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GEOMETRODYNAMICS FOR QUARKS AND HADRONS : THE DEFINITION OF CURRENTS

F. Csikor *)

CERN - Geneva

and

G. Preparata

CERN - Geneva
and
Istituto di Fisica
Università di Bari, Italy

A B S T R A C T

We show that the naïve definition of the "direct part" of the vector and axial vector currents does not obey the necessary conservation conditions. The reasons for this failure are clarified and the precise conditions for restoring conservation are stated. Finally, an improved definition of currents is proposed and the physical interpretation is discussed.

*) On leave from Institute for Theoretical Physics, Eötvös University, Budapest, Hungary.

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1. INTRODUCTION

The recently proposed geometrodynamical approach to hadrons and their interactions has already given many encouraging results[1-3]. The main idea of the approach is that hadrons are described by the simplest wave functions (w.f.) of quark-variables (coordinates) which are non-zero only over finite four-dimensional regions (bags) of the relative coordinates. Inside the bags, quarks are assumed to be "as free as possible". This latter requirement is expressed through simple wave equations (w.eq.) which the w.f.'s must satisfy. Furthermore, assuming local interaction between quark-antiquark pairs leads to a very natural definition of the three- (or four-) hadron couplings.

As for currents, it was shown that besides the vector-dominated piece, whose structure can be expressed in terms of current-hadron and three-hadron couplings, it is necessary to introduce a direct coupling of the current to the quark coordinates. This piece of the current represents the direct interaction of the current with the quarks building up the hadrons, and its expression should be derived from the simple form of the currents of free spin ½ objects. While this is possible, and it was indeed done in Ref. 2, the resulting expressions turn out to violate the conservation conditions.

The plan of the paper is as follows. In Section 2 we show the non-conservation of the vector currents. After having analysed that the currents are reasonably defined, we discuss the precise conditions for current conservation and find that in order to ensure conservation we must fulfil certain boundary conditions in addition to gauge invariance. In Section 3 the theorem is proved that the vector current previously defined is divergenceless when sandwiched between different helicity and momentum states of the same particle. Then we discuss the physical SU(3) [SU(4)] currents and list some special cases where the simple quark currents obey the necessary constraints. From this analysis we demonstrate that the general method of restoring the divergence conditions, developed in Section 4, is applicable to our currents, thus providing an improved set of vector and axial vector currents. The physical interpretation of the newly defined currents concludes the paper. Some useful formulae for the calculation of current matrix elements and three-bag vertices are collected in the Appendix.

2. THE ORIGIN OF CURRENT NON-CONSERVATION

For simplicity we shall discuss the spin-zero quark theory first [2]. Denoting the w.f. of a meson state of four-momentum p by ψ $(p;x_1,x_2)$ we have

$$\psi(p;x_1,x_2) = e^{ip\frac{(x_1+x_2)}{2}}\,\psi(p;x) \quad ; x = x_1 - x_2 \; . \tag{2.1}$$

where x_1, x_2 are the four-coordinates of quarks. In the equal quark mass (m) case, $\psi(p,x)$ is assumed to satisfy the equation [2]

$$\Box\psi(p,x) = \left(\frac{p^2}{4} - m^2\right)\psi(p,x)\,, \tag{2.2}$$

$$\left(p\cdot\frac{\partial}{\partial x}\right)^2\psi(p,x) = -\frac{p^2\omega^2}{4}\psi(p,x)\,, \tag{2.3}$$

with the boundary conditions

$$\psi(p;x)\Big|_{-x^2+\frac{(px)^2}{p^2}=R_s^2} = 0\,, \tag{2.4}$$

$$\psi(p;x)\Big|_{\frac{(px)^2}{p^2}=R_t^2} = 0\,. \tag{2.5}$$

The solutions of Eqs. (2.2) and (2.3) subjected to the boundary conditions (2.4) and (2.5) in the rest frame $[p = (M,\vec{0})]$ take the simple form:

$$\psi(p;x) = N_{n\ell}\cos\frac{\omega}{2}t\; j_\ell\left(\frac{\beta_{n\ell}}{R_s}|\vec{x}|\right)Y_\ell^m(\theta,\varphi) \tag{2.6}$$

with the conditions

$$\omega = \frac{\pi}{2R_t} \tag{2.7}$$

$$\left(M^2+\omega^2-4m^2\right)^{1/2} = \frac{2\beta_{n\ell}}{R_s}\,, \tag{2.8}$$

where $\beta_{n\ell}$ is the n^{th} positive zero of the spherical Bessel function $j_\ell(z)$, and $N_{n\ell}$ is a normalization factor. It should be noticed that for the solution (2.6) $\partial\psi/\partial x_\mu$ does not vanish at the boundary. Another relevant observation at this point is that in terms of the quark coordinates x_1 and x_2 the wave equation (2.2) is equivalent to:

