Loss of regularity for Kolmogorov equations

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Abstract

The celebrated Hörmander condition is a sufficient (and nearly necessary) condition for a second-order linear Kolmogorov partial differential equation (PDE) with smooth coefficients to be hypoelliptic. As a consequence, the solutions of Kolmogorov PDEs are smooth at all positive times if the coefficients of the PDE are smooth and satisfy Hörmander's condition even if the initial function is only continuous but not differentiable. First-order linear Kolmogorov PDEs with smooth coefficients do not have this smoothing effect but at least preserve regularity in the sense that solutions are smooth if their initial functions are smooth. In this article, we consider the intermediate regime of non-hypoelliptic second-order Kolmogorov PDEs with smooth coefficients. The main observation of this article is that there exist counterexamples to regularity preservation in that case. More precisely, we give an example of a second-order linear Kolmogorov PDE with globally bounded and smooth coefficients and a smooth initial function with compact support such that the unique globally bounded viscosity solution of the PDE is not even locally Hölder continuous and, thereby, we disprove the existence of globally bounded classical solutions of this PDE. From the perspective of probability theory, this observation has the consequence that there exists a stochastic differential equation (SDE) with globally bounded and smooth coefficients and a smooth function with compact support which is mapped by the transition semigroup of the SDE to a non-locally Hölder continuous function. In other words, degenerate noise can have a roughening effect. A further implication of this loss of regularity phenomenon is that numerical approximations may convergence slower than any arbitrarily small polynomial rate of convergence to the true solution of the SDE. More precisely, we prove for an example SDE with globally bounded and smooth coefficients that the standard Euler approximations converge to the exact solution of the SDE in the strong and numerically weak sense slower than any arbitrarily small polynomial rate of convergence.

1 Introduction and main results

The key observation of this article is to reveal the phenomenon of *loss of regularity* in Kolmogorov partial differential equations (PDEs). This observation has a direct consequence on the literature on *regularity analysis of linear PDEs*, on the literature on *regularity analysis of stochastic differential equations* (SDEs) and on the literature on *numerical approximations of SDEs*. We will illustrate the implications for each field separately.

Regularity analysis of linear partial differential equations Let $d, m \in \mathbb{N} := \{1, 2, \ldots\}$ be natural numbers, let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be infinitely often differentiable functions satisfying the coercivity property that there exists a real number $\rho \in \mathbb{R}$ such that $\langle x, \mu(x)\rangle \leq \rho\left(1 + \|x\|^2\right)$ and $\|\sigma(x)\|^2_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq \rho\left(1 + \|x\|^2\right)$ for all $x \in \mathbb{R}^d$ and let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a globally

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bounded and continuous function. Then we consider the second-order PDE

$$\tfrac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m}\sigma_{i,k}(x)\cdot\sigma_{j,k}(x)\cdot\tfrac{\partial^2}{\partial x_i\partial x_j}u(t,x) + \sum_{i=1}^{d}\mu_i(x)\cdot\tfrac{\partial}{\partial x_i}u(t,x) \tag{1a}$$

$$u(0,x) = \varphi(x) \tag{1b}$$

for $(t,x)\in(0,\infty)\times\mathbb{R}^d$. The PDE (1) is referred to as *Kolmogorov equation* in the literature (see, for example, Cerrai [5], Da Prato [10], Röckner [63], Röckner & Sobol [64]; it is also referred to as *Kolmogorov backward equation* or *Kolmogorov PDE*, see, e.g., Da Prato & Zabczyk [11], Øksendal [58]). It has a strong link to probability theory and appeared first (in a slightly different form; see display (125) in [44]) in Kolmogorov's celebrated paper [44]. Corollary 4.11 in Section 4 below implies that the PDE (1) admits a unique globally bounded viscosity solution. More precisely, Corollary 4.11 proves that there exists a unique globally bounded continuous function $u\colon[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ such that $u|_{(0,\infty)\times\mathbb{R}^d}$ is a viscosity solution of (1a) and such that $u(0,x)=\varphi(x)$ for all $x\in\mathbb{R}^d$. In this article, we are interested to know whether solutions u of the PDE (1a) *preserve regularity* in the sense that $u|_{(0,\infty)\times\mathbb{R}^d}$ is smooth if the initial function $u(0,\cdot)=\varphi(\cdot)$ is smooth. In particular, we will answer the question whether smoothness and global boundedness of the initial function $\varphi\colon\mathbb{R}^d\to\mathbb{R}$ implies the existence of a *classical solution* of the PDE (1).

In the case of first-order Kolmogorov PDEs with smooth coefficients, that is, $\sigma\equiv 0$ in (1a), regularity preservation of solutions of (1a) is well known. More precisely, if $\sigma(x)=0$ for all $x\in\mathbb{R}^d$ and if the initial function $\varphi\colon\mathbb{R}^d\to\mathbb{R}^d$ in (1b) is smooth, then it is well known that there exists a unique smooth classical solution of (1). In this sense, the PDE (1a) is *regularity preserving* in the purely first-order case $\sigma\equiv 0$. In the second-order case $\sigma\not\equiv 0$, the situation may be even better in the sense that the PDE (1a) often has a *smoothing effect*. More precisely, if the PDE (1a) is *hypoelliptic*, then by definition solutions u of the PDE (1a) are smooth in the sense that $u|_{(0,\infty)\times\mathbb{R}^d}$ is infinitely often differentiable even if the initial function $u(0,\cdot)=\varphi(\cdot)$ is only continuous but not differentiable. In the seminal paper [32], Hörmander gave a sufficient (and also nearly necessary; see the discussion before Theorem 1.1 in [32] and Section 2 in Hairer [27]) condition for (1a) to be hypoelliptic; see Theorem 1.1 in [32]. To formulate Hörmander's condition, define the vector field $\sigma_0\colon\mathbb{R}^d\to\mathbb{R}^d$ through $\sigma_0(x)=\mu(x)-\frac{1}{2}\sum_{k=1}^{m}\sigma_k'(x)\,\sigma_k(x)$ for all $x\in\mathbb{R}^d$. Then the *Hörmander condition* is fulfilled if

$$\operatorname{span}\big\{\sigma_{i_0}(x),[\sigma_{i_0},\sigma_{i_1}](x),\big[[\sigma_{i_0},\sigma_{i_1}],\sigma_{i_2}\big](x),\ldots\in\mathbb{R}^d\colon i_0,i_1,i_2,\ldots\in\{0,1,\ldots,m\},i_0\neq 0\big\}=\mathbb{R}^d \tag{2}$$

for all $x\in\mathbb{R}^d$ where $[f,g](x):=f'(x)g(x)-g'(x)f(x)$ for all $x\in\mathbb{R}^d$ is the Lie-bracket of two smooth vector fields $f,g\colon\mathbb{R}^d\to\mathbb{R}^d$. Consequently, if Hörmander's condition (2) is satisfied, then the PDE (1) admits a unique globally bounded smooth classical solution even if the initial function $\varphi\colon\mathbb{R}^d\to\mathbb{R}$ is assumed to be continuous and globally bounded only. Clearly, there are many cases where the Hörmander condition (2) fails to be fulfilled and where (1a) is not hypoelliptic, e.g., if $\sigma\equiv 0$. To the best of our knowledge, regularity of solutions of the PDE (1a) is in general unkown in the non-hypoelliptic case.

In this article, we address the question whether the second-order linear PDEs with smooth coefficients of the form (1a) at least preserve regularity in the non-hypoelliptic case. The following Theorem 1.1 answers this question to the negative. More precisely, the key observation of this article is to reveal the phenomenon of *loss of regularity* in the sense that the solution u of the PDE (1) starting in a smooth function $u(0,\cdot)\in\mathrm{C}^\infty_{cpt}(\mathbb{R}^d,\mathbb{R})$ with compact support may turn into a non-differentiable function $u(t,\cdot)\notin\mathrm{C}^1(\mathbb{R}^d,\mathbb{R})$ for every positive time $t\in(0,\infty)$. In analogy to the well-known "smoothing effect" in the hypoelliptic case, we will say in the case of loss of regularity that the PDE (1a) has a *roughening effect*. Here is a simple two-dimensional example with polynomial μ and linear σ which has this roughening effect. In the special case $d=2,m=1$ and $\mu(x)=\big(x_1\cdot x_2,-(x_1)^2\big)$ and $\sigma(x)=(0,x_2)$ for all $x=(x_1,x_2)\in\mathbb{R}^2$, the PDE (1a) reads as

$$\tfrac{\partial}{\partial t}u(t,x)=\tfrac{(x_2)^2}{2}\cdot\tfrac{\partial^2}{\partial x_2^2}u(t,x)+x_1x_2\cdot\tfrac{\partial}{\partial x_1}u(t,x)-(x_1)^2\cdot\tfrac{\partial}{\partial x_2}u(t,x) \tag{3}$$

for $(t,x)\in(0,\infty)\times\mathbb{R}^2$. Theorem 2.1 and Corollary 4.11 below imply that there exists an infinitely often differentiable function $\varphi\in\mathrm{C}^\infty_{cpt}(\mathbb{R}^d,\mathbb{R})$ with compact support such that the unique globally bounded viscosity solution $u\colon[0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ of the PDE (3) with $u(0,\cdot)=\varphi(\cdot)$ has the property that $u|_{(0,\infty)\times\mathbb{R}^d}$ is not differentiable and not locally Lipschitz continuous. In particular, we thereby disprove the existence of a globally bounded classical solution of the PDE (3) with $u(0,\cdot)=\varphi(\cdot)$. The drift coefficient μ of the PDE (3) grows superlinearly. One could wonder whether the roughening effect of example (3) is due to this superlinear growth of μ. To exclude this possibility, we prove for an example PDE with globally bounded and smooth coefficients that there exists a smooth initial function with compact support such that the solution u is not even locally Hölder continuous; see Theorem 1.1 below. In particular, Theorem 1.1 implies that, in general, the PDE (1) does not have a classical solution even if the coefficients and the initial function are *globally bounded* and *infinitely often differentiable.*

Theorem 1.1 (Disprove of the existence of classical solutions of the Kolmogorov PDE with smooth and globally bounded coefficients and initial function)**.** *There exists a natural number $d \in \mathbb{N}$, a globally bounded and infinitely often differentiable function $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, a symmetric nonnegative matrix $A = (A_{i,j})_{i,j\in\{1,2,\dots,d\}} \in \mathbb{R}^{d\times d}$ and an infinitely often differentiable function $\varphi \in \mathrm{C}^\infty_{cpt}(\mathbb{R}^d,\mathbb{R})$ with compact support such that there exists no globally bounded classical solution of the PDE*

$$\begin{aligned} \tfrac{\partial}{\partial t}u(t,x) &= \sum_{i,j=1}^{d} A_{i,j}\cdot \tfrac{\partial^2}{\partial x_i \partial x_j}u(t,x) + \sum_{i=1}^{d}\mu_i(x)\cdot \tfrac{\partial}{\partial x_i}u(t,x) \\ u(0,x) &= \varphi(x) \end{aligned} \tag{4}$$

for $(t,x)\in(0,\infty)\times\mathbb{R}^d$. In addition, there exists a unique globally bounded viscosity solution $u\colon[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ of (4) *and this function fails to be locally Hölder continuous.*

Theorem 1.1 follows immediately from Corollary 4.11 in Section 4 and from Theorem 3.1 in Section 3. More precisely, Corollary 4.11 and Theorem 3.1 imply that there exists an infinitely differentiable function $\varphi \in \mathrm{C}^\infty_{cpt}(\mathbb{R}^3,\mathbb{R})$ with compact support such that the unique globally bounded viscosity solution $u\colon[0,\infty)\times\mathbb{R}^3\to\mathbb{R}$ of the PDE

$$\tfrac{\partial}{\partial t}u(t,x) = \tfrac{\partial^2}{\partial x_2^2}u(t,x) + \cos\!\big(x_3\exp([x_2]^3)\big)\cdot\tfrac{\partial}{\partial x_1}u(t,x) \tag{5}$$

with initial condition $u(0,x)=\varphi(x)$ for $(t,x)=(t,x_1,x_2,x_3)\in(0,\infty)\times\mathbb{R}^3$ is not locally Hölder continuous. In particular, the PDE (5) with $u(0,\cdot)=\varphi(\cdot)$ has no globally bounded classical solution. The PDE (5) has a globally bounded and highly oscillating drift coefficient and a constant diffusion coefficient and serves as a counterexample to regularity preservation for Kolmogorov PDEs. An SDE with a globally bounded and highly oscillating diffusion coefficient and a vanishing drift coefficient has been presented in Li & Scheutzow [49] as a counterexample for strong completeness of SDEs. It is interesting to observe that the PDE (5) without the second-order term on the right-hand side of (5) preserves regularity and has a smooth classical solution, and that the PDE (5) without the first-order term on the right-hand side of (5) also preserves regularity and has a smooth classical solution. Thus, the roughening effect of the PDE (5) is a consequence of the interplay between the first-order and the second-order term in (5). We add that Theorem 3.4 in Section 3 is a stronger version of Theorem 1.1 in which the roughening effect appears on every arbitrarily small open subset of the state space; see Section 3 and also Theorem 1.2 below for more details. Note that the coefficients in our counterexample PDE (5) are analytic functions and that the initial function $\varphi\colon\mathbb{R}^d\to\mathbb{R}$ may be chosen to be analytic (see Theorem 3.1 for details). We emphasize that this does not contradict the classical Cauchy-Kovalevskaya Theorem (e.g., Theorem 4.6.2 in Evans [18]) proving existence, uniqueness and analyticity of solutions of PDEs with analytic coefficients as the Cauchy-Kovalevskaya Theorem applies to (4) in the case $A=0$ only. Moreover, we would like to point out that Theorem 1.1 does not contradict to Theorems 7.1.3, 7.1.4 and 7.1.7 in Evans [18] which show the existence of a unique classical solution of (4) if A is strictly positive (note that A in (5) is nonnegative but not strictly positive).

Theorem 1.1 shows that a general existence theorem for globally bounded classical solutions of the PDE (1) cannot be established. However, it is possible to ensure the existence of a viscosity solution of the PDE (1) under rather general assumptions on the coefficients. More precisely, one of our main results, Theorem 4.10 below, establishes the existence of a within a certain class unique viscosity solution for every second-order linear Kolmogorov PDE whose coefficients are locally Lipschitz continuous and satisfy the Lyapunov-type inequality (81). To the best of our knowledge, this is the first result in the literature proving existence and uniqueness of solutions of the Kolmogorov PDE (1) in the above generality; see also the discussion after Theorem 4.10 for a short review of existence and uniqueness results for Kolmogorov PDEs. A crucial result on the route to Theorem 4.10 is the uniqueness result of Corollary 4.7 for viscosity solutions of degenerate parabolic second-order linear PDEs.

The roughening effect of the PDE (1a) revealed in this first paragraph of this introduction has a direct consequence on the literature on regularity analysis of SDEs. This is subject of the next paragraph.

Regularity analysis of stochastic differential equations For the rest of this introduction, we use the following notation. Let $(\Omega,\mathcal{F},\mathbb{P})$ be an arbitrary probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ which supports a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion $W\colon[0,\infty)\times\Omega\to\mathbb{R}^m$ with continuous sample paths. It is a classical result that the above assumptions on μ and σ ensure the existence of a family $X^x=(X^x_1,\dots,X^x_d)\colon[0,\infty)\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, of up to indistinguishability unique solution processes (see, e.g., Theorem 3.1.1 in [62]) with continuous sample paths of the SDE

$$dX^x(t) = \mu(X^x(t))\,dt + \sigma(X^x(t))\,dW(t) \tag{6}$$

for $t\in(0,\infty)$ and $x\in\mathbb{R}^d$ and with $X^x(0)=x$ for all $x\in\mathbb{R}^d$ (see, e.g., Theorem 1 in Krylov [46]). Here, the function $\mu\colon\mathbb{R}^d\to\mathbb{R}^d$ is the infinitesimal mean and the function $\sigma\cdot\sigma^*\colon\mathbb{R}^d\to\mathbb{R}^{d\times d}$ is the infinitesimal covariance matrix of the SDE (6). It is also well known that the coercivity assumption on μ and the linear growth bound on σ additionally imply moment bounds $\sup_{x\in\{y\in\mathbb{R}^d\colon \|y\|\le p\}} \mathbb{E}\big[\sup_{t\in[0,p]} \|X^x(t)\|^p\big] < \infty$ for all $p \in [0,\infty)$ for the solution processes of the SDE (6). The *transition semigroup* $P_t\colon \mathrm{C}_b(\mathbb{R}^d,\mathbb{R}) \to \mathrm{C}_b(\mathbb{R}^d,\mathbb{R})$, $t \in [0,\infty)$, of the SDE (6) is defined through $(P_t\varphi)(x) := \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $t \in [0,\infty)$, $x \in \mathbb{R}^d$ and all $\varphi \in \mathrm{C}_b(\mathbb{R}^d,\mathbb{R})$ where $\mathrm{C}_b(\mathbb{R}^d,\mathbb{R})$ is as usual the space of globally bounded and continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. Note for every $\varphi \in \mathrm{C}_b(\mathbb{R}^d,\mathbb{R})$ that the function $\mathbb{R}^d \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ is continuous (see, e.g., Theorem 1.7 in Krylov [47]) and hence, the semigroup $(P_t)_{t\in[0,\infty)}$ is well-defined. Observe also that the function $\mathbb{R}^d \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ is continuous for every $\varphi \in \mathrm{C}_b(\mathbb{R}^d,\mathbb{R})$ although the SDE (6) is, in general, not strongly complete; see Li & Scheutzow [49] and see, e.g., also Elworthy [15], Kunita [48] and Fang, Imkeller & Zhan [19] for further results on strong completeness of SDEs.

Theorem 1.1 in Hörmander [32] and Proposition 4.12 below imply that if the Hörmander condition (2) is fulfilled, then the semigroup is smoothing in the sense that $P_t\big(\mathrm{C}_b(\mathbb{R}^d,\mathbb{R})\big) \subseteq \mathrm{C}_b^\infty(\mathbb{R}^d,\mathbb{R})$ for all $t \in (0,\infty)$. To the best of our knowledge, it remained an open question in the non-hypoelliptic case whether SDEs with infinitely often differentiable coefficients such as (6) preserve regularity in the sense that $P_t\big(\mathrm{C}_b^\infty(\mathbb{R}^d,\mathbb{R})\big) \subseteq \mathrm{C}_b^\infty(\mathbb{R}^d,\mathbb{R})$ for all $t \in (0,\infty)$. This article answers this question to the negative. More precisely, the following theorem reveals that smooth functions with compact support may be mapped to non-smooth functions by the transition semigroup of the SDE (6). In analogy to the well-known "smoothing effect" of many SDEs, we will say that the semigroup has a *roughening effect* in that case. Here is a simple two-dimensional example SDE with polynomial drift coefficient and linear diffusion coefficient which has this roughening effect. In the special case $d = 2$, $m = 1$ and $\mu(x) = \big(x_1 \cdot x_2, -(x_1)^2\big)$ and $\sigma(x) = (0, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, the SDE (6) reads as

$$\begin{aligned} dX_1^x(t) &= X_1^x(t) \cdot X_2^x(t)\, dt \\ dX_2^x(t) &= -\left(X_1^x(t)\right)^2\, dt + X_2^x(t)\, dW(t) \end{aligned} \tag{7}$$

for $t \in (0,\infty)$ and $x \in \mathbb{R}^2$. Observe that (3) is the Kolmogorov PDE of (7); see Corollary 4.11 for details. Moreoever, note that $\langle x, \mu(x)\rangle = 0$ for all $x \in \mathbb{R}^2$ in this example. Thus the solution process of the associated ordinary differential equation stays on the circle centered at $(0,0) \in \mathbb{R}^2$ going through the starting point. Theorem 2.1 in Section 2 shows for the SDE (7) that there exists an infinitely often differentiable function $\varphi \in \mathrm{C}_{cpt}^\infty(\mathbb{R}^d,\mathbb{R})$ with compact support such for every $t \in (0,\infty)$ the functions $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^2$ are continuous but not differentiable and not locally Lipschitz continuous. For every $t \in (0,\infty)$ we hence have the roughening effect $P_t\big(\mathrm{C}_{cpt}^\infty(\mathbb{R}^d,\mathbb{R})\big) \not\subseteq \mathrm{C}^1(\mathbb{R}^d,\mathbb{R})$ in the case of the SDE (7). The drift coefficient μ of the SDE (7) grows superlinearly. As above, the superlinear growth of μ is not necessary for the transition semigroup of the SDE to be roughening. This is subject of the next main result of this article.

Theorem 1.2 (A counterexample to regularity preservation with degenerate additive noise)**.** *There exists a natural number $d \in \mathbb{N}$, a globally bounded and infinitely often differentiable function $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and a constant function $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$, that is $\sigma(x) = \sigma(0)$ for all $x \in \mathbb{R}^d$, with the following properties. For every $t \in (0,\infty)$ the function $\mathbb{R}^d \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^d$ is continuous but nowhere locally Hölder continuous and for every open set $O \subset \mathbb{R}^d$ there exists an infinitely often differentiable function $\varphi \in \mathrm{C}_{cpt}^\infty(\mathbb{R}^d,\mathbb{R})$ with compact support such that the function $O \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ is continuous but not locally Hölder continuous. In particular, for every $t \in (0,\infty)$ we have $P_t\big(\mathrm{C}_{cpt}^\infty(\mathbb{R}^d,\mathbb{R})\big) \not\subseteq \cup_{\alpha\in(0,\infty)} \mathrm{C}^\alpha(\mathbb{R}^d,\mathbb{R})$.*

Theorem 1.2 follows immediately from Theorem 3.4 in Section 3. The roughening effect of some SDEs with smooth coefficients revealed through example (7) and Theorem 1.2 above, has a direct consequence on the literature on numerical approximations of SDEs. This is subject of the next paragraph.

Numerical approximations of stochastic differential equations Starting with Maruyama's adaptation of Euler's method to SDEs in 1955 (see [51]), an extensive literature on the numerical approximation of solutions of SDEs has been published in the last six decades; see, e.g., the books and overview articles [42, 43, 52, 24, 3, 53, 56, 38, 41] for extensive lists of references. A key objective in this field of research is to prove convergence of suitable numerical approximation processes to the solution process of the SDE and to establish a rate of convergence for the considered approximation scheme in the strong, in the almost sure or in the numerically weak sense.

