N'}, \sum_{N'} x_i)$ .

It follows from the definition of a rule that the pair  $(c_{N'}, \sum_{N'} x_i)$  is a well-defined problem, so it is meaningful to apply  $S$  to it.

In this section, what we call an “ICI rule” or a “CIC rule” is a rule whose components are rules as defined in Subsections 3.2 and 3.3 respectively. A characterization of the family of ICI rules that are *consistent* is available (Thomson, 2007a). Let  $e : \mathbb{R} \rightarrow \mathbb{R}$  be the identity function. Let  $\Gamma$  be the class of nowhere decreasing functions  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , the function  $e - \gamma$  being also nowhere decreasing, such that  $\gamma(0) = 0$ . (These requirements imply that  $\gamma$  is continuous.) To each  $\gamma \in \Gamma$  we associate the ICI rule  $S^\gamma$  defined by setting  $F(c) \equiv (n\gamma(\tilde{c}_1), \gamma(\tilde{c}_1) + (n-1)\gamma(\tilde{c}_2), \dots, \gamma(\tilde{c}_1) + \gamma(\tilde{c}_2) + \dots + \gamma(\tilde{c}_{n-2}) + 2\gamma(\tilde{c}_{n-1}))$ ,  $G(c)$  being obtained from  $F(c)$  by means of the ICI relations. This family of *consistent* ICI rules—we call them **ICI\* rules**—includes the constrained equal awards, constrained equal losses, and Talmud rules. In fact, it contains the entire Moreno-Ternero–Villar (2006a) family. (The minimal overlap rule is not *consistent* however.)

The form taken for ICI\* rules by the sufficiency criterion of Theorem 1 is particularly simple. Moreover, it becomes necessary.

**Theorem 3** *Let  $\gamma$  and  $\gamma' \in \Gamma$ . Then,  $S^\gamma$  Lorenz dominates  $S^{\gamma'}$  if and only if  $\gamma \geq \gamma'$ .*

**Proof:** The sufficiency part is a consequence of Theorem 1. Conversely, suppose that there is  $c_0 \in \mathbb{R}_+$  such that  $\gamma(c_0) < \gamma'(c_0)$ . Let  $N \equiv \{1, 2\}$  and  $c \in \mathbb{R}_+^N$  such that  $c_1 = c_0$  and  $c_2 > c_0$ . It is easy to see that the path of  $S^\gamma$  for  $c$  lies to the northwest of the path of  $S^{\gamma'}$  for  $c$ , so in fact, for each  $E \in [0, \sum c_i]$ ,  $S^{\gamma'}(c, E)$  Lorenz dominates  $S^\gamma(c, E)$ .  $\square$

A parallel characterization of the *consistent* members of the CIC family is available too (Thomson, 2007a). These rules—which we call **CIC\* rules**—are also indexed by the members of  $\Gamma$ : To each  $\gamma \in \Gamma$  we associate the CIC rule  $R^\gamma$  defined by setting  $F(c) \equiv (\gamma(\tilde{c}_n) - \gamma(\tilde{c}_{n-1}), \gamma(\tilde{c}_n) + \gamma(\tilde{c}_{n-1}) - 2\gamma(\tilde{c}_{n-2}), \dots, \gamma(\tilde{c}_n) + \gamma(\tilde{c}_{n-1}) + \dots + \gamma(\tilde{c}_2) - (n-1)\gamma(\tilde{c}_1))$ ,  $G(c)$  being obtained from  $F(c)$  by means of the CIC relations. We have the following counterpart of Theorem 3. We omit the proof.