$$(-\Box_1 - \Box_2)\,\psi(p;x_1,x_2) = 2m^2\psi(p;x_1,x_2)\,. \tag{2.9}$$

The "direct" contribution to the isospin current $\hat{J}^{\beta}_{\mu}$ is given by*) [2]

$$\langle p_1,\alpha | \hat{J}^{\beta}_{\mu}(x) | p_2,\gamma \rangle = i Z_0 \, \mathrm{Tr}\left[\frac{\tau^\beta}{2} [\tau^\alpha, \tau^\gamma] \right] \cdot \int dy \, \psi_2^*(p_2;x,y) \overleftrightarrow{\frac{\partial}{\partial x_\mu}} \psi_1(p_1;x,y) \quad , \tag{2.10}$$

where α,β,γ are isospin indices and Z_0 is a constant. The region of integration is the intersection of the two bags. If we could neglect the effects of the surface of the boundary of the integration region, current conservation would be ensured by the wave equation (2.9) **). However, owing to the non-vanishing of $\partial\psi_{1,2}/\partial x_\mu$ at the boundaries this may in some cases create trouble. In fact if we take these effects into account, we find that Eq. (2.10) is in general non-conserved.

In order to demonstrate this, we introduce as integration variable $z = x - y$, and convert Eq. (2.10) into

$$\langle p_1\alpha | \hat{J}^{\beta}_{\mu}(x) | p_2,\gamma \rangle = i Z_0 \, \mathrm{Tr}\left(\frac{\tau^\beta}{2} [\tau^\alpha, \tau^\gamma] \right) e^{i(p_2-p_1)x} \int dz \, e^{i(p_1+p_2)\frac{z}{2}} \psi_2^*(p_2,z) \left(\overleftrightarrow{\frac{\partial}{\partial z_\mu}} - \frac{i}{2}(p_1+p_2)_\mu \right) \psi_1(p_1,z) . \tag{2.11}$$

The current divergence is then obviously obtained from this expression by multiplying by $i(p_2-p_1)^\mu$. In the general case evaluation of Eq. (2.11) is a hard task; however, the integration can be carried out analytically if we choose $p_1 = (M_1,\vec{0})$, $p_2 = (M_2,\vec{0})$, and we find that for $\ell_1 = \ell_2 = 0$ the current is conserved only in the case $n_1 = n_2$. We have also carried out the numerical integration in a case where $q^2 = 0$, and found a similar result.

We now want to show explicitly the source of the non-conservation for the vector current. To do this in a general way we start from a bilocal field theory Lagrangian. The connection of the geometrodynamical approach with bilocal field theories has already been pointed out in the literature [4], [5]. As a matter of fact, Eq. (2.9) may be easily derived from the Lagrangian density (putting $L_{int} = 0$):

$$L = \frac{1}{2}\left(\partial^1_\mu \psi^+ \partial^{1\mu}\psi - m^2 \psi^+\psi \right) + \frac{1}{2}\left(\partial^2_\mu \psi^+ \partial^{2\mu}\psi - m^2 \psi^+\psi \right) + L_{int} , \tag{2.12}$$

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*) This is the contribution of the quark with coordinate x_1. There is a similar contribution for the other quark (i.e. antiquark).

**) This was done in Ref. [2].

where $\psi(x_1,x_2)$ is a bilocal field. The boundary condition as well as Eq. (2.3) may be imposed as additional subsidiary conditions. The quantized field is built up as a sum of creation and annihilation operators multiplied by w.f.'s *).

The invariance of L under global gauge transformations implies the following conservation law [5]

$$\partial^1_\mu j_1^\mu + \partial^2_\mu j_2^\mu = 0 \quad , \tag{2.13}$$

where

$$\begin{aligned} j_{1\mu} &= ie\, \psi^+ \overleftrightarrow{\partial}^1_\mu \psi , \\ j_{2\mu} &= ie\, \psi^+ \overleftrightarrow{\partial}^2_\mu \psi . \end{aligned} \tag{2.14}$$

The matrix element of the operator

$$\int d^4x_2 \;\; j_1^\mu (x_1,x_2)$$

coincides with Eq. (2.10). From Eq. (2.13) we obtain

$$\int \partial^1_\mu j_1^\mu (x_1,x_2)\, d^4x_2 = - \int \partial^2_\mu j_2^\mu \, d^4x_2 = - \int_S d\sigma_2^\mu \, j_{2\mu} \tag{2.15}$$

or

$$\partial^1_\mu \int j_1^\mu \, d^4x_2 = - \int_S d\sigma_2^\mu \, j_{2\mu} - I_2 \tag{2.16}$$

where I_2 results from our having brought the derivative ∂^1_μ in front of the integral. If the boundary is described by the equation $S(x_1,x_2) = 0$ **), we have

$$I_2 = \int_S d\sigma_2 \, \frac{\partial S}{\partial x_{1\mu}} \, j_{1\mu} \; , \tag{2.17}$$

and a sufficient condition for current conservation is given by

$$\frac{\partial S}{\partial x_{1\mu}} j_{1\mu} + \frac{\partial S}{\partial x_{2\mu}} j_{2\mu} = 0 . \tag{2.18}$$

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*) The term bilocal is here employed in a purely formal way. We do not imply that commutators obey some kind of multilocality condition.