Almost sure convergence rates of many numerical schemes such as the standard Euler method or the higher order Milstein method are well known for the SDE (6) and even for a much larger class of nonlinear SDEs; see Gyöngy [23] and Jentzen, Kloeden & Neuenkirch [39]. Many applications, however, require the numerical approximation of moments or other functionals of the solution process, for instance, the expected pay-off of an option in computational finance; see, e.g., Glasserman [22] for details. For this reason, applications are particularly interested in strong and numerically weak convergence rates. The vast majority of research results establishing strong and numerically weak convergence rates assume that the coefficients of the SDE are globally Lipschitz continuous or at least that they satisfy the *global monotonicity condition* that there exists a real number $\rho \in \mathbb{R}$ such that $\langle x-y, \mu(x)-\mu(y)\rangle + \frac{1}{2}\sum_{k=1}^m \|\sigma_k(x)-\sigma_k(y)\|^2 \le \rho\|x-y\|^2$ for all $x, y \in \mathbb{R}^d$ (see, e.g., Theorem 2.4 in Hu [34], Theorem 5.3 in Higham, Mao & Stuart [29], Schurz [66], Theorems 2 and 3 in Higham & Kloeden [28], Theorem 6.3 in Mao & Szpruch [50], Theorem 1.1 in Hutzenthaler, Jentzen & Kloeden [36], Theorem 3.2 in Gan & Wang [20]). Strong and numerically weak convergence rates without assuming global monotonicity are established in Göngy & Rasonyi [26] in the case of a class of scalar SDEs with globally Hölder continuous coefficients, in Dörsek [14] in the case of the two-dimensional stochastic Navier-Stokes equations and in Dereich, Neuenkirch & Szpruch [13], Alfonsi [1], Neuenkirch & Szpruch [57] in the case of a class of scalar SDEs (including, e.g., the Cox-Ingersoll-Ross process) that can be transformed in a suitable sense to SDEs that satisfy the global monotonicity assumption. The global monotonicity assumption is a serious restriction on the coefficients of the SDE and excludes many interesting SDEs in the literature (e.g., stochastic Lorenz equations, stochastic Duffing-van der Pol oscillators and the stochastic SIR model; see Section 4 in [35] for details and further examples). It remains an open problem to establish strong and numerically weak convergence rates in the general setting of the SDE (6).

In this article, we establish in the setting (6) the existence of an SDE with *globally bounded* and *infinitely often differentiable coefficients* for which the Euler approximations converge in the strong and in the numerically weak sense slower than any arbitrarily small polynomial rate of convergence. More precisely, our main result for the literature on the numerical approximation of SDEs is the following theorem.

Theorem 1.3 (A counterexample to the rate of convergence in the numerical approximation of nonlinear SDEs with additive noise)**.** *Let $T \in (0,\infty)$ and $x_0 \in \mathbb{R}^4$ be arbitrary. Then there exists a globally bounded and infinitely often differentiable function $\mu\colon \mathbb{R}^4 \to \mathbb{R}^4$ and a symmetric nonnegative matrix $B \in \mathbb{R}^{4\times 4}$ such that the stochastic process $X\colon [0,T]\times\Omega \to \mathbb{R}^4$ with continuous sample paths satisfying $X(t) = x_0 + \int_0^t \mu(X(s))\,ds + B\,W(t)$ for all $t \in [0,T]$ and its Euler-Maruyama approximations $Y^N\colon \{0, \frac{T}{N}, \frac{2T}{N}, \dots, T\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, satisfying $Y^N(0) = x_0$ and $Y^N(\frac{(n+1)T}{N}) = Y^N(\frac{nT}{N}) + \mu\big(Y^N(\frac{nT}{N})\big)\frac{T}{N} + B\,\big(W_{(n+1)T/N} - W_{nT/N}\big)$ for all $n \in \{0,1,\dots,N-1\}$, $N \in \mathbb{N}$, fulfill that*

$$\lim_{N\to\infty}\big(N^\alpha \cdot \mathbb{E}\big[\|X(T) - Y^N(T)\|\big]\big) = \lim_{N\to\infty}\big(N^\alpha \cdot \big\|\mathbb{E}[X(T)] - \mathbb{E}\big[Y^N(T)\big]\big\|\big) = \begin{cases} 0 & : \alpha = 0 \\ \infty & : \alpha > 0 \end{cases} \tag{8}$$

for all $\alpha \in [0,\infty)$. In particular, for every $\alpha \in (0,\infty)$ there exists no real number $c_\alpha \in (0,\infty)$ such that $\big\|\mathbb{E}\big[X(T)\big] - \mathbb{E}\big[Y^N(T)\big]\big\| \le c_\alpha \cdot N^{-\alpha}$ for all $N \in \mathbb{N}$.

Theorem 1.3 follows immediately from Theorem 5.1 in Section 5. In the deterministic case $\sigma \equiv 0$, it is well known that the Euler approximations converge to the solution process of (6) with the rate 1. In the stochastic case $\sigma \not\equiv 0$, this rate of convergence can often not be achieved. In particular, Clark & Cameron [6] proved for an SDE in the setting of (6) that a class of Euler-type schemes cannot, in general, converge strongly with a higher order than $\frac{1}{2}$. Since then, there have been many results on lower bounds of strong and numerically weak approximation errors for numerical approximation schemes of SDEs; see, e.g., [65, 4, 31, 30, 12, 54, 55, 56, 45] and the references therein. Now the observation of Theorem 1.3 is that there exist SDEs with smooth and globally bounded coefficients for which the standard Euler approximations converge in the strong and numerically weak sense *slower than any arbitrarily small polynomial rate of convergence*. To the best of our knowledge, Theorem 1.3 is the first result in the literature in which it has been established that Euler's method converges to the solution of an SDE with smooth coefficients in the strong and numerical weak sense slower than any arbitrarily small polynomial rate of convergence. Clearly, this lack of a rate of convergence is not a special property of the Euler scheme and holds for other schemes such as the Milstein scheme too. It is a consequence of the fact that the SDE (see (99)) to which we show that Euler's method converges in the strong and numerically weak sense slower than any arbitrarily small polynomial rate of convergence suffers under the *roughening effect* revealed in Theorems 1.1 and 1.2 (see Corollary 5.2 and Theorem 5.1 in Section 5 for details).

Comparing Theorem 5.1 with Theorem 2.4 in Gyöngy [23] reveals the remarkable difference that the Euler approximations for some SDEs have almost sure convergence rate $\frac{1}{2}-$ but no strong and no numerically weak rate of convergence. More formally, Theorem 2.4 in [23] shows in the setting of Theorem 1.3 that there exist finite random variables $C_\varepsilon\colon \Omega \to [0,\infty)$, $\varepsilon \in (0,\frac{1}{2})$, such that $\|X(T) - Y^N(T)\| \le C_\varepsilon \cdot N^{(\varepsilon - \frac{1}{2})}$ $\mathbb{P}$-a.s. for all $N \in \mathbb{N}$ and all $\varepsilon \in (0,\frac{1}{2})$. Taking expectation then results in $\mathbb{E}[\|X(T) - Y^N(T)\|] \le \mathbb{E}[C_\varepsilon] \cdot N^{(\varepsilon-\frac{1}{2})}$ for all $N \in \mathbb{N}$ and all $\varepsilon \in (0,\frac{1}{2})$ and from Theorem 1.3 we hence get that the error constants have infinite expectations, i.e., $\mathbb{E}[C_\varepsilon] = \infty$ for all $\varepsilon \in (0,\frac{1}{2})$. Finally, we emphasize that Monte Carlo simulations confirm the slow strong and numerically weak convergence phenomenon of Euler's method revealed in Theorem 1.3. For details, the reader is referred to Figure 1 in Section 5 below.

2 A counterexample to regularity preservation with linear multiplicative noise

In this section we establish the phenomenon of loss of regularity of the simple example SDE (7) with polynomial drift coefficient and linear diffusion coefficient. For this we consider the following setting. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $W\colon [0,\infty)\times\Omega\to\mathbb{R}$ be a one-dimensional standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion with continuous sample paths and let $X^x=(X_1^x,X_2^x)\colon [0,\infty)\times\Omega\to\mathbb{R}^2$, $x\in\mathbb{R}^2$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$\begin{aligned} dX_1^x(t) &= X_1^x(t)\cdot X_2^x(t)\,dt \\ dX_2^x(t) &= -\left(X_1^x(t)\right)^2\,dt + X_2^x(t)\,dW(t) \end{aligned} \tag{9}$$

for $t\in(0,\infty)$ and $x\in\mathbb{R}^2$ satisfying $X^x(0)=x$ for all $x\in\mathbb{R}^2$. The following Theorem 2.1 shows that the semigroup associated with the SDE (9) looses regularity in the sense that there exists an infinitely often differentiable function with compact support which is mapped to a non-smooth function by the semigroup.

Theorem 2.1 (A counterexample to regularity preservation with linear multiplicative noise)**.** *Let $X^x\colon [0,\infty)\times\Omega\to\mathbb{R}^2$, $x\in\mathbb{R}^2$, be solution processes of the SDE (9) with continuous sample paths and with $X^x(0)=x$ for all $x\in\mathbb{R}^2$. Then $\sup_{x\in\{y\in\mathbb{R}^2\colon\|y\|\leq p\}}\mathbb{E}\big[\sup_{t\in[0,p]}\|X^x(t)\|^p\big]<\infty$ for all $p\in[0,\infty)$ and there exists an infinitely often differentiable function $\varphi\in C^\infty_{cpt}(\mathbb{R}^2,\mathbb{R})$ with compact support such that for every $t,p\in(0,\infty)$ the mappings $\mathbb{R}^2\ni x\mapsto\mathbb{E}\big[X^x(t)\big]\in\mathbb{R}^2$, $\mathbb{R}^2\ni x\mapsto\mathbb{E}\big[\varphi(X^x(t))\big]\in\mathbb{R}$ and $\mathbb{R}^2\ni x\mapsto X^x(t)\in L^p(\Omega;\mathbb{R}^2)$ are continuous but not locally Lipschitz continuous and not differentiable.*

The proof of Theorem 2.1 is deferred to the end of this section. The proof of Theorem 2.1 uses the following simple lemma. Its proof is straightforward and therefore omitted.

Lemma 2.2 (Restricted exponential integrals of a geometric Brownian motion)**.** *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $W\colon[0,\infty)\times\Omega\to\mathbb{R}$ be a one-dimensional standard Brownian motion. Then*

$$\mathbb{E}\left[\mathbb{1}_{\{a\leq e^{W(t)}\leq b\}}\exp\!\left(c\cdot\int_0^t e^{W(s)}\,ds\right)\right]=\infty \tag{10}$$

for all $t,a,b,c\in(0,\infty)$ with $a<b$.

The proof of the following Lemma 2.3 makes use of Lemma 2.2. Using Lemma 2.3, the proof of Theorem 2.1 is then completed at the end of this section.

Lemma 2.3. *Let $X^x\colon[0,\infty)\times\Omega\to\mathbb{R}^2$, $x\in\mathbb{R}^2$, be solution processes of the SDE (9) with continuous sample paths and with $X^x(0)=x$ for all $x\in\mathbb{R}^2$. Then $\sup_{x\in\{y\in\mathbb{R}^2\colon\|y\|\leq p\}}\mathbb{E}\big[\sup_{t\in[0,p]}\|X^x(t)\|^p\big]<\infty$ for all $p\in[0,\infty)$ and*

$$\lim_{0\neq x_1\to 0}\left(\tfrac{1}{x_1}\cdot\mathbb{E}\Big[X_1^{(x_1,x_2)}(t)-X_1^{(0,x_2)}(t)\Big]\right)=\infty=\lim_{0\neq x_1\to 0}\left(\tfrac{1}{|x_1|}\cdot\left\|X_1^{(x_1,x_2)}(t)-X_1^{(0,x_2)}(t)\right\|_{L^p(\Omega;\mathbb{R})}\right) \tag{11}$$

for all $t,x_2,p\in(0,\infty)$ and there exists an infinitely often differentiable function $\varphi\in C^\infty_{cpt}(\mathbb{R}^2,\mathbb{R})$ with compact support such that $\lim_{0\neq x_1\to0}\left(\tfrac{1}{x_1}\cdot\mathbb{E}\big[\varphi\big(X^{(x_1,x_2)}(t)\big)-\varphi\big(X^{(0,x_2)}(t)\big)\big]\right)=\infty$ for all $t,x_2\in(0,\infty)$.

Proof of Lemma 2.3. Corollary 2.6 in Gyöngy & Krylov [25] guarantees the existence of a family of up to indistinguishability unique adapted stochastic processes $X^x\colon[0,\infty)\times\Omega\to\mathbb{R}^2$, $x\in\mathbb{R}^2$, with continuous sample paths satisfying (9). Moreover, the globally Lipschitz continuity of σ, the locally Lipschitz continuity of μ and $\langle x,\mu(x)\rangle=0$ for all $x\in\mathbb{R}^2$ imply that $\sup_{x\in\{y\in\mathbb{R}^2\colon\|y\|\leq p\}}\mathbb{E}\big[\sup_{t\in[0,p]}\|X^x(t)\|^p\big]<\infty$ for all $p\in[0,\infty)$. Next we disprove local Lipschitz continuity of the mapping $\mathbb{R}^2\ni x\mapsto X_1^x(t)\in L^p(\Omega;\mathbb{R})$ for every $t,p\in(0,\infty)$. More precisely, aiming at a contradiction, we assume that the second equality in (11) is false. Then there exist positive real numbers $t,x_2,p\in(0,\infty)$ and a sequence of real numbers $h_n\in\mathbb{R}\backslash\{0\}$, $n\in\mathbb{N}$, such that $\lim_{n\to\infty}h_n=0$ and such that $\limsup_{n\to\infty}\left(\frac{1}{|h_n|}\big\|X_1^{(h_n,x_2)}(t)-X_1^{(0,x_2)}(t)\big\|_{L^p(\Omega;\mathbb{R})}\right)<\infty$. Theorem 1.7 in Krylov [47] (see also Proposition 3.2.1 in Prévôt & Röckner [62]) yields that $\sup_{s\in[0,t]}\|X^{(h_n,x_2)}(s)-X^{(0,x_2)}(s)\|\to0$ in probability as $n\to\infty$. Hence, there exists a strictly increasing sequence $n_k\in\mathbb{N}$, $k\in\mathbb{N}$, of natural numbers such that $\lim_{k\to\infty}\sup_{s\in[0,t]}\|X^{(h_{n_k},x_2)}(s)-X^{(0,x_2)}(s)\|=0$ $\mathbb{P}$-a.s.; see, e.g., Corollary 6.13 in Klenke [40]. Applying this, Fatou's lemma and Lemma 2.2 implies

$$\begin{aligned}
\infty &> \limsup_{k\to\infty}\left(\tfrac{1}{|h_{n_k}|}\big\|X_1^{(h_{n_k},x_2)}(t)-X_1^{(0,x_2)}(t)\big\|_{L^p(\Omega;\mathbb{R})}\right)=\limsup_{k\to\infty}\left(\tfrac{1}{|h_{n_k}|}\big\|X_1^{(h_{n_k},x_2)}(t)\big\|_{L^p(\Omega;\mathbb{R})}\right)\\
&=\limsup_{k\to\infty}\left\|\exp\!\left(\int_0^t X_2^{(h_{n_k},x_2)}(s)\,ds\right)\right\|_{L^p(\Omega;\mathbb{R})}\geq\left\|\liminf_{k\to\infty}\left\{\exp\!\left(\int_0^t X_2^{(h_{n_k},x_2)}(s)\,ds\right)\right\}\right\|_{L^p(\Omega;\mathbb{R})}\\
&=\left\|\exp\!\left(\int_0^t X_2^{(0,x_2)}(s)\,ds\right)\right\|_{L^p(\Omega;\mathbb{R})}=\left\|\exp\!\left(\int_0^t e^{(W(s)-s/2)}\,ds\cdot x_2\right)\right\|_{L^p(\Omega;\mathbb{R})}\\
&\geq\left(\mathbb{E}\left[\exp\!\left(\int_0^t e^{W(s)}\,ds\cdot\frac{px_2}{e^{t/2}}\right)\cdot\mathbb{1}_{\{1\leq e^{W(t)}\leq 2\}}\right]\right)^{1/p}=\infty.
\end{aligned}\tag{12}$$

This contradiction implies that the second equality in (11) is true. The first equality in (11) follows from the second equality in (11). In the next step let $c \in (0,\infty)$ be an arbitrary fixed real number and let $\psi_1 \colon \mathbb{R} \to \mathbb{R}$ and $\psi_2 \colon \mathbb{R} \to [0,\infty)$ be two infinitely often differentiable functions with $x \cdot \psi_1(x) \geq 0$ for all $x \in \mathbb{R}$, with $\psi_1(x) = \psi_2(x) = 0$ for all $x \in \mathbb{R}\backslash[-c-1, c+1]$ and with $\psi_1(x) = x$ and $\psi_2(x) = 1$ for all $x \in [-c, c]$. Due to partition of unity, such functions indeed exist. Next let $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ be given by $\varphi(x_1,x_2) = \psi_1(x_1) \cdot \psi_2(x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Note that $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^2,\mathbb{R})$ is an infinitely often differentiable function with compact support. In addition, we have

$$\big\langle (\nabla\varphi)(0,x_2), y\big\rangle = \psi_1'(0) \cdot \psi_2(x_2) \cdot y_1 + \psi_1(0) \cdot \psi_2'(x_2) \cdot y_2 = \psi_2(x_2) \cdot y_1 \tag{13}$$

for all $x_2 \in \mathbb{R}$ and all $y = (y_1, y_2) \in \mathbb{R}^2$ and

$$\frac{\varphi(x_1,x_2) - \varphi(0,x_2)}{x_1} = \frac{\varphi(x_1,x_2)}{x_1} \in [0,\infty) \tag{14}$$

for all $x_1 \in \mathbb{R}\backslash\{0\}$ and all $x_2 \in (0,\infty)$. We now show that $\lim_{0\neq x_1\to 0}\left(\frac{1}{x_1} \cdot \mathbb{E}\big[\varphi\big(X^{(x_1,x_2)}(t)\big) - \varphi\big(X^{(0,x_2)}(t)\big)\big]\right) = \infty$ for all $t, x_2 \in (0,\infty)$. Aiming at a contradiction, assume that there exist positive real numbers $t, x_2 \in (0,\infty)$ and a sequence $h_n \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} h_n = 0$ and such that

$$\limsup_{n\to\infty}\left(\tfrac{1}{h_n} \cdot \mathbb{E}\Big[\varphi\big(X_1^{(h_n,x_2)}(t)\big) - \varphi\big(X_1^{(0,x_2)}(t)\big)\Big]\right) < \infty. \tag{15}$$

Theorem 1.7 in Krylov [47] yields that $\sup_{s\in[0,t]} \|X^{(h_n,x_2)}(s) - X^{(0,x_2)}(s)\| \to 0$ in probability as $n \to \infty$. Hence, there exists a strictly increasing sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, of natural numbers such that $\lim_{k\to\infty} \sup_{s\in[0,t]} \|X^{(h_{n_k},x_2)}(s) - X^{(0,x_2)}(s)\| = 0$ $\mathbb{P}$-a.s.; see, e.g., Corollary 6.13 in Klenke [40]. Applying this, Fatou's lemma and Lemma 2.2 then results in

$$\begin{aligned}
\infty &> \limsup_{k\to\infty}\left(\tfrac{1}{h_{n_k}}\mathbb{E}\Big[\varphi(X^{(h_{n_k},x_2)}(t)) - \varphi(X^{(0,x_2)}(t))\Big]\right) = \limsup_{k\to\infty}\mathbb{E}\left[\left|\tfrac{\varphi(X^{(h_{n_k},x_2)}(t))-\varphi(X^{(0,x_2)}(t))}{h_{n_k}}\right|\right] \\
&\geq \mathbb{E}\left[\liminf_{k\to\infty}\left|\tfrac{\varphi(X^{(h_{n_k},x_2)}(t))-\varphi(X^{(0,x_2)}(t))}{h_{n_k}}\right|\right] = \mathbb{E}\left[\liminf_{k\to\infty}\left(\tfrac{\varphi(X^{(h_{n_k},x_2)}(t))-\varphi(X^{(0,x_2)}(t))}{h_{n_k}}\right)\right] \\
&= \mathbb{E}\left[\psi_2\big(X^{(0,x_2)}(t)\big)\left(\liminf_{k\to\infty}\tfrac{X_1^{(h_{n_k},x_2)}(t)}{h_{n_k}}\right)\right] = \mathbb{E}\left[\psi_2\big(X_2^{(0,x_2)}(t)\big) \cdot \exp\!\left(\int_0^t e^{(W(s)-s/2)}\,ds \cdot x_2\right)\right] \\
&\geq \mathbb{E}\left[\mathbb{1}_{\left\{\frac{c}{2}\leq x_2\cdot\exp(W(t)-t/2)\leq c\right\}} \cdot \exp\!\left(\int_0^t e^{(W(s)-s/2)}\,ds \cdot x_2\right)\right] = \infty.
\end{aligned} \tag{16}$$

This contradiction implies that $\lim_{0\neq x_1\to 0}\left(\frac{1}{x_1} \cdot \mathbb{E}\big[\varphi\big(X^{(x_1,x_2)}(t)\big) - \varphi\big(X^{(0,x_2)}(t)\big)\big]\right) = \infty$ for all $t, x_2 \in (0,\infty)$. The proof of Lemma 2.3 is thus completed. ☐

Proof of Theorem 2.1. Theorem 1.7 in Krylov [47] (see also Proposition 3.2.1 in Prévôt & Röckner [62]), in particular, shows for every $t \in [0,\infty)$ that the mapping

$$\mathbb{R}^2 \ni x \mapsto X^x(t) \in L^0(\Omega;\mathbb{R}^2) \tag{17}$$

is continuous. This implies for every $\varphi \in \mathrm{C}_{cpt}^\infty(\mathbb{R}^2,\mathbb{R})$ and every $t \in [0,\infty)$ that the mapping $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ is continuous. Moreover, Lemma 2.3 proves that $\sup_{x\in\{y\in\mathbb{R}^2\colon \|y\|\leq p\}} \mathbb{E}\big[\sup_{t\in[0,p]}\|X^x(t)\|^p\big] < \infty$ for all $p \in [0,\infty)$. Combining this, (17), Corollary 6.21 in Klenke [40] and Theorem 6.25 in Klenke [40] shows for every $t, p \in [0,\infty)$ that the mappings $\mathbb{R}^2 \ni x \mapsto X^x(t) \in L^p(\Omega;\mathbb{R}^2)$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^2$ are continuous. Furthermore, Lemma 2.3 implies that there exists an infinitely often differentiable function $\varphi \in \mathrm{C}_{cpt}^\infty(\mathbb{R}^2,\mathbb{R})$ with compact support such that for every $t, p \in (0,\infty)$ the mappings $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^2$, $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ and $\mathbb{R}^2 \ni x \mapsto X^x(t) \in L^p(\Omega;\mathbb{R}^2)$ are not locally Lipschitz continuous and not differentiable. The proof of Theorem 2.1 is thus completed. ☐

In the remainder of this section, we briefly consider slightly modified versions of the SDE (9). The generator of the SDE (9) is nowhere elliptic. We remark that the phenomenon of loss of regularity may also appear for an SDE whose generator is in many points of the state space elliptic. For example, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $W = (W_1, W_2) \colon [0,\infty) \times \Omega \to \mathbb{R}^2$ be a two-dimensional standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion and let $X^x = (X_1^x, X_2^x) \colon [0,\infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$\begin{aligned}
dX_1^x(t) &= X_1^x(t) \cdot X_2^x(t)\,dt + X_1^x(t)\,dW_1(t) \\
dX_2^x(t) &= -\left(X_1^x(t)\right)^2\,dt + X_2^x(t)\,dW_2(t)
\end{aligned} \tag{18}$$

for $t \in (0,\infty)$ and $x \in \mathbb{R}^2$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^2$. The generator of the SDE (18) is in every point $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 \cdot x_2 \neq 0$ elliptic but there exists a function $\varphi \in C^\infty_{cpt}(\mathbb{R}^d, \mathbb{R})$ such that for every $t \in (0,\infty)$ the functions $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^2$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ are not locally Lipschitz continuous. The proof of this statement is completely analogous as in the case of the SDE (9). Furthermore, the same statement holds if the two independent standard Brownian motion in (18) are replaced by one and the same standard Brownian motion. More precisely, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and if $W\colon [0,\infty) \times \Omega \to \mathbb{R}$ is a one-dimensional standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, then the up to indistinguishability unique solution processes $X^x = (X^x_1, X^x_2)\colon [0,\infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, of the SDE

$$dX^x(t) = \begin{pmatrix} X^x_1(t) \cdot X^x_2(t) \\ -\left(X^x_1(t)\right)^2 \end{pmatrix} dt + X^x(t)\, dW(t) \tag{19}$$

for $t \in (0,\infty)$ and $x \in \mathbb{R}^2$ with continuous sample paths and with $X^x(0) = x$ for all $x \in \mathbb{R}^2$ fulfill that there exists a function $\varphi \in C^\infty_{cpt}(\mathbb{R}^2, \mathbb{R})$ such that for every $t \in (0,\infty)$ the functions $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^2$ and $\mathbb{R}^2 \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ are not locally Lipschitz continuous.