**Theorem 4** *Let  $\gamma$  and  $\gamma' \in \Gamma$ . Then,  $R^\gamma$  Lorenz dominates  $R^{\gamma'}$  if and only if for each pair  $\{a, b\} \in \mathbb{R}_+$  with  $a < b$ ,  $\gamma(b) - \gamma(a) \leq \gamma'(b) - \gamma'(a)$ .*

More can be said about Lorenz domination within the class of *consistent* rules. Since *consistency* relates the components of a rule across populations, one may hope that a behavior of a rule for low cardinalities would extend to higher cardinalities. This observation motivates the following definition. A property of a rule is **lifted** if whenever a rule satisfies it in the two-claimant case and is *consistent*, then it satisfies it in general (Hokari and Thomson, 2000). There is a practical advantage to a property being lifted: to prove that a *consistent* rule satisfies it, it suffices to prove that it does so in the usually less cumbersome, often trivial, case of two claimants.

Some properties are not lifted on their own but they are lifted if the rule satisfies some additional property (properties). **Resource monotonicity**, which says that for each claims vector, if the endowment increases, each claimant should receive at least as much as he did initially, has been most helpful in assisting the lifting of properties (again, see Hokari and Thomson, 2000).

We propose here to extend the lifting and assisted lifting ideas to order relations on pairs of rules. Given an order  $\succeq$  on the space of rules, the **order is lifted** if for each pair  $S, S'$  of rules, if  $S \succeq S'$  in the two-claimant case and both rules are *consistent*, then  $S \succeq S'$  in general. Our next theorem describes circumstances under which the Lorenz order is lifted. In fact, we only require rules to satisfy **bilateral consistency**, the property obtained from *consistency* by restricting the group of remaining agents to be of size two.

**Theorem 5** *Let  $S$  and  $S'$  be two rules satisfying order preservation of awards in the two-claimant case, resource monotonicity in the two-claimant case, and bilateral consistency.<sup>10</sup> Then, if  $S$  Lorenz dominates  $S'$  in the two-claimant*

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<sup>10</sup>These two properties are lifted by *bilateral consistency*.



case, in fact  $S$  Lorenz dominates  $S'$  in general.

**Proof:** Let  $N \in \mathcal{N}$  and  $(c, E) \in \mathcal{C}^N$ . Let  $x \equiv S(c, E)$  and  $x' \equiv S'(c, E)$ . To simplify notation, suppose  $N \equiv \{1, \dots, n\}$  and  $c_1 \leq \dots \leq c_n$ . Since  $S$  and  $S'$  satisfy *order preservation of awards*,  $x_1 \leq \dots \leq x_n$  and  $x'_1 \leq \dots \leq x'_n$ . Suppose that  $x$  does not Lorenz dominate  $x'$ . Then, there is  $i' \in N$  such that  $\sum_{\ell=1}^{\ell=i'} x'_\ell > \sum_{\ell=1}^{\ell=i'} x_\ell$ . Let  $i$  be the smallest index for which this is the case. Obviously,  $x'_i > x_i$ . Also,  $\sum_N x_k = \sum_N x'_k$ , and thus, there is  $j \in N$  with  $j > i$  such that  $x'_j < x_j$ . Let  $N' \equiv \{i, j\}$ . Since  $S$  is *bilaterally consistent*,  $(x_i, x_j) = S(c_i, c_j, x_i + x_j)$ . Since  $S'$  is *bilaterally consistent*,  $(x'_i, x'_j) = S(c_i, c_j, x'_i + x'_j)$ . Let  $y \equiv S'(c_i, c_j, x_i + x_j)$ . If  $x'_i + x'_j \leq x_i + x_j$ , then since  $S$  is *resource monotonic* in the two-claimant case,  $y_i \geq x'_i > x_i$ . If  $x'_i + x'_j > x_i + x_j$ , then since  $S$  is *resource monotonic* in the two-claimant case,  $y_j \leq x'_j < x_j$ . Since  $S$  and  $S'$  satisfy *order preservation of awards*,  $x_i \leq x_j$  and  $y_i \leq y_j$ . In either case, we conclude that  $(y_i, y_j) \succ_L (x_i, x_j)$ . Thus, it is not true that  $S \succ_L S'$  in the two-claimant case.  $\square$

