The Annals of Probability
2001, Vol. 29, No. 2, 902–937

LOSS NETWORK REPRESENTATION OF PEIERLS CONTOURS[1]

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We present a probabilistic approach for the study of systems with exclusions in the regime traditionally studied via cluster-expansion methods. In this paper we focus on its application for the gases of Peierls contours found in the study of the Ising model at low temperatures, but most of the results are general. We realize the equilibrium measure as the invariant measure of a loss network process whose existence is ensured by a subcriticality condition of a dominant branching process. In this regime the approach yields, besides existence and uniqueness of the measure, properties such as exponential space convergence and mixing, and a central limit theorem. The loss network converges exponentially fast to the equilibrium measure, without metastable traps. This convergence is faster at low temperatures, where it leads to the proof of an asymptotic Poisson distribution of contours. Our results on the mixing properties of the measure are comparable to those obtained with "duplicated-variables expansion," used to treat systems with disorder and coupled map lattices. It works in a larger region of validity than usual cluster-expansion formalisms, and it is not tied to the analyticity of the pressure. In fact, it does not lead to any kind of expansion for the latter, and the properties of the equilibrium measure are obtained without resorting to combinatorial or complex analysis techniques.

1. Introduction. In this paper we develop a probabilistic approach to the study of the equilibrium measure of systems with exclusions—such as hard-core gases, contours, polymers or animals—in the low-density or extreme-temperature regime. This regime has traditionally been studied via cluster-expansion methods, which relied either on sophisticated combinatorial estimations [Malyshev (1980), Seiler (1982) and Brydges (1984)] or on astute inductive hypotheses plus complex analysis [Kotecky and Preiss (1986) and Dobrushin (1996a, 1996b)].

In contrast, we realize the equilibrium measure as the invariant measure of a loss network process that can be studied using standard tools and notions from probabilistic models and processes. Loss networks, first introduced by Erlang in 1917, encompass a rather general family of processes as discussed in Kelly (1991) and references therein. Technically, we work with the so-called

Received July 1998; revised July 1999.

[1]Partially supported by FAPESP 95/0790-1 (Projeto Temático "Fenômenos críticos e processos evolutivos e sistemas em equilíbrio"), CNPq and FINEP (Núcleo de Excelência "Fenômenos críticos em probabilidade e processos estocásticos" PRONEX-177/96). The final version was written while P.A.F. was visiting professor at Laboratoire de Probabilités de l'Université de Paris VI.

AMS 2000 *subject classifications*. Primary 60K35, 82B, 82C.

Key words and phrases. Peierls contours, animal models, loss networks, Ising model, oriented percolation, central limit theorem, Poisson approximation.

fixed-routing loss networks. We build on ideas of Ferrari and Garcia (1998) to show (Section 3.4) that the existence of the loss network can be related to the absence of percolation in an oriented percolation process. This condition also yields other properties of the process and its invariant measure, such as uniqueness, convergence through sequences of finite volumes and mixing properties (Theorem 4.1). More precise results can be obtained by resorting to a dominant multitype branching process (Section 5). Roughly speaking, the mean number of branches of this process becomes the driving parameter: Subcriticality is a sufficient condition for the construction to work. Time and space rates of convergence and mixing rates are explicitly obtained in terms of this parameter (Theorem 2.2).

The approach of this paper was already exploited in Fernández, Ferrari and Garcia (1998). Here we refine and complete the theory presented there, extending their region of validity and including proofs of exponential mixing and convergence to a Poisson process.

For concreteness, we analyze in this paper the gases of Peierls contours used, for instance, for low-temperature studies of the Ising model [Peierls (1936), Dobrushin (1965) and Griffiths (1964)]. The subcriticality condition of the corresponding branching process is

$$\sup_{\gamma} \frac{1}{|\gamma|} \sum_{\theta:\, \theta \not\sim \gamma} |\theta|\, w(\theta) \;<\; 1\,, \tag{1.1}$$

where $|\gamma|$ indicates the length (perimeter, surface area) of a contour γ, $w(\gamma)$ is its weight and "$\sim$" stands for the volume-exclusion (=nonintersection) condition.

Condition (1.1) is considerably weaker than those obtained by the most developed cluster-expansion approaches [Kotecký and Preiss (1986) and Dobrushin (1996a, 1996b)] in which the factor $|\theta|$ on the right-hand side is replaced by a function that grows exponentially with $|\theta|$. The weakening of the condition has a price: Unlike previous approaches, ours does not yield analyticity properties of the expectations. Condition (1.1) is similar to conditions obtained in the study of systems for which nonanalyticity is known [von Dreifus, Klein and Perez (1995)] or suspected [Bricmont and Kupiainen (1996, 1997)]. Our results on space convergence and mixing rates can basically be obtained with the "duplicate-system" expansions of Bricmont and Kupiainen (1996, 1997).

The novelty of our approach lies in the following features: First, it offers a completely different framework for the study of hard-core measures, based on well-known stochastic processes. This can conceivably lead to new insights and stronger results. In particular, condition (1.1) can potentially be weakened via subcriticality estimates obtained directly for the associated oriented percolation process, without resorting to a dominating branching process. Similar improvements have been done for the contact process and oriented percolation; see Liggett (1995), for instance. Second, our construction involves a stochastic process (the loss network) that converges exponentially fast to the soughtafter measure. This process is, in principle, easy to simulate and its potential as a computational tool deserves to be explored [Fernández, Ferrari and Garcia (1999)]. Its rate of convergence to the equilibrium measure increases as the temperature decreases and, unlike spin-flip dynamics, it does not present metastable traps (any contour lives an exponential time of mean 1). Finally, the construction permits a rather straightforward proof of the asymptotic Poisson distribution of contours at low temperature (Theorem 2.3). It is not obvious to us how such a result can be obtained through the standard statistical mechanical expansions.

The present approach does not lead to a series expansion for the pressure (or free-energy density). In particular, it does not yield the "surface-tension bounds" that play such a crucial role in some applications of cluster expansions [see, e.g., Zahradníck (1984) and Borgs and Imbrie (1989)]. In fact, the approach is designed so as to *bypass* expansions of this type. It is a probabilistic approach designed to answer probabilistic questions—existence of expectations, properties of correlation functions—in a direct way, without combinatorial or complex-analysis techniques. It is not an alternative to cluster expansions: It has a different regime of validity and different aims.

While the ideas behind our results are natural and simple, their formalization requires many intermediate technical results that may obscure the development of our theory. Let us, therefore, present a sort of "road map" of the paper to guide the reader. The main results are presented, in a self-contained manner, in Section 2. The actual construction of the loss network is the subject of Section 3. We start with a reference "free" process of Poissonian births and exponentially distributed deaths, with respect to which the loss network is absolutely continuous. The novelty of our approach resides in the fact that, rather than independently generating birth times and death times, the lifetimes are associated to each birth time as a mark. Hence, unlike comparable constructions, each death time has an associated birthtime and defines a space–time "cylinder" representing the presence of a loss event (contour). This permits us to comb the process either backward or forward in time with equal ease. The "time backward" point of view leads to the notion of "backward oriented percolation," which will be our main conceptual and practical tool. The idea is to construct the loss network by erasing from the free process those cylinders that conflict with preexisting ones. This can only be done if the set of preexisting cylinders (the "clan of ancestors") is finite with probability 1. This is precisely the condition of absence of backward oriented percolation. The main technical result of Section 3 is the proof that this construction yields precisely the loss network. (This is contained in the proof of Theorem 3.1.) It is apparent that the same construction works for any other process absolutely continuous with respect to a Poisson birth-and-death process, for instance, point or Boolean processes [Baddeley and van Lieshout (1995) and Kendall (1997, 1998)].

Once the process is so constructed, the course of action is clear. First, we relate time and space mixing properties with the time and space size of the percolation clan. This is done in Section 4 (Theorem 4.1). The only slightly involved part of this section is related to the proof of the mixing properties. Indeed, we resort to a—standard but unusual—coupling between clans, together with a continuous-time construction (Section 4.5), to improve a little over results proved previously by the method of "duplicated variables" [von Dreifus, Klein and Perez (1995) and Bricmont and Kupiainen (1996)].

In order to present quantitative estimates in terms of the parameters of the problem, we follow the known technique [used, for instance, by Hall (1985)] of bounding percolation probabilities via a branching process. In particular, the size of the percolation clan is bounded by the number of branches. This is done in Section 5, where the only (very simple!) algebraic calculations of the paper are presented [(5.14), (5.37) and (5.40)]. As expected, subcriticality of the branching process implies lack of percolation and exponentially damped sizes of the percolation clan. The main estimations are contained in Theorem 5.1. Once again, we present a slightly unusual continuous-time construction (Section 5.2) to improve one of the estimations, namely, the time length of a clan [part (ii) of Theorem 5.1]. Readers can opt instead for the more direct, but slightly weaker, estimate presented in the first remark following the theorem.

The results of Theorem 2.2 are a direct consequence of the estimates of Section 5 applied to the percolation expressions of Section 4, as explained in Sections 6 and 7. The proof of the Poisson approximation of the loss network (Theorem 2.3) requires some further considerations presented in the final section (Section 8).

Our work was motivated in part by the posthumous review of Dobrushin (1996b), where he complained that while "perturbation methods are intensively used by mathematical physicists, they are not so popular as correlation inequalities among the probabilists." He called for a "systematical exposition oriented to the mathematicians." In this paper we follow his call: We concentrate on probabilistic issues and exploit probabilistic arguments.

2. Definitions and results.

2.1. *The contour model.* We consider the d-dimensional lattice $\mathbb{Z}^d$ and call *plaquettes* the $(d-1)$-dimensional unit cubes centered at points of $\mathbb{Z}^d$. We identify each plaquette with its center. A set of plaquettes is called a *surface*. Two plaquettes are *adjacent* if they share a $(d-2)$-dimensional face. This defines a notion of connection: A set γ of plaquettes forms a *connected* (hyper) surface if for every two plaquettes x, y there is a sequence of pairwise adjacent plaquettes starting at x and ending at y. A surface is *closed* if every $(d-2)$-dimensional face is shared by an even number of plaquettes in the surface. A *contour* γ is a connected and closed family of plaquettes. Two contours γ and θ are incompatible if they share some $(d-2)$-dimensional face. In this case we say that $\gamma \not\sim \theta$.

Denote by $\mathbf{G}^\Lambda$ the set of all contours in the volume Λ. The subset $\chi^\Lambda \subset \mathbb{N}^{\mathbf{G}^\Lambda}$ of *compatible* configurations is defined as

$$\chi^\Lambda = \big\{\eta \in \{0,1\}^{\mathbf{G}^\Lambda};\ \eta(\gamma)\,\eta(\theta) = 0 \text{ if } \gamma \not\sim \theta\big\}; \tag{2.1}$$

that is, a configuration of contours is compatible if it does not contain two incompatible contours. We denote $\chi = \chi^{\mathbb{Z}^d}$ and $\mathbf{G} = \mathbf{G}^{\mathbb{Z}^d}$. As usual, we endowed χ with the product topology.

For each fixed $\beta \in \mathbb{R}^+$, a parameter usually called the inverse temperature, and for each finite Λ, define the measure μ^Λ on χ^Λ by

$$\mu^\Lambda(\eta) = \frac{\exp(-\beta \sum_{\gamma:\,\eta(\gamma)=1} |\gamma|)}{Z^\Lambda}, \tag{2.2}$$

where $|\gamma|$ is the number of plaquettes in γ and Z^Λ is a renormalization constant making μ^Λ a probability.

2.2. *The loss network.* We introduce a Markov process called the (fixed routing) *loss network* in the set of compatible contours. This process was introduced by Erlang in 1917 [see Brockmeyer, Halstrøm and Jensen (1948), page 139]. An account of its properties can be found in Kelly (1991). In the traditional interpretation a contour γ represents the route taken up by a call. The plaquettes encompassing γ are the circuits held by the call. For a finite or infinite set $\Lambda \subset \mathbb{Z}^d$ and f a real continuous function on χ^Λ, the generator of the process is defined by

$$\begin{aligned} A^\Lambda f(\eta) = &\sum_{\gamma\in\mathbf{G}^\Lambda} e^{-\beta|\gamma|}\,\mathbf{1}\{\eta+\delta_\gamma \in \chi^\Lambda\}\,[f(\eta+\delta_\gamma) - f(\eta)] \\ &+ \sum_{\gamma\in\mathbf{G}^\Lambda} \eta(\gamma)\,[f(\eta-\delta_\gamma) - f(\eta)], \end{aligned} \tag{2.3}$$

where $\delta_\gamma(\theta) = \mathbf{1}\{\theta = \gamma\}$ and the sum of configurations is defined pointwisely in $\mathbb{N}^{\mathbf{G}^\Lambda}$: $(\eta+\xi)(\gamma) = \eta(\gamma)+\xi(\gamma)$. In words, each contour γ attempts to appear at rate $e^{-\beta|\gamma|}$ but it does so only if it is compatible with all present contours. Present contours disappear at rate 1. Our first result is the following sufficient condition for the existence of a process with generator A^Λ for any (infinite) Λ.

THEOREM 2.1. *If*

$$\alpha(\beta) = \sup_\gamma \frac{1}{|\gamma|} \sum_{\theta:\,\theta\not\sim\gamma} |\theta|\, e^{-\beta|\theta|} < \infty, \tag{2.4}$$

then for any (infinite) Λ the Markov process with generator A^Λ exists and admits at least one invariant measure.