3 A counterexample to regularity preservation with degenerate additive noise

In this section we show the roughening effect for an example SDE with globally bounded and infinitely often differentiable coefficients. For this, it suffices to consider the following counterexample to regularity preservation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion and let $X^x = (X^x_1, X^x_2, X^x_3)\colon [0,\infty) \times \Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$\begin{aligned} dX^x_1(t) &= \cos\!\big(X^x_3(t) \cdot \exp\!\big([X^x_2(t)]^3\big)\big)\, dt \\ dX^x_2(t) &= \sqrt{2}\, dW(t) \\ dX^x_3(t) &= 0\, dt \end{aligned} \tag{20}$$

for $t \in [0,\infty)$ and $x \in \mathbb{R}^3$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^3$. Observe that $X^x_1(t) = x_1 + \int_0^t \cos\big(x_3 \cdot \exp\big([x_2 + \sqrt{2}\, W_2(s)]^3\big)\big)\, ds$ $\mathbb{P}$-a.s. for all $t \in [0,\infty)$ and all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Theorem 3.1. *Let $T \in (0,\infty)$ and let $X^x\colon [0,\infty)\times\Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be solution processes of the SDE* (20) *satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^3$. Then there exists an infinitely often differentiable function $\varphi \in \mathrm{C}^\infty_{cpt}(\mathbb{R}^3, \mathbb{R})$ with compact support such that for every $t \in (0,T]$ the functions $\mathbb{R}^3 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^3$ and $\mathbb{R}^3 \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ are continuous but not locally Hölder continuous.*

In the following, regularity properties of the solution processes $X^x = (X^x_1, X^x_2, X^x_3)\colon [0,\infty)\times\Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, of the SDE (20) are investigated in order to prove Theorem 3.1. To do so, we first establish a few auxiliary results. We begin with a simple lemma on trigonometric integrals.

Lemma 3.2. *Let $a, b \in \mathbb{R}$ be real numbers with $a < b$, let $\psi\colon [a,b] \to [0,\infty)$ be a continuously differentiable function and let $\varphi\colon [a,b] \to \mathbb{R}$ be a twice continuously differentiable function with $e^{i\cdot\varphi(a)} = i$ and with $\varphi'(x) \geq 0$, $\varphi''(x) \geq 0$ and $\psi'(x) \leq 0$ for all $x \in [a,b]$. Then $\int_a^b \cos(\varphi(x))\, \psi(x)\, dx \leq 0$.*

Proof of Lemma 3.2. First, assume w.l.o.g. that $\varphi(b) \geq \varphi(a) + \pi$ (otherwise we have $\cos(\varphi(x)) \leq 0$ for all $x \in [a,b]$ and hence $\int_a^b \cos(\varphi(x))\, \psi(x)\, dx \leq 0$). Moreover, assme w.l.o.g. that $\varphi'(x) > 0$ for all $x \in (a,b]$ (otherwise consider $\varphi|_{[\tilde{a},b]}\colon [\tilde{a},b] \to \mathbb{R}$ where $\tilde{a} := \inf(\{x \in [a,b]\colon \varphi'(x) > 0\} \cup \{b\})$ and observe that $\int_a^b \cos(\varphi(x))\, \psi(x)\, dx = \int_{\tilde{a}}^b \cos(\varphi(x))\, \psi(x)\, dx$). In particular, $\varphi\colon [a,b] \to \mathbb{R}$ is strictly increasing and there exists a unique strictly increasing continuous function $\varphi^{-1}\colon [\varphi(a), \varphi(b)] \to [a,b]$ with $\varphi^{-1}(\varphi(x)) = x$ for all $x \in [a,b]$ and with $\varphi(\varphi^{-1}(x)) = x$ and $(\varphi^{-1})'(x) = \frac{1}{\varphi'(\varphi^{-1}(x))} > 0$ for all $x \in (\varphi(a), \varphi(b))$. The transformation rule and integration by parts therefore imply

$$\begin{aligned} &\int_a^b \cos(\varphi(x))\, \psi(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} \cos(x) \cdot \psi\big(\varphi^{-1}(x)\big) \cdot (\varphi^{-1})'(x)\, dx = \int_{\varphi(a)}^{\varphi(b)} \frac{\cos(x) \cdot \psi(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))}\, dx \\ &= \frac{[\sin(\varphi(b)) - 1]\, \psi\big(\varphi^{-1}(\varphi(b))\big)}{\varphi'\big(\varphi^{-1}(\varphi(b))\big)} - \int_{\varphi(a)}^{\varphi(b)} [\sin(x) - 1] \left[\frac{\psi'(\varphi^{-1}(x))}{[\varphi'(\varphi^{-1}(x))]^2} - \frac{\psi(\varphi^{-1}(x))\varphi''(\varphi^{-1}(x))}{[\varphi'(\varphi^{-1}(x))]^3} \right] dx \leq 0. \end{aligned} \tag{21}$$

This completes the proof of Lemma 3.2. ☐

The next lemma analyzes suitable regularity properties of the solution processes $X^x = (X_1^x, X_2^x, X_3^x)\colon [0,\infty)\times\Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, of the SDE (20) are investigated. Its proof is based on an application of Lemma 3.2.

Lemma 3.3 (A lower bound)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion. Then*

$$1 - \mathbb{E}\big[\cos\!\big(h \cdot \exp\!\big([x + W(t)]^3\big)\big)\big] \geq \exp\!\Big(\tfrac{-8}{t}\,\Big[|\ln(\tfrac{\pi}{2h})|^{2/3} + x^2\Big]\Big), \tag{22}$$

$$\int_0^t \mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\,\Big(1 - \cos\!\big(h \cdot e^{[x+W(s)]^3}\big)\Big)\Big]\,ds \geq \frac{t}{3}\cdot\mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}e^{\frac{-72|W(t)|^2}{t}}\Big]\cdot\exp\!\Big(\tfrac{-72}{t}\,\Big[|\ln(\tfrac{\pi}{2h})|^{2/3} + x^2\Big]\Big)$$

for all $h \in \big(0, \frac{\pi}{2}\exp\big(-|[\sqrt{t}+x]\vee 0|^3\big)\big]$, $t \in (0,\infty)$, $x \in \mathbb{R}$ and all Borel measurable sets $A \subset \mathbb{R}$.

Proof of Lemma 3.3. First of all, define a family $\varphi_{t,x,h}\colon \big[\frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}, \infty\big) \to \mathbb{R}$, $(t,x,h) \in \big\{(0,\infty)\times\mathbb{R}\times(0,\infty)\colon h \leq \frac{\pi}{2}\exp(-|x\vee 0|^3)\big\}$, of functions through $\varphi_{t,x,h}(y) := h\cdot\exp\!\big([x+\sqrt{t}y]^3\big)$ for all $y \in \big[\frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}, \infty\big)$, $t \in (0,\infty)$, $h \in \big(0, \frac{\pi}{2}\exp(-|x\vee 0|^3)\big]$ and all $x \in \mathbb{R}$. Observe that $\varphi'_{t,x,h}(y) = 3\sqrt{t}\,\big[x+\sqrt{t}y\big]^2\varphi_{t,x,h}(y) \geq 0$ and $\varphi''_{t,x,h}(y) = 6t\big[x+\sqrt{t}y\big]\varphi_{t,x,h}(y) + 9t\big[x+\sqrt{t}y\big]^4\varphi_{t,x,h}(y) \geq 0$ for all $y \in \big[\frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}, \infty\big)$, $t \in (0,\infty)$, $h \in \big(0, \frac{\pi}{2}\exp(-|x\vee 0|^3)\big]$ and all $x \in \mathbb{R}$. In addition, note that $\varphi_{t,x,h}\big(\frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}\big) = \frac{\pi}{2}$ for all $t \in (0,\infty)$, $h \in \big(0, \frac{\pi}{2}\exp(-|x\vee 0|^3)\big]$ and all $x \in \mathbb{R}$. We can thus apply Lemma 3.2 to obtain that

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}}^{\infty} \cos\!\Big(h\cdot\exp\!\big(\big[x+\sqrt{t}y\big]^3\big)\Big)\,e^{\frac{-y^2}{2}}\,dy \leq 0 \tag{23}$$

for all $t \in (0,\infty)$, $h \in \big(0, \frac{\pi}{2}\exp(-|x\vee 0|^3)\big]$ and all $x \in \mathbb{R}$. This implies

$$\begin{aligned}\mathbb{E}\big[\cos\!\big(h\cdot\exp\!\big([x+W(t)]^3\big)\big)\big] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos\!\Big(h\cdot\exp\!\big(\big[x+\sqrt{t}y\big]^3\big)\Big)\,e^{\frac{-y^2}{2}}\,dy\\ &\leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}}\cos\!\Big(h\cdot\exp\!\big(\big[x+\sqrt{t}y\big]^3\big)\Big)\,e^{\frac{-y^2}{2}}\,dy = 1 - \mathbb{P}\bigg[W_1 > \frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}\bigg]\end{aligned} \tag{24}$$

for all $t \in (0,\infty)$, $h \in \big(0, \frac{\pi}{2}\exp(-|x\vee 0|^3)\big]$ and all $x \in \mathbb{R}$. Moreover, Lemma 22.2 in Klenke [40] yields $\mathbb{P}[W_1 > y] \geq \frac{e^{-\frac{y^2}{2}}}{y\sqrt{2\pi}(1+y^{-2})} \geq \frac{e^{-\frac{y^2}{2}}}{y\sqrt{8\pi}} \geq e^{-4y^2}$ for all $y \in [1,\infty)$. Combining this and inequality (24) then shows

$$1 - \mathbb{E}\big[\cos\!\big(h\cdot\exp\!\big([x+W(t)]^3\big)\big)\big] \geq \mathbb{P}\bigg[W_1 > \frac{[\ln(\pi/(2h))]^{1/3}-x}{\sqrt{t}}\bigg] \geq \exp\!\bigg(\frac{-4\,\big|[\ln(\pi/(2h))]^{1/3}-x\big|^2}{t}\bigg) \tag{25}$$

for all $h \in \big(0, \frac{\pi}{2}\exp\big(-|[\sqrt{t}+x]\vee 0|^3\big)\big]$, $t \in (0,\infty)$ and all $x \in \mathbb{R}$ and the estimate $-|a+b|^2 \geq -2a^2 - 2b^2$ for all $a, b \in \mathbb{R}$ therefore results in the first inequality in (22). Next the first inequality in (22) implies

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\,\big|1-\cos\!\big(h\cdot\exp\!\big([x+W(s)]^3\big)\big)\big|\Big]\\ &= \mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\,\mathbb{E}\Big[1-\cos\!\big(h\cdot\exp\!\big(\big[x+\tfrac{s}{t}W(t)+W(s)-\tfrac{s}{t}W(t)\big]^3\big)\big)\big|W(t)\Big]\Big]\\ &\geq \mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\exp\!\Big(\tfrac{-8t}{s(t-s)}\,\Big[|\ln(\tfrac{\pi}{2h})|^{2/3} + \big[x+\tfrac{s}{t}W(t)\big]^2\Big]\Big)\Big]\end{aligned} \tag{26}$$

for all $h \in \big(0, \frac{\pi}{2}\exp\big(-|[\sqrt{t}+x]\vee 0|^3\big)\big]$, $x \in \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R})$ and all $s, t \in (0,\infty)$ with $s < t$. We denote here by $\mathcal{B}(\mathbb{R})$ the Borel sigma-algebra on $\mathbb{R}$. Hence, we get

$$\begin{aligned}&\int_0^t \mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\,\big|1-\cos\!\big(h\cdot\exp\!\big([x+W(s)]^3\big)\big)\big|\Big]\,ds\\ &\geq \int_{\frac{t}{3}}^{\frac{2t}{3}} \mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\,\big|1-\cos\!\big(h\cdot\exp\!\big([x+W(s)]^3\big)\big)\big|\Big]\,ds\\ &\geq \int_{\frac{t}{3}}^{\frac{2t}{3}} \mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\exp\!\Big(\tfrac{-8t}{s(t-s)}\,\Big[|\ln(\tfrac{\pi}{2h})|^{2/3} + \big[x+\tfrac{s}{t}W(t)\big]^2\Big]\Big)\Big]\,ds\\ &\geq \frac{t}{3}\cdot\mathbb{E}\Big[\mathbb{1}_{\{W(t)\in A\}}\exp\!\Big(\tfrac{-72}{t}\,\Big[|\ln(\tfrac{\pi}{2h})|^{2/3} + x^2 + |W(t)|^2\Big]\Big)\Big]\end{aligned} \tag{27}$$

for all $h \in \big(0, \frac{\pi}{2}\exp\big(-|[\sqrt{t}+x]\vee 0|^3\big)\big]$, $t \in (0,\infty)$, $x \in \mathbb{R}$ and all $A \in \mathcal{B}(\mathbb{R})$. This completes the proof of Lemma 3.3. ☐

We are now ready to prove Theorem 3.4 stated at the beginning of this section. Its proof uses the lower bound established in Lemma 3.3 above.

Proof of Theorem 3.1. First, note that Lemma 3.3 implies for every $t \in (0,\infty)$ that $\lim_{h\searrow 0} h^{-\varepsilon}\mathbb{E}\big[X_1^{(0,0,0)}(t) - X_1^{(0,0,h)}(t)\big] = \infty$ for all $\varepsilon \in (0,\infty)$. We hence get for every $t \in (0,\infty)$ that the function $\mathbb{R}^3 \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^3$ is not locally Hölder continuous. Moreover, let $\psi\colon \mathbb{R} \to [0,1]$ be an infinitely often differentiable function with compact support and with $\psi(x) = 1$ for all $x \in [-T,T]$ and let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ be a function given by $\varphi(x_1,x_2,x_3) = x_1\psi(x_1)\psi(x_2)\psi(x_3)$ for all $x_1,x_2,x_3 \in \mathbb{R}$. Again Lemma 3.3 then shows

$$
\begin{aligned}
&\lim_{h\searrow 0}\Big(h^{-\varepsilon}\cdot\mathbb{E}\big[\varphi\big(X^{(0,0,0)}(t)\big) - \varphi\big(X^{(0,0,h)}(t)\big)\big]\Big) = \lim_{h\searrow 0}\Big(h^{-\varepsilon}\cdot\mathbb{E}\Big[\big(X_1^{(0,0,0)}(t) - X_1^{(0,0,h)}(t)\big)\psi\big(\sqrt{2}W_2(t)\big)\Big]\Big)\\
&\geq \lim_{h\searrow 0}\Big(h^{-\varepsilon}\cdot\mathbb{E}\Big[\mathbb{1}_{\{|\sqrt{2}W_2(t)|\leq T\}}\big(X_1^{(0,0,0)}(t) - X_1^{(0,0,h)}(t)\big)\Big]\Big) = \infty
\end{aligned}
\tag{28}
$$

for all $t \in (0,T]$. The proof of Theorem 3.1 is thus completed. $\square$

In the remainder of this section, we briefly consider a slightly modified version of the SDE (20). More formally, let $(\mathbb{Z}_n)_{n\in\mathbb{N}_0}$ be a family of sets defined through $\mathbb{Z}_0 := \mathbb{Z} := \{\dots,-2,-1,0,1,2,\dots\}$ and through $\mathbb{Z}_n := \big\{z \in \mathbb{Z}\colon \frac{z}{2} \notin \mathbb{Z}\big\} = \{\dots,-3,-1,1,3,\dots\}$ for all $n \in \mathbb{N}$. Then let $\mu = (\mu_1,\mu_2,\mu_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $B \in \mathbb{R}^3$ be given by

$$
\mu(x) = \begin{pmatrix} \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}_n}\frac{1}{4^{(n+|m|)}}\cos\big((x_3 - \frac{m}{2^n})\exp\big([x_2]^3\big)\big) \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\tag{29}
$$

for all $x = (x_1,x_2,x_3) \in \mathbb{R}^3$. Note that $\mu\colon \mathbb{R}^3 \to \mathbb{R}^3$ is infinitely often differentiable and globally bounded by 2. Moreover, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion and let $X^x\colon [0,\infty)\times\Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be the up to indistinguishability unique solution processes with continuous sample paths of the SDE

$$
dX^x(t) = \mu(X^x(t))\,dt + B\,dW(t)
\tag{30}
$$

for $t \in [0,\infty)$ and $x \in \mathbb{R}^3$ satisfying $X^x(0) = x$ for all $x \in \mathbb{R}^3$. The following Theorem 3.4 establishes that the function $[0,\infty)\times\mathbb{R}^3 \to \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^3$ is nowhere locally Hölder continuous. Its proof is a straightforward consequence of Lemma 3.3 and therefore omitted.

Theorem 3.4 (A counterexample to regularity preservation with degenerate additive noise)**.** *Let $c,T \in (0,\infty)$ and let $X^x\colon [0,\infty)\times\Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be solution processes of the SDE* (30) *with continuous sample paths and with $X^x(0) = x$ for all $x \in \mathbb{R}^3$. Then for every $t \in (0,\infty)$ and every non-empty open set $O \subset \mathbb{R}^3$ the function $O \ni x \mapsto \mathbb{E}\big[X^x(t)\big] \in \mathbb{R}^3$ is continuous but not locally Hölder continuous. Moreover, there exists an infinitely often differentiable function $\varphi \in C^\infty_{cpt}(\mathbb{R}^3,\mathbb{R})$ with compact support such that for every $t \in (0,T]$ and every non-empty open set $O \subset (-c,c)^3$ the function $O \ni x \mapsto \mathbb{E}\big[\varphi(X^x(t))\big] \in \mathbb{R}$ is continuous but not locally Hölder continuous.*

4 Solutions of Kolmogorov equations

If the transition semigroup associated with an SDE is smooth, then it satisfies the Kolmogorov equation (which is a second-order linear PDE) corresponding to the SDE in the classical sense. The transition semigroups in our counterexamples are, however, not locally Lipschitz continuous and are therefore no classical solutions of the Kolmogorov equations of the corresponding SDEs. The purpose of this section is to verify that the non-smooth transition semigroup associated with such an SDE still satisfies the Kolmogorov equation but in a certain weak sense. More precisely, in Subsection 4.4, we show that the transition semigroups in our counterexamples are *viscosity solutions* of the associated Kolmogorov equations. Moreover, in Subsection 4.5, we show that the transition semigroups in our counterexamples are solutions of the associated Kolmogorov equations *in the distributional sense.*

4.1 Definition of viscosity solutions

Viscosity solutions were first introduced in Crandall & Lions [9] (see also [16, 17, 8]). The name *viscosity solution* is due to the method of vanishing viscosity; see the discussion in Section 10.1 in Evans [18]. For a review of the theory and for more references, we refer the reader to the well-known users's guide Crandall, Ishii & Lions [7].

Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $\mathbb{S}_d = \{A \in \mathbb{R}^{d\times d}\colon A = A^*\}$ be the set of all symmetric $d\times d$-matrices. Moreover, for every $A,B \in \mathbb{S}_d$ we write $A \leq B$ in the following if $\langle x,Ax\rangle \leq \langle x,Bx\rangle$ for all

$x \in \mathbb{R}^d$. A function $F\colon O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ is called *degenerate elliptic* (see, e.g., Crandall, Ishii & Lions [7]) if $F(x, r, p, A) \leq F(x, r, p, B)$ for all $x \in O$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$ and all $A, B \in \mathbb{S}_d$ with $A \geq B$. For convenience of the reader, we recall the definition of a viscosity solution (see, e.g., Crandall, Ishii & Lions [7] and also Definition 1.2 in Appendix C in Peng [61]).

Definition 4.1 (Viscosity solution)**.** *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $F\colon O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function. A function $u\colon O \to \mathbb{R}$ is said to be a* viscosity subsolution *of $F = 0$ (or, equivalently, a viscosity solution of $F \leq 0$) if u is upper semicontinuous and if*

$$F\big(x, \phi(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\big) \leq 0 \tag{31}$$

for all $\phi \in \mathrm{C}^2(O, \mathbb{R})$ with $\phi(x) = u(x)$ and $\phi \geq u$ and all $x \in O$. Similarly, a function $u\colon O \to \mathbb{R}$ is said to be a viscosity supersolution *of $F = 0$ (or, equivalently, a viscosity solution of $F \geq 0$) if u is lower semicontinuous and if*

$$F\big(x, \phi(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\big) \geq 0 \tag{32}$$

for all $\phi \in \mathrm{C}^2(O, \mathbb{R})$ with $\phi(x) = u(x)$ and $\phi \leq u$ and all $x \in O$. Finally, a function $u\colon O \to \mathbb{R}$ is said to be a viscosity solution *of $F = 0$ if u is both a viscosity subsolution and a viscosity supersolution of $F = 0$.*

Let us add two well-known remarks concerning viscosity solutions. We will use both of them in our proofs below.