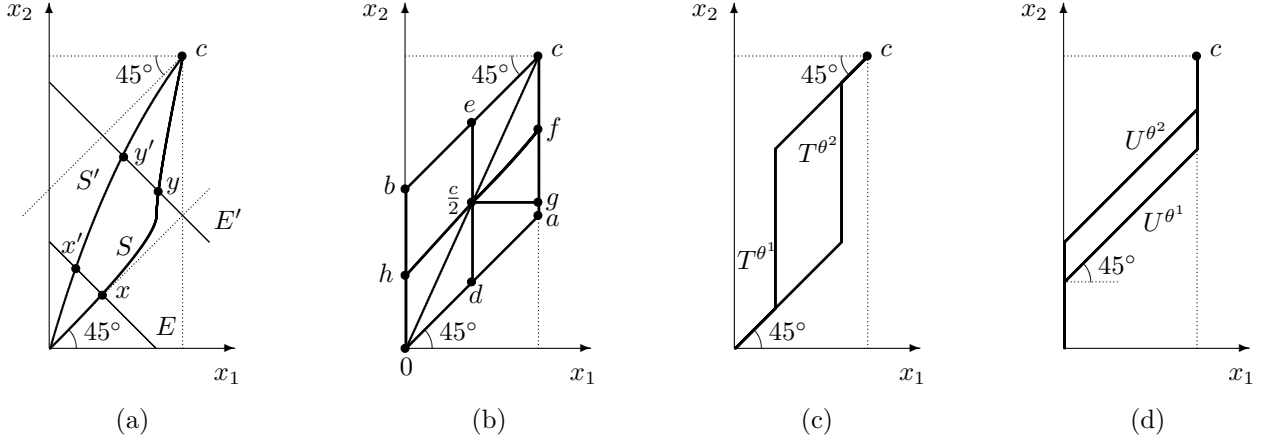
Next, we explain how to check Lorenz domination of various rules in the two-claimant case. The **path of awards of a rule for a claims vector** is the locus of the awards vector it chooses as the endowment varies from 0 to the sum of the claims. In the two-claimant case,  $S \succeq_L S'$  if for each  $N \equiv \{i, j\}$ , each  $c \in \mathbb{R}_+^N$  with  $c_i < c_j$ , and each  $0 \leq E \leq \sum c_i$ ,  $S(c, E)$  is closer than  $S'(c, E)$  to the intersection of the line of equation  $t_i + t_j = E$  with the 45° line. For rules satisfying *order preservation of awards*, and measuring the smaller claim and the award to its owner on the horizontal axis (thus, measuring the larger claim and the award to its owner on the vertical axis), this means that the path of awards of  $S$  for  $c$  lies to the southeast of the path of  $S'$  for  $c$  (Figure 2a).

Since many rules are *consistent*, we will derive a number of Lorenz rankings as corollaries of Theorem 5. Some pertain to rules not introduced yet. First is the rule for which awards are proportional to claims:

**Proportional rule,  $P$ :** For each  $(c, E) \in \mathcal{C}^N$ ,

$$P(c, E) \equiv \lambda c,$$

where  $\lambda \in \mathbb{R}_+$  is chosen so as to achieve efficiency.



**Figure 2: Illustrating Corollary 2.** Let  $N \equiv \{1, 2\}$ , and  $c \in \mathbb{R}_+^N$  be such that  $c_1 < c_2$ . (a) If  $S$  Lorenz dominates  $S'$  and both rules satisfy *order preservation of awards*, the path of  $S$  for  $c$  lies to the southeast of the path of  $S'$  for  $c$ . (b) Paths of awards of several standard rules. (c) Paths of two members of the family  $\{T^\theta\}_{\theta \in [0,1]}$ . (d) Paths of two members of the family  $\{U^\theta\}_{\theta \in [0,1]}$ .

Another rule is defined by applying the constrained equal awards rule “twice”, using the half-claims instead of the claims themselves in the formula (for the Talmud rule, there is a switch from the constrained equal awards formula to the constrained equal losses formula, instead of the same formula being used twice).