This theorem is proven in Section 6. We denote by $\eta_t^{\Lambda,\zeta}$ the corresponding process in Λ with initial configuration ζ. We omit the volume superindex when $\Lambda = \mathbb{Z}^d$:

$$\eta_t^\zeta = \eta_t^{\mathbb{Z}^d,\zeta}, \qquad A = A^{\mathbb{Z}^d}. \tag{2.5}$$

2.3. *Results on the invariant measure.* We say that f has *support* in $\Upsilon \subset \mathbb{Z}^d$ if f depends only on contours intersecting Υ (not necessarily contained in Υ). Let $|\mathrm{Supp}(f)| = \min\{|\Upsilon| : f$ has support in $\Upsilon\}$. When we write $\mathrm{Supp}(f)$ we mean some Υ such that $|\Upsilon| = |\mathrm{Supp}(f)|$ and f has support in Υ. For instance, if $f(\eta) = \eta(\gamma)$, $\mathrm{Supp}(f)$ may be set as $\{x\}$ for any $x \in \gamma$. The following results work for any such choice, so one can take the most favorable one in each case. Let

$$\beta^* \text{ solution of } \alpha(\beta) = 1, \tag{2.6}$$

$$\alpha_0(\beta) = \sum_{\gamma \ni 0} |\gamma|\, e^{-\beta|\gamma|}. \tag{2.7}$$

Let $|x|$ be some fixed norm, for instance, the one given by the Manhattan distance ($|x| = \sum_{i=1}^{d} |x_i|$, the sum of the coordinate lengths). Let the corresponding distance between two subsets of $\mathbb{Z}^d$ be

$$d(\Lambda, \Upsilon) = \min\{|x - y| : x \in \Lambda, y \in \Upsilon\}. \tag{2.8}$$

THEOREM 2.2. *If $\beta > \beta^*$ [i.e., $\alpha(\beta) < 1$], then the following statements hold*:

1. *Uniqueness. For any $\Lambda \subset \mathbb{Z}^d$, there is a unique process η_t^Λ with generator A^Λ. The process has a unique invariant measure denoted by μ^Λ. For Λ finite, this measure is precisely* (2.2). *For $\Lambda = \mathbb{Z}^d$, we denote $\eta_t = \eta_t^{\mathbb{Z}^d}$, $\mu = \mu^{\mathbb{Z}^d}$.*
2. *Exponential time convergence. For any $\Lambda \subset \mathbb{Z}^d$ and for measurable f on χ^Λ,*

$$\sup_{\zeta \in \chi^\Lambda} |\mu^\Lambda f - \mathbb{E} f(\eta_t^{\Lambda, \zeta})| \le 2\|f\|_\infty |\mathrm{Supp}(f)| \frac{\alpha_0}{\rho} e^{-\rho t}, \tag{2.9}$$

   where $\rho = (1 - \alpha)/(2 - \alpha)$.
3. *Exponential space convergence. Let Λ be a* (*finite or infinite*) *subset of $\mathbb{Z}^d$ and let f be a measurable function depending on contours contained in Λ. Then*

$$|\mu f - \mu^\Lambda f| \le 2\|f\|_\infty \alpha_0 M_2 \sum_{x \in \mathrm{Supp}(f)} \exp(-M_3 d(\{x\}, \Lambda^c)), \tag{2.10}$$

   where $M_2 = (1 - \alpha(\tilde\beta))^{-1}$ and $M_3 = (\beta - \tilde\beta)$, for any $\tilde\beta \in (\beta^*, \beta)$.
4. *Exponential mixing. For measurable functions f and g depending on contours contained in an arbitrary set $\Lambda \subset \mathbb{Z}^d$*:

$$\begin{aligned} &|\mu^\Lambda(fg) - \mu^\Lambda f \mu^\Lambda g| \\ &\quad \le 2\|f\|_\infty \|g\|_\infty (M_2)^2 \sum_{\substack{x \in \mathrm{Supp}(f), \\ y \in \mathrm{Supp}(g)}} |x - y| \exp(-M_3|x - y|), \end{aligned} \tag{2.11}$$

   where M_2 and M_3 are the same as in (2.10).

5. *Central limit theorem. Let f be a measurable function on χ with finite support such that $\mu f = 0$ and $\mu(|f|^{2+\delta}) < \infty$ for some $\delta > 0$. Let τ_x be the translation by x and assume $D = \sum_x \mu(f\tau_x f) > 0$. Then $D < \infty$ and*

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{x\in\Lambda} \tau_x f \underset{\Lambda\to\mathbb{Z}^d}{\Longrightarrow} Normal(0, D), \tag{2.12}$$

where the double arrow means convergence in distribution.

This theorem strengthens the results R1–R5 of Fernández, Ferrari and Garcia (1998). In that paper β^* was replaced by a value β_M defined as the solution of $\alpha_0(\beta_M) = 1/(d-1)$. This value is strictly larger than β^*. Item 5 generalizes (the central limit) Theorem 7.4 of Dobrushin (1996b), where only functions depending on a finite number of contours are considered. Theorem 2.10 will be proven in Section 7.

Finally, we prove a Poisson approximation. Consider the equivalence relation induced by the translation of contours. Let $\widetilde{\mathbf{G}}_j$ be the set formed by one representative containing the origin from each class of equivalence of contours with length j. For each Borel set $V \subset \mathbb{R}^d$ and $a \in \mathbb{R}$, let

$$V \cdot a = \left\{ x \in \mathbb{Z}^d : \left[\frac{x - 1/2}{a}, \frac{x + 1/2}{a}\right]^d \subset V \right\}. \tag{2.13}$$

Fix a contour length $j > 0$. For each $\gamma \in \widetilde{\mathbf{G}}_j$, let $M_{\gamma,\beta}$ be defined by

$$M_{\gamma,\beta}(V) = \sum_{x\in V\cdot e^{\beta|\gamma|/d}} \eta^\beta(\tau_x \gamma), \tag{2.14}$$

where η^β is distributed according to the invariant measure for the given β. Let $(M_{\gamma,\infty},\ \gamma \in \widetilde{\mathbf{G}}_j)$ be a family of independent unit Poisson processes in $\mathbb{R}^d$.

THEOREM 2.3. *For each contour length j, it is possible to jointly construct the d-dimensional processes $\{M_{\gamma,\beta} : \beta^* < \beta \le \infty,\ \gamma \in \widetilde{\mathbf{G}}_j\}$ in such a way that, for all regions V equal to a product of intervals,*

$$\begin{aligned} \mathbb{P}\big(M_{\gamma,\beta}(V) \neq M_{\gamma,\infty}(V)\big) &\le c(|\gamma|,\ V)\big(\alpha(\beta) + \exp(-\beta|\gamma|/d)\big) \\ &\sim \exp\big(-\beta \min\{2d,\ |\gamma|/d\}\big), \end{aligned} \tag{2.15}$$

where $c(|\gamma|, V)$ is a computable constant. As a consequence, $\{M_{\gamma,\beta} : \gamma \in \widetilde{\mathbf{G}}_j\}$ converges in distribution to a family consisting of $|\widetilde{\mathbf{G}}_j|$ independent Poisson processes with mean $e^{-\beta j}$.

Theorem 2.3 is proven in Section 8.

3. Graphical representation of loss networks. We construct the loss network as a function of stationary marked Poisson processes (à la Harris), each of which indicates the attempted birth times of a contour. A lifetime is associated to each attempted birth. The triple (contour, attempted birth, lifetime) is called a *cylinder*. The loss network is constructed by erasing cylinders which at birth violate the exclusion condition. The crucial point in this construction is the association of the lifetime to the birth time. This allows us to study the process backward in time by studying a Markovian oriented percolation process of cylinders. In contrast, the standard construction uses independent Poisson processes for the birth times and death times, respectively. In this case the backward construction looks hard.

3.1. *Marked Poisson processes.* To each contour $\gamma \in \mathbf{G}$ we associate an independent (of everything) marked Poisson process N_γ on $\mathbb{R}$ with rate $e^{-\beta|\gamma|}$. We call $T_k(\gamma) \in \mathbb{R}$, $\gamma \in \mathbf{G}$, the ordered time events of N_γ with the convention that $T_0(\gamma) < 0 < T_1(\gamma)$. For each occurrence time $T_i(\gamma)$ of the process N_γ, we choose an independent mark $S_i(\gamma)$ exponentially distributed with mean 1. At the Poisson time event $T_i(\gamma)$, a contour γ appears and it lasts $S_i(\gamma)$ time units.

The random family $\mathbf{C} = \big\{\{(\gamma, T_i(\gamma), S_i(\gamma)) : i \in \mathbb{Z}\} : \gamma \in \mathbf{G}\big\}$ consists of independent marked Poisson processes. A marked point $(\gamma, T_k(\gamma), S_k(\gamma)) \in \mathbf{C}$ is identified with $\gamma \times [T_k(\gamma), T_k(\gamma) + S_k(\gamma)]$, the *cylinder* with *basis* γ, *birth time* $T_k(\gamma)$ and *lifetime* $S_k(\gamma)$. The *life* of the cylinder is the time interval $[T_k(\gamma), T_k(\gamma) + S_k(\gamma)]$. For a generic cylinder $C = (\gamma, t, s)$, we use the notation

$$\begin{aligned} &\operatorname{Basis}(C) = \gamma, \qquad \operatorname{Birth}(C) = t,\\ &\qquad \operatorname{Death}(C) = t + s, \qquad \operatorname{Life}(C) = [t,\ t + s]. \end{aligned} \tag{3.1}$$

We define incompatibility between cylinders C and C' by

$$\begin{aligned} C' \not\sim C \quad &\text{if and only if } \operatorname{Basis}(C) \not\sim \operatorname{Basis}(C')\\ &\text{and } \operatorname{Life}(C) \cap \operatorname{Life}(C') \neq \varnothing; \end{aligned} \tag{3.2}$$

otherwise, $C' \sim C$ (compatible). We say that two sets of cylinders $\mathbf{A}$ and $\mathbf{A}'$ are *incompatible* if there is a cylinder in $\mathbf{A}$ incompatible with a cylinder in $\mathbf{A}'$:

$$\mathbf{A} \not\sim \mathbf{A}' \quad \text{if and only if } C \not\sim C' \text{ for some } C \in \mathbf{A} \text{ and } C' \in \mathbf{A}'. \tag{3.3}$$

Let $\mathbf{S} := (S_i^0(\theta) : \theta \in \mathbf{G},\ i \geq 1)$ be a (countable) family of iid exponential times of mean 1 independent of $\mathbf{C}$. These are the lifetimes that, when necessary, will be associated to the contours of the initial configuration. Indeed, we identify $\mathbf{S}$ with the set of cylinders $\{(\theta, 0, S_i^0(\theta)) : \theta \in \mathbf{G}, i \geq 1\}$. For $\xi \in \mathbb{N}^{\mathbf{G}}$, let

$$\mathbf{S}(\xi) = \bigcup_{\theta \in \mathbf{G}} \bigcup_{i=1}^{\xi(\theta)} \big\{\big(\theta,\ 0,\ S_i^0(\theta)\big)\big\}, \tag{3.4}$$

the family of cylinders associated to the initial configuration ξ, all with birth time 0. Notice that ξ may have more than one cylinder with the same basis. For $s < t$, define

$$\mathbf{C}[s,\ t] := \{C \in \mathbf{C} :\ \text{Birth}(C) \in [s,\ t]\}, \tag{3.5}$$

the set of cylinders born in the interval $[s,\ t]$.

REMARK 1. In Sections 2 to 7 of this paper we will work with the probability space given by the product of the spaces generated by $\mathbf{C}$ and $\mathbf{S}$. We call it $(\Omega, \mathscr{F}, \mathbb{P})$. We write $\mathbb{E}$ for the respective expectation. In Section 4.5.2 we use the direct product of this space by itself, while in Section 5 we need to consider countable products of this space. We use the same notation $\mathbb{P}$ and $\mathbb{E}$ for the corresponding probability and expectation in these enlarged spaces. In Section 8 we use a continuous-space Poisson process on which we simultaneously construct the process on $\mathbb{Z}^d$ for all values of β.

3.2. *The free network.* For $\xi \in \mathbb{N}^{\mathbf{G}}$, define

$$\xi_t^\xi(\gamma) = \sum_{C \in \mathbf{C}[0,t] \cup \mathbf{S}(\xi)} \mathbf{1}\{\text{Basis}(C) = \gamma,\ \text{Life}(C) \ni t\}. \tag{3.6}$$

The preceding process, called the *free network*, is a product of independent birth-and-death processes on $\mathbb{N}^{\mathbf{G}}$ with initial configuration ξ whose generator is given by

$$A^0 f(\xi) = \sum_{\gamma \in \mathbf{G}} e^{-\beta|\gamma|}[f(\xi + \delta_\gamma) - f(\xi)] + \sum_{\gamma \in \mathbf{G}} \xi(\gamma)[f(\xi - \delta_\gamma) - f(\xi)]. \tag{3.7}$$

The invariant (and reversible) measure for this process is the product measure μ^0 on $\mathbb{N}^{\mathbf{G}}$ with Poisson marginals

$$\mu^0\{\xi(\gamma) = k\} = \frac{(e^{-\beta|\gamma|})^k}{k!} \exp(e^{-\beta|\gamma|}). \tag{3.8}$$

In terms of loss networks, ξ_t^ξ is the process for which all the calls are accepted; that is, $\xi_t^\xi(\gamma)$ is the number of calls on route γ at time t when there is no restriction on the number of calls a circuit can accept and the initial configuration of calls is ξ.

3.3. *Finite-volume construction of a loss network.* In the construction of a loss network in a finite volume Λ with an initial condition $\zeta \in \chi^\Lambda$, we use only the finite set of Poisson processes $(N_\gamma : \gamma \subset \Lambda)$ and the finite family of initial lifetimes $(S_1^0(\theta) : \theta \subset \Lambda)$. Let $\mathbf{C}^\Lambda = \{C \in \mathbf{C} : \text{Basis}(C) \subset \Lambda\}$ and $\mathbf{S}(\zeta)$, defined as in (3.4), be such that all its cylinders are mutually compatible. We realize the dynamics $\eta_t^{\Lambda,\zeta}$ as a (deterministic) function of $\mathbf{C}^\Lambda$ and $\mathbf{S}(\zeta)$.

We construct inductively $\mathbf{K}_\zeta^\Lambda[0, t]$, the set of *kept* cylinders at time t. The complementary set corresponds to *erased* cylinders. At time 0 we include all cylinders of $\mathbf{S}(\zeta)$ in $\mathbf{K}_\zeta^\Lambda[0, t]$. Then, we move forward in time and consider the first Poisson mark: The corresponding cylinder is erased if it is incompatible with any of the cylinders already in $\mathbf{K}_\zeta^\Lambda[0, t]$; otherwise, it is kept. This procedure is successively performed mark by mark until all cylinders born before t are considered. Define $\eta_t^{\Lambda,\zeta} \in \chi^\Lambda$ as

$$\begin{aligned}\eta_t^{\Lambda,\zeta}(\gamma) &= \sum_{C\in\mathbf{K}_\zeta^\Lambda[0,t]} \mathbf{1}\{\mathrm{Basis}(C) = \gamma, \mathrm{Life}(C) \ni t\} \\ &= \mathbf{1}\Big\{\gamma \in \{\mathrm{Basis}(C) : C \in \mathbf{K}_\zeta^\Lambda[0, t],\ \mathrm{Life}(C) \ni t\}\Big\};\end{aligned} \tag{3.9}$$

that is, $\eta_t^{\Lambda,\zeta}$ signals all contours which are bases of a kept cylinder that is alive at time t. We show in Section 3.4 that $\eta_t^{\Lambda,\zeta}$ has generator A^Λ defined as in (2.3) restricting the sums to the set of contours contained in Λ. It is immediate that μ^Λ defined in (2.2) is reversible for this process. Since we are dealing with an irreducible Markov process in a finite state space, $\eta_t^{\Lambda,\zeta}$ converges in distribution to μ^Λ for any initial configuration ζ. This, in particular, implies that μ^Λ is the unique invariant measure for this process. Later in the paper we determine the speed of convergence.

Using the same $\mathbf{C}$ and $\mathbf{S}$ in the construction of $\eta_t^{\Lambda,\zeta}$ and ξ_t^ξ, we have that if $\zeta(\gamma) \le \xi(\gamma)$ for all $\gamma \subset \Lambda$, then

$$\eta_t^{\Lambda,\zeta}(\gamma) \le \xi_t^\xi(\gamma) \quad \text{for all } \gamma \subset \Lambda, \tag{3.10}$$

because in the free network ξ_t^ξ all cylinders are kept.