Remark 4.1. *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $F\colon O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function. Then an upper semicontinuous function $u\colon O \to \mathbb{R}$ is a viscosity subsolution of $F = 0$ if and only if u satisfies*

$$F\big(x, u(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\big) \leq 0 \tag{33}$$

for all $\phi \in \{\psi \in \mathrm{C}^2(O, \mathbb{R})\colon x$ is a local maximum of $(u - \psi)\colon O \to \mathbb{R}\}$ and all $x \in O$. Moreover, a lower semicontinuous function $u\colon O \to \mathbb{R}$ is a viscosity supersolution of $F = 0$ if and only if u satisfies

$$F\big(x, u(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\big) \geq 0 \tag{34}$$

for all $\phi \in \{\psi \in \mathrm{C}^2(O, \mathbb{R})\colon x$ is a local minimum of $(u - \psi)\colon O \to \mathbb{R}\}$ and all $x \in O$.

Remark 4.2 (Classical solutions are viscosity solutions)**.** *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $F\colon O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function and let $u \in \mathrm{C}^2(O, \mathbb{R})$ be a classical subsolution of $F = 0$, i.e., suppose that*

$$F\big(x, u(x), (\nabla u)(x), (\operatorname{Hess} u)(x)\big) \leq 0 \tag{35}$$

for all $x \in O$. Then u is also a viscosity subsolution of $F = 0$. Indeed, for every $x \in O$ and every $\phi \in \{\psi \in \mathrm{C}^2(O, \mathbb{R})\colon x$ is a local maximum of $(u - \psi)\colon O \to \mathbb{R}\}$ we have $(\nabla(u - \phi))(x) = 0$ and $(\operatorname{Hess}(u - \phi))(x) \leq 0$ and therefore

$$\begin{aligned} F\big(x, u(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\big) &= F\big(x, u(x), (\nabla u)(x), (\operatorname{Hess}\phi)(x)\big) \\ &\leq F\big(x, u(x), (\nabla u)(x), (\operatorname{Hess} u)(x)\big) \leq 0 \end{aligned} \tag{36}$$

due to (35) *and due to the degenerate ellipticity assumption on F. The corresponding statement holds for classical supersolutions and classical solutions of $F = 0$.*

4.2 An approximation result for viscosity solutions

The following approximation result for viscosity solutions is essentially well known (see Proposition 1.2 in Ishii [37] which refers to the first order case in Theorem A.2 in Barles & Perthame [2]; see also Lemma 6.1 in Crandall, Ishii & Lions [7] and the remarks thereafter). For completeness we give the proof here following the line of arguments for the first order case in Theorem A.2 in Barles & Perthame [2]. In the remainder of this article we use the notation $\operatorname{dist}(x, A) := \inf(\{\|x - y\| \in [0, \infty)\colon y \in A\} \cup \{\infty\}) \in [0, \infty]$ for all $x \in \mathbb{R}^d$, all $A \subset \mathbb{R}^d$ and all $d \in \mathbb{N}$.

Lemma 4.2. *Let $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $u_n\colon O \to \mathbb{R}$, $n \in \mathbb{N}_0$, be functions and let $F_n\colon O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$, $n \in \mathbb{N}_0$, be degenerate elliptic functions such that F_0 is continuous. Moreover, assume that*

$$\limsup_{n\to\infty} \sup_{(x,r,p,A)\in K} |F_n(x, r, p, A) - F_0(x, r, p, A)| = 0 = \limsup_{n\to\infty} \sup_{x\in\bar{K}} |u_n(x) - u_0(x)| \tag{37}$$

for all compact sets $K \subset O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$ and all compact sets $\bar{K} \subset O$ and assume for every $n \in \mathbb{N}$ that u_n is a viscosity solution of $F_n = 0$. Then u_0 is a viscosity solution of $F_0 = 0$.

Proof of Lemma 4.2. The proof is divided into two steps. *Step 1:* In this first step assume that there exists an $x_0 \in O$ and a function $\phi \in C^2(O, \mathbb{R})$ such that x_0 is a strict maximum of $u_0 - \phi$, i.e.,

$$u_0(x) - \phi(x) < u_0(x_0) - \phi(x_0) \tag{38}$$

for all $x \in O \backslash \{x_0\}$. In the next step we define $r := \min\bigl(1, \frac{1}{2}\operatorname{dist}(x_0, \mathbb{R}^d \backslash O)\bigr) \in [0, 1]$. Since $O \subset \mathbb{R}^d$ is an open set, we obtain that $r \in (0, 1]$. Furthermore, continuity of the function ϕ and of the functions u_n, $n \in \mathbb{N}$, together with compactness of the set $\{y \in \mathbb{R}^d : \|y - x_0\| \leq r\} \subset O$ proves that there exists a sequence $x_n \in \{y \in \mathbb{R}^d : \|y - x_0\| \leq r\} \subset O$, $n \in \mathbb{N}$, of vectors such that

$$u_n(x) - \phi(x) \leq u_n(x_n) - \phi(x_n) \tag{39}$$

for all $x \in \{y \in \mathbb{R}^d : \|y - x_0\| \leq r\}$ and all $n \in \mathbb{N}$. In the next step we prove that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to x_0. Aiming at a contraction, we assume that the sequence $(x_n)_{n \in \mathbb{N}}$ does not converge to x_0. Due to compactness of $\{y \in \mathbb{R}^d : \|y - x_0\| \leq r\}$, there exists a vector $\bar{x}_0 \in \{y \in \mathbb{R}^d : 0 < \|y - x_0\| \leq r\} \subset O$ and an increasing sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, such that $\lim_{k \to \infty} x_{n_k} = \bar{x}_0$. In particular, we obtain that the set $\{\bar{x}_0\} \cup (\cup_{k \in \mathbb{N}} \{x_{n_k}\})$ is compact. Assumption (37), inequality (39) and inequality (38) hence imply that

$$u_0(x_0) - \phi(x_0) = \lim_{k \to \infty} \left(u_{n_k}(x_0) - \phi(x_0)\right) \leq \limsup_{k \to \infty} \left(u_{n_k}(x_{n_k}) - \phi(x_{n_k})\right) = u_0(\bar{x}_0) - \phi(\bar{x}_0) < u_0(x_0) - \phi(x_0).$$

From this contradiction we infer that $\lim_{n \to \infty} x_n = x_0$. Assumption (37) and continuity of $\nabla\phi : O \to \mathbb{R}^d$ and of $\operatorname{Hess}\phi : O \to \mathbb{S}_d$ hence imply that

$$\lim_{n \to \infty} \bigl(x_n, u_n(x_n), (\nabla\phi)(x_n), (\operatorname{Hess}\phi)(x_n)\bigr) = \bigl(x_0, u(x_0), (\nabla\phi)(x_0), (\operatorname{Hess}\phi)(x_0)\bigr). \tag{40}$$

In addition, $\lim_{n \to \infty} x_n = x_0$ and (39) show that there exists a natural number $n_0 \in \mathbb{N}$ such that we have for all $n \in \{n_0, n_0 + 1, \ldots\}$ that $\|x_n - x_0\| < r$ and that $x_n \in O$ is a local maximum of the function $(u_n - \phi) : O \to \mathbb{R}$. Hence, Remark 4.1 and the assumption that u_n is a viscosity solution of $F_n = 0$ show that

$$F_n\bigl(x_n, u_n(x_n), (\nabla\phi)(x_n), (\operatorname{Hess}\phi)(x_n)\bigr) \leq 0 \tag{41}$$

for all $n \in \{n_0, n_0 + 1, \ldots\}$. Continuity of F_0, equation (40), assumption (37), inequality (41) and compactness of the set $\cup_{n \in \mathbb{N}_0} \{(x_n, u_n(x_n), (\nabla\phi)(x_n), (\operatorname{Hess}\phi)(x_n))\}$ therefore yield that

$$\begin{aligned} F_0\bigl(x_0, u(x_0), (\nabla\phi)(x_0), (\operatorname{Hess}\phi)(x_0)\bigr) &= \lim_{n \to \infty} F_0\bigl(x_n, u_n(x_n), (\nabla\phi)(x_n), (\operatorname{Hess}\phi)(x_n)\bigr) \\ &= \lim_{n \to \infty} F_n\bigl(x_n, u_n(x_n), (\nabla\phi)(x_n), (\operatorname{Hess}\phi)(x_n)\bigr) \leq 0. \end{aligned} \tag{42}$$

We thus have proved that $F\bigl(x, u(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\bigr) \leq 0$ for all $\phi \in \{\psi \in C^2(O, \mathbb{R}) : x$ is a strict maximum of $(u - \psi) : O \to \mathbb{R}\}$ and all $x \in O$. *Step 2:* In this second step assume that there exists an $x_0 \in O$ and a function $\phi \in C^2(O, \mathbb{R})$ such that $\phi(x_0) = u(x_0)$ and $\phi \geq u$. Define functions $\phi_\varepsilon : O \to \mathbb{R}$, $\varepsilon \in (0, 1)$, through $\phi_\varepsilon(x) = \phi(x) + \varepsilon\|x - x_0\|^2$ for all $x \in O$ and all $\varepsilon \in (0, 1)$. Note that x_0 is a strict maximum of the function $(u - \phi_\varepsilon) : O \to \mathbb{R}$ for every $\varepsilon \in (0, 1)$. Step 1 can thus be applied to obtain

$$F\bigl(x_0, u(x_0), (\nabla\phi_\varepsilon)(x_0), (\operatorname{Hess}\phi_\varepsilon)(x_0)\bigr) \leq 0 \tag{43}$$

for all $\varepsilon \in (0, 1)$. Moreover, observe that $(\nabla\phi_\varepsilon)(x_0) = (\nabla\phi)(x_0)$ and that $(\operatorname{Hess}\phi_\varepsilon)(x_0) = (\operatorname{Hess}\phi)(x_0) + 2\varepsilon I_d$ for all $\varepsilon \in (0, 1)$ where $I_d \in \mathbb{S}^d$ is the $d \times d$-unit matrix. Consequently, we see that $\lim_{\varepsilon \searrow 0} (\nabla\phi_\varepsilon)(x_0) = (\nabla\phi)(x_0)$ and that $\lim_{\varepsilon \searrow 0} (\operatorname{Hess}\phi_\varepsilon)(x_0) = (\operatorname{Hess}\phi)(x_0)$. Continuity of F_0 and inequality (43) hence yield

$$F_0\bigl(x_0, u(x_0), (\nabla\phi)(x_0), (\operatorname{Hess}\phi)(x_0)\bigr) = \lim_{\varepsilon \searrow 0} F_0\bigl(x_0, u(x_0), (\nabla\phi_\varepsilon)(x_0), (\operatorname{Hess}\phi_\varepsilon)(x_0)\bigr) \leq 0. \tag{44}$$

We thus have proved that $F\bigl(x, u(x), (\nabla\phi)(x), (\operatorname{Hess}\phi)(x)\bigr) \leq 0$ for all $\phi \in C^2(O, \mathbb{R})$ with $\phi(x) = u(x)$ and $\phi \geq u$ and all $x \in O$. This shows that u is a viscosity subsolution of $F = 0$. In the same way, it can be shown that u is a viscosity supersolution of $F = 0$ and we thereby obtain that u is a viscosity solution of $F = 0$. The proof of Lemma 4.2 is thus completed. □

4.3 Uniqueness of viscosity solutions of Kolmogorov equations

The main result of this subsection (Corollary 4.6 below) establishes uniqueness of viscosity solutions of a second-order linear PDE within a certain class of functions and is apparently new. This uniqueness result is based on the well-known concept of superharmonic functions or – in the PDE language – on the idea of dominating supersolutions. More precisely, let $d \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$. For solution processes $X^x : [0, \infty) \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, of many SDEs, there exists a function

$V \in \mathrm{C}^2\big(\mathbb{R}^d, (0,\infty)\big)$ (often $\mathbb{R}^d \ni x \mapsto 1 + \|x\|^2 \in (0,\infty)$) and a real number $\rho \in \mathbb{R}$ such that the stochastic processes $[0,\infty) \times \Omega \ni (t,\omega) \to e^{-\rho t} \cdot V(X^x(t)(\omega)) \in (0,\infty)$, $x \in \mathbb{R}^d$, are non-negative supermartingales (so that $\mathbb{E}[V(X^x(t))] \leq e^{\rho t} \cdot V(x)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$); see, e.g., the examples in Section 4 in [35]. For these stochastic processes to be supermartingales, it suffices that the Lyapunov-type function $(0,\infty) \times \mathbb{R}^d \ni (t,x) \mapsto e^{\rho t} \cdot V(x) \in (0,\infty)$ satisfies

$$\tfrac{\partial}{\partial t}\big(e^{\rho t} \cdot V(x)\big) - \big\langle \nabla_x\big(e^{\rho t} \cdot V(x)\big), \mu(x)\big\rangle - \tfrac{1}{2}\operatorname{tr}\Big(\sigma(x)\,[\sigma(x)]^* \operatorname{Hess}_x\big(e^{\rho t} \cdot V(x)\big)\Big) \geq 0 \tag{45}$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$ where the function $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ is the infinitesimal mean and where the function $\sigma\sigma^*\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is the infinitesimal covariance matrix of the solution processes X^x, $x \in \mathbb{R}^d$ of the considered SDE. In other words, it suffices that $(0,\infty) \times \mathbb{R}^d \ni (t,x) \to e^{\rho t} \cdot V(x) \in (0,\infty)$ is a classical supersolution of the Kolmogorov equation. For $T \in (0,\infty)$, $d \in \mathbb{N}$ and an open set $O \subset \mathbb{R}^d$, a function $G\colon (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ is here called *degenerate elliptic* if $G(t,x,r,p,A) \leq G(t,x,r,p,B)$ for all $t \in (0,T)$, $x \in O$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$ and all $A, B \in \mathbb{S}_d$ with $A \leq B$ (see, e.g., inequality (1.2) in Appendix C in Peng [61] and compare also with Subsection 4.1 above). The next lemma is a slightly modified special case of Theorem 2.2 in Appendix C in Peng [61].

Lemma 4.3 (A domination result for viscosity subsolutions)**.** *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $G\colon (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function and let $u\colon [0,T] \times O \to \mathbb{R}$ be an upper semicontinuous function such that $u|_{(0,T)\times O}$ is a viscosity subsolution of*

$$\tfrac{\partial}{\partial t}u(t,x) - G\big(t,x,u(t,x),(\nabla_x u)(t,x),(\operatorname{Hess}_x u)(t,x)\big) = 0 \tag{46}$$

for $(t,x) \in (0,T) \times O$. Moreover, assume that

$$\lim_{n\to\infty}\left[\sup_{x \in \{y \in O\colon \operatorname{dist}(y,\mathbb{R}^d \backslash O) < \frac{1}{n} \text{ or } \|y\| > n\}} \sup_{t \in [0,T]} u(t,x)\right] \leq 0 \tag{47}$$

and that

$$G(t,x,r,0,0) \leq 0 \qquad \text{and} \qquad u(0,x) \leq 0 \tag{48}$$

for all $(t,x,r) \in (0,T) \times O \times (0,\infty)$. Then $u(t,x) \leq 0$ for all $(t,x) \in [0,T) \times O$.

Proof of Lemma 4.3. First of all, we define a family $S_\delta \in (-\infty,\infty]$, $\delta \in [0,1]$, of extended real numbers through $S_\delta := \sup_{(t,x)\in[0,T]\times O}\big(u(t,x) - \frac{\delta}{(T-t)}\big) \in (-\infty,\infty]$ for all $\delta \in [0,1]$. Assumption (47) implies that there exists a natural number $n_0 \in \mathbb{N}$ such that $u(t,x) \leq 1$ for all $t \in [0,T]$ and all $x \in \big\{y \in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O) < \frac{1}{n_0} \text{ or } \|y\| > n_0\big\}$. Moreover, the upper semicontinuous function u is bounded from above on the compact set $\{(t,x) \in [0,T] \times O\colon \operatorname{dist}(x,\mathbb{R}^d\backslash O) \geq \frac{1}{n_0} \text{ and } \|x\| \leq n_0\}$. Therefore, we infer that $S_0 < \infty$ and hence, we obtain that $S_\delta \leq S_0 < \infty$ for all $\delta \in [0,1]$. In the next step we show that $S_\delta \leq 0$ for all $\delta \in (0,1]$. Aiming at a contradiction, we assume that there exists a real number $\delta \in (0,1]$ such that $S_\delta > 0$. Assumption (47) implies that there exists a natural number $n_\delta \in \mathbb{N}$ such that $u(t,x) - \frac{\delta}{(T-t)} \leq u(t,x) \leq \frac{S_\delta}{2}$ for all $t \in [0,T]$ and all $x \in \big\{y \in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O) < \frac{1}{n_0} \text{ or } \|y\| > n_0\big\}$. This together with upper semicontinuity of u implies that the function $[0,T] \times O \ni (t,x) \mapsto u(t,x) - \frac{\delta}{(T-t)} \in [-\infty,\infty)$ attains its supremum S_δ in the compact set $\{(t,x) \in [0,T]\times O\colon \operatorname{dist}(x,\mathbb{R}^d\backslash O) \geq \frac{1}{n_0} \text{ and } \|x\| \leq n_0\}$. Let $(t_0,x_0) \in [0,T]\times O$ be such that $u(t_0,x_0) - \frac{\delta}{(T-t_0)} = S_\delta$. Note that $u(T,x_0) - \frac{\delta}{(T-T)} = -\infty < S_\delta$ implies that $t_0 \neq T$. Moreover, observe that $u(0,x_0) - \frac{\delta}{T} \leq u(0,x_0) \leq 0 < S_\delta$ implies that $t_0 \neq 0$. So, we see that $(t_0,x_0) \in (0,T) \times O$. The function $\phi\colon (0,T) \times O \to \mathbb{R}$ defined through $\phi(t,x) := S_\delta + \frac{\delta}{(T-t)}$ for all $(t,x) \in (0,T) \times O$ is twice continuously differentiable and satisfies $\phi(t_0,x_0) = S_\delta + \frac{\delta}{(T-t_0)} = u(t_0,x_0) \in (0,\infty)$, $(\nabla_x\phi)(t,x) = 0$, $(\operatorname{Hess}_x\phi)(t,x) = 0$ and

$$u(t,x) = u(t,x) - \tfrac{\delta}{(T-t)} + \tfrac{\delta}{(T-t)} \leq S_\delta + \tfrac{\delta}{(T-t)} = \phi(t,x) \tag{49}$$

for all $(t,x) \in (0,T) \times O$. As $u|_{(0,T)\times O}$ is a viscosity subsolution of (46), we conclude that

$$\begin{aligned} 0 < \tfrac{\delta}{(T-t_0)^2} &= \big(\tfrac{\partial}{\partial t}\phi\big)(t_0,x_0) \leq G\big(t_0,x_0,\phi(t_0,x_0),(\nabla_x\phi)(t_0,x_0),(\operatorname{Hess}_x\phi)(t_0,x_0)\big) \\ &= G\big(t_0,x_0,\phi(t_0,x_0),0,0\big) \leq 0 \end{aligned} \tag{50}$$

where the last inequality follows from assumption (48). This contradiction implies that $S_\delta \leq 0$ for all $\delta \in (0,1]$. Therefore, we obtain that $u(t,x) \leq \frac{\delta}{(T-t)}$ for all $\delta \in (0,1]$ and all $(t,x) \in [0,T) \times O$ and hence, we get that $u(t,x) \leq 0$ for all $(t,x) \in [0,T) \times O$. This finishes the proof of Lemma 4.3. ☐

Lemma 4.4 (Scaling of viscosity subsolutions)**.** *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $V \in \mathrm{C}^2((0,T)\times O,(0,\infty))$, let $G\colon (0,T)\times O\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function, let $u\colon (0,T)\times O \to \mathbb{R}$ be a viscosity subsolution of (46) and let $\tilde{G}\colon (0,T)\times O\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d \to \mathbb{R}$ be a function defined through*

$$\begin{aligned}&\tilde{G}(t,x,r,p,A) := \\ &\tfrac{1}{V(t,x)}\, G\Big(t,x,r\,V(t,x), p\,V(t,x) + r\,(\nabla_x V)(t,x), A\,V(t,x) + 2p\,\big[(\nabla_x V)(t,x)\big]^* + r\,(\mathrm{Hess}_x V)(t,x)\Big) - r\,\tfrac{\frac{\partial}{\partial t}V(t,x)}{V(t,x)}\end{aligned} \tag{51}$$

for all $(t,x,r,p,A) \in (0,T)\times O\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d$. Then $\tilde{G}$ is degenerate elliptic and the function $\tilde{u}\colon (0,T)\times O \to \mathbb{R}$ defined through $\tilde{u}(t,x) = \frac{u(t,x)}{V(t,x)}$ for all $(t,x) \in (0,T)\times O$ is a viscosity subsolution of

$$\tfrac{\partial}{\partial t}\tilde{u}(t,x) - \tilde{G}\big(t,x,\tilde{u}(t,x),(\nabla_x\tilde{u})(t,x),(\mathrm{Hess}_x\tilde{u})(t,x)\big) = 0 \tag{52}$$

for $(t,x) \in (0,T)\times O$.