**Piniles’ rule, *Pin*** (Piniles, 1861) : For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,

$$Pin_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \frac{c_i}{2} + \min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case,  $\lambda \in \mathbb{R}_+$  is chosen so as to achieve efficiency.

We refer to the rule defined in a similar way but instead applying the constrained equal losses formula twice, as the **reverse of Piniles’ rule**. We omit the formal definition. Like the constrained equal awards rule, our final rule is motivated by egalitarian objectives, but under tighter constraints:

**Constrained egalitarian rule, *CE*** (Chun, Schummer, and Thomson, 2001) : For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,

$$CE_i(c, E) \equiv \begin{cases} \min\{\frac{c_i}{2}, \lambda\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \max\{\frac{c_i}{2}, \min\{c_i, \lambda\}\} & \text{otherwise,} \end{cases}$$

where in each case,  $\lambda \in \mathbb{R}_+$  is chosen so as to achieve efficiency.

**Corollary 2** (a) *The constrained equal awards rule Lorenz dominates all bilaterally consistent rules.*

(b) *The constrained equal losses rule is Lorenz dominated by all bilaterally consistent rules satisfying order preservation.*

(c) *Piniles' rule Lorenz dominates the Talmud, proportional, and reverse Talmud rules.*

(d) *The constrained egalitarian rule Lorenz dominates Piniles' rule.*

(e) *The Talmud, proportional, and reverse Talmud rules Lorenz dominate the reverse of Piniles' rule.*

(f) *For each pair  $\{\theta^1, \theta^2\}$  of elements of  $[0, 1]$ , if  $\theta^1 < \theta^2$ , then  $T^{\theta^2}$  Lorenz dominates  $T^{\theta^1}$ .*

(g) *For each pair  $\{\theta^1, \theta^2\}$  of elements of  $[0, 1]$ , if  $\theta^1 < \theta^2$ , then  $U^{\theta^1}$  Lorenz dominates  $U^{\theta^2}$ .*

Given  $x^1, x^2, \dots, x^k \in \mathbb{R}^N$ , let  $\mathbf{bro.seg}[x^1, x^2, \dots, x^k]$  be the broken segment connecting these points in that order.

**Proof:** The proof that all of the rules mentioned in the corollary satisfy the properties of Theorem 5 is direct and we omit it. (Some of these facts are proved by Young, 1987, and some by Thomson, 2007c.) To establish the Lorenz domination statements for two claimants, we inspect paths of awards. They are illustrated in Figures 2b,c,d for  $N \equiv \{1, 2\}$  and  $c \in \mathbb{R}_+^N$  with  $c_1 < c_2$ .

- The path of the constrained equal awards rule is  $\mathbf{bro.seg}[0, a, c]$ . It is closer to the  $45^\circ$  line than the path of any rule, *consistent* or not. This proves (a).

- The path of the constrained equal losses rule is  $\mathbf{bro.seg}[0, a, c]$ . It lies further from the  $45^\circ$  line than the path of any rule satisfying *order preservation*, *consistent* or not. This proves (b).<sup>11</sup>

For statements (c)-(e), and as each of the rules involved satisfies *order preservation of awards*, to prove that some rule  $S$  Lorenz dominates some other rule  $S'$ , it suffices to check, as we noted earlier, that the path of  $S$  lies to the southeast of the path of  $S'$ . We simply enumerate the paths, leaving to the reader the trivial verification that this geometric relation between them does hold.

- The path of Piniles' rule is  $\mathbf{bro.seg}[0, d, \frac{c}{2}, f, c]$  (Figure 2b).

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<sup>11</sup>It would not suffice to restrict attention to the class of rules satisfying *order preservation of losses*.