Since Λ is finite, there exists a sequence of random times $t_i = t_i(\mathbf{C}^\Lambda)$ with $t_i \to \pm\infty$ as $i \to \pm\infty$ such that $\xi_{t_i}(\gamma) = 0$ for all $\gamma \in \Lambda$. We can, in particular, consider t_i as the entrance times of ξ_t in the set $\{\xi : \xi(\gamma) = 0 \text{ for all } \gamma \in \Lambda\}$. Since this process has a unique invariant measure which gives positive probability to this set, (t_i) is a stationary renewal process with interrenewal time with finite mean. We extend the construction of a set of kept cylinders to $t \in \mathbb{R}$, forgetting the set $\mathbf{S}(\zeta)$, by doing the preceding procedure in each time interval $[t_i, t_{i+1}]$ with the cylinders of $\mathbf{C}[t_i, t_{i+1}]$. This can be done because no cylinder intersects $\{t_i : i \in \mathbb{Z}\}$. Let us denote by $\mathbf{K}^\Lambda$ the resulting set of kept cylinders and η_t^Λ its projection in the sense of (3.9). By construction, $\mathbf{K}^\Lambda$ has a time translation-invariant distribution. The process η_t^Λ has generator A^Λ and distribution independent of t, hence given by μ^Λ. This implies that, for any $f : \{0,1\}^{\mathbf{G}^\Lambda} \to \mathbb{R}$ and any $t \in \mathbb{R}$,

$$\mu^\Lambda f = \mathbb{E} f(\eta_t^\Lambda). \tag{3.11}$$

Since $\eta_t^\Lambda(\gamma) \le \xi_t(\gamma)$ for all $\gamma \in \Lambda$ and $\xi_t(\gamma)$ has Poisson distribution with mean $e^{-\beta|\gamma|}$, we have, taking $f(\eta) = \eta(\gamma)$ in (3.11), that

$$\mu^\Lambda\big\{\eta : \eta(\gamma) = 1\big\} \le e^{-\beta|\gamma|}. \tag{3.12}$$

3.4. *Infinite-volume construction. Backward oriented percolation.* If we try to perform an analogous construction in infinite volume, we are confronted with the problem that there is no first mark. To overcome this, we follow the original approach of Harris (1972) [see also Durrett (1995)] and introduce the notion of percolation. The goal is to partition the set of cylinders into finite subsets to which the previous mark-by-mark construction can be applied.

We come back to the infinite-volume construction of Section 3.1. For an arbitrary space–time point (x, t), define the set of cylinders containing the point (x, t) by

$$\mathbf{A}_1^{x,t} = \{C \in \mathbf{C};\ \mathrm{Basis}(C) \ni x,\ \mathrm{Life}(C) \ni t\}. \tag{3.13}$$

For any cylinder C, define the set of *ancestors* of C as the set of cylinders born before C that are incompatible with C:

$$\begin{aligned}\mathbf{A}_1^{C} &= \{C' \in \mathbf{C};\ C' \not\sim C, \mathrm{Birth}(C') < \mathrm{Birth}(C)\}\\ &= \textstyle\bigcup_{\mathrm{x}\in\mathrm{Basis(C)}} \mathbf{A}^{x,\,\mathrm{Birth}(C)}.\end{aligned} \tag{3.14}$$

The definition of ancestor of C does not depend on the lifetime of C. Recursively, for $n \geq 2$, the nth generation of ancestors of (x, t) is defined as

$$\mathbf{A}_n^{x,t} = \{C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^{x,t}\}, \tag{3.15}$$

and, for a given cylinder C,

$$\mathbf{A}_n^{C} = \{C'' : C'' \in \mathbf{A}_1^{C'} \text{ for some } C' \in \mathbf{A}_{n-1}^{C}\}. \tag{3.16}$$

We say that there is *backward oriented percolation* in $\mathbf{C}$ if there exists a space–time point (x, t) such that $\mathbf{A}_n^{x,t} \neq \emptyset$ for all n; that is, there exists a point with infinitely many generations of ancestors. Let the *clan* of the space–time point (x, t) be the union of its ancestors:

$$\mathbf{A}^{x,t} = \bigcup_{n\geq 1} \mathbf{A}_n^{x,t}. \tag{3.17}$$

In the next theorem we give a sufficient condition for the existence of the infinite-volume process in any finite time interval in terms of backward percolation.

THEOREM 3.1. *If, with probability* 1, $\mathbf{A}^{x,t} \cap \mathbf{C}[0, t]$ *is finite for any* $x \in \mathbb{Z}^d$ *and* $t \geq 0$, *then for any* (*possibly infinite*) $\Lambda \subset \mathbb{Z}^d$, *the process with generator* A^Λ *is well defined for any initial configuration* $\zeta \in \chi^\Lambda$*and has at least one invariant measure* μ^Λ.

PROOF. We construct the process for $\Lambda = \mathbb{Z}^d$; the construction for other regions Λ is analogous. The initial configuration is denoted by $\zeta \in \chi$ and the initial cylinders are given by $\mathbf{S}(\zeta)$, defined in (3.4). Note that that all the cylinders of $\mathbf{S}(\zeta)$ are mutually compatible. We then partition $\mathbf{S}(\zeta) \bigcup \mathbf{C}[0, t]$ into a set of *kept*, cylinders, denoted by $\mathbf{K}$, and a set of *erased* cylinders, denoted by $\mathbf{D}$.

The construction is as follows. First, all cylinders in $\mathbf{S}(\zeta)$ are kept. Second, for each $x \in \Lambda$, the percolation clan of $(x,\ t)$ in $[0, t]$, $\mathbf{A}^{x,t} \cap \mathbf{C}[0, t]$, is partitioned in kept and deleted cylinders as in the finite-volume case. To do so, we order the cylinders of $\mathbf{A}^{x,t} \cap \mathbf{C}[0, t]$ by birth-time. This can be done because by hypothesis $\mathbf{A}^{x,t} \cap \mathbf{C}[0, t]$ has a finite number of cylinders. Then we successively classify each cylinder as kept if it is compatible with all cylinders already classified as kept [including those in $\mathbf{S}(\zeta)$]; if not, we classify it as erased. We denote the resulting sets $\mathbf{K}_\zeta^{x,t}[0, t]$ and $\mathbf{D}_\zeta^{x,t}[0, t]$, respectively.

Denoting

$$\mathbf{K}_\zeta[0, t] := \bigcup_{x\in\Lambda} \mathbf{K}_\zeta^{x,t}[0, t], \qquad \mathbf{D}_\zeta[0, t] := \bigcup_{x\in\Lambda} \mathbf{D}_\zeta^{x,t}[0, t], \tag{3.18}$$

we have that

$$\mathbf{K}_\zeta[0, t] \,\dot\cup\, \mathbf{D}_\zeta[0, t] = \mathbf{C}[0, t] \cup \mathbf{S}(\zeta). \tag{3.19}$$

Indeed, the classification of any given cylinder $C \in \mathbf{C}[0, t]$ depends only on (a) its ancestors in $[0, t]$, $\mathbf{A}^C \cap \mathbf{C}[0, t]$, and (b) the finite subset of $\mathbf{S}(\zeta)$ of cylinders that are incompatible with some of the ancestors of C in $[0, t]$. Therefore there is no inconsistency: $\mathbf{K}_\zeta^{x,t}[0, t] \cap \mathbf{D}_\zeta^{y,t}[0, t] = \varnothing$ for all $x \neq y$ and $\mathbf{K}_\zeta^{x,t}[0, t] \subset \mathbf{K}_\zeta^{x',t'}[0, t']$ for $t < t'$ if $(x,\ t) \in C$ for some $C \in \mathbf{K}_\zeta^{x',t'}[0, t']$.

The process is now defined as in (3.9) by

$$\eta_t^\zeta(\gamma) = \mathbf{1}\Big\{\gamma \in \big\{\mathrm{Basis}(C) : C \in \mathbf{K}_\zeta[0, t],\ \mathrm{Life}(C) \ni t\big\}\Big\}. \tag{3.20}$$

The reader can check that for finite Λ the preceding construction is equivalent to that of Section 3.3. Applied to the set of cylinders of $\mathbf{C}^\Lambda[0, t]$, it yields the set $\mathbf{K}_\zeta^\Lambda[0, t]$ defined in the paragraph preceding formula (3.9).

To show that η_t^ζ has generator A, denote $\eta_t = \eta_t^\zeta$ and $\mathbf{K} = \mathbf{K}_\zeta$ and write

$$\begin{aligned} &\big[f(\eta_{t+h}) - f(\eta_t)\big] \\ &\qquad = \sum_{C\in\mathbf{K}[0,\,t+h]} \mathbf{1}\big\{\mathrm{Birth}(C) \in [t, t+h]\big\}\big[f(\eta_t + \delta_{\mathrm{Basis}(C)}) - f(\eta_t)\big] \\ &\qquad\quad + \sum_{C\in\mathbf{K}[0,\,t]} \mathbf{1}\big\{\mathrm{Life}(C) \ni t, \mathrm{Life}(C) \not\ni t+h\big\}\big[f(\eta_t - \delta_{\mathrm{Basis}(C)}) - f(\eta_t)\big] \\ &\qquad\quad + \{\text{other things}\}, \end{aligned} \tag{3.21}$$

where {other things} refers to events with more than one Poisson mark in the time interval $[t,\ t+h]$ for the contours in the (finite) support of f. Since the total rate of the Poisson marks in this set is finite, the event {other things} has a probability of order h^2. Now, denoting

$$N_\gamma(t,\ s) = \#\big\{k : T_k(\gamma) \in (t,\ s)\big\}, \tag{3.22}$$

we have

$$\begin{aligned}
&\sum_{C\in\mathbf{C}} \mathbf{1}\{\mathrm{Birth}(C)\in[t,\ t+h]\}\mathbf{1}\{C\in\mathbf{K}[0,t+h]\}[f(\eta_t+\delta_{\mathrm{Basis}(C)})-f(\eta_t)]\\
&\quad=\sum_{C\in\mathbf{C}} \mathbf{1}\{N_{\mathrm{Basis}(C)}[t,\ t+h]=1\}\\
&\qquad\times \mathbf{1}\{\mathrm{Basis}(C)\sim\mathrm{Basis}(C'),\forall C'\in\mathbf{K}[0,t]:\mathrm{Life}(C')\ni t\}\\
&\qquad\times [f(\eta_t+\delta_{\mathrm{Basis}(C)})-f(\eta_t)]\\
&\quad=\sum_{\gamma}\mathbf{1}\{N_\gamma[t,\ t+h]=1\}\mathbf{1}\{\eta_t+\delta_\gamma\in\chi\}[f(\eta_t+\delta_\gamma)-f(\eta_t)].
\end{aligned}\tag{3.23}$$

To compute the second term of (3.21), observe that $\mathrm{Life}(C)$ is independent of $\mathrm{Birth}(C)$ and both the event $\{C\in\mathbf{K}[0,t]\}$ and η_t are $\mathscr{F}_t$-measurable. Here $\mathscr{F}_t$ is the σ-algebra generated by the births and deaths occurring before t. Hence

$$\begin{aligned}
&\mathbb{P}(\mathrm{Life}(C)\ni t,\ \mathrm{Life}(C)\not\ni t+h\,|\,\mathscr{F}_t)\\
&\qquad=\mathbb{P}(\mathrm{Life}(C)\not\ni t+h\,|\,\mathrm{Life}(C)\ni t)\mathbf{1}\{\mathrm{Life}(C)\ni t\}
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\sum_C \mathbf{1}\{C\in\mathbf{K}[0,t]\}\,\mathbf{1}\{\mathrm{Life}(C)\ni t,\mathrm{Life}(C)\not\ni t+h\}\\
&\qquad\qquad\times[f(\eta_t-\delta_{\mathrm{Basis}(C)})-f(\eta_t)]\Big]\\
&\quad=\mathbb{E}\Big[\sum_C \mathbb{P}(\mathrm{Life}(C)\not\ni t+h\,|\,\mathrm{Life}(C)\ni t)\mathbf{1}\{C\in\mathbf{K}[0,t],\mathrm{Life}(C)\ni t\}\\
&\qquad\qquad\times[f(\eta_t-\delta_{\mathrm{Basis}(C)})-f(\eta_t)]\Big].
\end{aligned}\tag{3.25}$$

Since $\mathrm{Life}(C)$ is exponentially distributed with mean 1,

$$\mathbb{P}(\mathrm{Life}(C)\not\ni t+h\,|\,\mathrm{Life}(C)\ni t)=h+o(h).\tag{3.26}$$

Taking the expectation of (3.21) and substituting (3.23)–(3.26), we get

$$\begin{aligned}
&\mathbb{E}[f(\eta_{t+h})-f(\eta_t)]\\
&\quad=\sum_\gamma he^{-\beta|\gamma|}\mathbb{E}\Big(\mathbf{1}\{\eta_t+\delta_\gamma\in\chi\}\,[f(\eta_t+\delta_\gamma)-f(\eta_t)]\Big)+o(h)\\
&\qquad+\sum_\gamma h\,\mathbb{E}\Big(\eta_t(\gamma)\,[f(\eta_t-\delta_\gamma)-f(\eta_t)]\Big)+o(h),
\end{aligned}\tag{3.27}$$

which, dividing by h and taking limit, gives,

$$\frac{d\mathbb{E}f(\eta_t^\zeta)}{dt}=A\mathbb{E}f(\eta_t^\zeta).\tag{3.28}$$

The existence of an invariant measure follows by compactness as our process is defined in the compact space χ. See Chapter 1 of Liggett (1985). □

We show in the next theorem that under a stronger hypothesis the process can be constructed for times in the whole real line. Since the construction is time translation invariant, the distribution of η_t will be invariant.

THEOREM 3.2. *If, with probability* 1, *there is no backward oriented percolation in* $\mathbf{C}$, *then the process with generator* A *can be constructed in* $(-\infty, \infty)$ *in such a way that the marginal distribution of* η_t *is invariant.*

PROOF. The lack of percolation allows us to construct a set $\mathbf{K} \subset \mathbf{C}$ as $\mathbf{K}_\zeta[0, t]$ was constructed from $\mathbf{C}[0, t] \bigcup \mathbf{S}(\zeta)$ in the proof of the previous theorem. We just proceed clan by clan and simply ignore the cylinders of $\mathbf{S}$. Note that $\mathbf{K}$ is both space and time translation invariant by construction. Analogously to the previous theorem we define η_t as the section of $\mathbf{K}$ at time t:

$$\eta_t(\gamma) = \mathbf{1}\big\{\gamma \in \big\{\text{Basis}(C) : C \in \mathbf{K},\ \text{Life}(C) \ni t\big\}\big\}. \tag{3.29}$$

By construction, the distribution of η_t does not depend on t; hence its distribution is an invariant measure for the process. □

Let us denote by μ the distribution of η_t, in anticipation of the fact that this is precisely the measure of Theorem 2.2.