Proof of Lemma 4.4. First, observe that $\tilde{u}$ is upper semicontinuous and that $\tilde{G}$ is degenerate elliptic. In the next step assume that there exist a vector $(t,x) \in (0,T)\times O$ and a function $\phi \in \mathrm{C}^2((0,T)\times O,\mathbb{R})$ satisfying $\phi(t,x) = \tilde{u}(t,x)$ and $\phi \geq \tilde{u}$. Then the function $(0,T)\times O \ni (s,y) \mapsto \phi(s,y)\,V(s,y) \in \mathbb{R}$ is in $\mathrm{C}^2((0,T)\times O,\mathbb{R})$ and satisfies $\phi(t,x)\,V(t,x) = \tilde{u}(t,x)\,V(t,x) = u(t,x)$ and $\phi\cdot V \geq \tilde{u}\cdot V = u$. As u is a viscosity subsolution of (46), we get

$$\begin{aligned}&V(t,x)\cdot\tfrac{\partial}{\partial t}\phi(t,x) + \phi(t,x)\cdot\tfrac{\partial}{\partial t}V(t,x) \\ &= \tfrac{\partial}{\partial t}\big(\phi(t,x)\,V(t,x)\big) \leq G\Big(t,x,\phi(t,x)\,V(t,x),\big(\nabla_x(\phi V)\big)(t,x),\big(\mathrm{Hess}_x(\phi V)\big)(t,x)\Big).\end{aligned} \tag{53}$$

Rearranging this inequality results in

$$\begin{aligned}&\tfrac{\partial}{\partial t}\phi(t,x) \\ &\leq \tfrac{1}{V(t,x)}\, G\Big(t,x,\phi(t,x)V(t,x),\big(\nabla_x(\phi V)\big)(t,x),\big(\mathrm{Hess}_x(\phi V)\big)(t,x)\Big) - \phi(t,x)\,\tfrac{\frac{\partial}{\partial t}V(t,x)}{V(t,x)} \\ &= \tfrac{1}{V(t,x)}\, G\Big(t,x,\phi(t,x)V(t,x),(\nabla_x\phi)(t,x)\,V(t,x) + \phi(t,x)\,(\nabla_x V)(t,x),(\mathrm{Hess}_x\phi)(t,x)\,V(t,x) \\ &\qquad + (\nabla_x\phi)(t,x)\big[(\nabla_x V)(t,x)\big]^* + (\nabla_x V)(t,x)\big[(\nabla_x\phi)(t,x)\big]^* + \phi(t,x)\,(\mathrm{Hess}_x V)(t,x)\Big) - \phi(t,x)\,\tfrac{\frac{\partial}{\partial t}V(t,x)}{V(t,x)} \\ &= \tilde{G}\big(t,x,\phi(t,x),(\nabla_x\phi)(t,x),(\mathrm{Hess}_x\phi)(t,x)\big).\end{aligned} \tag{54}$$

This proves inequality (54) for all $\phi \in \{\psi \in \mathrm{C}^2((0,T)\times O,\mathbb{R})\colon \psi(t,x) = \tilde{u}(t,x) \text{ and } \psi \geq \tilde{u}\}$ and all $(t,x) \in (0,T)\times O$. Therefore, $\tilde{u}$ is a viscosity subsolution of (52) and the proof of Lemma 4.4 is completed. ☐

Corollary 4.5 (A comparison result for viscosity solutions)**.** *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $G\colon (0,T)\times O\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function which satisfies the linearity property*

$$G(t,x,cr_1 + r_2, cp_1 + p_2, cA_1 + A_2) = c\,G(t,x,r_1,p_1,A_1) + G(t,x,r_2,p_2,A_2) \tag{55}$$

for all $(t,x) \in (0,T)\times O$, $r_1,r_2 \in \mathbb{R}$, $p_1,p_2 \in \mathbb{R}^d$, $c \in \mathbb{R}$ and all $A_1,A_2 \in \mathbb{S}_d$. Moreover, let $u_1,u_2\colon [0,T]\times O \to \mathbb{R}$ and $V\colon [0,T]\times O \to (0,\infty)$ be continuous functions such that $u_1|_{(0,T)\times O}$ is a viscosity subsolution of (46), such that $u_2|_{(0,T)\times O}$ is viscosity supersolution of (46) and such that $V|_{(0,T)\times O} \in \mathrm{C}^2((0,T)\times O,(0,\infty))$ is a classical supersolution of (46). Furthermore, assume that $u_1(0,x) \leq u_2(0,x)$ for all $x \in O$ and that

$$\lim_{n\to\infty}\left[\sup_{x\in\left\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\setminus O)<\frac{1}{n} \text{ or } \|y\|>n\right\}}\ \sup_{t\in[0,T]}\frac{(u_1(t,x) - u_2(t,x))}{V(t,x)}\right] \leq 0. \tag{56}$$

Then $u_1 \leq u_2$, i.e., it holds that $u_1(t,x) \leq u_2(t,x)$ for all $(t,x) \in [0,T]\times O$.

Proof of Corollary 4.5. Define the function $u\colon [0,T]\times O \to \mathbb{R}$ through $u(t,x) := \frac{u_1(t,x) - u_2(t,x)}{V(t,x)}$ for all $(t,x) \in [0,T]\times O$. Due to assumption (55), the function $(u_1 - u_2)|_{(0,T)\times O}$ is a viscosity subsolution of (46). Lemma 4.4 thus yields that the function $u|_{(0,T)\times O}$ is a viscosity subsolution of (52) with $\tilde{G}\colon (0,T)\times O\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_d \to \mathbb{R}$ defined as in (51). For applying Lemma 4.3, observe that assumption (56) implies inequality (47). Moreover, note that assumption (55) and the assumption that V is a classical supersolution of (46) ensure that

$$\begin{aligned}\tilde{G}(t,x,r,0,0) &= \tfrac{1}{V(t,x)}\, G\big(t,x,r\,V(t,x), r\,(\nabla_x V)(t,x), r\,(\mathrm{Hess}_x V)(t,x)\big) - r\,\tfrac{\frac{\partial}{\partial t}V(t,x)}{V(t,x)} \\ &= \tfrac{r}{V(t,x)}\,\underbrace{\Big(G\big(t,x,V(t,x),(\nabla_x V)(t,x),(\mathrm{Hess}_x V)(t,x)\big) - \tfrac{\partial}{\partial t}V(t,x)\Big)}_{\leq 0 \text{ since } V \text{ is assumed to be a classical supersolution of (46)}} \leq 0\end{aligned} \tag{57}$$

for all $(t,x,r) \in (0,T)\times O\times(0,\infty)$. Therefore, the assumptions of Lemma 4.3 are satisfied for the function u and Lemma 4.3 together with continuity of $u_1 - u_2$ hence yields that $u_1(t,x) \leq u_2(t,x)$ for all $(t,x) \in [0,T]\times O$. This finishes the proof of Corollary 4.5. ☐

The next corollary proves uniqueness of viscosity solutions for a class of second-order linear PDEs. It follows immediately from Corollary 4.5 above and its proof is therefore omitted.

Corollary 4.6 (Uniqueness of viscosity solutions). *Let $T \in (0, \infty)$, $d \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set and let $G\colon (0,T) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ be a degenerate elliptic function which satisfies the linearity property* (55). *Moreover, let $u_1, u_2\colon [0,T] \times O \to \mathbb{R}$ and $V\colon [0,T] \times O \to (0,\infty)$ be continuous functions such that $u_1|_{(0,T)\times O}$ and $u_2|_{(0,T)\times O}$ are viscosity solutions of* (46) *and such that $V|_{(0,T)\times O} \in \mathrm{C}^2((0,T) \times O, (0,\infty))$ is a classical supersolution of* (46)*. Furthermore, assume that $u_1(0,x) = u_2(0,x)$ for all $x \in O$ and that*

$$\lim_{n\to\infty} \left[\sup_{x \in \left\{y \in O\colon \operatorname{dist}(y, \mathbb{R}^d \setminus O) < \frac{1}{n} \text{ or } \|y\| > n\right\}} \sup_{t\in[0,T]} \frac{|u_1(t,x) - u_2(t,x)|}{V(t,x)} \right] = 0. \tag{58}$$

Then $u_1 = u_2$, i.e., it holds that $u_1(t,x) = u_2(t,x)$ for all $(t,x) \in [0,T] \times O$.

The next result, Corollary 4.7, specifies Corollary 4.6 to the case where the Lyapunov-type function $V\colon [0,T]\times O \to (0,\infty)$ in Corollary 4.6 is of the form $V(t,x) = e^{\rho t} \cdot \tilde{V}(x)$ for all $(t,x) \in [0,T] \times O$ where $\rho \in \mathbb{R}$ is a real number and where $\tilde{V}\colon O \to (0,\infty)$ is a twice continuously differentiable function. Corollary 4.7 is thus an immediate consequence of Corollary 4.6 and its proof is therefore omitted.

Corollary 4.7 (Uniqueness of viscosity solutions). *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, $\rho \in \mathbb{R}$, let $O \subset \mathbb{R}^d$ be an open set, let $\varphi\colon O \to \mathbb{R}$ be a continuous function, let $v\colon (0,T) \times O \to \mathbb{R}$, $\mu\colon (0,T) \times O \to \mathbb{R}^d$ and $A\colon (0,T) \times O \to \{B \in \mathbb{S}_d\colon B \geq 0\}$ be functions, let $V\colon O \to (0,\infty)$ be a twice continuously differentiable function satisfying*

$$v(t,x)\, V(x) + \langle \mu(t,x), (\nabla V)(x)\rangle + \operatorname{tr}(A(t,x)\, (\operatorname{Hess} V)(x)) \leq \rho \cdot V(x) \tag{59}$$

for all $x \in O$. Then there exists at most one continuous function $u\colon [0,T] \times O \to \mathbb{R}$ which fulfills $u(0,x) = \varphi(x)$ for all $x \in O$, which fulfills $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\left\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\setminus O)<1/n \text{ or } \|y\|>n\right\}} \frac{|u(t,x)|}{V(x)} = 0$ and which fulfills that $u|_{(0,T)\times O}$ is a viscosity solution of

$$\tfrac{\partial}{\partial t} u(t,x) - v(t,x)\, u(t,x) - \langle \mu(t,x), (\nabla_x u)(t,x)\rangle - \operatorname{tr}(A(t,x)\, (\operatorname{Hess}_x u)(t,x)) = 0 \tag{60}$$

for $(t,x) \in (0,T) \times O$.

4.4 Viscosity solutions of Kolmogorov equations

The main result of this subsection, Theorem 4.10 below, establishes that the transition semigroup associated with a suitable SDE with locally Lipschitz continuous coefficients is the unique viscosity solution of the Kolmogorov equation of the SDE. To establish this result, we first prove two auxiliary results, Lemma 4.8 and Lemma 4.9. The proof of Lemma 4.8 makes use of Lemma 4.2 above.

Lemma 4.8 (Existence of viscosity solutions of Kolmogorov equations with globally Lipschitz continuous coefficients with compact support). *Let $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $W\colon [0,\infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $O \subset \mathbb{R}^d$ be an open set, let $\varphi\colon O \to \mathbb{R}$ be a continuous function and let $\mu\colon O \to \mathbb{R}^d$ and $\sigma\colon O \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions with compact support. Then there exists a family $X^x\colon [0,\infty) \times \Omega \to O$, $x \in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying*

$$X^x(t) = x + \int_0^t \mu(X^x(s))\, ds + \int_0^t \sigma(X^x(s))\, dW(s) \tag{61}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s. and all $x \in O$ and the function $u\colon (0,\infty) \times O \to \mathbb{R}$ given by $u(t,x) = \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $(t,x) \in (0,\infty) \times O$ is a viscosity solution of

$$\tfrac{\partial}{\partial t} u(t,x) - \big\langle (\nabla_x u)(t,x), \mu(x)\big\rangle - \tfrac{1}{2} \operatorname{tr}\Big(\sigma(x)\, [\sigma(x)]^*\, (\operatorname{Hess}_x u)(t,x)\Big) = 0 \tag{62}$$

for $(t,x) \in (0,\infty) \times O$.

Proof of Lemma 4.8. First of all, observe that the local Lipschitz continuity of μ and σ and the assumption that μ and σ have compact supports show that μ and σ are also globally Lipschitz continuous. This implies the existence of a family $X^x\colon [0,\infty)\times\Omega \to O$, $x \in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying (61) (see, e.g., Theorem 1 in Krylov [46]). It thus remains to show that the function $u\colon (0,\infty) \times O \to \mathbb{R}$ introduced above is a viscosity solution of (62). For this we use the notation $\operatorname{supp}(f) := \overline{\{x \in O\colon f(x) \neq 0\}}^{O} \subset O$ for all functions $f\colon O \to \mathbb{R}^k$ and all $k \in \mathbb{N}$. In the next step let $U \subset O$ be a relatively compact open set in O with the property that $\big(\operatorname{supp}(\mu) \cup \operatorname{supp}(\sigma)\big) \subset U$. By assumption $\operatorname{supp}(\mu)$

and $\mathrm{supp}(\sigma)$ are compact sets and hence, such a set does indeed exist. Next, let $\mu^{(n)} \in C^{\infty}_{cpt}(O, \mathbb{R}^d)$, $n \in \mathbb{N}$, and $\sigma^{(n)} \in C^{\infty}_{cpt}(O, \mathbb{R})$, $n \in \mathbb{N}$, be sequences of smooth functions satisfying $\lim_{n\to\infty} \sup_{x\in U} \|\mu(x) - \mu^{(n)}(x)\| = \lim_{n\to\infty} \sup_{x\in U} \|\sigma(x) - \sigma^{(n)}(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} = 0$ and $\mathrm{supp}\big(\mu^{(n)}\big) \cup \mathrm{supp}\big(\sigma^{(n)}\big) \subset U$ for all $n \in \mathbb{N}$. By using the fact that $\mathrm{dist}\big(\partial U, \mathrm{supp}(\mu) \cup \mathrm{supp}(\sigma)\big) := \inf_{u\in\partial U} \inf_{x\in \mathrm{supp}(\mu)\cup\mathrm{supp}(\sigma)} \|u - x\| > 0$ and by using the convolution of two functions with compact support, it can be shown that such sequences do indeed exist. Furthermore, observe that the global Lipschitz continuity of $\mu^{(n)}$, $n \in \mathbb{N}$, and $\sigma^{(n)}$, $n \in \mathbb{N}$, implies the existence of a family $X^{x,n} \colon [0,\infty) \times \Omega \to O$, $x \in O$, $n \in \mathbb{N}$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying

$$X^{x,n}(t) = x + \int_0^t \mu^{(n)}(X^{x,n}(s))\, ds + \int_0^t \sigma^{(n)}(X^{x,n}(s))\, dW(s) \tag{63}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s., $x \in O$ and all $n \in \mathbb{N}$. Moreover, let $K_k \subset O$, $k \in \mathbb{N}$, be a sequence of compact sets such that $O = \cup_{k\in\mathbb{N}} K_k$ and such that $K_k \subset K_{k+1}$ for $k \in \mathbb{N}$. Then the Stone-Weierstrass theorem implies that there exists a smooth function $\varphi_k \in C^{\infty}(O, \mathbb{R})$ satisfying $\sup_{x\in K_k} |\varphi(x) - \varphi_k(x)| < \frac{1}{k}$ for each $k \in \mathbb{N}$. Consequently we have that $\lim_{k\to\infty} \sup_{x\in K} |\varphi(x) - \varphi_k(x)| = 0$ for all compact sets $K \subset O$. Now we define functions $u^{n,k} \colon (0,\infty) \times O \to \mathbb{R}$, $n, k \in \mathbb{N}$, and $u^{(n)} \colon (0,\infty) \times O \to \mathbb{R}$, $n \in \mathbb{N}$, through $u^{n,k}(t,x) := \mathbb{E}\big[\varphi_k(X^{x,n}(t))\big]$ and $u^{(k)}(t,x) := \mathbb{E}\big[\varphi_k(X^{x}(t))\big]$ for all $t \in (0,\infty)$, $x \in O$ and all $n, k \in \mathbb{N}$. The fact that φ_k, $k \in \mathbb{N}$, are smooth functions and the fact that $\mu^{(n)}$, $n \in \mathbb{N}$, and $\sigma^{(n)}$, $n \in \mathbb{N}$, are smooth functions with compact support imply that the functions $u^{n,k} \colon (0,\infty) \times O \to \mathbb{R}$, $n, k \in \mathbb{N}$, are infinitely often differentiable and globally Lipschitz continuous (see, e.g., Corollary 2.8.1 and Theorem 2.8.1 in [21]) and Theorem 4.3 in [59] hence yields that

$$\big(\tfrac{\partial}{\partial t} u^{n,k}\big)(t,x) - \big\langle \big(\nabla_x u^{n,k}\big)(t,x), \mu^{(n)}(x) \big\rangle - \tfrac{1}{2} \operatorname{tr}\Big(\sigma^{(n)}(x) \left[\sigma^{(n)}(x)\right]^{*} \big(\mathrm{Hess}_x u^{n,k}\big)(t,x)\Big) = 0 \tag{64}$$

for all $(t,x) \in (0,\infty) \times O$ and all $n, k \in \mathbb{N}$. In the next step we define functions $F_n \colon (0,\infty) \times O \times \mathbb{R} \times \mathbb{R}^{d+1} \times \mathcal{S}_{d+1} \to \mathbb{R}$, $n \in \mathbb{N}$, and $F \colon (0,\infty) \times O \times \mathbb{R} \times \mathbb{R}^{d+1} \times \mathcal{S}_{d+1} \to \mathbb{R}$ through

$$\begin{aligned} F(t,x,r,p,A) &:= p_1 - \big\langle (p_i)_{i\in\{2,\dots,d+1\}}, \mu(x) \big\rangle - \tfrac{1}{2} \operatorname{tr}\Big(\sigma(x) \left[\sigma(x)\right]^{*} (A_{i,j})_{i,j\in\{2,\dots,d+1\}}\Big), \\ F_n(t,x,r,p,A) &:= p_1 - \Big\langle (p_i)_{i\in\{2,\dots,d+1\}}, \mu^{(n)}(x) \Big\rangle - \tfrac{1}{2} \operatorname{tr}\Big(\sigma^{(n)}(x) \big[\sigma^{(n)}(x)\big]^{*} (A_{i,j})_{i,j\in\{2,\dots,d+1\}}\Big) \end{aligned} \tag{65}$$

for all $(t,x) \in (0,\infty) \times O \subset \mathbb{R}^{d+1}$, $r \in \mathbb{R}$, $p = (p_1,\dots,p_{d+1}) \in \mathbb{R}^{d+1}$ and all $A = (A_{i,j})_{i,j\in\{1,\dots,d+1\}} \in \mathbb{S}_{d+1}$. Combining this definition with (64) shows for every $n, k \in \mathbb{N}$ that the function $u^{n,k}$ is a classical solution of $F_n = 0$. Remark 4.2 hence shows for every $n, k \in \mathbb{N}$ that the function $u^{n,k}$ is also a viscosity solution of $F_n = 0$. Furthermore, observe that the smoothness of the functions $\varphi_k \in C^{\infty}(O, \mathbb{R})$, $k \in \mathbb{N}$, and the global Lipschitz continuity of the functions $(\mu^{(n)})_{n\in\mathbb{N}}$, $(\sigma^{(n)})_{n\in\mathbb{N}}$, μ and σ imply that

$$\begin{aligned} &\lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in O} \big|u^{(k)}(t,x) - u^{n,k}(t,x)\big| = \lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in\bar{U}} \big|\mathbb{E}\big[\varphi_k(X^{x,n}(t))\big] - \mathbb{E}\big[\varphi_k(X^{x}(t))\big]\big| \\ &\le \lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in\bar{U}} \mathbb{E}\Big[\big|\varphi_k(X^{x,n}(t)) - \varphi_k(X^{x}(t))\big|\Big] \\ &\le \underbrace{\left(\sup_{x\in\bar{U}} \|\varphi_k'(x)\|_{L(\mathbb{R}^d,\mathbb{R})}\right)}_{<\infty} \cdot \underbrace{\left(\lim_{n\to\infty} \sup_{t\in(0,T]} \sup_{x\in\bar{U}} \mathbb{E}\Big[\big|X^{x,n}(t) - X^{x}(t)\big|\Big]\right)}_{=0} = 0 \end{aligned} \tag{66}$$

for all $T \in (0,\infty)$ and all $k \in \mathbb{N}$. Combining (66), the fact that for every $n, k \in \mathbb{N}$ the function $u^{n,k}$ is a viscosity solution of $F_n = 0$, definition (65) and Lemma 4.2 shows for every $k \in \mathbb{N}$ that $u^{(k)}$ is a viscosity solution of $F = 0$. In addition, note that

$$\begin{aligned} &\lim_{k\to\infty} \sup_{(t,x)\in(0,\infty)\times K} \big|u(t,x) - u^{(k)}(t,x)\big| \le \lim_{k\to\infty} \sup_{(t,x)\in(0,\infty)\times K} \mathbb{E}\big[\,|\varphi(X^{x}(t)) - \varphi_k(X^{x}(t))|\,\big] \\ &\le \lim_{k\to\infty} \sup_{y\in\overline{U}\cup K} |\varphi(y) - \varphi_k(y)| = 0 \end{aligned} \tag{67}$$

for all compact sets $K \subset O$. Combining (67), the fact that for every $k \in \mathbb{N}$ the function $u^{(k)}$ is a viscosity solution of $F = 0$ and Lemma 4.2 finally shows that u is a viscosity solution of $F = 0$. This completes the proof of Lemma 4.8. □

The next result, Lemma 4.9, is a generalization and a consequence of Lemma 4.8 above. Its proof makes use of both Lemma 4.2 and Lemma 4.8.

Lemma 4.9 (Existence of viscosity solutions of Kolmogorov equations with locally Lipschitz continuous coefficients)**.** *Let $d, m \in \mathbb{N}$, $\rho \in \mathbb{R}$, let $O \subset \mathbb{R}^d$ be an open set, let $\varphi\colon O \to \mathbb{R}$ be a continuous function, let $\mu\colon O \to \mathbb{R}^d$ and $\sigma\colon O \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions and let $V \in \mathrm{C}^2(O,(0,\infty))$ be such that $\limsup_{p\nearrow 1}\sup_{x\in O}\frac{|\varphi(x)|}{(1+V(x))^p} < \infty$, such that*

$$\big\langle (\nabla V)(x), \mu(x)\big\rangle + \tfrac{1}{2}\operatorname{tr}\!\Big(\sigma(x)\,[\sigma(x)]^*\,\big(\operatorname{Hess} V\big)(x)\Big) \le \rho\cdot V(x) \tag{68}$$

for all $x \in O$ and such that $\lim_{n\to\infty}\inf\big(V(x)\in(0,\infty)\colon x\in\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\setminus O)\le \frac{1}{n} \text{ or } \|y\|\ge n\}\big) = \infty$. Moreover let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and let $W\colon [0,\infty)\times\Omega\to\mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion. Then there exists a family $X^x\colon [0,\infty)\times\Omega\to O$, $x\in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying

$$X^x(t) = x + \int_0^t \mu(X^x(s))\,ds + \int_0^t \sigma(X^x(s))\,dW(s) \tag{69}$$

for all $t\in[0,\infty)$ $\mathbb{P}$-a.s. and all $x\in O$ and the function $u\colon (0,\infty)\times O\to\mathbb{R}$ given by $u(t,x) = \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $(t,x)\in(0,\infty)\times O$ is a viscosity solution of

$$\tfrac{\partial}{\partial t}u(t,x) - \big\langle (\nabla_x u)(t,x), \mu(x)\big\rangle - \tfrac{1}{2}\operatorname{tr}\!\Big(\sigma(x)\,[\sigma(x)]^*\,(\operatorname{Hess}_x u)(t,x)\Big) = 0 \tag{70}$$

for $(t,x)\in(0,\infty)\times O$ and satisfies $\lim_{n\to\infty}\sup_{(t,x)\in(0,T]\times\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\setminus O)<1/n \text{ or } \|y\|>n\}}\frac{|u(t,x)|}{V(x)} = 0$ for all $T\in(0,\infty)$.