- The path of the Talmud rule is  $\text{bro.seg}[0, d, e, c]$  (Figure 2b).
- The path of the proportional rule is  $\text{seg}[0, c]$  (Figure 2b).
- The path of the reverse Talmud rule is  $\text{seg}[0, h, f, c]$  (Figure 2b).
- For the constrained egalitarian rule, two cases have to be distinguished. If  $2c_1 < c_2$ , its path is  $\text{bro.seg}[0, d, \frac{c}{2}, g, c]$ . This is the case represented in Figure 2b. If  $2c_1 > c_2$ , let  $m$  be the point where the horizontal segment emanating from  $\frac{c}{2}$  meets the  $45^\circ$  line and  $p$  be the point where this line meets the vertical segment passing through  $c$ . Then, the path is  $\text{bro.seg}[0, d, \frac{c}{2}, m, p, c]$ .
- The path of the reverse of Piniles' rule is  $\text{bro.seg}[0, h, \frac{c}{2}, e, c]$  (Figure 2b).

For statements (f) and (g), we note the following:

- The path of  $T^\theta$  is  $\text{bro.seg}[\theta(c_1, c_1), (\theta c_1, c_2 - (1 - \theta)c_1, c]$  (Figure 2c).
- The path of  $U^\theta$  is  $\text{bro.seg}[(0, \theta(c_2 - c_1), (c_1, (1 - \theta)c_1 + \theta c_2, c]$  (Figure 2d).

□

Statements (a) and (b) of Corollary 2 are not the interesting ones since, as we have seen, the constrained equal awards rule Lorenz dominates all rules, whether they are *consistent* or not, and the constrained equal losses rule is dominated by all rules satisfying *order preservation*, whether they are *consistent* or not, but we have included them for completeness. Bosmans and Lauwers (2007) obtain the first two statements in (c) and (d).<sup>12</sup> We noted earlier that Moreno-Tertero and Villar (2006b) provide a direct proof of (f). So, together with Corollary 1d, we now have three proofs of this statement.

Consider the requirement on a rule that when the endowment is equal to the half-sum of the claims, it should select the vector of half-claims (Chun, Schummer, and Thomson, 2001). This **midpoint property** is obviously implied by *self-duality*. In fact, it is considerably weaker as it applies, for each claims vector, to only one endowment, instead of to infinitely many pairs of endowments. Using the logic of Theorem 5, one can show that if two rules  $S$  and  $S'$  satisfy the *midpoint property* in addition to the hypotheses of this theorem, then if  $S$  Lorenz dominates  $S'$  on the subclass of two-claimant problems in which the endowment is at most as large as the half-sum of the claims, then  $S$  Lorenz dominates  $S'$  on this “lower half-domain” for

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<sup>12</sup>Bosmans and Lauwers (2007) also offer Lorenz comparisons involving the random arrival rule (O'Neill, 1982) and the adjusted proportional rule (Curiel, Maschler and Tijs, 1987). They are not covered here, since neither rule is a member of the three families we consider (except for two claimants because in that case, these rules coincide with the Talmud rule).

arbitrarily many claimants. A similar statement holds for the “higher half-domain” defined in parallel manner. As corollaries, one obtains that on the lower half-domain, Piniles’ rule and the Talmud rule Lorenz dominate the proportional rule, which itself Lorenz dominates the reverse Talmud rule. On the higher half-domain, the reverse Talmud rule and Piniles’ rule Lorenz dominate the proportional rule, which itself Lorenz dominates the Talmud rule. Moreno-Ternero and Villar (2006b) and Bosmans and Lauwers (2007) also establish Lorenz rankings on these half-domains.