As in the finite case,

$$\eta_t(\gamma) \le \xi_t(\gamma) \tag{3.30}$$

for all $\gamma \in \mathbf{G}$. This implies that the distribution μ inherits property (3.12):

$$\mu\big\{\eta_t^\Lambda(\gamma) = 1\big\} = \mathbb{E}\eta_t(\gamma) \le \mathbb{E}\xi_t(\gamma) = e^{-\beta|\gamma|}. \tag{3.31}$$

Let

$$\mathbf{A}(\Upsilon) = \bigcup_{x \in \Upsilon} \mathbf{A}^{x,0} \tag{3.32}$$

be the *clan* of $\Upsilon \subset \mathbb{Z}^d$ (at time 0).

REMARKS. (i) It follows from (3.29) that, for any $t \in \mathbb{R}$ and continuous f,

$$\mu f = \mathbb{E} f(\eta_t). \tag{3.33}$$

(ii) The presence/absence of a contour γ at time t depends only on the clan of ancestors of (x, t) for any $x \in \gamma$ through a certain function. More generally, for each f there exists a function Φ such that

$$f(\eta_t) = \Phi\bigg(\bigcup_{x \in \text{Supp}(f)} \mathbf{A}^{x,t}\bigg). \tag{3.34}$$

For instance, $\eta_t(\gamma) = 1$ if and only if $\mathbf{A}_1^{x,t}$ contains a cylinder in $\mathbf{K}$ with basis γ whose life contains t. This depends only on the set $\mathbf{A}^{x,t}$. In particular, with the notation (3.32),

$$f(\eta_0) = \Phi\Big(\mathbf{A}\big(\mathrm{Supp}(f)\big)\Big). \tag{3.35}$$

Analogous statements are true for the process starting with a fixed configuration at time 0:

$$f(\eta_t^\zeta) = \Phi\Big(\mathbf{A}_\zeta\big(\mathrm{Supp}(f),\ [0,t]\big)\Big), \tag{3.36}$$

where

$$\begin{aligned}\mathbf{A}_\zeta(\Upsilon,\ [0,t]) = {} & \Bigg[\bigg(\bigcup_{x\in\Upsilon} \mathbf{A}^{x,t}\bigg) \cap \mathbf{C}[0,t]\Bigg] \\ & \cup \Bigg\{C \in \mathbf{S}(\zeta) : \{C\} \not\sim \bigg(\bigcup_{x\in\Upsilon} \mathbf{A}^{x,t}\bigg) \cap \mathbf{C}[0,t]\Bigg\}\end{aligned} \tag{3.37}$$

is the set of cylinders in $\mathbf{C}[0,t] \cup \mathbf{S}(\zeta)$ which determines the value of $f(\eta_t^\zeta)$ when f has support Υ.

4. Percolation, space–time convergence and mixing. In this section we exploit the relationship between the loss network process and the absence of percolation to prove a more precise version of Theorem 2.2. In the proof of the mixing properties we shall need a continuous-time construction of the backward percolation clan.

4.1. *The key theorem.* The precise statement of the next theorem requires the notion of nonoriented percolation in a time interval. For any time interval $(s,\ t)$ and any space–time point $(x,\ t')$, define

$$\mathbf{G}_0^{x,t'}[s,\ t] = \big\{C \in \mathbf{C}[s,\ t] \ :\ \mathrm{Basis}(C) \ni x, \mathrm{Life}(C) \ni t'\big\} \tag{4.1}$$

and

$$\begin{aligned}\mathbf{G}_n^{x,t'}[s,\ t] = \big\{C \in \mathbf{C}[s,\ t] : {} & \mathrm{Basis}(C) \not\sim \mathrm{Basis}(C') \\ & \text{for some } C' \in \mathbf{G}_{n-1}^{x,t'}[s,\ t]\big\}.\end{aligned} \tag{4.2}$$

Notice that in the definition of $\mathbf{G}_n$ there is no exigency that the birth time of C' be prior to the birth time of C or that the lifetimes intersect. Let

$$\mathbf{G}^{x,t'}[s,\ t] = \bigcup_{k\geq 0} \mathbf{G}_k^{x,t'}[s,\ t]. \tag{4.3}$$

We say that there is no (nonoriented) percolation in $[s,\ t]$ if, for any space–time point $(x,\ t')$, $\mathbf{G}^{x,t'}[s,\ t]$ contains a finite number of cylinders. We will show later that the condition $\alpha < \infty$ is sufficient for the existence of an h such that the probability that there is no nonoriented percolation in $[0, h]$ is 1.

In addition, we need the following definitions.

- The *time length* (TL) and the *space width* (SW) of the family of cylinders $\mathbf{A}^{x,t}$ are, respectively,

$$TL(\mathbf{A}^{x,t}) = t - \sup\{s : \text{Life}(C) \ni s, \text{ for some } C \in \mathbf{A}^{x,t}\}, \tag{4.4}$$

$$SW(\mathbf{A}^{x,t}) = \Big|\bigcup_{C\in\mathbf{A}^{x,t}} \text{Basis}(C)\Big|. \tag{4.5}$$

In words, the space width is the number of sites occupied by the projection of the bases of the cylinders in the family. The time length is the length of the time interval between t and the first birth in the family of ancestors of (x, t).

- $\mathbf{A}^{\Lambda}(\text{Supp}(f))$ is the set of ancestors of $\text{Supp}(f)$ constructed from $\mathbf{C}^{\Lambda}$ as $\mathbf{A}(\text{Supp}(f))$ was constructed from $\mathbf{C}$. [Notice that this is *not* the same as $\mathbf{A}(\text{Supp}(f)) \cap \mathbf{C}^{\Lambda}$.]

In item 4 of the next theorem we enlarge our probability space to the direct product of our working space with itself: $(\Omega, \mathbf{F}, \mathbb{P}) \times (\Omega, \mathbf{F}, \mathbb{P})$. As noted before, we continue to use $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation of this space.

THEOREM 4.1. *Assume that there is no backward oriented percolation with probability* 1. *Then*:

1. *Uniqueness. The measure* μ *is the unique invariant measure for the process* η_t.
2. *Time convergence. For any function* f *with finite support*,

$$\lim_{t\to\infty} \sup_{\eta\in\chi} \Big|\mathbb{E}f(\eta_t^{\zeta}) - \mu f\Big| = 0. \tag{4.6}$$

Furthermore,

$$\begin{aligned}
&\sup_{\eta\in\chi}\big|\mathbb{E}f(\eta_t^{\zeta}) - \mu f\big| \\
&\quad\le 2\,\|f\|_\infty\, \mathbb{P}\Big(\bigcup_{x\in\text{Supp}(f)} \{\mathbf{A}^{x,t} \not\sim \mathbf{S}(\zeta) \text{ or } \text{TL}(\mathbf{A}^{x,t}) > t\}\Big) &(4.7)\\
&\quad\le 2\,\|f\|_\infty \sum_{x\in\text{Supp}(f)} \Big[\mathbb{P}\big(\text{TL}(\mathbf{A}^{x,0}) > bt\big) + e^{-(1-b)t}\,\mathbb{E}\big(\text{SW}(\mathbf{A}^{x,0})\big)\Big] &(4.8)
\end{aligned}$$

for any $b \in (0, 1)$.

3. *Space convergence. As* $\Lambda \to \mathbb{Z}^d$, μ^{Λ} *converges weakly to* μ. *More precisely, if* f *is a function depending on contours contained in a finite set* Λ, *then*

$$|\mu f - \mu^{\Lambda} f| \le 2\,\|f\|_\infty\, \mathbb{P}\Big(\mathbf{A}\big(\text{Supp}(f)\big) \ne \mathbf{A}^{\Lambda}\big(\text{Supp}(f)\big)\Big). \tag{4.9}$$

4. *Mixing. If, in addition, there exists a value h such that there is no (nonoriented) percolation in $(0, h)$ with probability 1, then, for f and g with finite support,*

$$\lim_{|x|\to\infty} |\mu(f\tau_x g) - \mu f\, \mu g| = 0, \tag{4.10}$$

where τ_x is translation by x. More precisely, for f and g with arbitrary support,

$$|\mu(fg) - \mu f\, \mu g| \le 2\,\|f\|_\infty \|g\|_\infty\, \mathbb{P}\big(\mathbf{A}(\mathrm{Supp}(f)) \not\sim \widehat{\mathbf{A}}(\mathrm{Supp}(g))\big), \tag{4.11}$$

where $\widehat{\mathbf{A}}(\mathrm{Supp}(g))$ has the same distribution as $\mathbf{A}(\mathrm{Supp}(g))$ but is independent of $\mathbf{A}(\mathrm{Supp}(f))$.

The existence of μ has been proven in Theorem 3.2. In the rest of this section we prove the other properties.

4.2. *Time convergence and uniqueness.* We use the same Poisson marks to construct simultaneously the stationary process η_t and a process starting at time 0 with an arbitrary initial configuration ζ. The second process is denoted by η_t^ζ (as before). This is what in the literature is known as *coupling*. By construction [cf. (3.18) and (3.20)], the process η_t^ζ ignores the cylinders in $\mathbf{C}$ with birth times less than 0 but takes into account the set of cylinders with basis given by the contours of the initial configuration ζ and birth time 0, $\mathbf{S}(\zeta) = \{(\theta, 0, S_1^0(\theta)) \in \mathbf{S} : \eta(\theta) = 1\}$. Recall that the times $S_k^0(\theta)$ are exponentially distributed with mean 1 and independent of everything.

By (3.33) and (3.11),

$$\sup_{\zeta\in\chi} \big|\mathbb{E} f(\eta_t^\zeta) - \mu f\big| = \sup_{\zeta\in\chi} \Big|\mathbb{E}\Big(f(\eta_t^\zeta) - f(\eta_t)\Big)\Big|. \tag{4.12}$$

Since we are using $\mathbf{C}$ to construct η_t and $\mathbf{C}[0,t] \cup \mathbf{S}(\zeta)$ to construct η_t^ζ, it follows from (3.34) and (3.36) that

$$\begin{aligned} \big|f(\eta_t^\zeta) - f(\eta_t)\big| &= \bigg|\Phi\big(\mathbf{A}_\zeta(\mathrm{Supp}(f), [0,t])\big) - \Phi\bigg(\bigcup_{x\in \mathrm{Supp}(f)} \mathbf{A}^{x,t}\bigg)\bigg| \\ &\le 2\,\|f\|_\infty\, \mathbf{1}\bigg\{\bigg(\mathbf{S}(\zeta) \not\sim \bigcup_{x\in\mathrm{Supp}(f)} \mathbf{A}^{x,t}\bigg) \text{ or } \mathrm{TL}(\mathbf{A}^{x,t}) > t\bigg\}. \end{aligned} \tag{4.13}$$

To see this, notice that $\{\mathbf{A}^{x,t} \subset \mathbf{C}[0,t]\} = \{\mathrm{TL}(\mathbf{A}^{x,t}) < t\}$ and that $\bigcup_{x\in\mathrm{Supp}(f)} \mathbf{A}^{x,t} \sim \mathbf{S}(\zeta)$ and $\bigcup_{x\in\mathrm{Supp}(f)} \mathbf{A}^{x,t} \subset \mathbf{C}[0,t]$ if and only if $\bigcup_{x\in\mathrm{Supp}(f)} \mathbf{A}^{x,t} = \mathbf{A}_\zeta(\mathrm{Supp}(f), [0,t])$. Equation (4.13) shows (4.7).

To prove the weak convergence (4.6), we fix $b \in [0,1]$ and bound the indicator function on the right-hand side of (4.13) by

$$\mathbf{1}\big\{\mathrm{TL}(\mathbf{A}^{x,t}) > bt\big\} + \mathbf{1}\big\{\mathrm{TL}(\mathbf{A}^{x,t}) < bt, \mathbf{A}^{x,t} \not\sim \mathbf{S}(\zeta)\big\}. \tag{4.14}$$

The expected value of the first term in (4.14) goes to 0 because $\mathbf{A}^{x,t}$ has a finite number of cylinders with probability 1. The second term in (4.14) is bounded above by

$$\mathbf{1}\big\{\max\big\{S_1^0(\gamma) : \zeta(\gamma) = 1 \text{ and } \gamma \not\sim \mathrm{Basis}(C) \text{ for some } C \in \mathbf{A}^{x,t}\big\} > (1-b)t\big\}. \tag{4.15}$$

Since $\mathbf{S}$ and $\mathbf{A}^{x,t}$ are independent and S_i^0 are iid exponentially distributed random variables of mean 1,

$$\begin{aligned} &\mathbb{E}\Big(\max\big\{S_1^0(\gamma) : \zeta(\gamma) = 1 \text{ and } \gamma \not\sim \mathrm{Basis}(C) \text{ for some } C \in \mathbf{A}^{x,t}\big\} > (1-b)t \,\Big|\, \mathbf{A}^{x,t}\Big) \\ &\quad = 1 - (1 - e^{-(1-b)t})^{|\{\gamma : \zeta(\gamma)=1 \text{ and } \gamma \not\sim \mathrm{Basis}(C) \text{ for some } C \in \mathbf{A}^{x,t}\}|}. \end{aligned} \tag{4.16}$$

Since ζ is a configuration of compatible contours, it contains at most one contour per site; that is, $|\{\gamma \ni x : \zeta(\gamma) = 1\}| \le 1$ for all $x \in \mathbb{Z}^d$. This implies that at most $\mathrm{SW}(\mathbf{A}^{x,t})$ cylinders of $C(\zeta)$ can be incompatible with cylinders in $\mathbf{A}^{x,t}$. Hence (4.16) is bounded by

$$1 - (1 - e^{-(1-b)t})^{\mathrm{SW}(\mathbf{A}^{x,t})}. \tag{4.17}$$

The expectation of (4.17) is given by

$$\sum_{n \ge 1} \Big[1 - (1 - e^{-(1-b)t})^n\Big]\, \mathbb{P}\Big(\mathrm{SW}(\mathbf{A}^{x,0}) = n\Big) \tag{4.18}$$

because the distribution of $\mathbf{A}^{x,t}$ does not depend on t. Our hypothesis of no backward oriented percolation implies that $\mathbf{A}^{x,0}$ contains a finite number of (finite) contours. Hence $\sum_{n\ge1} \mathbb{P}(\mathrm{SW}(\mathbf{A}^{x,0}) = n) = 1$ and by dominated convergence (4.18) goes to 0 as $t \to \infty$. This proves (4.6).