Proof of Lemma 4.9. First, note that Corollary 2.6 in Gyöngy & Krylov [25] guarantees the existence of a family $X^x\colon [0,\infty)\times\Omega\to O$, $x\in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying (63). It thus remains to show that the function $u\colon (0,\infty)\times O\to\mathbb{R}$ introduced above is a viscosity solution of (70) and that u satisfies $\lim_{n\to\infty}\sup_{(t,x)\in[0,T]\times\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\setminus O)<1/n \text{ or } \|y\|>n\}}\frac{|u(t,x)|}{V(x)} = 0$ for all $T\in(0,\infty)$. For this we first observe that the function $u\colon (0,\infty)\times O\to\mathbb{R}$ introduced above is well-defined. Indeed, the assumption $\limsup_{p\nearrow 1}\sup_{x\in O}\frac{|\varphi(x)|}{(1+V(x))^p}<\infty$ implies the existence of a real number $p\in(0,1)$ such that $c := \sup_{x\in O}\frac{|\varphi(x)|}{(1+V(x))^p} < \infty$ and combining this and assumption (81) with Jensen's inequality shows that

$$\mathbb{E}\big[|\varphi(X^x(t))|\big] \le c\,\big(1+\mathbb{E}\big[V(X^x(t))\big]\big)^p \le c\,\big(1+e^{\rho t}\,V(x)\big)^p \le c\,e^{\max(\rho,0)t}\,(1+|V(x)|^p) < \infty \tag{71}$$

for all $(t,x)\in(0,\infty)\times O$. The assumption $\lim_{n\to\infty}\inf\big\{V(x)\in(0,\infty)\colon x\in\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O)\le\frac{1}{n} \text{ or } \|y\|\ge n\}\big\} = \infty$ and the condition $p<1$ hence yield that

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sup_{(t,x)\in(0,T]\times\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O)<1/n \text{ or } \|y\|>n\}}\frac{|u(t,x)|}{V(x)}\right)\\
&\le \lim_{n\to\infty}\left(\sup_{(t,x)\in(0,T]\times\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O)<1/n \text{ or } \|y\|>n\}}\left[\frac{c\,e^{\max(\rho,0)t}\,(1+|V(x)|^p)}{V(x)}\right]\right)\\
&= c\cdot e^{\max(\rho,0)T}\cdot\lim_{n\to\infty}\left(\sup_{x\in\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O)<1/n \text{ or } \|y\|>n\}}\left[\frac{1}{V(x)}+\frac{1}{|V(x)|^{(1-p)}}\right]\right) = 0
\end{aligned}\tag{72}$$

for all $T\in(0,\infty)$. It thus remains to show that u is a viscosity solution of (70). For this let $O_n\subset O$, $n\in\mathbb{N}$, be a sequence of open sets defined through $O_n := \big\{x\in O\colon \operatorname{dist}(x,\mathbb{R}^d\backslash O)>\frac{1}{n} \text{ and } \|x\|<n\big\}$ for all $n\in\mathbb{N}$. Then let $\mu^{(n)}\colon O\to\mathbb{R}^d$, $n\in\mathbb{N}$, and $\sigma^{(n)}\colon O\to\mathbb{R}^{d\times m}$, $n\in\mathbb{N}$, be sequences of locally Lipschitz continuous functions with compact support and with $\mu^{(n)}|_{O_n} = \mu|_{O_n}$ and $\sigma^{(n)}|_{O_n} = \sigma|_{O_n}$ for all $n\in\mathbb{N}$. Partition of the unity shows that such sequences do indeed exist. Partition of the unity also shows that there exists a sequence $\varphi_k\colon O\to\mathbb{R}$, $k\in\mathbb{N}$, of continuous functions with compact support, with $\varphi|_{O_k} = \varphi_k|_{O_k}$ for all $k\in\mathbb{N}$ and with $|\varphi_k(x)|\le|\varphi(x)|$ for all $x\in O$ and all $k\in\mathbb{N}$. In the next step we define functions $F_n\colon (0,\infty)\times O\times\mathbb{R}\times\mathbb{R}^{d+1}\times\mathcal{S}_{d+1}\to\mathbb{R}$, $n\in\mathbb{N}$, and $F\colon (0,\infty)\times O\times\mathbb{R}\times\mathbb{R}^{d+1}\times\mathcal{S}_{d+1}\to\mathbb{R}$ through

$$\begin{aligned}
F(t,x,r,p,A) &:= p_1 - \big\langle (p_i)_{i\in\{2,\dots,d+1\}}, \mu(x)\big\rangle - \tfrac{1}{2}\operatorname{tr}\!\Big(\sigma(x)\,[\sigma(x)]^*\,(A_{i,j})_{i,j\in\{2,\dots,d+1\}}\Big)\\
F_n(t,x,r,p,A) &:= p_1 - \Big\langle (p_i)_{i\in\{2,\dots,d+1\}}, \mu^{(n)}(x)\Big\rangle - \tfrac{1}{2}\operatorname{tr}\!\Big(\sigma^{(n)}(x)\big[\sigma^{(n)}(x)\big]^*\,(A_{i,j})_{i,j\in\{2,\dots,d+1\}}\Big)
\end{aligned}\tag{73}$$

for all $(t,x)\in(0,\infty)\times O\subset\mathbb{R}^{d+1}$, $r\in\mathbb{R}$, $p=(p_1,\dots,p_{d+1})\in\mathbb{R}^{d+1}$ and all $A=(A_{i,j})_{i,j\in\{1,\dots,d+1\}}\in\mathbb{S}_{d+1}$. This definition together with the fact that $\mu^{(n)}|_{O_n} = \mu|_{O_n}$ and $\sigma^{(n)}|_{O_n} = \sigma|_{O_n}$ for all $n\in\mathbb{N}$ proves that

$$\lim_{n\to\infty}\sup_{(t,x,r,p,A)\in K}|F(t,x,r,p,A) - F_n(t,x,r,p,A)| = 0 \tag{74}$$

for all compact sets $K \subset (0,\infty) \times O \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d$. Furthermore, combining the fact that $\mu^{(n)}\colon O \to \mathbb{R}^d$, $n \in \mathbb{N}$, and $\sigma^{(n)}\colon O \to \mathbb{R}^{d\times m}$, $n \in \mathbb{N}$, are locally Lipschitz continuous functions with compact support and Lemma 4.8 shows the existence of a family of up to indistinguishability unique adapted stochastic processes $X^{x,n}\colon [0,\infty) \times \Omega \to O$, $x \in O$, $n \in \mathbb{N}$, with continuous sample paths satisfying

$$X^{x,n}(t) = x + \int_0^t \mu^{(n)}(X^{x,n}(s))\,ds + \int_0^t \sigma^{(n)}(X^{x,n}(s))\,dW(s) \tag{75}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s., all $x \in O$ and all $n \in \mathbb{N}$. In addition, Lemma 4.8 implies that the functions $u^{n,k}\colon (0,\infty)\times O \to \mathbb{R}$, $n,k \in \mathbb{N}$, defined through $u^{n,k}(t,x) := \mathbb{E}\big[\varphi_k(X^{x,n}(t))\big]$ for all $(t,x) \in (0,\infty)\times O$ and all $n,k \in \mathbb{N}$ satisfy that for every $n,k \in \mathbb{N}$ the function $u^{n,k}$ is a viscosity solution of $F_n = 0$. Define functions $u^k\colon (0,\infty)\times O \to \mathbb{R}$, $k \in \mathbb{N}$, through $u^k(t,x) := \mathbb{E}\big[\varphi_k(X^x(t))\big]$ for all $(t,x) \in (0,\infty)\times O$ and all $k \in \mathbb{N}$. Next we show for every $k \in \mathbb{N}$ that $u^{n,k}$ converges uniformly on compact subsets of $(0,\infty)\times O$ to $u^{(k)}$ as $n \to \infty$. To do so, we define a family $\tau_n^x\colon \Omega \to [0,\infty]$, $x \in O$, $n \in \mathbb{N}$, of stopping times through $\tau_n^x(\omega) := \inf(\{t \in [0,\infty)\colon X^x(t,\omega) \in O\backslash O_n\} \cup \{\infty\})$ for all $x \in O$, $n \in \mathbb{N}$ and all $\omega \in \Omega$. Then observe that

$$\begin{aligned}
&\big|u^{n,k}(t,x) - u^{(k)}(t,x)\big| = \big|\mathbb{E}\big[\varphi_k(X^{x,n}(t)) - \varphi_k(X^x(t))\big]\big| \leq \mathbb{E}\Big[\big|\varphi_k(X^{x,n}(t)) - \varphi_k(X^x(t))\big|\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{t\leq\tau_n^x\}}\big|\varphi_k(X^{x,n}(t)) - \varphi_k(X^x(t))\big|\Big] + \mathbb{E}\Big[\mathbb{1}_{\{t>\tau_n^x\}}\big|\varphi_k(X^{x,n}(t)) - \varphi_k(X^x(t))\big|\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{t>\tau_n^x\}}\big|\varphi_k(X^{x,n}(t)) - \varphi_k(X^x(t))\big|\Big] \leq 2\left(\sup_{y\in O}|\varphi_k(y)|\right)\mathbb{P}\big[\tau_n^x < t\big] < \infty
\end{aligned} \tag{76}$$

for all $t \in [0,\infty)$, $x \in O$ and all $n,k \in \mathbb{N}$. To bound the right-hand side, we employ assumption (68) to obtain that $\mathbb{E}\big[V(X^x(t\wedge\tau))\big] \leq e^{\max(\rho,0)t}\cdot V(x)$ for all $t \in [0,\infty)$, $x \in O$ and all stopping times $\tau\colon \Omega \to [0,\infty]$. Using this estimate together with Lemma 1.4 in Krylov [47], we get

$$\begin{aligned}
\mathbb{P}\big[\tau_n^x < t\big] &= \mathbb{P}\Big[\exists\, s \in [0,t)\colon X^x(s) \in O\backslash O_n\Big] \leq \mathbb{P}\left[\sup_{s\in[0,t]} V(X^x(s)) \geq \inf_{y\in O\backslash O_n} V(y)\right] \\
&\leq \frac{\sup_{\tau\colon \Omega\to[0,\infty]\text{ is stopping time}} \mathbb{E}\big[V(X^x(t\wedge\tau))\big]}{\inf_{y\in O\backslash O_n} V(y)} \leq \frac{e^{\max(\rho,0)t}\cdot V(x)}{\inf_{y\in O\backslash O_n} V(y)}
\end{aligned} \tag{77}$$

for all $t \in [0,\infty)$, $x \in O$ and all $n \in \mathbb{N}$. Inserting this inequality on the right-hand side of (76) implies

$$\begin{aligned}
\limsup_{n\to\infty}\left(\sup_{x\in K}\sup_{t\in(0,T]}\Big|u^{n,k}(t,x) - u^{(k)}(t,x)\Big|\right) &\leq 2\left(\sup_{y\in O}|\varphi_k(y)|\right)\left(\limsup_{n\to\infty}\sup_{x\in K}\mathbb{P}[\tau_n^x < T]\right) \\
&\leq 2\left(\sup_{y\in O}|\varphi_k(y)|\right)\frac{e^{\max(\rho,0)T}\,(\sup_{x\in K} V(x))}{\liminf_{n\to\infty}\big(\inf_{y\in O\backslash O_n} V(y)\big)} = 0
\end{aligned} \tag{78}$$

for all compact sets $K \subset O$, $k \in \mathbb{N}$ and all $T \in (0,\infty)$. Combining (74), (78), the fact that for every $n,k \in \mathbb{N}$ the function $u^{n,k}$ is a viscosity solution of $F_n = 0$ and Lemma 4.2 shows for every $k \in \mathbb{N}$ that $u^{(k)}$ is a viscosity solution of $F = 0$. In the next step we observe that the identity $\varphi|_{O_k} = \varphi_k|_{O_k}$ for all $k \in \mathbb{N}$, the inequality $|\varphi_k(x)| \leq |\varphi(x)|$ for all $x \in O$ and all $k \in \mathbb{N}$ and the Hölder inequality imply

$$\begin{aligned}
&\big|u(t,x) - u^{(k)}(t,x)\big| \leq \mathbb{E}\Big[\big|\varphi(X^x(t)) - \varphi_k(X^x(t))\big|\Big] = \mathbb{E}\Big[\mathbb{1}_{\{\tau_k^x<t\}}\big|\varphi(X^x(t)) - \varphi_k(X^x(t))\big|\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{\tau_k^x<t\}}\big(|\varphi(X^x(t))| + |\varphi_k(X^x(t))|\big)\Big] \leq 2\cdot\mathbb{E}\big[\mathbb{1}_{\{\tau_k^x<t\}}\,|\varphi(X^x(t))|\big] \\
&\leq 2c\mathbb{E}\Big[\mathbb{1}_{\{\tau_k^x<t\}}\,(1 + V(X^x(t)))^p\Big] \leq 2c\,(\mathbb{P}[\tau_k^x < t])^{(1-p)}\,\big(1 + \mathbb{E}\big[V(X^x(t))\big]\big)^p \\
&\leq 2c\left(\frac{e^{\max(\rho,0)t}\cdot V(x)}{\inf_{y\in O\backslash O_k} V(y)}\right)^{(1-p)}\Big(1 + e^{\max(\rho,0)t}\,V(x)\Big)^p
\end{aligned} \tag{79}$$

for all $(t,x) \in (0,\infty)\times O$ and all $k \in \mathbb{N}$ where we used estimate (77) for the last step. Therefore we conclude

$$\limsup_{k\to\infty}\sup_{t\in(0,T]}\sup_{x\in K}\big|u(t,x) - u^{(k)}(t,x)\big| \leq 2c\frac{1 + e^{\max(\rho,0)T}\,(\sup_{x\in K} V(x))}{\liminf_{k\to\infty}\big(\inf_{y\in O\backslash O_k} V(y)\big)^{(1-p)}} = 0 \tag{80}$$

for all $T \in (0,\infty)$ and all compact sets $K \subset O$. Combining (80) and that for every $k \in \mathbb{N}$ the function $u^{(k)}$ is a viscosity solution of $F = 0$ with Lemma 4.2 shows that u is a viscosity solution of $F = 0$. This completes the proof of Lemma 4.9. ☐

The next Theorem 4.10 is the main result of this section. It is a direct consequence of Lemma 4.9 and Corollary 4.7.

Theorem 4.10 (Existence and uniqueness of viscosity solutions of Kolmogorov equations)**.** *Let $d, m \in \mathbb{N}$, $\rho \in \mathbb{R}$, let $O \subset \mathbb{R}^d$ be an open set, let $\varphi\colon O \to \mathbb{R}$ be a continuous function, let $\mu\colon O \to \mathbb{R}^d$ and $\sigma\colon O \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions and let $V \in \mathrm{C}^2(O, (0,\infty))$ be such that $\limsup_{p\nearrow 1} \sup_{x\in O} \frac{|\varphi(x)|}{(1+V(x))^p} < \infty$, such that*

$$\big\langle \big(\nabla V\big)(x), \mu(x)\big\rangle + \tfrac{1}{2}\operatorname{tr}\Big(\sigma(x)\,[\sigma(x)]^*\,\big(\operatorname{Hess} V\big)(x)\Big) \leq \rho \cdot V(x) \tag{81}$$

for all $x \in O$ and such that $\lim_{n\to\infty} \inf\big(V(x) \in (0,\infty)\colon x \in \{y \in O\colon \operatorname{dist}(y, \mathbb{R}^d\backslash O) \leq \frac{1}{n} \text{ or } \|y\| \geq n\}\big) = \infty$. Then there exists a unique continuous function $u\colon [0,\infty)\times O \to \mathbb{R}$ which fulfills $u(0,x) = \varphi(x)$ for all $x \in O$, which fulfills $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\{y\in O\colon \operatorname{dist}(y,\mathbb{R}^d\backslash O)<1/n \text{ or } \|y\|>n\}} \frac{|u(t,x)|}{V(x)} = 0$ for all $T \in (0,\infty)$ and which fulfills that $u|_{(0,\infty)\times O}$ is a viscosity solution of

$$\tfrac{\partial}{\partial t}u(t,x) - \big\langle (\nabla_x u)(t,x), \mu(x)\big\rangle - \tfrac{1}{2}\operatorname{tr}\Big(\sigma(x)\,[\sigma(x)]^*\,(\operatorname{Hess}_x u)(t,x)\Big) = 0 \tag{82}$$

for $(t,x) \in (0,\infty)\times O$. Moreover, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and if $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ is a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, then there exists a family $X^x\colon [0,\infty)\times\Omega \to O$, $x \in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying

$$X^x(t) = x + \int_0^t \mu(X^x(s))\,ds + \int_0^t \sigma(X^x(s))\,dW(s) \tag{83}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s. and all $x \in O$ and in that case, the unique function $u\colon [0,\infty)\times O \to \mathbb{R}$ enjoys the probabilistic representation $u(t,x) = \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $(t,x) \in [0,\infty)\times O$.

Proof of Theorem 4.10. Lemma 4.9 implies the existence of a function $u\colon [0,\infty)\times O \to \mathbb{R}$ satisfying the desired properties. Uniqueness of such a function follows from Corollary 4.7. The probabilistic representation of u also follows from Lemma 4.9. The proof of Theorem 4.10 is thus completed. □

In the literature, there are many results proving an assertion similar to Theorem 4.10 under various assumptions on the functions μ and σ. Theorem 4.3 in Pardoux & Peng [59] implies that the transition semigroup associated with the SDE (83) is a viscosity solution of (82) if μ and σ are globally Lipschitz continuous; see also Peng [60]. Theorem C.2.4 in Peng [61] can be applied if μ is locally Lipschitz continuous and if σ is constant and then proves the same assertion. Uniqueness of the viscosity solution of (82) with given initial function follows from Thereom 8.2 in the User's guide Crandall, Ishii & Lions [7] if μ is globally one-sided Lipschitz continuous, that is, if there exists a constant $c \in \mathbb{R}$ such that $\langle x-y, \mu(x)-\mu(y)\rangle \leq c\|x-y\|^2$ for all $x, y \in \mathbb{R}^d$, and if σ is globally Lipschitz continuous. Moreover, Theorem 5.13 in Krylov [47] with implies that the transition semigroup solves the Kolmogorov equation (82) in the sense of distributions if μ and σ are globally Lipschitz continuous. In addition, Theorem 7.1.3 and Theorem 7.1.4 in Evans [18] show that there exists a unique weak solution of the PDE (82) if the coefficients μ and σ are bounded and if the PDE (82) is uniformly parabolic.

In many situations the open set $O \subset \mathbb{R}^d$ and the Lyapunov-type function $V\colon O \to \mathbb{R}$ in Theorem 4.10 satisfy $O = \mathbb{R}^d$ and $V(x) = \big(1 + \|x\|^2\big)^p$ for all $x \in \mathbb{R}^d$ where $p \in [1,\infty)$ is an arbitrary real number. This is subject of the following Corollary 4.11. It is a direct consequence of Theorem 4.10 and its proof is therefore omitted.

Corollary 4.11 (Existence and uniqueness of at most polynomially growing viscosity solutions of Kolmogorov equations)**.** *Let $d, m \in \mathbb{N}$, let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous and at most polynomially growing function, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions with $\sup_{x\in\mathbb{R}^d} \frac{\langle x, \mu(x)\rangle}{(1+\|x\|^2)} < \infty$ and $\sup_{x\in\mathbb{R}^d} \frac{\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}}{(1+\|x\|)} < \infty$. Then there exists a unique continuous function $u\colon [0,\infty)\times O \to \mathbb{R}$ which fulfills $u(0,x) = \varphi(x)$ for all $x \in O$, which fulfills $\limsup_{p\to\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{|u(t,x)|}{(1+\|x\|)^p} < \infty$ and which fulfills that $u|_{(0,\infty)\times O}$ is a viscosity solution of*

$$\tfrac{\partial}{\partial t}u(t,x) - \big\langle (\nabla_x u)(t,x), \mu(x)\big\rangle - \tfrac{1}{2}\operatorname{tr}\Big(\sigma(x)\,[\sigma(x)]^*\,(\operatorname{Hess}_x u)(t,x)\Big) = 0 \tag{84}$$

for $(t,x) \in (0,\infty)\times O$. Moreover, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and if $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ is a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, then there exists a family $X^x\colon [0,\infty)\times\Omega \to O$, $x \in O$, of up to indistinguishability unique adapted stochastic processes with continuous sample paths satisfying

$$X^x(t) = x + \int_0^t \mu(X^x(s))\,ds + \int_0^t \sigma(X^x(s))\,dW(s) \tag{85}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s. and all $x \in O$ and in that case, the unique function $u\colon [0,\infty)\times O \to \mathbb{R}$ enjoys the probabilistic representation $u(t,x) = \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $(t,x) \in [0,\infty)\times O$.

Note that all examples in this article fulfill the assumptions of Corollary 4.11. In particular, observe that μ and σ from the SDE (9) in Section 2, μ and σ from the SDE (18) in Section 2, μ and σ in Theorem 3.4 in Section 3, μ and σ from the SDE (20) in Section 3 as well as μ and σ from the SDE (99) in Section 5 all fulfill the assumptions of Corollary 4.11.

4.5 Solutions of Kolmogorov equations in the distributional sense

Roughly speaking, Proposition 4.12 proves that the semigroup associated to an SDE with smooth coefficients solves the corresponding Kolmogorov equation in the distributional sense. For formulating this result, we introduce more notation. More precisely, for $d \in \mathbb{N}$, an open set $O \subset \mathbb{R}^d$ and a locally Lebesgue integrable function $u\colon O \to \mathbb{R}$, we denote by $\nu_u\colon \mathrm{C}^\infty_{cpt}(O,\mathbb{R}) \to \mathbb{R}$ the distribution associated with the function u, i.e., the linear mapping from $\mathrm{C}^\infty_{cpt}(O,\mathbb{R})$ to $\mathbb{R}$ given by $\nu_u(\psi) = \int_O u(x)\cdot\psi(x)\,dx$ for all $\psi \in \mathrm{C}^\infty_{cpt}(O,\mathbb{R})$. The following result is well-known in the case of globally Lipschitz continuous coefficients defined on the whole space; see Theorem 5.13 (ii) in Krylov [47].