## 6 Lorenz comparisons and operators

An operator on the space of rules is a mapping that associates with each rule another one. A systematic investigation of operators is undertaken by Thomson and Yeh (2001). Central are the following three. Given a rule  $S$ , the **claims truncation operator** associates with  $S$  the rule  $\mathbf{S}^t$  defined by first truncating claims at the endowment, then applying  $S$ : calling  $t_i(c, E) \equiv \min\{c_i, E\}$  and  $t(c, E) \equiv (t_i(c, E))_{i \in N}$ , we have  $\mathbf{S}^t(c, E) \equiv S(t(c, E), E)$ . The **attribution of minimal rights operator** associates with  $S$  the rule  $\mathbf{S}^m$  defined by first assigning to each claimant the difference between the endowment and the sum of the claims of the other claimants, if this difference is non-negative, and 0 otherwise, then applying  $S$ : calling  $m_i(c, E) \equiv \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\}$  and  $m(c, E) \equiv (m_i(c, E))_{i \in N}$ , we have  $\mathbf{S}^m(c, E) \equiv m(c, E) + S(c - m(c, E), E - \sum m_i(c, E))$ . The **duality operator** associates with  $S$  the rule  $\mathbf{S}^d$  that treats the endowment as  $S$  treats the deficit:  $\mathbf{S}^d(c, E) \equiv c - S(c, \sum c_i - E)$ .

Given an operator  $p$  and a rule  $S$ , let  $S^p$  be the rule obtained by subjecting  $S$  to  $p$ . Given an order  $\succeq$  on the space of rules, **operator  $p$  preserves the order** if for each pair of rules  $S, S'$  such that  $S \succeq S'$ , then  $S^p \succeq S'^p$ . It **reverses it** if for each pair of rules  $S, S'$  such that  $S \succeq S'$ , then  $S'^p \succeq S^p$ .

We have the following theorem. Part (c) is also noted by Bosmans and Lauwers (2007).

**Theorem 6** (a) *The claims truncation operator preserves the Lorenz order.*

(b) *So does the attribution of minimal rights operator for any two rules satisfying order preservation.*

(c) *The duality operator reverses it.*

**Proof:** Let  $S$  and  $S'$  be two rules such that  $S \succeq_L S'$ . Let  $N \in \mathcal{N}$  and  $(c, E) \in \mathcal{C}^N$ .

(a) We have  $S^t(c, E) = S(t(c, E), E) \succeq_L S'(t(c, E), E) = S'^t(c, E)$ .

(b) Suppose  $N \equiv \{1, \dots, n\}$  and  $c_1 \leq \dots \leq c_n$ . We need to compare  $S^m(c, E) = m(c, E) + S(c - m(c, E), \sum c_i - E)$  and  $S'^m(c, E) = m(c, E) + S'(c - m(c, E), \sum c_i - E)$ . Obviously,  $m_1(c, E) \leq \dots \leq m_n(c, E)$  and a simple calculation shows that  $c_1 - m_1(c, E) \leq \dots \leq c_n - m_n(c, E)$  (Thomson, 2007c). Thus, since  $S$  and  $S'$  satisfy *order preservation*,  $S_1(c - m(c, E), E - \sum m_i(c, E)) \leq \dots \leq S_n(c - m(c, E), E - \sum m_i(c, E))$  and  $S'_1(c - m(c, E), E - \sum m_i(c, E)) \leq \dots \leq S'_n(c - m(c, E), E - \sum m_i(c, E))$ . Altogether then,  $m_1(c, E) + S_1(c - m(c, E), E - \sum m_i(c, E)) \leq \dots \leq m_n(c, E) + S_n(c - m(c, E), E - \sum m_i(c, E))$  and similarly,  $m_1(c, E) + S'_1(c - m(c, E), E - \sum m_i(c, E)) \leq \dots \leq m_n(c, E) + S'_n(c - m(c, E), E - \sum m_i(c, E))$ . Since  $S \succeq_L S'$ , then for each  $k = 1, \dots, n$ ,  $\sum_{i=1}^k S_i(c - m(c, E), E - \sum m_i(c, E)) \geq \sum_{i=1}^k S'_i(c - m(c, E), E - \sum m_i(c, E))$ . For each  $k = 1, \dots, n$ , the  $k$ -th partial sums for  $S^m(c, E)$  and  $S'^m(c, E)$  are obtained from these partial sums by adding the same quantity  $\sum_{i=1}^k m_i(c, E)$ . Thus, the inequalities between them are preserved and  $S^m \succeq_L S'^m$ .