To prove (4.8), we start from the expectation of (4.14) and use (4.18) to bound the expected value of the second term by

$$\begin{aligned} & e^{-(1-b)t} \sum_{n\ge1} \mathbb{P}\big(\mathrm{SW}(\mathbf{A}^{x,0}) = n\big) \sum_{k=0}^{n-1} \big(1 - e^{-(1-b)t}\big)^k \\ &\quad \le e^{-(1-b)t} \sum_{n\ge1} n\, \mathbb{P}\big(\mathrm{SW}(\mathbf{A}^{x,0}) = n\big) \\ &\quad \le e^{-(1-b)t}\, \mathbb{E}\big(\mathrm{SW}(\mathbf{A}^{x,0})\big). \end{aligned} \tag{4.19}$$

The preceding arguments prove that the process converges, uniformly in the initial configuration, to the invariant measure μ. An immediate consequence is that μ is the unique invariant measure. This concludes the proof of items 1 and 2 of Theorem 4.1. □

4.3. *Finite-volume effects.* To prove inequality (4.9), we use (3.33), (3.11) and (3.34) to get

$$\begin{aligned} \mu f - \mu^\Lambda f &= \mathbb{E} f(\eta_0) - \mathbb{E} f(\eta^{\Lambda_0}) \\ &= \mathbb{E}\big[\Phi\big(\mathbf{A}\big(\mathrm{Supp}(f)\big)\big) - \Phi\big(\mathbf{A}^\Lambda\big(\mathrm{Supp}(f)\big)\big)\big], \end{aligned} \tag{4.20}$$

where Φ is the function referred to in (3.35). By definition,

$$\Phi\big(\mathbf{A}\big(\mathrm{Supp}(f)\big)\big) \le \|f\|_\infty . \tag{4.21}$$

Hence inequality (4.9) follows from (4.20).

Since the spatial projections of the set of ancestors of $\mathrm{Supp}(f)$ are finite, the right-hand side of (4.9) goes to 0, proving, in particular, the weak convergence of μ^Λ to μ. □

4.4. *Mixing: Its relation with a coupling construction.* The proof of item 4 of Theorem 4.1 is very similar in spirit to the preceding proof but it requires a somewhat more delicate argument based on the coupling of two continuous-time versions of the backward percolation process. We first notice that (4.10) is a straightforward consequence of (4.11), because, in the absence of backward percolation, the spatial projections of the set of ancestors of $\mathrm{Supp}(f)$ and $\mathrm{Supp}(g)$ are finite. This implies that the right-hand side of (4.11) goes to 0.

To prove (4.11), we use (3.33) and (3.34) to get

$$\begin{aligned} |\mu(fg) - \mu f\, \mu g| &= \mathbb{E}(f(\eta_0) g(\eta_0)) - \mathbb{E} f(\eta_0)\, \mathbb{E} g(\eta_0) \\ &= \mathbb{E}\big[\Phi\big(\mathbf{A}\big(\mathrm{Supp}(f)\big)\big)\, \Phi\big(\mathbf{A}\big(\mathrm{Supp}(g)\big)\big)\big] \\ &\quad - \mathbb{E}\big[\Phi\big(\widehat{\mathbf{A}}\big(\mathrm{Supp}(f)\big)\big)\, \Phi\big(\widehat{\mathbf{A}}\big(\mathrm{Supp}(g)\big)\big)\big] , \end{aligned} \tag{4.22}$$

where Φ is the function referred to in (3.34) and $(\widehat{\mathbf{A}}(\mathrm{Supp}(f)), \widehat{\mathbf{A}}(\mathrm{Supp}(g)))$ has the *same* marginal distributions as $(\mathbf{A}(\mathrm{Supp}(f)), \mathbf{A}(\mathrm{Supp}(g)))$ but its marginals are *independent*.

Identity (4.22) shows that to obtain (4.10) it is enough to construct a coupling (joint construction) of the four processes

$$\big(\mathbf{A}\big(\mathrm{Supp}(f)\big),\, \mathbf{A}\big(\mathrm{Supp}(g)\big),\, \widehat{\mathbf{A}}\big(\mathrm{Supp}(f)\big),\, \widehat{\mathbf{A}}\big(\mathrm{Supp}(g)\big)\big),$$

such that

$$\mathbf{A}\big(\mathrm{Supp}(f)\big) = \widehat{\mathbf{A}}\big(\mathrm{Supp}(f)\big) \tag{4.23}$$

and

$$\widehat{\mathbf{A}}\big(\mathrm{Supp}(g)\big) \sim \mathbf{A}\big(\mathrm{Supp}(f)\big) \quad \text{implies} \quad \mathbf{A}\big(\mathrm{Supp}(g)\big) = \widehat{\mathbf{A}}\big(\mathrm{Supp}(g)\big). \tag{4.24}$$

Indeed, from (4.21) and (4.23)–(4.24) we obtain that the last line of (4.22) is bounded above by the right-hand side of (4.11).

In the remainder of this section we discuss the construction of the coupling with properties (4.23)–(4.24). The construction is natural and straightforward, but unavoidably technical. As an alternative, we mention the approach based on "duplicated variables" [von Dreifus, Klein and Perez (1995) and Bricmont and Kupiainen (1996)], which is probabilistically simpler but requires some combinatorial input.

4.5. *Construction of a four-clan coupling.* We need to couple two clans in the same random set of cylinders with two independent copies with the same marginal distribution. Moreover, to strengthen our results, we need to ensure that the marginal realizations remain the same as much as possible. The coupling (Section 4.5.2) is based on a construction of backward percolation clans as nonhomogeneous continuous-time Markov processes (Section 4.5.1). The hypothesis on the absence of nonoriented percolation for some time interval $(0, h)$ is needed for the infinite-volume construction of the coupling.

4.5.1. *A continuous-time construction of the backward percolation clan.* For $\Upsilon \subset \mathbb{Z}^d$, define

$$\begin{aligned}\mathbf{A}_t(\Upsilon) &= \{C' \in \mathbf{A}(\Upsilon) : 0 > \mathrm{Birth}(C') > -t\} \\ &= \mathbf{A}(\Upsilon) \cap \mathbf{C}[-t, 0],\end{aligned} \tag{4.25}$$

that is, the set of cylinders in $A(\Upsilon)$ with birth time posterior to $-t$. The inclusion of a new cylinder in the time interval $[t,\ t+h]$ depends on the existence of a birth Poisson mark in $[-t-h,\ -t]$ whose corresponding cylinder is incompatible with some $C' \in \mathbf{A}_t(\Upsilon)$. That is, if C is a cylinder with $\mathrm{Basis}(C) \not\sim \mathrm{Basis}(C')$ then for some $C' \in \mathbf{A}_t$,

$$\begin{aligned}&\mathbb{P}\big(\mathbf{A}_{t+h} = \widetilde{\mathbf{A}} \cup C | \mathbf{A}_t = \widetilde{\mathbf{A}},\ \mathbf{A}_{t'} = \widetilde{\mathbf{A}}_{t'},\ t' \in [0, t)\big) \\ &\qquad = \mathbb{P}\{C \in \mathbf{C} : \mathrm{Birth}(C) \in [-t-h,\ -t], \\ &\qquad\qquad \mathrm{Death}(C) > t - \mathrm{TI}(\widetilde{\mathbf{A}},\ \mathrm{Basis}(C))\} + o(h).\end{aligned} \tag{4.26}$$

We have denoted

$$\mathrm{TI}(\mathbf{A}_t,\ \gamma) = \min\{\mathrm{Birth}(C') : C' \in \mathbf{A}_t, \mathrm{Basis}(C') \not\sim \gamma\} \tag{4.27}$$

and abbreviated $\mathbf{A}_t(\Upsilon) = \mathbf{A}_t$. The remainder $o(h)$ is the correction related to the probability that C is not the only cylinder born in $[-t-h,\ -t]$. Since the birth time is independent of the lifetime, which is exponentially distributed with rate 1,

$$\begin{aligned}&\mathbb{P}\Big(\mathbf{A}_{t+h} = \widetilde{\mathbf{A}} \cup C | \mathbf{A}_t = \widetilde{\mathbf{A}},\ \mathbf{A}_{t'} = \widetilde{\mathbf{A}}_{t'},\ t' \in [0, t)\Big) \\ &\qquad = \mathbb{P}\Big\{C \in \mathbf{C} \ :\ \mathrm{Birth}(C) \in [-t-h,\ -t]\Big\} \\ &\qquad\quad \times \mathbb{P}\big(\mathrm{Life}(C) > t - \mathrm{TI}(\widetilde{\mathbf{A}},\ \mathrm{Basis}(C))\big) + o(h) \\ &\qquad = he^{-\beta|\mathrm{Basis}(C)|}e^{-t+\mathrm{TI}(\widetilde{\mathbf{A}},\ \mathrm{Basis}(C))} + o(h).\end{aligned} \tag{4.28}$$

This implies that when the configuration at time t^- is $\widetilde{\mathbf{A}}$, a new cylinder with basis γ is included in $\mathbf{A}_t(\Upsilon)$ at rate

$$e^{-\beta|\gamma|}e^{-t+\mathrm{TI}(\widetilde{\mathbf{A}},\ \gamma)}. \tag{4.29}$$

From (4.28), as in the computation of the forward Kolmogorov equations, we get

$$\mathbb{E}\left(\frac{df(\mathbf{A}_t)}{dt}\middle|\mathbf{A}_s,\ 0 \le s \le t\right) = \sum_{\gamma} \int_{t-\mathrm{TI}(\mathbf{A}_t,\,\gamma)}^{\infty} ds\, e^{-s}\, e^{-\beta|\gamma|}\big[f\big(\mathbf{A}_t \cup (\gamma,\ t,\ s)\big) - f(\mathbf{A}_t)\big], \tag{4.30}$$

where the sum is over the set $\{\gamma \in \mathbf{G} : \gamma \not\sim \mathrm{Basis}(C') \text{ for some } C' \in \mathbf{A}_t\}$. This equation characterizes the law of the process $\mathbf{A}_t(\Upsilon)$ as a nonhomogeneous Markov process.

We now construct $\mathbf{A}_t(\Upsilon)$ by combing the Poisson marks backward in time in a continuous manner.

Finite-volume case. If we only consider contours contained in a finite set $\Lambda \subset \mathbb{Z}^d$, there is only a finite set of possible bases for the cylinders and the Poisson marks are well ordered with probability 1. The construction proceeds mark by mark backward in time. Set $\mathbf{A}_0(\Upsilon) = \Upsilon$. If there is a Poisson birth mark at time $-t$ whose corresponding cylinder is called C'', then

- if $C'' \not\sim C'$ for some $C' \in \mathbf{A}_{t-}(\Upsilon)$, set $\mathbf{A}_t(\Upsilon) = \mathbf{A}_{t-}(\Upsilon) \cup \{C''\}$;
- if $C'' \sim C'$ for all $C' \in \mathbf{A}_{t-}(\Upsilon)$, set $\mathbf{A}_t(\Upsilon) = \mathbf{A}_{t-}(\Upsilon)$,

where the incompatibility between cylinders was defined in (3.2).

Infinite-volume case. In an infinite volume the construction can be performed using a percolation argument as in Section 3.4. By hypothesis, there exists an h such that each cylinder born in the interval $[-h,\ 0]$ belongs to a finite nonoriented clan. Hence the set of cylinders born in the interval $[-h,\ 0]$ can be partitioned in connected families:

$$\mathbf{C}[-h,\ 0] = \dot{\bigcup_{k\ge 0}} \mathbf{H}_k[-h,\ 0], \tag{4.31}$$

where the sets $\mathbf{H}_k[s,\ t]$ are the maximal sets of cylinders with the property that cylinders in different $\mathbf{H}_k$'s are compatible. We can then order the birth time of the cylinders inside each $\mathbf{H}_k$ and proceed as for the finite-volume case. This yields the process $\mathbf{A}_t(\Upsilon)$ for $t \in [0,\ h]$. To extend the construction for arbitrary $t > 0$, we simply repeat the previous procedure in $[h,\ 2h]$, $[2h,\ 3h]$, etc.

4.5.2. *A coupling between two interacting and two independent clans.* We take two independent marked Poisson process whose marks and cylinders we respectively call blue and red. We enlarge our probability space and continue using $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation with respect to the space generated by the product of the blue and red Poisson processes. Using these marks, we construct simultaneously the processes $(\mathbf{A}_t(\Lambda_1), \mathbf{A}_t(\Lambda_2), \widehat{\mathbf{A}}_t(\Lambda_1), \widehat{\mathbf{A}}_t(\Lambda_2))$, for $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$, in the following way:

1. The processes $\mathbf{A}_t(\Lambda_1)$ and $\mathbf{A}_t(\Lambda_2)$ are constructed using only the blue marks, as described in Section 4.5.1, and ignoring the red marks. Hence they are the clans of Λ_1 and Λ_2, respectively.
2. The process $\widehat{\mathbf{A}}_t(\Lambda_1)$ is also constructed only with the blue marks; hence it coincides with $\mathbf{A}_t(\Lambda_1)$.
3. The process $\widehat{\mathbf{A}}_t(\Lambda_2)$ is constructed with a precise combination of blue and red marks in such a way that (a) it coincides with $\mathbf{A}_t(\Lambda_2)$ for a time interval that is as long as possible, (b) it is independent of $\widehat{\mathbf{A}}_t(\Lambda_1)$ and (c) it has the same marginal distribution as $\mathbf{A}_t(\Lambda_2)$.

Property 3 is achieved in the following way.

Finite-volume case. If both Λ_1 and Λ_2 are finite sets, we order the marks by appearance and introduce a flag variable, $\mathrm{Flag}(t) \in \{0, 1\}$, which indicates if some cylinder of $\widehat{\mathbf{A}}_t(\Lambda_1)$ is incompatible with some cylinder of $\widehat{\mathbf{A}}_t(\Lambda_2)$:

$$\mathrm{Flag}(t) = \mathbf{1}\{C' \not\sim C'' \text{ for some } C' \in \widehat{\mathbf{A}}_t(\Lambda_1), C'' \in \widehat{\mathbf{A}}_t(\Lambda_2)\}. \tag{4.32}$$

We now proceed as follows, mark by mark backward in time. First, we set $\mathrm{Flag}(0) = 0$. The construction guarantees that $\mathrm{Flag}(t) = 0$ implies $\mathbf{A}_t(\Lambda_i) = \widehat{\mathbf{A}}_t(\Lambda_i)$ for $i = 1, 2$.