Proposition 4.12. *Let $d,m \in \mathbb{N}$, let $O \subset \mathbb{R}^d$ be an open set, let $\mu = (\mu_1,\dots,\mu_d) \in \mathrm{C}^\infty(O,\mathbb{R}^d)$, $\sigma = (\sigma_{i,j})_{i\in\{1,\dots,d\},j\in\{1,\dots,m\}} \in \mathrm{C}^\infty(O,\mathbb{R}^{d\times m})$, let $\varphi \in \mathrm{C}_b(O,\mathbb{R})$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $X^x\colon [0,\infty)\times\Omega \to O$, $x \in O$, be a family of adapted stochastic processes with continuous sample paths satisfying*

$$X^x(t) = x + \int_0^t \mu(X^x(s))\,ds + \int_0^t \sigma(X^x(s))\,dW(s) \tag{86}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s. and all $x \in O$. Then the distribution $\nu_u\colon \mathrm{C}^\infty((0,\infty)\times O,\mathbb{R}) \to \mathbb{R}$ corresponding to the continuous function $u\colon (0,\infty)\times O \to \mathbb{R}$ given by $u(t,x) = \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $(t,x) \in (0,\infty)\times O$ satisfies

$$\tfrac{\partial}{\partial t}\nu_u = \sum_{i=1}^d \Big(\tfrac{\partial}{\partial x_i}\nu_u\Big)\cdot\mu_i + \frac{1}{2}\sum_{l=1}^m\sum_{i,j=1}^d \Big(\tfrac{\partial^2}{\partial x_i \partial x_j}\nu_u\Big)\cdot\sigma_{i,l}\cdot\sigma_{j,l}. \tag{87}$$

In other words, the continuous function $u\colon (0,\infty)\times O \to \mathbb{R}$ satisfies

$$\begin{aligned}
&-\int_O\int_0^\infty u(t,x)\cdot\big(\tfrac{\partial}{\partial t}\psi\big)(t,x)\,dt\,dx = -\sum_{i=1}^d \int_O\int_0^\infty u(t,x)\cdot\tfrac{\partial}{\partial x_i}\big(\psi(t,x)\cdot\mu_i(x)\big)\,dt\,dx \\
&\quad + \frac{1}{2}\sum_{l=1}^m\sum_{i,j=1}^d \int_O\int_0^\infty u(t,x)\cdot\tfrac{\partial^2}{\partial x_i\partial x_j}\big(\psi(t,x)\cdot\sigma_{i,l}(x)\cdot\sigma_{j,l}(x)\big)\,dt\,dx
\end{aligned} \tag{88}$$

for all $\psi \in \mathrm{C}^\infty_{cpt}((0,\infty)\times O,\mathbb{R})$.

Proof of Proposition 4.12. First of all, we observe that Theorem 1.7 in Krylov [47] combined with Lebesgue's theorem of dominated convergence proves that the function $u\colon (0,\infty)\times O \to \mathbb{R}$ given by $u(t,x) = \mathbb{E}\big[\varphi(X^x(t))\big]$ for all $(t,x) \in (0,\infty)\times O$ is continuous (see also Remark 5.9 in Krylov [47]). Next, let $O_n \subset O$, $n \in \mathbb{N}$, be a sequence of open sets defined through $O_n := \big\{x \in O\colon \mathrm{dist}(x,\mathbb{R}^d\backslash O) > \frac{1}{n} \text{ and } \|x\| < n\big\}$ for all $n \in \mathbb{N}$. Furthermore, let $\mu^{(n)} = (\mu^{(n)}_1,\dots,\mu^{(n)}_d) \in \mathrm{C}^\infty_{cpt}(O,\mathbb{R}^d)$, $n \in \mathbb{N}$, and $\sigma^{(n)} = (\sigma^{(n)}_{i,j})_{i\in\{1,\dots,d\},j\in\{1,\dots,m\}} \in \mathrm{C}^\infty_{cpt}(O,\mathbb{R}^{d\times m})$, $n \in \mathbb{N}$, be sequences of functions with $\mu^{(n)}|_{O_n} = \mu|_{O_n}$ and $\sigma^{(n)}|_{O_n} = \sigma|_{O_n}$ for all $n \in \mathbb{N}$. The fact that $\mu^{(n)}$, $n \in \mathbb{N}$, and $\sigma^{(n)}$, $n \in \mathbb{N}$, are globally Lipschitz continuous functions then ensures that there exists a family $X^{x,n}\colon [0,\infty)\times\Omega \to O$, $x \in O$, $n \in \mathbb{N}$, of up to indistinguishability unique adapted stochastic processes with continuous sample satisfying

$$X^{x,n}(t) = x + \int_0^t \mu^{(n)}(X^{x,n}(s))\,ds + \int_0^t \sigma^{(n)}(X^{x,n}(s))\,dW(s) \tag{89}$$

for all $t \in [0,\infty)$ $\mathbb{P}$-a.s., all $x \in O$ and all $n \in \mathbb{N}$. Next, let $\varphi_k \in C^\infty(O,\mathbb{R})$, $k \in \mathbb{N}$, be a sequence of smooth functions with $c := \sup_{k\in\mathbb{N}}\sup_{x\in O}\|\varphi_k(x)\| < \infty$ and $\lim_{k\to\infty}\sup_{x\in K}\|\varphi(x)-\varphi_k(x)\| = 0$ for all compact sets $K \subset O$. Such a sequence does indeed exist since the function $\varphi\colon O \to \mathbb{R}$ is assumed to be continuous and globally bounded. In the next step let $u^{(k)}\colon (0,\infty)\times O \to \mathbb{R}$, $k \in \mathbb{N}$, and $u^{n,k}\colon (0,\infty)\times O \to \mathbb{R}$, $n,k \in \mathbb{N}$, be functions defined through $u^{(k)}(t,x) := \mathbb{E}\big[\varphi_k(X^x(t))\big]$ and $u^{n,k}(t,x) := \mathbb{E}\big[\varphi_k(X^{x,n}(t))\big]$ for all $(t,x) \in (0,\infty)\times O$ and all $n,k \in \mathbb{N}$. Note that the functions $u^{n,k}$, $n,k \in \mathbb{N}$, satisfy

$$\begin{aligned}
&\int_O\int_0^\infty u^{n,k}(t,x)\cdot\big(\tfrac{\partial}{\partial t}\psi\big)(t,x)\,dt\,dx - \sum_{i=1}^d \int_O\int_0^\infty u^{n,k}(t,x)\cdot\Big(\tfrac{\partial}{\partial x_i}\big(\psi(t,x)\cdot\mu^{(n)}_i(x)\big)\Big)\,dt\,dx \\
&\quad + \frac{1}{2}\sum_{l=1}^m\sum_{i,j=1}^d \int_O\int_0^\infty u^{n,k}(t,x)\cdot\Big(\tfrac{\partial^2}{\partial x_i\partial x_j}\big(\psi(t,x)\cdot\sigma^{(n)}_{i,l}(x)\cdot\sigma^{(n)}_{j,l}(x)\big)\Big)\,dt\,dx = 0
\end{aligned} \tag{90}$$

for all $\psi \in \mathrm{C}^{\infty}_{cpt}((0,\infty)\times O,\mathbb{R})$ and all $n,k \in \mathbb{N}$ according to Theorem 5.13 (ii) in Krylov [47]. In addition, observe that the estimate $|\varphi_k(X^x(t,\omega)) - \varphi_k(X^{x,n}(t,\omega))| \le 2c < \infty$ for all $\omega \in \Omega$, $x \in O$ and all $t \in [0,\infty)$ together with Lebesgue's theorem of dominated convergence implies

$$\limsup_{n\to\infty} \left| u^{(k)}(t,x) - u^{n,k}(t,x) \right| \le \limsup_{n\to\infty} \mathbb{E}\big[\, |\varphi_k(X^x(t)) - \varphi_k(X^{x,n}(t))| \,\big] = 0 \tag{91}$$

for all $(t,x) \in (0,\infty)\times O$ and all $k \in \mathbb{N}$. Next, note that

$$\begin{aligned}
&\limsup_{k\to\infty} \left| u^{(k)}(t,x) - u(t,x) \right| \le \limsup_{k\to\infty} \mathbb{E}\big[\, |\varphi_k(X^x(t)) - \varphi(X^x(t))| \,\big] \\
&\le 2c \cdot \mathbb{P}[X^x(t) \notin K] + \limsup_{k\to\infty} \mathbb{E}\big[\mathbb{1}_{\{X^x(t)\in K\}} \, |\varphi_k(X^x(t)) - \varphi(X^x(t))| \big] \\
&\le 2c \cdot \mathbb{P}[X^x(t) \notin K] + \limsup_{k\to\infty} \sup_{y\in K} |\varphi_k(y) - \varphi(y)| = 2c \cdot \mathbb{P}[X^x(t) \notin K]
\end{aligned} \tag{92}$$

for all $(t,x) \in (0,\infty)\times O$ and all compact sets $K \subset O$ and hence, we obtain

$$\limsup_{k\to\infty} \left| u^{(k)}(t,x) - u(t,x) \right| \le 2c \cdot \inf_{K \subset\subset O} \mathbb{P}[X^x(t) \notin K] = 0 \tag{93}$$

for all $(t,x) \in (0,\infty)\times O$. In the next step we fix an arbitrary $\psi \in \mathrm{C}^{\infty}_{cpt}((0,\infty)\times O,\mathbb{R})$ and we now show (88) for this ψ. Let $\chi \in \mathrm{C}^{\infty}_{cpt}((0,\infty)\times O,\mathbb{R})$ and $\chi^{(n)} \in \mathrm{C}^{\infty}_{cpt}((0,\infty)\times O,\mathbb{R})$, $n \in \mathbb{N}$, be given by

$$\begin{aligned}
\chi(t,x) &= \big(\tfrac{\partial}{\partial t}\psi\big)(t,x) - \sum_{i=1}^{d} \tfrac{\partial}{\partial x_i}\big(\psi(t,x)\cdot\mu_i(x)\big) + \frac{1}{2}\sum_{l=1}^{m}\sum_{i,j=1}^{d} \tfrac{\partial^2}{\partial x_i \partial x_j}\big(\psi(t,x)\cdot\sigma_{i,l}(x)\cdot\sigma_{j,l}(x)\big) \\
\chi^{(n)}(t,x) &= \big(\tfrac{\partial}{\partial t}\psi\big)(t,x) - \sum_{i=1}^{d} \tfrac{\partial}{\partial x_k}\big(\psi(t,x)\cdot\mu_i^{(n)}(x)\big) + \frac{1}{2}\sum_{l=1}^{m}\sum_{i,j=1}^{d} \tfrac{\partial^2}{\partial x_i \partial x_j}\big(\psi(t,x)\cdot\sigma_{i,l}^{(n)}(x)\cdot\sigma_{j,l}^{(n)}(x)\big)
\end{aligned} \tag{94}$$

for all $(t,x) \in (0,\infty)\times O$ and all $n \in \mathbb{N}$. Using this notation, (90) reads as

$$\int_O \int_0^\infty u^{n,k}(t,x)\cdot\chi^{(n)}(t,x)\,dt\,dx = 0 \tag{95}$$

for all $n,k \in \mathbb{N}$. Moreover, observe that the fact that ψ has compact support ensures that there exists a natural number $n_0 \in \mathbb{N}$ such that $\chi^{(n)} = \chi$ for all $n \in \{n_0, n_0+1, \dots\}$. Combining this, (91), the estimate $\left|u^{(k)}(t,x) - u^{n,k}(t,x)\right| \le 2c < \infty$ for all $(t,x) \in (0,\infty)\times O$, $n,k \in \mathbb{N}$ and Lebesgue's theorem of dominated convergence implies that

$$\begin{aligned}
&\lim_{n\to\infty} \left[\int_O \int_0^\infty u^{n,k}(t,x)\cdot\chi^{(n)}(t,x)\,dt\,dx \right] = \int_O \int_0^\infty \lim_{n\to\infty} \left[u^{n,k}(t,x)\cdot\chi^{(n)}(t,x) \right] dt\,dx \\
&= \int_O \int_0^\infty u^{(k)}(t,x)\cdot\chi(t,x)\,dt\,dx
\end{aligned} \tag{96}$$

for all $k \in \mathbb{N}$. Equation (95) hence shows that $\int_O \int_0^\infty u^{(k)}(t,x)\cdot\chi(t,x)\,dt\,dx = 0$ for all $k \in \mathbb{N}$. Equation (93), the estimate $\left|u^{(k)}(t,x)\right| \le c$ for all $(t,x) \in (0,\infty)\times O$, $k \in \mathbb{N}$ and again Lebesgue's theorem of dominated convergence therefore prove that

$$\begin{aligned}
&\int_O \int_0^\infty u(t,x)\cdot\chi(t,x)\,dt\,dx = \int_O \int_0^\infty \left[\lim_{k\to\infty} u^{(k)}(t,x) \right]\cdot\chi(t,x)\,dt\,dx \\
&= \lim_{k\to\infty} \left[\int_O \int_0^\infty u^{(k)}(t,x)\cdot\chi(t,x)\,dt\,dx \right] = 0.
\end{aligned} \tag{97}$$

This shows (88) and the proof of Proposition 4.12 is thus completed. □

5 A counterexample to the rate of convergence of the Euler-Maruyama method

In this section we use the results of Section 3 to establish the existence of an SDE with smooth and globally bounded coefficients for which the Euler-Maruyama method convergences *slower than any arbitrarily small* *polynomial rate of convergence.* Thereby, we will prove Theorem 1.3 in the introduction. We consider the following setting. Let $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)\colon \mathbb{R}^4 \to \mathbb{R}^4$ and $B \in \mathbb{R}^{4\times 4}$ be given by

$$\mu(x) = \begin{pmatrix} \mathbb{1}_{(1,\infty)}(x_4) \cdot \exp\Big(\frac{-1}{((x_4)^2-1)}\Big) \cdot \cos\Big(\big(x_3 - \int_0^1 e^{\frac{-1}{(1-u^2)}}\,du\big) \cdot \exp\big((x_2)^3\big)\Big) \\ 0 \\ \mathbb{1}_{(-1,1)}(x_4) \cdot \exp\Big(\frac{-1}{(1-(x_4)^2)}\Big) \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{98}$$

for all $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The function $\mathbb{R} \ni x \mapsto \mathbb{1}_{(-1,1)}(x) \cdot \exp(-1/(1-x^2)) \in [0,1]$ that appears in μ has been suggested as a mollifier function in Lemma 1.2.3 in Hörmander [33]. Note that $\mu\colon \mathbb{R}^4 \to \mathbb{R}^4$ is infinitely often differentiable and globally bounded. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space supporting a four-dimensional standard Brownian motion $W\colon [0,\infty) \times \Omega \to \mathbb{R}^4$ with continuous sample paths. Then there exists a unique stochastic process $X\colon [0,\infty) \times \Omega \to \mathbb{R}^4$ with continuous sample paths which fulfills $X(t) = \int_0^t \mu(X(s))\,ds + B\,W(t)$ for all $t \in [0,\infty)$. The stochastic process $X = (X_1, X_2, X_3, X_4)\colon [0,\infty) \times \Omega \to \mathbb{R}^4$ is thus a solution process of the SDE

$$\begin{aligned} dX_1(t) &= \mathbb{1}_{(1,\infty)}(X_4(t)) \cdot \exp\Big(\tfrac{-1}{([X_4(t)]^2-1)}\Big) \cdot \cos\Big(\Big(X_3(t) - \textstyle\int_0^1 e^{-1/(1-u^2)}\,du\Big) \cdot \exp\big(\,[X_2(t)]^3\,\big)\Big)\,dt \\ dX_2(t) &= dW_2(t) \\ dX_3(t) &= \mathbb{1}_{(-1,1)}(X_4(t)) \cdot \exp\Big(\tfrac{-1}{(1-[X_4(t)]^2)}\Big)\,dt \\ dX_4(t) &= 1\,dt \end{aligned} \tag{99}$$

for $t \in [0,\infty)$ satisfying $X(0) = 0$. In the next step we define the Euler-Maruyama approximations for the SDE (99) using the following notation. Let $\lfloor\cdot\rfloor_h\colon [0,\infty) \to [0,\infty)$, $h \in (0,\infty)$, be a family of mappings defined through

$$\lfloor t \rfloor_h := \max\{s \in \{0, h, 2h, \dots\}\colon s \le t\} \tag{100}$$

for all $t \in [0,\infty)$ and all $h \in (0,\infty)$. Then let $Y^h = (Y_1^h, Y_2^h, Y_3^h, Y_4^h)\colon [0,\infty) \times \Omega \to \mathbb{R}^4$, $h \in (0,\infty)$, be Euler-Maruyama approximation processes defined recursively through

$$Y^h(0) := 0 \qquad \text{and} \qquad Y^h(t) := \mu\big(Y^h(\lfloor t \rfloor_h)\big) \cdot (t - \lfloor t \rfloor_h) + B\,(W(t) - W(\lfloor t \rfloor_h)) \tag{101}$$

for all $t \in \big(nh, (n+1)h\big]$, $n \in \{0, 1, \dots\}$ and all $h \in (0,\infty)$. Observe that this definition ensures that

$$Y_1^h(t) = \int_1^t \mathbb{1}_{(1,\infty)}(\lfloor s \rfloor_h)\, e^{\frac{-1}{(|\lfloor s \rfloor_h|^2-1)}} \cos\bigg(\bigg(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u \rfloor_h)\, e^{\frac{-1}{(1-|\lfloor u \rfloor_h|^2)}}\,du - \int_0^1 e^{\frac{-1}{(1-u^2)}}\,du\bigg)\, e^{(\{W_2(\lfloor s \rfloor_h)\}^3)}\bigg)\,ds \tag{102}$$

for all $t \in [1,\infty)$ and all $h \in (0,\infty)$. The following Theorem 5.1 proves that the Euler-Maruyama method (101) for the SDE (99) convergences *slower than any arbitrarily small polynomial rate of convergence.* Theorem 5.1 together with an elementary transformation argument implies then Theorem 1.3.

Theorem 5.1 (A counterexample to the rate of convergence of the Euler-Maruyama method)**.** *Let $X = (X_1, X_2, X_3, X_4)\colon [0,\infty) \times \Omega \to \mathbb{R}^4$ be a solution process of the SDE* (99) *with continuous sample paths and with $X(0) = 0$. Then $\mathbb{E}\big[X_1(t)\big] - \mathbb{E}\big[Y_1^h(t)\big] \ge \exp\big(-14|\ln(h)|^{2/3}\big)$ for all $h \in (0, \frac{1}{22}]$ and all $t \in [2,\infty)$ and therefore, we obtain*

$$\lim_{h \searrow 0} \bigg(\frac{\mathbb{E}\big[\|X(t) - Y^h(t)\|\big]}{h^\alpha}\bigg) = \lim_{h \searrow 0} \bigg(\frac{\big\|\mathbb{E}[X(t)] - \mathbb{E}[Y^h(t)]\big\|}{h^\alpha}\bigg) = \begin{cases} 0 & : \alpha = 0 \\ \infty & : \alpha > 0 \end{cases} \tag{103}$$

for all $\alpha \in [0,\infty)$ and all $t \in [2,\infty)$. In particular, for every $t \in [2,\infty)$ and every $\alpha, C, h_0 \in (0,\infty)$ there exists a real number $h \in (0, h_0)$ such that $\big\|\mathbb{E}\big[X(t)\big] - \mathbb{E}\big[Y^h(t)\big]\big\| > C \cdot h^\alpha$.

The proof of Theorem 5.1 is deferred to the end of this section. To the best of our knowledge, the SDE (99) is the first SDE with smooth coefficients in the literature for which it has been established that the Euler-Maruyama scheme converges in the strong and numerical weak sense *slower than any arbitrarily small rate of convergence.* Using the results of Section 3, one can show that the SDE (99) is not locally Hölder continuous with respect to the initial value. This is summarized in the next corollary. Its proof is a straightforward consequence of Lemma 3.3 in Subsection 3 and is therefore omitted.

Corollary 5.2. *Let $X^x\colon [0,\infty) \times \Omega \to \mathbb{R}^4$, $x \in \mathbb{R}^4$, be solution processes of the SDE* (99) *with continuous sample paths and with $X^x(0) = x$ for all $x \in \mathbb{R}^4$. Then for every $t \in (0,\infty)$ the function $\mathbb{R}^4 \ni x \mapsto \mathbb{E}[X^x(t)] \in \mathbb{R}^4$ is not locally Hölder continuous.*

In the following, the size of the quantity $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^h(T)]\| \in [0,\infty)$ is analyzed for sufficiently small $h \in (0,\infty)$ and thereby Theorem 5.1 is established. To do so, we first establish a few auxiliary results. We begin with an elementary estimate for the numerical integration of concave functions.

Lemma 5.3 (Numerical integration of concave functions)**.** *Let $\lfloor \cdot \rfloor_h \colon [0,\infty) \to [0,\infty)$, $h \in (0,\infty)$, be given by* (100)*, let $b \in (0,\infty)$ be a real number and let $\psi \colon [0,b] \to \mathbb{R}$ be a continuously differentiable function with a non-increasing derivative. Then*

$$\int_0^b (\psi(s) - \psi(\lfloor s \rfloor_h))\, ds \leq \frac{1}{2}\left[\psi'(0)\cdot h^2 + \Big(\psi(\lfloor b \rfloor_h - h) - \psi(0)\Big)\cdot h + \psi'(\lfloor b \rfloor_h)\cdot (b - \lfloor b \rfloor_h)^2\right] \tag{104}$$

for all $h \in (0,b]$.

Proof of Lemma 5.3. The fundamental theorem of calculus and monotonicity of ψ' imply

$$\begin{aligned}
&\int_0^b (\psi(s) - \psi(\lfloor s \rfloor_h))\, ds = \int_0^b \int_{\lfloor s \rfloor_h}^s \psi'(u)\, du\, ds \leq \int_0^b \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h)\, du\, ds \\
&= \int_0^h \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h)\, du\, ds + \int_h^{\lfloor b \rfloor_h} \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h)\, du\, ds + \int_{\lfloor b \rfloor_h}^b \int_{\lfloor s \rfloor_h}^s \psi'(\lfloor s \rfloor_h)\, du\, ds \\
&\leq \psi'(0)\cdot \frac{h^2}{2} + \frac{h}{2}\left(\sum_{\substack{n\in\mathbb{N}\\ nh<\lfloor b \rfloor_h}} \int_{(n-1)h}^{nh} \psi'(s)\, ds\right) + \psi'(\lfloor b \rfloor_h)\cdot \frac{(b - \lfloor b \rfloor_h)^2}{2} \\
&= \psi'(0)\cdot \frac{h^2}{2} + (\psi(\lfloor b \rfloor_h - h) - \psi(0))\cdot \frac{h}{2} + \psi'(\lfloor b \rfloor_h)\cdot \frac{(b - \lfloor b \rfloor_h)^2}{2}
\end{aligned} \tag{105}$$

for all $h \in (0,b]$. This finishes the proof of Lemma 5.3. □

Using Lemma 5.3, we establish in the next lemma a simple lower bound for the numerical integration of the function $\mathbb{1}_{(-1,1)}(x)\cdot \exp(-1/(1-x^2))$, $x \in \mathbb{R}$, in the third component of $\mu \colon \mathbb{R}^4 \to \mathbb{R}^4$.