**) The boundary depends in general on the matrix element considered.

Noticing that at any particular point of S = 0 we are almost always at the boundary of only one "bag", we can immediately conclude that Eq. (2.18) implies that both the w.f.'s and their normal derivatives should vanish at the boundary. Thus we see that surface effects do in general spoil conservation.

This conclusion cannot be avoided in a spinning quark theory. This can be checked by starting from a bilocal Lagrangian [3]:

$$L = \frac{1}{2} \mathrm{Tr}\left\{ -\bar{\Psi}\overleftarrow{\partial\!\!\!/}_1 \Psi \partial\!\!\!/_2 + m\bar{\Psi}\left[i\partial\!\!\!/_1 \Psi - i\Psi\partial\!\!\!/_2 + m\Psi \right] + \mathrm{h.c} \right\} + L_{int} \quad (2.19)$$

where $\overleftarrow{\partial}_\mu$ acts on $\bar{\psi}$, and all other derivatives act on ψ. Gauge invariance provides us with the currents

$$j_{1\mu} = -ie\,\mathrm{Tr}\left\{ im\bar{\Psi}\gamma_\mu\Psi - \frac{1}{2}(\partial\!\!\!/_2\bar{\Psi})\gamma_\mu\Psi + \frac{1}{2}\bar{\Psi}\gamma_\mu(\Psi\partial\!\!\!/_2) \right\}$$

$$j_{2\mu} = -ie\,\mathrm{Tr}\left\{ im\bar{\Psi}\gamma_\mu\Psi + \frac{1}{2}(\bar{\Psi}\partial\!\!\!/_1)\gamma_\mu\Psi - \frac{1}{2}\bar{\Psi}\gamma_\mu(\partial\!\!\!/_1\Psi) \right\} . \quad (2.20)$$

The currents of the quark (antiquark) are given by the integrals of j_1^μ (j_2^μ) with respect to x_2 (x_1) and coincide with those given in Ref. [3]. Proceeding as above, we arrive at the sufficient condition for conservation (2.18), which in this case also cannot be obeyed.

Thus we see that in a bilocal theory current conservation is by no means trivial, the trouble arising from the surface terms related to the complicated boundary conditions that "quark confinement" requires. It may well turn out that a systematic treatment of surface effects would cure the above diseases by including appropriate surface terms in the expressions for the current matrix elements. In the following sections we will show that one can find an appropriate surface integral which, added to the "naïve" current, restores current conservation.

3. THE VECTOR CURRENT OF IDENTICAL PARTICLES IS CONSERVED

The conservation of the vector currents between identical particles follows from quite general assumptions. The following theorem is true:

Assuming Lorentz covariance and parity conservation, the matrix elements of the divergence of any translation invariant local vector current $\hat{J}_\mu(x)$, taken between arbitrary momentum and polarization states of the same (integer spin) particle, satisfy:

$$\langle p_1, \varepsilon_1 | \, i\partial_\mu \hat{J}^\mu(0) | p_2, \varepsilon_2 \rangle = \langle p_2, \varepsilon_2 | \, i\partial_\mu \hat{J}^\mu(0) | p_1, \varepsilon_1 \rangle \quad (3.1)$$

where p_i and ε_i denote the four momenta and polarization vectors.

The proof of the theorem is elementary. One must write down the invariant decomposition of the current-matrix element (see, for example, Ref. [6]), and notice that the form factors are invariant under the interchange $p_1 \leftrightarrow p_2$.

From the above theorem it follows that if $\langle p_1,\varepsilon_1|\hat{J}_\mu(0)|p_1,\varepsilon_2\rangle$ is symmetric under the interchange $p_1,\varepsilon_1 \leftrightarrow p_2,\varepsilon_2$, then it is also conserved. Such a symmetry property for identical particles is in fact satisfied by a current which is Hermitian and first class with respect to time reversal (i.e. has real form factors). Thus we have only to prove that our currents are Hermitian and have (for real polarization tensor) real matrix elements. For the zero quark spin current these properties can be easily verified starting from Eq. (2.11). For the spinning quark case we must write down the current matrix elements in more detail.