- If at time $-t$ a (blue or red) mark is present and $\mathrm{Flag}(t-) = 0$, then
  - — If the mark is blue and the corresponding cylinder can be included in $\mathbf{A}_{t-}(\Lambda_1)$ but not in $\mathbf{A}_t(\Lambda_2)$, then include it in $\mathbf{A}_t(\Lambda_1)$ and $\widehat{\mathbf{A}}_t(\Lambda_1)$. Analogously, if it can be included in $\mathbf{A}_{t-}(\Lambda_2)$ but not in $\mathbf{A}_t(\Lambda_1)$, then include it in $\mathbf{A}_t(\Lambda_2)$ and $\widehat{\mathbf{A}}_t(\Lambda_2)$. Keep the Flag $= 0$.
  - — If the mark is blue and the corresponding cylinder can be included in both $\mathbf{A}_{t-}(\Lambda_1)$ and $\mathbf{A}_{t-}(\Lambda_2)$, then include it in both $\mathbf{A}_t(\Lambda_1)$ and $\mathbf{A}_t(\Lambda_2)$ but include it only in $\widehat{\mathbf{A}}_t(\Lambda_1)$. Set the Flag $= 1$.
  - — If the mark is red and the corresponding cylinder can be included in both $\mathbf{A}_{t-}(\Lambda_1)$ and $\mathbf{A}_{t-}(\Lambda_2)$, then include it only in $\widehat{\mathbf{A}}_t(\Lambda_2)$. Set the Flag $= 1$.
  - — If the mark is red and the corresponding cylinder can be included in either $\mathbf{A}_{t-}(\Lambda_1)$ or $\mathbf{A}_{t-}(\Lambda_2)$ but not in both of them, then ignore the mark. Keep the Flag $= 0$.
  - — If the mark is red or blue but the corresponding cylinder can be included in neither $\mathbf{A}_{t-}(\Lambda_1)$ nor $\mathbf{A}_{t-}(\Lambda_2)$, then ignore the mark. Keep the Flag $= 0$.
- If at time $-t$ a (blue or red) mark appears and $\mathrm{Flag}(t-) = 1$, then use blue marks for $\mathbf{A}_t(\Lambda_1)$, $\mathbf{A}_t(\Lambda_2)$ and $\widehat{\mathbf{A}}_t(\Lambda_1)$ and red marks for $\widehat{\mathbf{A}}_t(\Lambda_2)$. Keep the Flag $= 1$.

To verify that the previous coupling has the right marginals, it suffices to notice that the rate of inclusion of cylinders in each one of the marginals is precisely given by (4.29).

Infinite-volume case. In the case in which at least one of Λ_1 and Λ_2 is infinite, we consider families $\overline{\mathbf{H}}_k$ analogous to the $\mathbf{H}_k$ given in (4.31) but defined using the time interval $[-h/2, 0]$ and both red and blue marks. Therefore in the combined set of cylinders there is no nonoriented percolation and we can construct the coupling working in a finite set of $\overline{\mathbf{H}}_k$'s at a time. We then continue working in time intervals of length $h/2$ to reach arbitrary times t.

By construction, the coupling satisfies (4.23). It also satisfies (4.24) because the flag changes from 0 to 1 (and remains 1 forever) the first time a cylinder of $\mathbf{A}_t(\mathrm{Supp}(g))$ is incompatible with a cylinder of $\widehat{\mathbf{A}}_t(\mathrm{Supp}(f))$. This implies

$$\mathrm{Flag}(\infty) = \mathbf{1}\{\mathbf{A}(\mathrm{Supp}(f)) \not\sim \widehat{\mathbf{A}}(\mathrm{Supp}(g))\}. \tag{4.33}$$

5. Branching processes: Time length and space width. In this section we estimate the time length and space width of the families of ancestors $\mathbf{A}^{x,t}$. We follow the well-known approach of introducing a branching process that dominates the backward percolation process [see e.g., Hall (1985)], though we must consider *multitype* branching. The main result of this section is Theorem 5.1, which shows that the hypotheses of Theorem 2.2 lead to exponential upper bounds of both $\mathrm{TL}(\mathbf{A}^{x,t})$ and $\mathrm{SW}(\mathbf{A}^{x,t})$.

5.1. *Multitype branching processes.* We introduce a multitype branching process $\mathbf{B}_n$, in the set of cylinders, which dominates $\mathbf{A}_n$. To do this, we look "backward in time" and let "ancestors" play the role of "branches." In particular, births in the original marked Poisson process correspond to the disappearance of branches. We reserve the words "birth" and "death" for the original forward-time Poisson process.

We start by enlarging our probability space and defining, for any given set $\{C_1, \dots, C_k\}$, *independent* random sets $\mathbf{B}_1^{C_i}$ with the same marginal distribution as $\mathbf{A}_1^{C_i}$. The important point here is that

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{C_i} \subset \bigcup_{i=1}^{k} \mathbf{B}_1^{C_i}. \tag{5.1}$$

The proof of this fact relies on fixing a way to distribute common ancestors. For example, consider the total order $\prec$ in the set of cylinders induced by the birth times. That is $C \prec C'$ if and only if $\mathrm{Birth}(C) \leq \mathrm{Birth}(C')$. For any finite set of cylinders $\{C_1, \dots, C_k\}$ such that $C_i \prec C_{i+1}$, $i = 1, \dots, k-1$, define

$$\widetilde{\mathbf{A}}_1^{C_j} = \mathbf{A}_1^{C_j} \Big\backslash \left(\bigcup_{\ell=1}^{j-1} \mathbf{A}_1^{C_\ell} \right). \tag{5.2}$$

This ensures that $\widetilde{\mathbf{A}}_1^{C_j}$ are independent sets and

$$\bigcup_{i=1}^{k} \mathbf{A}_1^{C_i} = \bigcup_{i=1}^{k} \widetilde{\mathbf{A}}_1^{C_i}. \tag{5.3}$$

On the other hand, for any C, $\widetilde{\mathbf{A}}_1^C$ is stochastically dominated by $\mathbf{A}_1^C$ [i.e., there exists a joint realization $(\widetilde{\mathbf{A}}_1^C, \mathbf{A}_1^C)$ such that $\mathbb{P}(\widetilde{\mathbf{A}}_1^C \subset \mathbf{A}_1^C) = 1$]. From this observation and (5.3) we get (5.1).

The procedure defined by $\mathbf{B}_1$ naturally induces a multitype branching process in the space of cylinders. We define the nth generation of the branching process by

$$\mathbf{B}_n^C = \{\mathbf{B}_1^{C'} : C' \in \mathbf{B}_{n-1}^C\}, \tag{5.4}$$

where, for all C', $\mathbf{B}_1^{C'}$ has the same distribution as $\mathbf{A}_1^{C'}$ and is an independent random set depending only on C'. Inductively,

$$\mathbf{A}_n^C \subset \mathbf{B}_n^C. \tag{5.5}$$

Indeed,

$$\mathbf{A}_n^C = \bigcup_{C' \in \mathbf{A}_{n-1}^C} \mathbf{A}_1^{C'} = \bigcup_{C' \in \mathbf{A}_{n-1}^C} \widetilde{\mathbf{A}}_1^{C'}, \tag{5.6}$$

where in the definition of $\widetilde{\mathbf{A}}_1^{C'}$ we use $\{C_1, \dots, C_k\} = \bigcup_{i=0}^{n-1} \mathbf{A}_i^C$. Hence the inductive hypothesis $\mathbf{A}_i^C \subset \mathbf{B}_i^C$, for $i = 1, \dots, n-1$, yields (5.5).

Consistent with our previous notation, we denote by

$$\mathbf{B}^C = \bigcup_{n \geq 0} \mathbf{B}_n^C, \qquad \mathbf{B}^{x,t} = \bigcup_{n \geq 0} \mathbf{B}_n^{x,t}, \qquad \mathbf{B}^{\Upsilon} = \bigcup_{x \in \Upsilon} \mathbf{B}^{x,0} \tag{5.7}$$

the branching clans of C, (x, t) and Υ (at time 0), respectively. By (5.5),

$$\mathbf{A}^C \subset \mathbf{B}^C, \qquad \mathbf{A}^{x,t} \subset \mathbf{B}^{x,t}, \qquad \mathbf{A}^{\Upsilon} \subset \mathbf{B}^{\Upsilon}. \tag{5.8}$$

Defining the time length and space width of this clan as in (4.4) and (4.5), we get

$$\mathrm{TL}(\mathbf{A}^C) \leq \mathrm{TL}(\mathbf{B}^C), \quad \mathrm{TL}(\mathbf{A}^{x,t}) \leq \mathrm{TL}(\mathbf{B}^{x,t}), \quad \mathrm{TL}(\mathbf{A}^{\Upsilon}) \leq \mathrm{TL}(\mathbf{B}^{\Upsilon}), \tag{5.9}$$

and similarly for the respective space widths.

The (multitype) branching process $\mathbf{B}_n$ induces naturally a multitype branching process in the set of contours. For a cylinder C with basis γ and birth time 0, define $\mathbf{b}_n^\gamma \in \mathbb{N}^{\mathbf{G}}$ as the number of cylinders in the nth generation of ancestors of C with basis θ:

$$\mathbf{b}_n^\gamma(\theta) = \left|\left\{C' \in \mathbf{B}_n^C : \mathrm{Basis}(C') = \theta\right\}\right|. \tag{5.10}$$

This process will be useful in estimating the space properties of the clans of ancestors. We have the following relationship: If $\mathrm{Basis}(C) = \gamma$, then

$$\sum_\theta \mathbf{b}_n^\gamma(\theta) = |\mathbf{B}_n^C|. \tag{5.11}$$

The process $\mathbf{b}_n$ is a multitype branching process whose offspring distributions are Poisson with means

$$m(\gamma,\ \theta) = \mathbf{1}\{\gamma \not\sim \theta\}\ e^{-\beta|\theta|} \int_0^\infty e^{-t}\,dt = \mathbf{1}\{\gamma \not\sim \theta\} e^{-\beta|\theta|}. \tag{5.12}$$

To see this, notice that the cylinders C' with basis θ that are potential ancestors of C (with basis γ) form a Poisson process of rate $\mathbf{1}\{\gamma \not\sim \theta\}\, e^{-\beta|\theta|}$. Each of those cylinders is an ancestor of C if its lifetime is larger than the difference between the birth time of C and C'. The lifetimes of different cylinders are independent exponentially distributed random variables of rate 1. The probability that the lifetime of any given cylinder is larger than t is given by e^{-t}. Hence the birth times of the ancestors of C with basis θ form a (nonhomogeneous) Poisson process of rate depending on t given by $\mathbf{1}\{\gamma \not\sim \theta\}\, e^{-\beta|\theta|}\, e^{-t}$. The mean number of births is therefore given by (5.12).

LEMMA 5.1. *The means* (5.12) *satisfies*

$$\sum_\theta m^n(\gamma,\ \theta) \le \sum_\theta |\theta| m^n(\gamma,\ \theta) \le |\gamma|\alpha^n, \tag{5.13}$$

where m^n is the nth power of the matrix m and α is defined in (2.4).

PROOF. The first inequality is immediate. For the second,

$$\begin{aligned} &\sum_\theta |\theta| m^n(\gamma,\ \theta) \\ &\quad = \sum_{\gamma_1 : \gamma_1 \not\sim \gamma} e^{-\beta|\gamma_1|} \sum_{\gamma_2 : \gamma_2 \not\sim \gamma_1} e^{-\beta|\gamma_2|} \cdots \sum_{\theta : \theta \not\sim \gamma_{n-1}} |\theta| e^{-\beta|\theta|} \\ &\quad = |\gamma| \sum_{\gamma_1 : \gamma_1 \not\sim \gamma} \frac{|\gamma_1|}{|\gamma|} e^{-\beta|\gamma_1|} \sum_{\gamma_2 : \gamma_2 \not\sim \gamma_1} \frac{|\gamma_2|}{|\gamma_1|} e^{-\beta|\gamma_2|} \cdots \sum_{\theta : \theta \not\sim \gamma_{n-1}} \frac{|\theta|}{|\gamma_{n-1}|} e^{-\beta|\theta|} \\ &\quad \le |\gamma| \left(\sup_\gamma \sum_{\theta : \theta \not\sim \gamma} \frac{|\theta|}{|\gamma|} e^{-\beta|\theta|} \right)^n. \end{aligned} \tag{5.14}$$

□

This lemma shows, in particular, that the branching process $\mathbf{b}_n$ is subcritical if $\alpha < 1$.

5.2. *Continuous-time branching process.* Let C be a cylinder with basis γ and birth time 0. Combing backwards continuously in time the branching clan $\mathbf{B}^C$, we define a continuous-time multitype branching process $\psi_t^\gamma(\theta) =$ number of contours of type θ present at time t (of this process) whose initial configuration is δ_γ. Each $C' \in \mathbf{B}^C$ is a branch, that is, belongs to the first generation of ancestors of a unique cylinder $U(C')$ in $\mathbf{B}^C$. In the branching process ψ_t all the branches (ancestors) of $U(C')$ appear simultaneously at the birth of $U(C')$, that is, when $U(C')$ disappears if we look backward in time.

Therefore the part of C' in the interval $[\mathrm{Birth}(U(C')), \mathrm{Death}(C')]$ is ignored. Formally,

$$\psi_t^\gamma(\theta) = \left|\{C' \in \mathbf{B}^C : \mathrm{Basis}(C') = \theta, \mathrm{Birth}(C') < -t < \mathrm{Birth}\big(U(C')\big)\}\right|. \tag{5.15}$$

In the process ψ_t each contour γ lives a mean-one exponential time after which it dies and gives birth to k_θ contours θ, $\theta \in \mathbf{G}$, with probability

$$\prod_\theta \frac{e^{m(\gamma,\,\theta)}\, m(\gamma,\ \theta)^{k_\theta}}{k_\theta!} \tag{5.16}$$

for $k_\theta \ge 0$. These are independent Poisson distributions of mean $m(\gamma,\ \theta)$. The infinitesimal generator of the process is given by

$$Lf(\psi) = \sum_{\gamma\in\mathbf{G}} \psi(\gamma) \sum_{\eta\in\mathscr{Y}_0(\gamma)} \prod_{\theta:\eta(\theta)\ge 1} \frac{e^{m(\gamma,\theta)} m(\gamma,\theta)^{\eta(\theta)}}{\eta(\theta)!}\big[f(\psi+\eta-\delta_\gamma)-f(\psi)\big], \tag{5.17}$$

where $\psi,\ \eta \in \mathscr{Y}_0 = \{\psi \in \mathbb{N}^{\mathbf{G}};\ \sum_\theta \psi(\theta) < \infty\}$ and $\mathscr{Y}_0(\gamma) = \{\psi \in \mathscr{Y}_0;\ \psi(\theta) \ge 1$ implies $\theta \not\prec \gamma\}$ and $f : \mathscr{Y}_0 \to \mathbb{N}$.

The branching process ψ_t^γ allows us to estimate the time length of a clan, due to the obvious fact:

$$\sum_\theta \psi_t^\gamma(\theta) = 0 \quad \text{implies} \quad \mathrm{TL}(\mathbf{B}^C) < t. \tag{5.18}$$

Let $M_t(\gamma,\ \theta)$ be the mean number of contours of type θ in ψ_t and $R_t(\gamma)$ its sum over θ:

$$M_t(\gamma,\ \theta) = \mathbb{E}\psi_t^\gamma(\theta), \qquad R_t(\gamma) = \sum_\theta M_t(\gamma,\ \theta). \tag{5.19}$$

The bound we need is given in the next lemma.