(c) We omit the obvious proof for the duality operator.  $\square$

Given any rule, one may also ask whether it can be Lorenz compared to the rule obtained by subjecting it to a particular operator. The following theorem provides an answer. **Claims monotonicity** of a rule says that whenever an agent's claim decreases, he should receive at most as much as he did initially. We may be interested in restricting the impact this increase has on the other agents' awards. In the aggregate, they cannot receive less, but we may require that in fact, each should receive at least as much as he did initially. The requirement is considered by Thomson (2007c) under the name of **others-oriented claims monotonicity**. Generalizing the idea to possible decreases in the claims of several agents at once, we require that each of the other agents should receive at least as much as he did initially (which implies that at least one of the agents whose claim decreases should receive at most as much as he did initially).<sup>13</sup> Let us call this requirement **strong others-oriented claims monotonicity**. Most rules satisfy it. Also, say that two properties are **dual** if whenever a rule satisfies one of them, the

<sup>13</sup>The requirement that each of the agents whose claim decreases should receive at most as much as he did initially is very strong, as it covers the case when one of these agents experiences a very small decrease in his claim. His award should be allowed to increase.

dual rule satisfies the other. The *dual of strong others-oriented monotonicity* (we do not attempt to find a name for it) says that if the claims of several agents decrease and the endowment decreases by the sum of the decreases of these claims, then each of the agents whose claims do not change receives at least as much as he did initially. Most rules satisfy this requirement too.

**Theorem 7** (a) *Let  $S$  be a rule satisfying order preservation and strong others-oriented claims monotonicity. Then,  $S$  is Lorenz dominated by the rule obtained by subjecting  $S$  to the claims truncation operator.*

(b) *Let  $S$  be a rule satisfying order preservation and the dual of strong others-oriented claims monotonicity. Then,  $S$  Lorenz dominates the rule obtained by subjecting  $S$  to the attribution of minimal rights operator.*

**Proof:** (a) Let  $S$  be rule satisfying the hypotheses. Let  $N \in \mathcal{N}$ ,  $(c, E) \in \mathcal{C}^N$ ,  $x \equiv S(c, E)$ , and  $N' \equiv \{i \in N: c_i > E\}$ . In the problem  $(t(c, E), E)$ , the claim of each agent  $i \in N'$  is truncated at  $E$ . Let  $y \equiv S(t(c, E), E)$ . By *strong others-oriented monotonicity*, for each  $i \in N \setminus N'$ ,  $y_i \geq x_i$ , which implies that there is  $i \in N'$  such that  $y_i \leq x_i$ . By *equal treatment of equals*, which *order preservation* implies, for each pair  $i, j \in N'$ ,  $y_i = y_j \equiv a$ . Thus, there is  $j \in N'$  such that for each  $k \in N'$  such that  $c_k < c_j$ —let  $N''$  denote this set, which could be empty— $y_k = a > x_k$ , and for each  $k \in N'$  such that  $c_k \geq c_j$ ,  $y_k = a \leq x_k$ . As the order of claims is not reversed by truncation, the successive sums of ordered coordinates of  $x$  and  $y$  involve the same sets of claimants. The agents listed first are the members of  $(N \setminus N') \cup N''$ , whose awards can only increase, and the agents listed next are the members of  $N' \setminus N''$ , whose awards can only decrease. Then, an obvious calculation shows that  $y \succ_L x$ .

(b) This statement follows from (a) by dual reasoning. It exploits the fact that after attribution of minimal rights, the revised claims of all agents whose minimal rights are positive are equal.  $\square$

## 7 Conclusion

A wide range of properties have been formulated to evaluate rules. The results described here provide us with another tool in this evaluation process. The Lorenz order on the space of vectors is incomplete—that is, two awards vectors for a given problem may not be comparable in this order—yet we have found a large number of pairs of rules that can be Lorenz ordered uniformly,

independently of the number of claimants, the values of their claims, and the endowment. More importantly, we hope that the techniques we developed to obtain these results throw useful light on the reasons for these rankings, and that they will help obtain others.