Starting from Eq. (2.20) and substituting the solutions of the wave equations [3] we get (assuming unequal quark masses),

$$\begin{aligned}\langle p_1,\lambda_1|\hat{J}_\mu(0)|p_2,\lambda_2\rangle = \frac{1}{2}\int d^4x\, e^{i\frac{x}{2}(p_1-p_2)}\, \mathrm{Tr}\Big\{\Big[\Big(\Big(\frac{p_1}{2}-\frac{\partial}{\partial x}\Big)^2-m_2^2\Big)\\ \Gamma_1^{\lambda_1 *}(\theta_{p_1},\varphi_{p_1})\Big(-\frac{\not p_1}{2}-i\not\partial_x+m_1\Big)\phi_1^*(x_{p_1})\Big]\gamma_\mu\Big[\Big(-\frac{\not p_2}{2}+i\not\partial_x+m_3\Big)\Gamma_2^{\lambda_2}(\theta_{p_2},\varphi_{p_2})\,\cdot\\ \cdot\Big(\frac{\not p_2}{2}+i\not\partial_x+m_2\Big)\phi_2(x_{p_2})\Big]\Big\} + \mathrm{Tr}\Big\{\Big[\Big(\frac{\not p_1}{2}-i\not\partial_x+m_2\Big)\Gamma_1^{\lambda_1 *}(\theta_{p_1},\varphi_{p_1})\\ \Big(-\frac{\not p_1}{2}-i\not\partial_x+m_1\Big)\phi_1^*(x_{p_1})\Big]\gamma_\mu\Big[\Big(-\frac{\not p_2}{2}+i\not\partial_x+m_3\Big)\Gamma_2^{\lambda_2}(\theta_{p_2},\varphi_{p_2})\\ \Big(\Big(\frac{p_2}{2}+i\frac{\partial}{\partial x}\Big)^2-m_2^2\Big)\phi_2(x_{p_2})\Big]\Big\}\end{aligned} \qquad (3.2)$$

where m_1, m_2, and m_3 are the masses of the quarks building up the mesons. In this section only the diagonal currents shall be considered, i.e. we shall take $m_1 = m_3$. For the pseudoscalar family,

$$\Gamma_\ell^m(\theta,\varphi) = \gamma_5 Y_\ell^m(\theta,\varphi) \qquad (3.3)$$

and for the vector family

$$\Gamma_J^M(\theta,\varphi) = \sum C(\ell m, 1m'|JM)\, Y_\ell^m(\theta,\varphi)\, \not\varepsilon^{m'} \qquad (3.4)$$

where C(...) are Clebsch-Gordan coefficients and $\varepsilon_\mu^m(p)$ are spin-one polarization vectors. In Eq. (3.2) the star indicates complex conjugation for the numerical coefficients only, and $x_p = \Lambda_p \cdot x$, Λ_p being the Lorentz transformation taking the four momentum p to rest. Choosing $\vec{p}_1$, $\vec{p}_2$ parallel to the z-axis and taking Λ_p as pure boosts, λ_1 and λ_2 are simply helicity indices. Furthermore, we have explicitly:

$$\phi(x_p) = N\, j_\ell\left(\frac{\beta_{n\ell}|\vec{x}_p|}{R_s}\right) \cos \frac{\pi}{2R_t} x_p^0 \; e^{i\frac{\Omega}{2} x_p^0} , \tag{3.5}$$

where N is a normalization factor, βn_ℓ is again the n^{th} positive zero of j_ℓ, and

$$\Omega = \sqrt{\frac{\beta_{n\ell}^2}{R_s^2} + m_1^2} - \sqrt{\frac{\beta_{n\ell}^2}{R_s^2} + m_2^2} . \tag{3.6}$$

The values of the space (R_s) and time (R_t) radii depend on the particular state chosen and are related to its mass through the eigenvalue equation:

$$M = -\frac{\pi}{R_t} + \sqrt{\frac{\beta_{n\ell}^2}{R_s^2} + m_1^2} + \sqrt{\frac{\beta_{n\ell}^2}{R_s^2} + m_2^2} . \tag{3.7}$$

Finally the x-integration in Eq. (3.2) spans the intersection of the two bags, whose boundaries are given by

$$\frac{(px)^2}{p^2} = R_t^2(p^2) \quad , \quad \frac{(px)^2}{p^2} - x^2 = R_s^2(p^2) . \tag{3.8}$$

The hermiticity and the reality of form factors are most conveniently proven in the Fourier-transformed space. Thus integrals of the type

$$I = \int_{\text{All space}} d^4x \, [\theta_1(x)\chi_1(x)][\theta_2(x)\chi_2(x)] \, e^{i\frac{x}{2}(p_1-p_2)} , \tag{3.9}$$

where the θ's are the characteristic functions of the two bags, by denoting the Fourier transform (FT) by

$$\begin{aligned} \tilde{\chi}_1(k) &= FT\,[\theta_1(x)\chi_1(x)] \\ \tilde{\chi}_2(k) &= FT\,[\theta_2(x)\chi_2(x)] \end{aligned} \tag{3.10}$$

take the form

$$I = \int d^4k \, \tilde{\chi}_1(k) \tilde{\chi}_2^*\left(k + \frac{p_1 - p_2}{2}\right) . \tag{3.11}$$