LEMMA 5.2. *The mean number of branches $R_t(\gamma)$ satisfies*

$$\mathbb{P}\Big(\sum_\theta \psi_t^\gamma(\theta) > 0\Big) \ \le\ R_t(\gamma) \ \le\ |\gamma|\, e^{(\alpha-1)t}. \tag{5.20}$$

PROOF. The first inequality is immediate because $\sum_\theta \psi_t^\gamma(\theta)$ assumes nonnegative integer values and $R_t(\gamma)$ is its mean value.

To show the second inequality, we first use the generator given by (5.17) to get the Kolmogorov backward equations for $R_t(\gamma)$:

$$\frac{d}{dt} R_t(\gamma) = \sum_{\gamma'} m(\gamma,\ \gamma')R_t(\gamma') - R_t(\gamma). \tag{5.21}$$

Since $R_0(\gamma') \equiv 1$, the solution is

$$R_t(\gamma) = \sum_{\gamma'} \big[\exp\big[t(m - I)\big]\big](\gamma,\ \gamma'), \tag{5.22}$$

where m is the matrix with entries $m(\gamma,\ \gamma')$ and I is the identity matrix. This can be rewritten as

$$
\begin{aligned}
R_t(\gamma) &= e^{-t} \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\gamma'} m^n(\gamma,\ \gamma') \\
&\leq e^{-t} \sum_{n\geq 0} \frac{t^n}{n!} |\gamma| \alpha^n,
\end{aligned}
\tag{5.23}
$$

where m^n is the nth power of the matrix m; the last bound is just the leftmost inequality in (5.13). □

5.3. *Time length and space width.* We are now ready to provide bounds for the time length and space width of the percolation clan.

THEOREM 5.1. *If $\beta > \beta^*$ [i.e., $\alpha(\beta) < 1$], then*

(i) *The probability of backward oriented percolation is* 0.

(ii) *For any positive b,*

$$\mathbb{P}\big(\mathrm{TL}(\mathbf{A}^{x,\,t}) > bt\big) \leq \alpha_0 e^{-(1-\alpha)bt}. \tag{5.24}$$

(iii)

$$\mathbb{E}\big(\mathrm{SW}(\mathbf{A}^{x,\,t})\big) \leq \frac{\alpha_0(\beta)}{1-\alpha(\beta)}. \tag{5.25}$$

(iv)

$$\mathbb{E}\big(\exp\big[a\mathrm{SW}(\mathbf{A}^{x,\,t})\big]\big) \leq \frac{\alpha_0(\beta - a)}{1-\alpha(\beta - a)}. \tag{5.26}$$

(v)

$$\mathbb{P}\big(\mathrm{SW}(\mathbf{A}^{x,\,t}) \geq \ell\big) \leq \frac{\alpha_0(\tilde\beta)}{1-\alpha(\tilde\beta)} e^{-(\beta-\tilde\beta)\ell} \tag{5.27}$$

for any $\tilde\beta \in (\beta^*,\ \beta)$.

PROOF. (i) We follow an idea of Hall (1985). For each $C \in \mathbf{C}$, we use the domination (5.5) and the identity (5.11). Therefore, to prove that there is no backward oriented percolation, it is enough to prove that, for fixed γ,

$$\mathbb{P}\bigg(\sum_\theta \mathbf{b}_n^\gamma(\theta) \neq 0 \text{ for infinitely many } n\bigg) = 0. \tag{5.28}$$

Since $\mathbf{b}_n^\gamma(\theta)$ assumes nonnegative integer values, by the Borel–Cantelli lemma, a sufficient condition for (5.28) is

$$\sum_n \sum_\theta m^n(\gamma,\ \theta) < \infty. \tag{5.29}$$

But this follows from Lemma 5.1.

(ii) By (5.18) and (5.9), for each $x \in \gamma$ and $s \le t$,

$$\sum_{\theta} \psi_t^{\gamma}(\theta) = 0 \text{ implies } \mathrm{TL}(\mathbf{A}^{x,0}) \le t. \tag{5.30}$$

Hence

$$\begin{aligned} \mathbb{P}\big(\mathrm{TL}(\mathbf{A}^{x,0}) > t\big) &\le \sum_{\gamma \ni x} \mathbb{P}\big(\eta_0(\gamma) = 1\big) R_t(\gamma) \\ &\le \sum_{\gamma \ni x} e^{-\beta|\gamma|}\, R_t(\gamma) \le \alpha_0 e^{-(1-\alpha)t} \end{aligned} \tag{5.31}$$

by the rightmost inequality in (5.20).

(iii) We find an upper bound for the space diameter of the backward percolation clan through and upper bound for the total number of occupied points by the multitype branching process $\mathbf{b}_n$ defined by (5.10). In fact,

$$\mathrm{SW}(\mathbf{A}^{x,0}) \le \sum_{\gamma \ni x} \eta_0(\gamma) \sum_{n} \sum_{\theta} |\theta|\, \mathbf{b}_n^{\gamma}(\theta). \tag{5.32}$$

By (3.31), $\mathbb{E}\eta_0(\gamma) \le e^{-\beta|\gamma|}$. Hence, by (5.13),

$$\begin{aligned} &\mathbb{E}\Big(\sum_{\gamma \ni x} \eta_0(\gamma) \sum_{n} \sum_{\theta} |\theta| \mathbf{b}_n^{\gamma}(\theta)\Big) \\ &\quad \le \sum_{\gamma \ni x} e^{-\beta|\gamma|} \sum_{n} \sum_{\theta} |\theta| m^n(\gamma,\ \theta) \\ &\quad \le \alpha_0 \sum_{n} \alpha^n. \end{aligned} \tag{5.33}$$

(iv) Write

$$\begin{aligned} \mathbb{E}(e^{aSW}) &= \sum_{\ell} e^{a\ell}\, \mathbb{P}(SW = \ell) \\ &\le \sum_{\ell} e^{a\ell} \sum_{k} \sum_{\gamma_1,\dots,\gamma_k} \mathbf{1}\big\{|\gamma_1 \cup \cdots \cup \gamma_k| = \ell\big\} \\ &\qquad \times \mathbb{P}\Big(\gamma_1 \ni 0, \mathbf{b}_1^{\gamma_1}(\gamma_2) \ge 1, \dots, \mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \ge 1\Big). \end{aligned} \tag{5.34}$$

By the Markovian property of $\mathbf{b}_n$, we get

$$\begin{aligned} &\mathbb{P}\Big(\gamma_1 \ni 0, \mathbf{b}_1^{\gamma_1}(\gamma_2) \ge 1, \dots, \mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \ge 1\Big) \\ &\quad = \mathbb{P}(\gamma_1 \ni 0)\, \mathbb{P}\Big(\mathbf{b}_1^{\gamma_1}(\gamma_2) \ge 1\Big) \cdots \mathbb{P}\Big(\mathbf{b}_1^{\gamma_{k-1}}(\gamma_k) \ge 1\Big) \\ &\quad \le \mathbf{1}\big\{\gamma_1 \ni 0,\ \gamma_1 \not\sim \gamma_2, \dots, \gamma_{k-1} \not\sim \gamma_k\big\} \prod_{i=1}^{k} \exp(-\beta|\gamma_i|). \end{aligned} \tag{5.35}$$

Substituting this in (5.34) and using

$$e^{a\ell}\, \mathbf{1}\big\{|\gamma_1 \cup \cdots \cup \gamma_k| = \ell\big\} \le \mathbf{1}\big\{|\gamma_1 \cup \cdots \cup \gamma_k| = \ell\big\} \exp\Big(a \sum_{i=1}^{k} |\gamma_i|\Big), \tag{5.36}$$

we get

$$
\begin{aligned}
\mathbb{E}(e^{aSW}) &\le \sum_k \sum_{\gamma_1,\dots,\gamma_k} \mathbf{1}\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \dots, \gamma_{k-1} \not\sim \gamma_k\} \exp\Biggl(-(\beta-a)\sum_{i=1}^{k}|\gamma_i|\Biggr) \\
&= \sum_k \sum_{\gamma_1 \ni 0} |\gamma_1| \exp(-(\beta-a)|\gamma_1|) \\
&\quad\times \frac{1}{|\gamma_1|} \sum_{\gamma_2 \not\sim \gamma_1} |\gamma_2| \exp(-(\beta-a)|\gamma_2|) \cdots \frac{1}{|\gamma_{k-1}|} \sum_{\gamma_k \not\sim \gamma_{k-1}} \exp(-(\beta-a)|\gamma_k|) \\
&\le \alpha_0(\beta-a) \sum_{k\ge 0} \alpha(\beta-a)^k.
\end{aligned}
\tag{5.37}
$$

(v) It suffices to use (iv) and the exponential Chebyshev inequality and to notice that a must be less than $\beta - \beta^*$ to avoid a 0 in the denominator of (5.26). □

REMARKS. (i) Part (ii) can, in fact, be proven by a more elementary argument not requiring the continuous-time construction of Section 5.2. The argument gives the same rate of decay as in (5.24) but a worse leading constant. Let us sketch it:

$$
\begin{aligned}
&\mathbb{P}\Bigl(\mathrm{TL}(\mathbf{A}^{x,0}) > t\Bigr) \\
&\quad\le \sum_k \sum_{\gamma_1,\dots,\gamma_k} \mathbf{1}\{\gamma_1 \ni 0, \gamma_1 \not\sim \gamma_2, \dots, \gamma_{k-1} \not\sim \gamma_k\} \mathbb{P}(S_1 + \cdots + S_k > t),
\end{aligned}
\tag{5.38}
$$

where S_i are independent mean-one exponentially distributed random variables and independent of γ_i. The time S_i represents the period between the birth of γ_i and γ_{i+1}. As the sum of independent exponentials is a gamma distribution with parameters k and 1,

$$
\mathbb{P}(S_1 + \cdots + S_k > t) = e^{-t} \sum_{i=0}^{k} \frac{t^i}{i!}. \tag{5.39}
$$

Therefore (5.38) is bounded by

$$
\begin{aligned}
&e^{-t} \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{k\ge i} \sum_{\gamma_1 \ni 0} |\gamma_1| \exp(-\beta|\gamma_1|) \\
&\qquad\qquad \times \frac{1}{|\gamma_1|} \sum_{\gamma_2 \not\sim \gamma_1} |\gamma_2| \exp(-\beta|\gamma_2|) \cdots \frac{1}{|\gamma_{k-1}|} \sum_{\gamma_k \not\sim \gamma_{k-1}} \exp(-\beta|\gamma_k|) \\
&\quad\le e^{-t} \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{k\ge i} \alpha_0 \alpha^{k-1} \\
&\quad= \frac{\alpha_0}{\alpha(1-\alpha)} e^{-(1-\alpha)t}.
\end{aligned}
\tag{5.40}
$$

(ii) In Fernández, Ferrari and Garcia (1998) we offered an alternative proof of part (iii), based on a computation of the exponential moment of the total population of a subcritical single-type branching process, which dominates the space width. However, this proof works in a smaller range of β.

6. Proof of Theorem 2.1. The following theorem shows that the condition $\alpha < \infty$ implies the hypothesis of Theorem 3.1. This proves Theorem 2.1.

THEOREM 6.1. *If $\alpha < \infty$, then, for all x and positive t, the set $\mathbf{A}^{x,t} \cap \mathbf{C}[0,t]$ has a finite number of cylinders with probability* 1.

PROOF. By time translation invariance it is sufficient to prove that $\mathbf{A}^{x,0} \cap \mathbf{C}[-t,0]$ is finite with probability 1. Let C be a cylinder with basis γ and birth time 0. Recall the definition of $U(C')$ just before (5.15) and define

$$\tilde{\psi}_t^{\gamma}(\theta) = \big|\{C' \in \mathbf{B}^C : \text{Basis}(C') = \theta, -t < \text{Birth}(U(C'))\}\big|. \tag{6.1}$$

The process $\tilde{\psi}_t^{\gamma}$ signals all contours born in $[0,t]$ in the process ψ_t^{γ}. Notice that, for $x \in \gamma$,

$$\big|\mathbf{A}^{x,0} \cap \mathbf{C}[-t,0]\big| \le \big|\{C \in \mathbf{B}^{x,0} : \text{Birth}(C) \in [-t,0]\}\big| \le \sum_{\theta} \tilde{\psi}_t^{\gamma}(\theta). \tag{6.2}$$

We prove that this is finite with probability 1 by showing it has a finite mean. Indeed, reasoning as in the previous section,

$$\mathbf{E}\bigg(\sum_{\theta} \tilde{\psi}_t^{\gamma}(\theta)\bigg) = \sum_{\theta} (e^{mt})(\gamma,\theta) \le |\gamma| e^{t\alpha} < \infty \tag{6.3}$$

if $\alpha < \infty$. □

7. Proof of Theorem 2.2. We prove that the hypotheses of Theorem 2.2 imply those of Theorem 4.1. The different parts of Theorem 2.2 follow by combining Theorem 4.1 and the space-width and time-length estimations of Theorem 5.1.

Existence and uniqueness. In Theorem 5.1(i) the condition $\alpha(\beta) < 1$ was shown to imply lack of backward oriented percolation. This, plus part 1 of Theorem 4.1, proves part 1 of Theorem 2.2.

Exponential time convergence. Inequality (2.9) follows from (5.24), (5.25) and (4.8) by choosing $b = (2-\alpha)^{-1}$.

Exponential space convergence. In view of (4.20) it suffices to bound

$$\begin{aligned} &\mathbb{P}\big(\text{SW}\big(\mathbf{A}\big(\text{Supp}(f)\big)\big) \ge d\big(\text{Supp}(f), \Lambda^c\big)\big) \\ &\qquad \le \sum_{x \in \text{Supp}(f)} \mathbb{P}\big(\text{SW}(\mathbf{A}^{x,0}) \ge d\big(\{x\}, \Lambda^c\big)\big). \end{aligned} \tag{7.1}$$

By (5.27), this is bounded by

$$\sum_{x\in\mathrm{Supp}(f)} \frac{\alpha_0(\tilde\beta)}{1-\alpha(\tilde\beta)} \exp(-(\beta-\tilde\beta)\,d(\{x\},\Lambda^c)). \tag{7.2}$$

Exponential mixing. We shall use part 4 of Theorem 4.1. We first show, in the next lemma, that $\alpha < \infty$ implies the existence of an h such that there is nonoriented percolation in the interval $(0, h)$.

LEMMA 7.1. *For all* $h > 0$ *such that*

$$\alpha h < 1, \tag{7.3}$$

the probability that there is no (*nonoriented*) *percolation in* $(0, h)$ *is* 1.