Lemma 5.4 (Numerical integration of the function $\mathbb{1}_{(-1,1)}(x)\cdot \exp(-1/(1-x^2))$, $x \in \mathbb{R}$)**.** *Let $\lfloor \cdot \rfloor_h \colon [0,\infty) \to [0,\infty)$, $h \in (0,\infty)$, be given by* (100)*. Then*

$$\frac{h}{20} \leq \int_0^\infty \mathbb{1}_{[0,1)}(\lfloor s \rfloor_h)\cdot \exp\!\Big(\tfrac{-1}{(1-|\lfloor s \rfloor_h|^2)}\Big)\, ds - \int_0^1 \exp\!\Big(\tfrac{-1}{(1-s^2)}\Big)\, ds \leq 2h \tag{106}$$

for all $h \in (0,\frac{1}{8}]$.

Proof of Lemma 5.4. First of all, observe that

$$\frac{d}{dx}\Big(e^{-1/(1-x^2)}\Big) = \frac{-2x\cdot e^{-1/(1-x^2)}}{(1-x^2)^2} \qquad \text{and} \qquad \frac{d^2}{dx^2}\Big(e^{-1/(1-x^2)}\Big) = \frac{6\cdot e^{-1/(1-x^2)}}{(1-x^2)^4}\Big(x^4 - \frac{1}{3}\Big) \tag{107}$$

for all $x \in (-1,1)$. We hence obtain that the function $[0,3^{-1/4}] \ni s \mapsto e^{-1/(1-s^2)} \in \mathbb{R}$ has a non-increasing derivative. Applying Lemma 5.3 and using that the function $[0,\infty) \ni s \mapsto \mathbb{1}_{[0,1)}(s)\cdot e^{-1/(1-s^2)} \in \mathbb{R}$ is non-increasing therefore results in

$$\begin{aligned}
&\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor s \rfloor_h)\cdot \exp\!\Big(\tfrac{-1}{(1-|\lfloor s \rfloor_h|^2)}\Big)\, ds - \int_0^1 \exp\!\Big(\tfrac{-1}{(1-s^2)}\Big)\, ds \geq \int_0^{3^{-1/4}} \exp\!\Big(\tfrac{-1}{(1-|\lfloor s \rfloor_h|^2)}\Big) - \exp\!\Big(\tfrac{-1}{(1-s^2)}\Big)\, ds \\
&\geq \frac{h}{2}\cdot\left(\exp\!\Big(\tfrac{-1}{(1-0^2)}\Big) - \exp\!\Big(\tfrac{-1}{(1-|\lfloor 3^{-1/4} \rfloor_h - h|^2)}\Big)\right) + \tfrac{2\cdot \lfloor 3^{-1/4} \rfloor_h \cdot e^{-1/(1-|\lfloor 3^{-1/4} \rfloor_h|^2)}}{\left[1-|\lfloor 3^{-1/4} \rfloor_h|^2\right]^2}\cdot \tfrac{(3^{-1/4} - \lfloor 3^{-1/4} \rfloor_h)^2}{2} \\
&\geq \frac{h}{2}\cdot\left(e^{-1} - \exp\!\Big(\tfrac{-1}{(1-[3^{-1/4}-2h]^2)}\Big)\right) \geq \frac{h}{2}\cdot\left(e^{-1} - \exp\!\Big(\tfrac{-1}{(1-[\frac{1}{2}]^2)}\Big)\right) = h\cdot \frac{(e^{-1} - e^{-4/3})}{2} > \frac{h}{20}
\end{aligned} \tag{108}$$

for all $h \in (0,\frac{1}{8}]$. Moreover, note that (107) implies that

$$\begin{aligned}
&\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor s \rfloor_h)\cdot \exp\!\Big(\tfrac{-1}{(1-|\lfloor s \rfloor_h|^2)}\Big)\, ds - \int_0^1 \exp\!\Big(\tfrac{-1}{(1-s^2)}\Big)\, ds \\
&\leq h + \int_0^1 \left|\exp\!\Big(\tfrac{-1}{(1-|\lfloor s \rfloor_h|^2)}\Big) - \exp\!\Big(\tfrac{-1}{(1-s^2)}\Big)\right| ds \leq h + \sup_{x\in(0,1)}\left[\frac{2x\cdot e^{-1/(1-x^2)}}{(1-x^2)^2}\right]\cdot h \\
&= h + \left[\frac{2\cdot 3^{-1/4}\cdot e^{-1/(1-3^{-1/2})}}{(1-3^{-1/2})^2}\right]\cdot h = h + \left[\frac{6}{3^{1/4}\cdot(\sqrt{3}-1)^2\cdot e^{\sqrt{3}/(\sqrt{3}-1)}}\right]\cdot h \leq 2h
\end{aligned} \tag{109}$$

for all $h \in (0,\infty)$. Combining (108) and (109) completes the proof of Lemma 5.4. $\square$

We are now ready to prove Theorem 5.1. Its proof uses Lemma 5.4 as well as Lemma 3.3 in Section 3 above.

Proof of Theorem 5.1. First of all, note that

$$\begin{aligned}
&\mathbb{E}\big[X_1(t)\big] - \mathbb{E}\big[Y_1^h(t)\big] = \int_1^t \exp\!\Big(\tfrac{-1}{(s^2-1)}\Big) - \mathbb{1}_{(1,\infty)}(\lfloor s\rfloor_h)\cdot \exp\!\Big(\tfrac{-1}{(|\lfloor s\rfloor_h|^2-1)}\Big)\, ds \\
&+ \int_1^t \mathbb{1}_{(1,\infty)}(\lfloor s\rfloor_h)\, e^{\frac{-1}{(|\lfloor s\rfloor_h|^2-1)}}\, \mathbb{E}\bigg[1-\cos\bigg(\bigg(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u\rfloor_h)\, e^{\frac{-1}{(1-|\lfloor u\rfloor_h|^2)}}\, du - \int_0^1 e^{\frac{-1}{(1-u^2)}}\, du\bigg)\, e^{\left(\{W_2(\lfloor s\rfloor_h)\}^3\right)}\bigg)\bigg]\, ds \\
&\geq \int_{3/2}^t \mathbb{1}_{(1,\infty)}(\lfloor s\rfloor_h)\, e^{\frac{-1}{(|\lfloor s\rfloor_h|^2-1)}}\, \mathbb{E}\bigg[1-\cos\bigg(\bigg(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u\rfloor_h)\, e^{\frac{-1}{(1-|\lfloor u\rfloor_h|^2)}}\, du - \int_0^1 e^{\frac{-1}{(1-u^2)}}\, du\bigg)\, e^{\left(\{W_2(\lfloor s\rfloor_h)\}^3\right)}\bigg)\bigg]\, ds
\end{aligned}$$

for all $t \in [\frac{3}{2},\infty)$ and all $h \in (0,\infty)$. The estimate $\lfloor s\rfloor_h \geq \lfloor \frac{3}{2}\rfloor_h \geq \frac{3}{2} - h \geq \frac{11}{8}$ for all $s \in [\frac{3}{2},\infty)$, $h \in (0,\frac{1}{8}]$, Lemma 5.4 and Lemma 3.3 therefore show that

$$\begin{aligned}
&\mathbb{E}\big[X_1(t)\big] - \mathbb{E}\big[Y_1^h(t)\big] \\
&\geq \exp\!\Big(\tfrac{-1}{(\frac{121}{64}-1)}\Big) \int_{\frac{3}{2}}^v \mathbb{E}\bigg[1-\cos\bigg(\bigg(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u\rfloor_h)\, e^{-1/(1-|\lfloor u\rfloor_h|^2)}\, du - \int_0^1 e^{-1/(1-u^2)}\, du\bigg)\, e^{\left(\{W_2(\lfloor s\rfloor_h)\}^3\right)}\bigg)\bigg]\, ds \\
&\geq e^{\frac{-64}{57}} \int_{\frac{3}{2}}^v \exp\!\left(\frac{-8}{\lfloor s\rfloor_h}\left|\ln\!\left(\frac{\pi}{2\left(\int_0^\infty \mathbb{1}_{[0,1)}(\lfloor u\rfloor_h)\cdot e^{-1/(1-|\lfloor u\rfloor_h|^2)}\, du - \int_0^1 e^{-1/(1-u^2)}\, du\right)}\right)\right|^{2/3}\right) ds \\
&\geq \frac{(v-\frac{3}{2})}{4}\cdot \exp\!\Big(\tfrac{-64}{11}\left|\ln\!\big(\tfrac{10\pi}{h}\big)\right|^{2/3}\Big)
\end{aligned}$$

for all $h \in \big(0, \min\{\frac{1}{8}, \frac{\pi}{4}\exp(-v^{3/2})\}\big]$, $t \in [v,\infty)$ and all $v \in [\frac{3}{2},\infty)$. Hence, we finally obtain that $\mathbb{E}\big[X_1(t)\big] - \mathbb{E}\big[Y_1^h(t)\big] \geq \exp\!\Big(-\ln(8) - \frac{64}{11}\left|\ln(10\pi)\right|^{2/3} - \frac{64}{11}\left|\ln(h)\right|^{2/3}\Big)$ for all $h \in (0,\frac{1}{22}]$ and all $t \in [2,\infty)$ and this completes the proof of Theorem 5.1. $\square$

In the next step we illustrate the lower bound on the weak approximation error in Theorem 5.1 by a numerical simulation. More precisely, we ran Monte Carlo simulations and approximatively calculated the quantity $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^{\frac{T}{N}}(T)]\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{29}, 2^{30}\}$. We approximated these differences of expectations with an average over 100 000 independent Monte Carlo realizations. Moreover, we discretized the integrals $X_1(2) = \int_1^2 \exp\big(\frac{-1}{(s^2-1)}\big)\, ds$ and $X_3(2) = \int_0^1 \exp\big(\frac{-1}{(1-s^2)}\big)\, ds$ in the exact solution with a uniform grid and mesh size $\frac{2}{2^{31}} = 2^{-30}$. Figure 1 depicts the resulting graph.

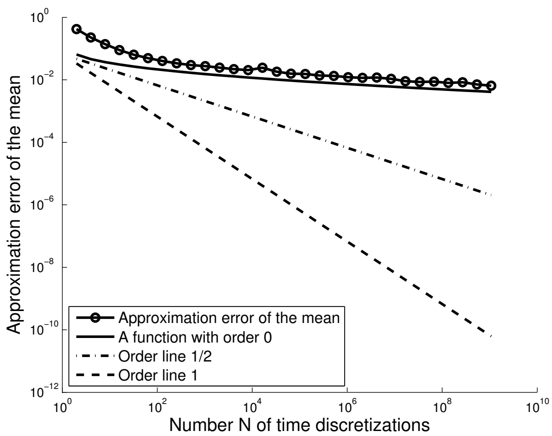

Figure 1: The norm $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^{\frac{T}{N}}(T)]\|$ of the difference between the mean of the solution of the SDE (99) and the mean of the Euler-Maruyama approximations (101) for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{29}, 2^{30}\}$. The function with convergence order 0 is given by (110).

In addition to the weak approximation error $\|\mathbb{E}[X(T)] - \mathbb{E}[Y^{\frac{T}{N}}(T)]\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{29}, 2^{30}\}$, we also plotted the function

$$\{2^1, 2^2, \ldots, 2^{30}\} \ni N \mapsto \frac{1}{15\cdot(\ln(N))^{\frac{1}{3}}} \exp\left(-\tfrac{1}{2T}\left(\ln(N) - \tfrac{1}{2T}\left(\ln(N)\right)^{\frac{2}{3}}\right)^{\frac{2}{3}}\right) \in (0,1] \tag{110}$$

(a function with order 0), the function $\{2^1, 2^2, \dots, 2^{30}\} \ni N \mapsto \frac{1}{15 \cdot \sqrt{N}} \in (0, 1]$ (order line $\frac{1}{2}$) and the function $\{2^1, 2^2, \dots, 2^{30}\} \ni N \mapsto \frac{1}{15 \cdot N} \in (0, 1]$ (order line 1) in Figure 1. In the standard literature in computational stochastics (see, e.g., Kloeden & Platen [42]) the Euler-Maruyama scheme is shown to converge in the numerically weak sense with order 1 if the coefficients of the SDE are smooth and globally Lipschitz continuous (see Chapter 8 in Kloeden & Platen [42] for the precise assumptions) and therefore, the order line 1 is plotted in Figure 1. Moreover, the function with order 0 is included in Figure 1 so that one can compare the graph visually with a function which has convergence order 0. According to our simulations, the approximation error for the mean $\mathbb{E}[X(2)]$ does not drop far below $\frac{1}{100}$ even for $N = 2^{30} > 10^9$ time discretisations. This indicates that calculating the mean $\mathbb{E}[X(T)]$ with the Euler-Maruyama method up to a high precision requires a huge computational effort. In particular, this suggests for applications that an approximation cannot, in general, be assumed to be very close to the exact value even after a very high computational effort.

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References

[1] Alfonsi, A. Strong convergence of some drift implicit Euler scheme. Application to the CIR process. *arXiv:1206.3855* (2012), 8 pages.

[2] Barles, G., and Perthame, B. Discontinuous solutions of deterministic optimal stopping time problems. *RAIRO Modél. Math. Anal. Numér. 21*, 4 (1987), 557–579.

[3] Burrage, K., Burrage, P. M., and Tian, T. Numerical methods for strong solutions of stochastic differential equations: an overview. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 460*, 2041 (2004), 373–402. Stochastic analysis with applications to mathematical finance.

[4] Cambanis, S., and Hu, Y. Exact convergence rate of the Euler-Maruyama scheme, with application to sampling design. *Stochastics Stochastics Rep. 59*, 3-4 (1996), 211–240.

[5] Cerrai, S. *Second order PDE's in finite and infinite dimension*, vol. 1762 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. A probabilistic approach.

[6] Clark, J. M. C., and Cameron, R. J. The maximum rate of convergence of discrete approximations for stochastic differential equations. In *Stochastic differential systems (Proc. IFIP-WG 7/1 Working Conf., Vilnius, 1978)*, vol. 25 of *Lecture Notes in Control and Information Sci.* Springer, Berlin, 1980, pp. 162–171.

[7] Crandall, M. G., Ishii, H., and Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.) 27*, 1 (1992), 1–67.

[8] Crandall, M. G., and Lions, P.-L. Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre. *C. R. Acad. Sci. Paris Sér. I Math. 292*, 3 (1981), 183–186.

[9] Crandall, M. G., and Lions, P.-L. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc. 277*, 1 (1983), 1–42.

[10] Da Prato, G. *Kolmogorov equations for stochastic PDEs.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2004.

[11] Da Prato, G., and Zabczyk, J. *Second order partial differential equations in Hilbert spaces*, vol. 293 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2002.

[12] Davie, A. M., and Gaines, J. G. Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. *Math. Comp. 70*, 233 (2001), 121–134 (electronic).

[13] Dereich, S., Neuenkirch, A., and Szpruch, L. An Euler-type method for the strong approximation of the Cox-Ingersoll-Ross process. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 468*, 2140 (2012), 1105–1115.

[14] Dörsek, P. Semigroup splitting and cubature approximations for the stochastic Navier-Stokes equations. *SIAM J. Numer. Anal. 50*, 2 (2012), 729–746.

[15] Elworthy, K. D. Stochastic dynamical systems and their flows. In *Stochastic analysis (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1978)*. Academic Press, New York, 1978, pp. 79–95.

[16] Evans, L. C. A convergence theorem for solutions of nonlinear second-order elliptic equations. *Indiana Univ. Math. J. 27*, 5 (1978), 875–887.

[17] Evans, L. C. On solving certain nonlinear partial differential equations by accretive operator methods. *Israel J. Math. 36*, 3-4 (1980), 225–247.

[18] Evans, L. C. *Partial differential equations*, second ed., vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010.

[19] Fang, S., Imkeller, P., and Zhang, T. Global flows for stochastic differential equations without global Lipschitz conditions. *Ann. Probab. 35*, 1 (2007), 180–205.

[20] Gan, S., and Wang, X. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *arXiv:1102.0662v4* (2011), 23 pages.

[21] Gīhman, Ĭ. Ī., and Skorohod, A. V. *Stochastic differential equations*. Springer-Verlag, New York, 1972. Translated from the Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 72.

[22] Glasserman, P. *Monte Carlo methods in financial engineering*, vol. 53 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2004. Stochastic Modelling and Applied Probability.

[23] Gyöngy, I. A note on Euler's approximations. *Potential Anal. 8*, 3 (1998), 205–216.

[24] Gyöngy, I. Approximations of stochastic partial differential equations. In *Stochastic partial differential equations and applications (Trento, 2002)*, vol. 227 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 2002, pp. 287–307.

[25] Gyöngy, I., and Krylov, N. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields 105*, 2 (1996), 143–158.

[26] Gyöngy, I., and Rásonyi, M. A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. *Stochastic Process. Appl. 121*, 10 (2011), 2189–2200.

[27] Hairer, M. On Malliavin's proof of Hörmander's theorem. *Bull. Sci. Math. 135*, 6-7 (2011), 650–666.

[28] Higham, D. J., and Kloeden, P. E. Strong convergence rates for backward Euler on a class of nonlinear jump-diffusion problems. *J. Comput. Appl. Math. 205*, 2 (2007), 949–956.

[29] Higham, D. J., Mao, X., and Stuart, A. M. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal. 40*, 3 (2002), 1041–1063 (electronic).

[30] Hofmann, N., Müller-Gronbach, T., and Ritter, K. Optimal approximation of stochastic differential equations by adaptive step-size control. *Math. Comp. 69*, 231 (2000), 1017–1034.

[31] Hofmann, N., Müller-Gronbach, T., and Ritter, K. Step size control for the uniform approximation of systems of stochastic differential equations with additive noise. *Ann. Appl. Probab. 10*, 2 (2000), 616–633.

[32] Hörmander, L. Hypoelliptic second order differential equations. *Acta Math. 119* (1967), 147–171.

[33] Hörmander, L. *The analysis of linear partial differential operators. I*, second ed. Springer Study Edition. Springer-Verlag, Berlin, 1990. Distribution theory and Fourier analysis.

[34] Hu, Y. Semi-implicit Euler-Maruyama scheme for stiff stochastic equations. In *Stochastic analysis and related topics, V (Silivri, 1994)*, vol. 38 of *Progr. Probab.* Birkhäuser Boston, Boston, MA, 1996, pp. 183–202.

[35] Hutzenthaler, M., and Jentzen, A. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *arXiv:1203.5809* (2012), 59 pages.

[36] Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. *Ann. Appl. Probab. 22*, 4 (2012), 1611–1641.

[37] Ishii, H. A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 16*, 1 (1989), 105–135.

[38] Jentzen, A., and Kloeden, P. E. The numerical approximation of stochastic partial differential equations. *Milan J. Math. 77*, 1 (2009), 205–244.

[39] Jentzen, A., Kloeden, P. E., and Neuenkirch, A. Pathwise approximation of stochastic differential equations on domains: higher order convergence rates without global Lipschitz coefficients. *Numer. Math. 112*, 1 (2009), 41–64.

[40] Klenke, A. *Probability theory.* Universitext. Springer-Verlag London Ltd., London, 2008. A comprehensive course, Translated from the 2006 German original.

[41] Kloeden, P. E., and Neuenkirch, A. Convergence of numerical methods for stochastic differential equations in mathematical finance. *preprint* (2012), 1–28.

[42] Kloeden, P. E., and Platen, E. *Numerical solution of stochastic differential equations*, vol. 23 of *Applications of Mathematics (New York).* Springer-Verlag, Berlin, 1992.

[43] Kloeden, P. E., Platen, E., and Schurz, H. *Numerical solution of SDE through computer experiments.* Universitext. Springer-Verlag, Berlin, 1994. With 1 IBM-PC floppy disk (3.5 inch; HD).

[44] Kolmogorov, A. N. Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung (in german). *Math. Ann. 104* (1931), 415–458.

[45] Kruse, R. Characterization of bistability for stochastic multistep methods. *BIT 52*, 2 (2010), 389–407.

[46] Krylov, N. V. A simple proof of the existence of a solution to the Itô equation with monotone coefficients. *Theory Probab. Appl. 35*, 3 (1990), 583–587.

[47] Krylov, N. V. On Kolmogorov's equations for finite-dimensional diffusions. In *Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998)*, vol. 1715 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 1–63.

[48] Kunita, H. *Stochastic flows and stochastic differential equations*, vol. 24 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.

[49] Li, X.-M., and Scheutzow, M. Lack of strong completeness for stochastic flows. *Ann. Probab. 39*, 4 (2011), 1407–1421.

[50] Mao, X., and Szpruch, L. Strong convergence rates for backward Euler-Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients. *Stochastics Stochastics Rep.* (2012), 1–28.

[51] Maruyama, G. Continuous Markov processes and stochastic equations. *Rend. Circ. Mat. Palermo (2) 4* (1955), 48–90.

[52] Milstein, G. N. *Numerical integration of stochastic differential equations*, vol. 313 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1995. Translated and revised from the 1988 Russian original.

[53] Milstein, G. N., and Tretyakov, M. V. *Stochastic numerics for mathematical physics.* Scientific Computation. Springer-Verlag, Berlin, 2004.

[54] Müller-Gronbach, T. The optimal uniform approximation of systems of stochastic differential equations. *Ann. Appl. Probab. 12*, 2 (2002), 664–690.

[55] Müller-Gronbach, T., and Ritter, K. Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. *Found. Comput. Math. 7*, 2 (2007), 135–181.

[56] Müller-Gronbach, T., and Ritter, K. Minimal errors for strong and weak approximation of stochastic differential equations. In *Monte Carlo and quasi-Monte Carlo methods 2006.* Springer, Berlin, 2008, pp. 53–82.

[57] Neuenkirch, A., and Szpruch, L. First order strong approximations of scalar SDEs with values in a domain. *arXiv:1209.0390* (2012), 27 pages.

[58] Oeksendal, B. *Stochastic Differential Equations: An Introduction with Applications.* Springer-Verlag, Berlin, 2000.

[59] Pardoux, É., and Peng, S. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, vol. 176 of *Lecture Notes in Control and Inform. Sci.* Springer, Berlin, 1992, pp. 200–217.

[60] Peng, S. Backward stochastic differential equations and applications to optimal control. *Appl. Math. Optim. 27*, 2 (1993), 125–144.

[61] Peng, S. Nonlinear expectations and stochastic calculus under uncertainty. *arXiv:1002.4546v1* (2010), 149 pages.

[62] Prévôt, C., and Röckner, M. *A concise course on stochastic partial differential equations*, vol. 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[63] Röckner, M. L^p-analysis of finite and infinite dimensional diffusions. In *Stochastic PDEs and Kolmogorov Equations in Infinite Dimensions*, vol. 1715 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 65–116.

[64] Röckner, M., and Sobol, Z. Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations. *Ann. Probab. 34*, 2 (2006), 663–727.

[65] Rümelin, W. Numerical treatment of stochastic differential equations. *SIAM J. Numer. Anal. 19*, 3 (1982), 604–613.

[66] Schurz, H. An axiomatic approach to numerical approximations of stochastic processes. *Int. J. Numer. Anal. Model. 3*, 4 (2006), 459–480.