$\chi_{1,2}(x)$ are general linear combinations of $Y_\ell^m\phi$ and their derivatives. Owing to the non-vanishing of the derivatives of $Y_\ell^m\phi$ at the boundaries, in general

$$FT\left(\theta(x)\,\partial_\mu\partial_\nu\cdots Y_\ell^m\varphi\right) \neq (-ik_\mu)(-ik_\nu)\cdots\, FT\left(\theta(x)Y_\ell^m\varphi\right), \tag{3.12}$$

equality holding only for a single-derivative. The relevant formulae for calculating the FT's are given in the Appendix. Thus in Fourier space, Eq. (3.2) may be written down as follows:

$$\begin{aligned}
\langle p_1\lambda_1 | \hat{J}_\mu(0) | p_2\lambda_2\rangle &= \frac{1}{2}\int d^4k_1 \,. \\
&\mathrm{Tr}\Big\{\Big[\Big(\Big(\frac{p_1}{2}+k_1\Big)^2-m_2^2\Big)\Gamma_1^{\lambda_1^*}(\theta_{p_1},\varphi_{p_1})\Big(-\frac{\not{p}_1}{2}+\not{k}_1+m_1\Big)\tilde{\phi}_1^*(p_1,k_{1p_1})\Big] \\
&\quad \gamma_\mu\Big[\Big(-\frac{\not{p}_2}{2}+\not{k}_2+m_1\Big)\Gamma_2^{\lambda_2}(\theta_{p_2},\varphi_{p_2})\Big(\frac{\not{p}_2}{2}+\not{k}_2+m_2\Big)\tilde{\phi}_2(p_2,k_{2p_2})\Big]\Big\} \\
&+\mathrm{Tr}\Big\{\Big[\Big(\frac{\not{p}_1}{2}+\not{k}_1+m_2\Big)\Gamma_1^{\lambda_1^*}(\theta_{p_1},\varphi_{p_1})\Big(-\frac{\not{p}_1}{2}+\not{k}_1+m_1\Big)\tilde{\phi}_1^*(p_1,k_{1p_1})\Big] \\
&\quad \gamma_\mu\Big[\Big(-\frac{\not{p}_2}{2}+\not{k}_2+m_1\Big)\Gamma_2^{\lambda_2}(\theta_{p_2},\varphi_{p_2})\Big(\Big(\frac{p_2}{2}+k_2\Big)^2-m_2^2\Big)\tilde{\phi}_2(p_2,k_{2p_2})\Big]\Big\},
\end{aligned} \tag{3.13}$$

where our notation is such that

$$FT\left[\theta Y_\ell^m\phi\right] = Y_\ell^m\tilde{\phi}$$

and

$$k^\mu k^\nu\cdots FT\left[\theta Y_\ell^m\phi\right] = FT\left[\theta\left(i\partial^\mu i\partial^\nu\cdots Y_\ell^m\phi\right)\right]. \tag{3.14}$$

Using the hermiticity of γ-matrices it is a trivial matter to derive the hermiticity of the current taken between any state. To prove the reality of form factors, it is enough for the pseudoscalar family to prove that

$$FT\left[\theta\left(i\partial^\mu i\partial^\nu\cdots \mathrm{Re}\,Y_\ell^m\phi\right)\right]$$

and

$$FT\left[\theta\left(i\partial^\mu i\partial^\nu\cdots \mathrm{Im}\,Y_\ell^m\phi\right)\right]$$

are both given by $i^{\ell+1}$ times a real function. For the vector family it is necessary to prove that

$$FT[\theta(i\partial^\kappa i\partial^\nu \ldots \; \mathrm{Re}\, \Gamma_J^M \phi)]$$

and

$$FT[\theta(i\partial^\mu i\partial^\nu \ldots \; \mathrm{Im}\, \Gamma_J^M \phi)]$$

are i^k times a real function ($k = J + 1$ for $J = \ell \pm 1$ and $k = J$ for $J = \ell$).

The above properties are easily checked from the formulae given in the Appendix and from the symmetry properties of the Clebsch-Gordan coefficients.

Having found a case where the divergence condition is satisfied, it is natural to look for other cases where the free quark model (divergence) conditions hold. In a broken SU(3) [SU(4)] theory we require the divergence of the vector current (3.2) to be given by an expression similar to (3.2) with γ_μ replaced by $(m_1 - m_3)$. Similarly, for an axial vector current [defined by replacing γ_μ by $\gamma_5\gamma_\mu$ in Eq. (3.2)] we should get a divergence whose expression is given by changing in Eq. (3.2) γ_μ by $(m_1 + m_3)\gamma_5$.

Assuming $m_a = m_d$ we see that the isospin and hypercharge (and the other diagonal) currents are conserved when taken between equal mass particles of the same broken SU(3) [SU(4)] multiplets. Turning to the axial vector currents we could prove that the divergence equations are satisfied for two arbitrary particles of the PS family. On the other hand, for the vector meson family the divergence equation fails even for equal mass particles. Thus we have to face the general problem of finding a suitable solution of the equation,

$$\begin{aligned} q^\mu J_\mu^{\mathrm{add}} &= -S \\ S &= q^\mu J_\mu^{\mathrm{original}} + D \end{aligned} \tag{3.15}$$

where D is the matrix element of the free quark model divergence. Clearly the new current

$$J_\mu = J_\mu^{\mathrm{original}} + J_\mu^{\mathrm{add}} \tag{3.16}$$

will have the required properties.