## References

- Alcalde, J., M. Marco, and J. A. Silva, “Bankruptcy problems and the Ibn Ezra’s proposal”, *Economic Theory* 26 (2005), 103-114.
- , ——, and ——, “The minimal overlap rule revisited”, *Social Choice and Welfare*, forthcoming, 2007.
- Aumann, R. and M. Maschler, “Game theoretic analysis of a bankruptcy problem from the Talmud”, *Journal of Economic Theory* 36 (1985), 195-213.
- Bergantiños, G. and L. Méndez-Naya, “Additivity in bankruptcy problems and in allocation problems”, *Spanish Economic Review* 3 (2001), 223-229.
- Bosmans, K. and L. Lauwers, “Lorenz comparisons of nine rules for the adjudication of conflicting claims”, mimeo, 2007.
- Chun, Y., J. Schummer, and W. Thomson, “Constrained egalitarianism: A new solution to bankruptcy problems”, *Seoul Journal of Economics* 14 (2001), 269-297.
- Chun, Y. and W. Thomson, “Convergence under replication of rules to adjudicate conflicting claims”, *Games and Economic Behavior* 50 (2005), 129-142.
- Curiel, I., M. Maschler and S. H. Tijs, “Bankruptcy games”, *Zeitschrift für Operations Research* 31 (1987), A143-A159.
- Dominguez, D. and W. Thomson, “A new solution to the problem of adjudicating conflicting claims”, *Economic Theory* 28 (2006), 283-307.
- Herrero, C. and A. Villar, “The three musketeers: four classical solutions to bankruptcy problems”, *Mathematical Social Sciences* 39 (2001), 307-328.
- Hokari, T and W. Thomson, “On properties of division rules lifted by bilateral consistency”, mimeo 2000, revised 2006, *Journal of Mathematical Economics*, forthcoming.
- Hougaard, J. L. and L. P. Østerdal, “Inequality preserving rationing”, *Economics Letters* 87 (2005), 355-360.
- Ju, B.-G. and J. Moreno-Tertero, “Progressivity, inequality reduction and merging-proofness in taxation”, mimeo, 2006.
- and ——, “On the equivalence between progressive taxation and inequality reduction”, *Social Choice and Welfare*, forthcoming, 2007.
- Moreno-Tertero, J. and A. Villar, “The Talmud rule and the securement of agents’ awards”, *Mathematical Social Sciences* 47 (2004), 245-257.

- and —, “The TAL-family of rules for bankruptcy problems”, *Social Choice and Welfare* 27 (2006a), 231-249.
- and —, “On the relative equitability of a family of taxation rules”, *Journal of Public Economic Theory* 8 (2006b), 283-291.
- O’Neill, B., “A problem of rights arbitration from the Talmud”, *Mathematical Social Sciences* 2 (1982), 345-371.
- Piniles, H. M., *Darkah shel Torah*, 1861, Forester, Vienna.
- Schummer, J. and W. Thomson, “Two derivations of the uniform rule”, *Economics Letters* 55 (1997), 333-337.
- Thomson, W., Condorcet Lecture, Fourth International Meeting of the Society for Social Choice and Welfare, Alicante, 2000.
- , “Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey”, *Mathematical Social Sciences* 45 (2003), 249-297.
- , “Two families of rules for the adjudication of conflicting claims”, mimeo, 2007a.
- , “Consistent allocation rules”, mimeo, 2007b.
- , *How to Divide When there Isn’t Enough*, mimeo, 2007c.
- Thomson, W. and C.-H. Yeh, “Operators for the adjudication of conflicting claims”, mimeo, 2001, revised 2006, *Journal of Economic Theory*, forthcoming.
- Young, P., “On dividing an amount according to individual claims or liabilities”, *Mathematics of Operations Research* 12 (1987), 398-414.