PROOF. Analogously to Section 5.1, we dominate the construction of the set ancestors of the nonoriented percolation process by a multitype branching process. In this branching process, the mean number of ancestors θ of a contour γ is

$$\overline{m}(\gamma,\theta) = h\,\mathbf{1}\{\theta \not\sim \gamma\} e^{-\beta|\theta|}. \tag{7.4}$$

As in Lemma 5.1, this branching process is subcritical if $\overline{\alpha} = \alpha h < 1$. □

We now prove (2.11). From (3.33) and (3.11)

$$\mu_\Lambda(fg) - \mu_\Lambda f\,\mu_\Lambda g = \mathbb{E}\big(f(\eta_0)g(\eta_0)\big) - \mathbb{E}f(\eta_0)\mathbb{E}g(\eta_0). \tag{7.5}$$

By (4.11) it is enough to bound

$$\mathbb{P}\big(\mathbf{A}\big(\mathrm{Supp}(f)\big) \not\sim \widehat{\mathbf{A}}\big(\mathrm{Supp}(g)\big)\big), \tag{7.6}$$

where $\widehat{\mathbf{A}}(\mathrm{Supp}(g))$ has the same distribution as $\mathbf{A}(\mathrm{Supp}(g))$ but is independent of $\mathbf{A}(\mathrm{Supp}(f))$. This is bounded by

$$\sum_{\substack{x\in\mathrm{Supp}(f),\\ y\in\mathrm{Supp}(g)}} \mathbb{P}\big(\mathbf{A}^{x,0}\not\sim\widehat{\mathbf{A}}^{y,0}\big) \le \sum_{\substack{x\in\mathrm{Supp}(f),\\ y\in\mathrm{Supp}(g)}} \mathbb{P}\big(\mathrm{SW}(\mathbf{A}^{x,0})+\mathrm{SW}(\widehat{\mathbf{A}}^{y,0})\ge|x-y|\big). \tag{7.7}$$

Using the following inequality, which is valid for independent random variables S_1 and S_2,

$$\mathbb{P}(S_1+S_2\ge\ell) \le \sum_{j=1}^{\ell} \mathbb{P}(S_1\ge j)\mathbb{P}(S_2\ge \ell-j) \tag{7.8}$$

and the exponential decay of (5.27), we get the decay stated in (2.11).

Central limit theorem. We apply the central limit theorem for stationary mixing random fields proven by Bolthausen (1982). Let $X_x = \tau_x f$. Let $\mathscr{A}_\Lambda$ be the σ-algebra generated by $\{X_x : x \in \Lambda\}$. Define

$$\alpha_{k,\ell}(n) = \sup\bigl\{|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)| : A_1 \in \mathscr{A}_{\Lambda_1}, A_2 \in \mathscr{A}_{\Lambda_2}, |\Lambda_1| \le k, |\Lambda_2| \le \ell, d(\Lambda_1, \Lambda_2) \ge n\bigr\}. \tag{7.9}$$

The simplified version of the Bolthausen theorem stated in Remark 1, page 1049, of his paper says that if there exists a $\delta > 0$ such that $||X_x||_{2+\delta} < \infty$ and

$$\sum_{n=1}^{\infty} n^{d-1}\bigl(\alpha_{2,\infty}(n)\bigr)^{\delta/(2+\delta)} < \infty, \tag{7.10}$$

then $D < \infty$ and (2.12) holds. Hence it suffices to show that $\alpha_{2,\infty}(n)$ decays exponentially fast with n. We can write

$$\alpha_{2,\infty}(n) = \sup_{a, g_1, g_2} \bigl|\mu(g_1 g_2) - \mu g_1 \mu g_2\bigr|, \tag{7.11}$$

where the supremum is taken over the set of $a \in \mathbb{Z}^d$, g_1 in the set of indicator functions with support on $\mathrm{Supp}(f) \bigcup \tau_a \mathrm{Supp}(f)$ and g_2 in the set of indicator functions with support in

$$\bigcup\bigl\{\tau_y \mathrm{Supp}(f) : y \in \mathbb{Z}^d \text{ and } |y - x| \ge n \ \forall x \in \mathrm{Supp}(f)\bigr\}. \tag{7.12}$$

By (2.12),

$$\begin{aligned} \alpha_{2,\infty}(n) &\le 2(M_2)^2 \sum_{\substack{x \in \mathrm{Supp}(g_1),\\ y \in \mathrm{Supp}(g_2)}} |x - y| \exp(-M_3|x - y|) \\ &\le 4(M_2)^2 |\mathrm{Supp}(f)| \sum_{|y| \ge n - 2|\mathrm{Supp}(f)|} \exp(-M_3|y|) \end{aligned} \tag{7.13}$$

because $|\mathrm{Supp}(g_1)| \le 2|\mathrm{Supp}(f)|$ and $\|g_1\|_\infty = \|g_2\|_\infty = 1$. Hence $\alpha_{2,\infty}(n)$ decreases exponentially fast with n. □

8. Proof of Theorem 2.3: Poisson approximation. We define first a common probability space where all processes $\{\eta^\beta : \beta^* < \beta \le \infty\}$ can simultaneously be constructed. For each $\gamma \in \widetilde{\mathbf{G}} = \bigcup_j \widetilde{\mathbf{G}}_j$, let $\mathbf{N}_\gamma$ be a marked Poisson process on $\mathbb{R}^{d+2}$ of rate 1. The event points of this process are denoted by (u, t, r, s), where $u \in \mathbb{R}^d$, $t, r \in \mathbb{R}$ and $s \in \mathbb{R}^+$. The coordinate t is interpreted as time while the coordinate r is later used to tune the rate of the projected process (u, t). The coordinate s—the *mark*—is an exponential random variable with mean 1 independent of everything (used later to determine the lifetime of the corresponding point/cylinder). Denote by $\mathbb{P}$ the product measure generated by $(\mathbf{N}_\gamma : \gamma \in \widetilde{\mathbf{G}})$, and by $\mathbb{E}$ the corresponding expectation. We identify the random counting measure $\mathbf{N}_\gamma$ with the corresponding discrete random subset of $R^{d+1} \times \mathbb{R}^+$.

Fix a contour length j and an inverse temperature β. By counting only those points in $\mathbf{N}_\theta$ whose r coordinate is in $[0, e^{-\beta(|\theta|-j)}]$, we generate the $(d+1)$-dimensional marked process

$$\mathbf{N}_{\theta,\beta} = \text{ marked Poisson process of rate } e^{-\beta(|\theta|-j)}. \tag{8.1}$$

The life of each point $(u, t, s) \in \mathbf{N}_{\theta,\beta} \times \mathbb{R}^+$ is the interval $[t, t+s]$.

Define a family of marked point processes indexed by β and $\tau_x\theta$, $x \in \mathbb{Z}^d$, $\theta \in \widetilde{G}$ for Borel sets $I \subset \mathbb{R}$ by

$$N_{\tau_x\theta,\beta}(I) = \mathbf{N}_{\theta,\beta}\big(O(xe^{-\beta j/d},\ e^{-\beta j/d}/2) \times I\big), \tag{8.2}$$

where, for $y = (y_1, \dots, y_d) \in \mathbb{Z}^d$, O is the d-dimensional “rectangle”

$$O(y, \rho) = [y_1 - \rho, y_1 + \rho] \times \cdots \times [y_d - \rho, y_d + \rho]. \tag{8.3}$$

Since the volume of $O(xe^{-\beta j/d},\ e^{-\beta j/d}/2)$ is $e^{-\beta j}$ and $\mathbf{N}_{\theta,\beta}$ has rate $e^{-\beta(|\theta|-j)}$, the resulting process $N_{\tau_x\theta,\beta}(I)$ is a one-dimensional marked Poisson process of rate $e^{-\beta|\theta|}$. The marks are the independent exponentially distributed random variables of mean 1, inherited from $\mathbf{N}_{\theta,\beta}$. The point of this construction is that all these Poisson processes are constructed simultaneously as a function of the original $(d+2)$-dimensional Poisson processes.

Now for each β we use the processes $N_{\tau_x\theta,\beta}$ to perform the graphical construction of Section 3.1. We call $\mathbf{C}^\beta$ the family of cylinders so obtained. Let ξ_t^β be the free network of Section 3.2 and η_t^β the loss networks of (3.29). As in (3.8) and (3.33) these processes have invariant distributions μ_β^0 and μ_β, respectively.

Let V be a d-dimensional rectangle as in the statement of the theorem. Let

$$\begin{aligned} M^0_{\gamma,\beta}(V) &= \sum_{x \in V \cdot e^{\beta|\gamma|/d}} \xi_0^\beta(\tau_x\gamma) \\ &= \sum_{x \in V \cdot e^{\beta|\gamma|/d}} \sum_{C \in \mathbf{C}^\beta} \mathbf{1}\big\{\mathrm{Basis}(C) = \tau_x\gamma, \mathrm{Life}(C) \ni 0\big\} \end{aligned} \tag{8.4}$$

as in (3.6). The superscript 0 on the left-hand side indicates that we are dealing with the free process ξ_t^β, while the subscript 0 on the right-hand side indicates time 0. The family $(M^0_{\gamma,\beta}(V) : \gamma \in \widetilde{G}_j)$ consists of $|\widetilde{G}_j|$ independent Poisson random variables with mean

$$\mathbb{E}M^0_{\gamma,\beta}(V) = |V \cdot e^{\beta|\gamma|/d}|\, e^{-\beta|\gamma|}. \tag{8.5}$$

By (3.29), $\eta_0^\beta(\gamma)$ constructed with the cylinders in $\mathbf{C}^\beta$ is μ_β distributed. Thus we can use $\eta_0^\beta(\gamma)$ in the definition (2.14) of $M_{\gamma,\beta}$. By (3.30),

$$\eta_0^\beta(\gamma) \le \xi_0^\beta(\gamma). \tag{8.6}$$

Hence $M_{\gamma,\beta}(V) \le M^0_{\gamma,\beta}(V)$.

The joint construction also implies that

$$\begin{aligned}
&\mathbb{P}\Big(M^0_{\gamma,\beta}(V) - M_{\gamma,\beta}(V) \ge 1\Big)\\
&\quad\le \sum_{x\in V\cdot e^{\beta|\gamma|/d}} \mathbb{P}\Big(\xi^\beta_0(\tau_x\gamma) - \eta^\beta_0(\tau_x\gamma) \ge 1\Big)\\
&\quad\le \sum_{x\in V\cdot e^{\beta|\gamma|/d}} \Big[\mathbb{P}\Big(\xi^\beta_0(\tau_x\gamma) \ge 1,\, \eta^\beta_0(\tau_x\gamma) = 0\Big)\\
&\qquad\qquad + \mathbb{P}\Big(\xi^\beta_0(\tau_x\gamma) \ge 2,\, \eta^\beta_0(\tau_x\gamma) = 1\Big)\Big].
\end{aligned} \tag{8.7}$$

From the construction, for any $\theta \in \mathbf{G}$,

$$\begin{aligned}
&\big\{\xi^\beta_0(\theta) \ge 1,\, \eta^\beta_0(\theta) = 0\big\}\\
&\qquad\subset \big\{\mathbf{C}^\beta : \mathbf{C}^\beta \ni C \text{ with } \mathrm{Basis}(C) = \theta, \mathrm{Life}(C) \ni 0, \mathbf{A}^C_1 \ne \varnothing\big\}.
\end{aligned} \tag{8.8}$$

The probability of this last event is bounded by

$$\mathbb{P}\big(\xi^\beta_0(\theta) \ge 1\big)\mathbb{P}\bigg(\sum_{\theta'} \mathbf{b}^\theta_1(\theta') \ge 1\bigg). \tag{8.9}$$

Since $\mathbb{P}(\xi^\beta_0(\theta) \ge 1) = 1 - \exp(e^{-\beta|\theta|}) \le e^{-\beta|\theta|}$, (8.9) is bounded above by

$$e^{-\beta|\theta|} \sum_{\theta' :\, \theta' \not\sim \theta} e^{-\beta|\theta'|} \le e^{-\beta|\theta|}|\theta|\alpha(\beta). \tag{8.10}$$

On the other hand,

$$\mathbb{P}\Big(\xi^\beta_0(\theta) \ge 2,\, \eta^\beta_0(\theta) = 1\Big) \le \mathbb{P}\Big(\xi^\beta_0(\theta) \ge 2\Big) \le \tfrac{1}{2}e^{-2\beta|\theta|}. \tag{8.11}$$

From (8.7)–(8.11) we get

$$\begin{aligned}
&\mathbb{P}(M^0_{\gamma,\beta}(V) - M_{\gamma,\beta}(V) \ge 1)\\
&\quad\le \Big|V\cdot e^{\beta|\gamma|/d}\Big| e^{-\beta|\gamma|}\Big(|\gamma|\alpha(\beta) + \frac{1}{2}e^{-\beta|\gamma|}\Big) \sim e^{-2d\beta}.
\end{aligned} \tag{8.12}$$

To finish the proof of (2.15), we must show that $M^0_{\gamma,\beta}$ is close to a Poisson process. For $|\gamma| = j$, let $M_{\gamma,\infty}$ count those points of the Poisson process $\mathbf{N}_{\gamma,\infty}$ whose life contains the origin. The process $M_{\gamma,\infty}$ is a Poisson process in $\mathbb{R}^d$ of rate 1. This is because the lifetimes are independent exponentials of mean 1 and, for $|\gamma| = j$, $\mathbf{N}_{\gamma,\infty}$ is a Poisson process of rate 1. The family $\{M_{\gamma,\infty} : \gamma \in \widetilde{G}_j\}$ inherits independence from $\{\mathbf{N}_\gamma : \gamma \in \widetilde{\mathbf{G}}\}$. For $J \subset \mathbb{Z}^d$, let

$$J : a = \big\{r \in \mathbb{R}^d : ra \in J + [1/2, 1/2]^d\big\} \subset \mathbb{R}^d. \tag{8.13}$$

By definition,

$$M^0_{\gamma,\beta}(V) = M_{\gamma,\infty}\big((V\cdot e^{\beta j/d}) : e^{\beta j/d}\big). \tag{8.14}$$

Then, as $(V \cdot e^{\beta j/d}) : e^{\beta j/d} \subset V$,

$$
\begin{aligned}
\mathbb{P}\Big(M^0_{\gamma,\beta}(V) - M_{\gamma,\infty}(V) \neq 0\Big) &\leq \mathbb{P}\Big(M_{\gamma,\infty}\Big(V \backslash \big[(V \cdot e^{\beta j/d}) : e^{\beta j/d}\big]\Big) > 0\Big) \\
&\leq \Big|V \backslash \big[(V \cdot e^{\beta j/d}) : e^{\beta j/d}\big]\Big| \\
&\leq 2d|V|^{(d-1)/d} e^{-\beta j/d}.
\end{aligned} \tag{8.15}
$$

Inequality (2.15) follows from (8.12) and (8.15).

Proposition I.2 of Neveu (1977) and the comments following the statement of the proposition say that the distribution on finite unions of d-dimensional finite-volume rectangles is enough to characterize a point process. Since the estimates (2.15) can be easily extended to finite unions of rectangles, the weak convergence follows. □

Acknowledgments. We thank Jean Bricmont, Joel L. Lebowitz, Enzo Olivieri, Errico Presutti, Timo Seppäläinen, Alan Sokal and Bernard Ycart for instructive discussions and criticism.

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