4. THE NEW CURRENT

In this last section we shall find a general solution to Eq. (3.15). As we know, the term S is given by an integral over the surface of the intersection of the two "bags" considered; consequently, also J_μ^{add} should be given by a surface integral. An important property of S is that the form factors appearing in its invariant decomposition are entire functions of q^2.

A formal solution to

$$q^{\mu} J_{\mu}^{add} = - S \tag{4.1}$$

is given by

$$J_{\mu}^{add} = - \frac{\partial}{\partial q_{\mu}} \left(q^{\alpha} \frac{\partial}{\partial q^{\alpha}} \right)^{-1} S . \tag{4.2}$$

We shall show that Eq. (4.2) has a well-defined meaning in our case, that J_{μ}^{add} is given by surface integrals only, and that the form factors in an invariant decomposition of J_{μ}^{add} have correct analytic properties (actually they are entire functions). Thus Eq. (4.2) is an acceptable candidate for J_{μ}^{add}.

Let us write down the invariant decomposition of S as

$$S = \sum_{k,\alpha} t^{\alpha,k}_{\mu_1 \cdots \mu_k} q^{\mu_1} \cdots q^{\mu_k} S_k^{\alpha}(q^2) , \tag{4.3}$$

where $S_k^{\alpha}(q^2)$ are scalar functions and $t^{\alpha,k}_{\mu_1 \ldots \mu_k}$ are tensors containing polarization vectors and the four-vector $P_{\mu} = (p_1 + p_2)_{\mu}$. It is obvious that $t^{\alpha,k}_{\mu_1 \ldots \mu_k} q^{\mu_1} \ldots q^{\mu_k}(q^2)^{\ell}$ is an eigenfunction of the Euler operator $q^{\alpha}(\partial/\partial q_{\alpha})$ with eigenvalue $(k + 2\ell)$. From the analyticity properties of $S_k^{\alpha}(q^2)$ we can write the Taylor expansion convergent everywhere on the finite complex q^2-plane:

$$S_k^{\alpha}(q^2) = \sum_{\ell=0}^{\infty} S^{\alpha}_{k,\ell} (q^2)^{\ell} , \tag{4.4}$$

hence $[q^{\alpha}(\partial/\partial q_{\alpha})]^{-1}$ S is well defined provided we have

$$S^{\alpha}_{0,0} \neq 0 . \tag{4.5}$$

Thus we get

$$J_{\mu}^{add} = - \frac{\partial}{\partial q_{\mu}} \sum_{k \alpha \ell} t^{\alpha,k}_{\mu_1 \cdots \mu_k} q^{\mu_1} \cdots q^{\mu_k} S^{\alpha}_{k,\ell} \frac{(q^2)^{\ell}}{2\ell + k} =$$

$$= - \frac{\partial}{\partial q_{\mu}} \sum_{k \alpha} t^{\alpha,k}_{\mu_1 \cdots \mu_k} q^{\mu_1} \cdots q^{\mu_k} \frac{1}{2 (q^2)^{k/2}} \int_0^{q^2} \tau^{k/2 - 1} S_k^{\alpha}(\tau) \, d\tau . \tag{4.6}$$

S being defined by surface integrals, J_{μ} is also expressible in such a way. It should be clear that the definition of J_{μ}^{add} is necessarily not unique, due to the fact that a conserved piece may always be added to it. This fact is fully reflected by our formulae (4.3)-(4.6) when we explore different possibilities for the analytic continuation in q_μ of S. The simplest analytic continuation in q_μ which avoids the case $S^\alpha_{0,0} = 0$ is the following. The form factors S^α_k are taken to be functions of q^2 only. For the "polynomials" $t^\alpha_{\mu_1 \ldots \mu_k} q^{\mu_1} \ldots q^{\mu_k}$ the prescription is to put any vector multiplying the polarization tensors equal to P_μ. Finally, to avoid the singularity arising from $S^\alpha_{0,0} = 0$ we multiply the k = 0 invariant tensors by $Pq/(M_1^2-M_2^2)$ (which is unity for physical values of q_μ), in order to make them of order k = 1. This additional factor is also taken to be unity in the equal mass case, but in this case, from the results of Section 3, we know that the form factor multiplying it vanishes identically.

To illustrate the above construction, let us consider the matrix element of a vector current between a scalar and a vector particle. We have

$$\begin{aligned} J_\mu^{\text{original}} &= \varepsilon_\mu A + (\varepsilon.P)P_\mu B + (\varepsilon.P)q_\mu C \\ S &= (\varepsilon.P)\left[A + B(M_1^2-M_2^2) + Cq^2\right] = (\varepsilon.P)\,D \\ D &= \sum_{\ell=0}^{\infty} d_\ell\,(q^2)^\ell . \end{aligned} \tag{4.7}$$

According to the above discussion, we write

$$(\varepsilon.P) \rightarrow (\varepsilon.P)\,\frac{P.q}{M_1^2-M_2^2}$$

and get

$$J_\mu^{\text{add}} = -(\varepsilon.P)\left[\frac{P_\mu}{M_1^2-M_2^2}\sum_{\ell=0}^{\infty}\frac{d_\ell\,(q^2)^\ell}{2\ell+1} + q_\mu\sum_{\ell=1}^{\infty} d_\ell\,\frac{2\ell}{2\ell+1}\,(q^2)^{\ell-1}\right]$$

To summarize, we have given an improved definition of all the vector and axial vector currents of the geometrodynamical approach. The additional piece of the current is given by Eq. (4.6) supplemented by the simplest analytic continuation discussed above. The new currents do satisfy all the free quark model divergence conditions. They have correct analyticity properties, and the additional terms contain only <u>surface</u> charge densities, whose neglect caused the troubles of current non-conservation.

As explained in Refs. 2 and 3, the wave functions are normalized by requiring the electric charge of the particles to be given correctly. This was precisely the procedure adopted in Refs. 2 and 3, which on the basis of our results of Sections 3 and 4 is by no means modified by adding J_μ^{add} to the naïve current. The new currents are defined only up to a conserved piece; however it has been possible to define the simplest choice. It remains to be seen by comparison with experiments whether this choice is really the right one.

APPENDIX

We collect here some formulae which are useful for calculating current matrix elements as well as three-bag vertices. As explained in Section 3, we must calculate the Fourier transforms (FT) of

$$\theta(\mathrm{bag})\ \partial_{\mu_1}\cdots\partial_{\mu_n}\ \psi(p;x), \tag{A.1}$$

where the scalar wave function $\psi(p;x)$ is given by

$$\psi(p;x) = N\, Y_\ell^m(\theta_p,\varphi_p)\ j_\ell\left(\beta_{n\ell}\frac{|\vec{x}_p|}{R_s}\right)\cos\left(\frac{\pi}{2R_t}x_p^0\right)e^{i\frac{\Omega}{2}x_p^0 t}, \tag{A.2}$$

and $\theta(\mathrm{bag})$ is the characteristic function of the bag; $\psi(p;x)$ depends only on $x_p = \Lambda_p \cdot x$, and $\theta(\mathrm{bag})$ is defined in an invariant way, so that it is sufficient to calculate the FT's in the rest frame of p. To obtain the FT in any other frame we must simply perform a Lorentz transformation.

In the rest frame of p, both $\theta(\mathrm{bag})$ and $\psi(p;x)$ factorize into a time- and space-dependent part

$$\begin{gathered}\theta(\mathrm{bag}) = \theta_s\,\theta_t\\ \psi(p;x) = N\,\psi_s\,\psi_t\\ \theta_s = \theta(R_s^2 - |\vec{x}|^2)\ ,\quad \theta_t = \theta(R_t^2 - x_0^2)\\ \psi_s = N\,Y_\ell^m\, j_\ell\left(\beta_{n\ell}\frac{|\vec{x}|}{R_s}\right),\quad \psi_t = \cos\frac{\pi}{2R_t}x_0\ e^{i\frac{\Omega}{2}x_0}\ .\end{gathered} \tag{A.3}$$

The wave equations take the form:

$$\begin{gathered}\frac{\partial^2}{\partial x_0^2}\psi_t = \left[-\frac{\frac{\pi^2}{R_t^2}+\Omega^2}{4} + i\,\Omega\frac{\partial}{\partial x_0}\right]\psi_t\\ \vec{\nabla}^2\psi_s = -\frac{\beta_{n\ell}^2}{R_s^2}\psi_s\ .\end{gathered} \tag{A.4}$$

With the help of Eq. (A.4) all the higher time-derivatives may be reduced to linear combinations of ψ_t and $(\partial/\partial x_0)\phi_t$. One gets obviously

$$\mathrm{FT}(\theta_t\psi_t) = i\pi R_t^{1/2}\ \frac{\cos R_t\left(k_0+\frac{\Omega}{2}\right)}{\frac{\pi^2}{4} - R_t^2\left(k_0+\frac{\Omega}{2}\right)^2} \tag{A.6}$$

$$\mathrm{FT}\left(\theta_t\frac{\partial}{\partial x_0}\psi_t\right) = ik_0\ \mathrm{FT}(\theta_t\psi_t)\ . \tag{A.7}$$

Notice that Eq. (A.7) is valid in the usual form owing to the vanishing of ψ_t at the boundaries.

Equation (A.5) makes a similar reduction possible for higher space derivatives. However in practice we need also $FT(\theta_S\ \partial x_i\ \partial x_j\ \psi_s)$ for i and j not summed over. We have obviously

$$\partial_{x_i}\partial_{x_j}\cdots\partial_{x_n}\psi_s = \sum_{\ell' m'} a_{\ell' m'} Y_{\ell'}^{m'}(\theta,\varphi)\ j_{\ell'}\left(\beta_{n\ell}\frac{|\vec{x}|}{R_s}\right), \qquad \text{(A.8)}$$

as any derivative of ψ_s obeys Eq. (A.5) and is not singular for $|\vec{x}| = 0$. The coefficients $a_{\ell' m'}$ may be obtained by repeated use of the formulae given below:

$$\frac{\partial}{\partial x} Y_\ell^m j_\ell(|\vec{x}|) = \frac{1}{2}\left\{ j_{\ell+1}(|\vec{x}|)\left[\frac{(\ell+m+1)^{1/2}(\ell+m+2)^{1/2}}{(2\ell+1)^{1/2}(2\ell+3)^{1/2}} Y_{\ell+1}^{m+1} - \frac{(\ell-m+1)^{1/2}(\ell-m+2)^{1/2}}{(2\ell+1)^{1/2}(2\ell+3)^{1/2}} Y_{\ell+1}^{m-1}\right]\right.$$
$$\left. + j_{\ell-1}(|\vec{x}|)\left[\frac{(\ell-m)^{1/2}(\ell-m-1)^{1/2}}{(2\ell-1)^{1/2}(2\ell+1)^{1/2}} Y_{\ell-1}^{m+1} - \frac{(\ell+m)^{1/2}(\ell+m-1)^{1/2}}{(2\ell-1)^{1/2}(2\ell+1)^{1/2}} Y_{\ell-1}^{m-1}\right]\right\} \qquad \text{(A.9)}$$

$$\frac{\partial}{\partial y} Y_\ell^m j_\ell(|\vec{x}|) = \frac{1}{2i}\left\{ j_{\ell+1}(|\vec{x}|)\left[\frac{(\ell+m+1)^{1/2}(\ell+m+2)^{1/2}}{(2\ell+1)^{1/2}(2\ell+3)^{1/2}} Y_{\ell+1}^{m+1} + \frac{(\ell-m+1)^{1/2}(\ell-m+2)^{1/2}}{(2\ell+1)^{1/2}(2\ell+3)^{1/2}} Y_{\ell+1}^{m-1}\right]\right.$$
$$\left. + j_{\ell-1}(|\vec{x}|)\left[\frac{(\ell-m)^{1/2}(\ell-m-1)^{1/2}}{(2\ell-1)^{1/2}(2\ell+1)^{1/2}} Y_{\ell-1}^{m+1} + \frac{(\ell+m)^{1/2}(\ell+m-1)^{1/2}}{(2\ell-1)^{1/2}(2\ell+1)^{1/2}} Y_{\ell-1}^{m-1}\right]\right\} \qquad \text{(A.10)}$$

$$\frac{\partial}{\partial z} Y_\ell^m j_\ell(|\vec{x}|) = - j_{\ell+1}(|\vec{x}|)\frac{(\ell+m+1)^{1/2}(\ell-m+1)^{1/2}}{(2\ell+1)^{1/2}(2\ell+3)^{1/2}} Y_{\ell+1}^m + j_{\ell-1}(|\vec{x}|)\frac{(\ell-m)^{1/2}(\ell+m)^{1/2}}{(2\ell+1)^{1/2}(2\ell-1)^{1/2}} Y_{\ell-1}^m$$

$$\text{(A.11)}$$

The FT-s may then be obtained using

$$FT(\theta_s Y_{\ell'}^{m'} j_{\ell'}) = -(4\pi)(-i)^{\ell'} Y_{\ell'}^{m'}(\theta_k,\varphi_k)\ R_s^3$$
$$\frac{\beta_{n\ell}\, j_{\ell'}(R_s|\vec{k}|)\, j_{\ell'-1}(\beta_{n\ell}) - R_s|\vec{k}|\, j_{\ell'-1}(R_s|\vec{k}|)\, j_{\ell'}(\beta_{n\ell})}{R_s^2|\vec{k}|^2 - \beta_{n\ell}^2} . \qquad \text{(A.12)}$$

For $\ell = \ell'$, Eq. (A.12) simplifies as $j_\ell(\beta_{n\ell}) = 0$. In practice, only the first and second derivatives are necessary. Owing to the vanishing of ψ_s at the boundary, we have the usual result:

$$FT\left(\theta_s \, i\partial_i \psi_s\right) = -k_i \, FT\left(\theta_s \psi_s\right) , \tag{A.13}$$

but this is no longer true for second derivatives

$$FT\left(\theta_s \, i\partial_i \, i\partial_j \psi_s\right) \neq k_i \, k_j \, FT\left(\theta_s \psi_s\right) . \tag{A.14}$$

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