The connection between our construction and the original Hilton-Pedersen construction is that for an acute rational angle $\theta_k = p_k\pi/n$ with odd denominator n, the succeeding angle $\theta_{k+1} = \Lambda(\theta_k)$ has even numerator $p_{k+1} = 2p_k$ if $\pi/4 \geq \theta_k = F(\theta_{k+1})$, and has odd numerator $p_{k+1} = n - 2p_k$ if $\pi/4 \leq \theta_k = S(\theta_{k+1})$. Thus our criterion becomes the Hilton-Pedersen criterion reversed in time: $p_k = A_{j+1}$ if $p_{k+1} = A_j$. Moreover, the sequence $\{\sigma_k\}$ of folds and switchfolds generated by our construction is periodic with some period N, and is the same as the Hilton-Pedersen sequence, but reversed in order. If $\alpha_0 = \phi_0$, then the angles α_k generated by the Hilton-Pedersen construction and ϕ_k generated by our construction are equal when k is a multiple of N. Our result then implies the Hilton-Pedersen result that $\{\phi_{Nk}\}$ converges to β.

Note also: From one of the referees we learned that the Hilton-Pedersen construction in some respects resembles an earlier, much-studied method of S. Fujimoto for folding a length of paper into n equal parts. This method is summarized in English in Hull [**5**].

REFERENCES

1. P. Hilton, D. Holton, and J. Pedersen, *Mathematical Reflections: In a Room with Many Mirrors*, Springer-Verlag, New York, 1997.
2. ———, *Mathematical Vistas: From a Room with Many Windows*, Springer-Verlag, New York, 2000.
3. P. Hilton and J. Pedersen, Approximating any regular polygon by folding paper, *Math. Mag.* **56** (1983) 141–155.
4. ———, Folding regular star polygons and number theory, *Math. Intelligencer* **7** (1983) 15–26.
5. T. Hull, *Project Origami: Activities for Exploring Mathematics*, A. K. Peters, Wellesley, MA, 2006.
6. B. Polster, Variations on a theme in paper folding, this MONTHLY **111** (2004) 39–47.

Department of Mathematics, United States Naval Academy, Annapolis, MD 21402
wdw@usna.edu

Lost (and Found) in Translation: André's *Actual* Method and Its Application to the Generalized Ballot Problem

Marc Renault

1. INTRODUCTION. Although Désiré André is widely credited with creating the "reflection method" to solve the ballot problem, in fact, André never employed this method. Furthermore, while the reflection method does not provide a solution to the generalized ballot problem, André's original solution can be modified to produce such a solution. We examine André's true solution and its generalization, and explore the connection between André and the reflection method in the mathematics literature.

2. THE BALLOT PROBLEM. In 1887 Joseph Bertrand [**4**] introduced the ballot problem and its solution:

> Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a - b)/(a + b)$.

Bertrand sketched a proof based on induction, and asked if a direct proof could be given.

In that same year (and only 38 pages after Bertrand's article in the same journal), Émile Barbier [**3**] implied a generalization of Bertrand's result: if $a > kb$ for some positive integer k, then the probability that A stays ahead of B by more than a factor of k throughout the counting is $(a - kb)/(a + b)$. Barbier provided no proof of his result, direct or otherwise.

Still in 1887 (and 29 pages after Barbier), Désiré André [**2**] offered a direct proof of the solution to Bertrand's ballot problem. A translation of André's solution appears in section 4.

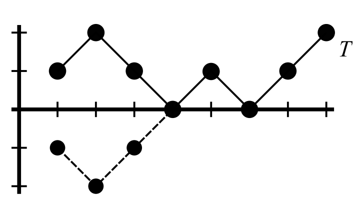
Many excellent articles and reputable sources cite André [**2**] claiming that he used the "reflection method" (or "reflection principle") to solve the ballot problem; this is in fact not the case. To be fair, the reflection method is a variation of André's method of proof. Comtet [**6**, p. 22] writes when introducing the method, "We first formulate the *principle of reflection*, which essentially is due to André." However, it is not accurate to say that André actually employed the reflection method in his proof.

We first look at the reflection method and see why it is not readily applicable to the generalized ballot problem. Then, we present a translation of André's original proof, and based on that, we provide a proof of the generalized ballot problem. Finally, we consider the occurrence of the reflection principle throughout the literature and its association with André.

3. THE REFLECTION METHOD. We can express ballot permutations as lattice paths in the Euclidean plane by thinking of votes for A as upsteps $(1, 1)$ and votes for B as downsteps $(1, -1)$. Ballot permutations (or paths) that satisfy the ballot problem are called "good," while those that do not are called "bad." Feller [**9**, ch. III] gives a nice account of the reflection method.

The Ballot Problem. Given positive integers a, b with $a > b$, find the number of lattice paths starting at the origin and consisting of a upsteps $(1, 1)$ and b downsteps $(1, -1)$ such that no step ends on the x-axis.

Solution via reflection. Let T denote the terminal point of the path, $(a + b, a - b)$. Since every good path must start with an upstep, there are as many good paths as there are paths from $(1, 1)$ to T that never touch the x-axis.



The set of paths from $(1, 1)$ to T that *do* touch the x-axis somewhere is in one-to-one correspondence with the set of all paths from $(1, -1)$ to T; this is seen by reflecting across the x-axis the initial segment of the path that ends with the step that first touches the x-axis. Subtracting the number of these paths from the number of all paths from $(1, 1)$ to T produces the number of good paths:

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}. \qquad \blacksquare$$

In the generalized ballot problem, votes for A are upsteps $(1, 1)$, votes for B are downsteps $(1, -k)$ and we wish to find how many of these paths stay above the x-axis after the origin.

The difficulty in generalizing the reflection method is that when one reflects such a lattice path, the result does not have the required types of upsteps and downsteps. Hilton and Pedersen [**12**] solve the general case by a completely different method after commenting, "André's Reflection Method does not seem to be readily applicable to obtaining a formula corresponding to [the general case]" (see also [**11**, pp. 220–233]). Goulden and Serrano [**10**] also note that while there are many solutions to the generalized ballot problem in the literature, "there appears to be no solution which is in the spirit of the reflection principle." They go on to provide a nice proof that *rotates* the initial path segment by 180°; even so, they acknowledge that their proof does not specialize to the reflection method when $k = 1$. (As an interesting side note, rotating a path is equivalent to writing a ballot permutation in reverse order, and Uspensky [**14**, p. 152] used this method to solve the ballot problem in 1937.)

4. ANDRÉ'S ACTUAL METHOD. André was the first to solve the ballot problem by subtracting the number of bad ballot permutations from the number of all possible ballot permutations. This, of course, is also the approach taken by the reflection method. However, whereas the reflection method modifies an initial segment of a lattice path (equivalently, a ballot permutation), André uses no geometric reasoning and he *interchanges* two portions of a ballot permutation. A translation of his solution follows; the two phrases in brackets are clarifying remarks not found in the original paper.

> The number of possible outcomes is obviously the number of permutations one can form with a letters A and b letters B.
>
> Let $Q_{a,b}$ be the number of *unfavorable* outcomes. The permutations corresponding to them are of two kinds: those that start with B, and those that start with A.
>
> The number of unfavorable permutations starting with B equals the number of all permutations which one can form with a letters A and $b - 1$ letters B, because it is obviously enough to suppress the initial letter B to obtain the remaining letters.
>
> The number of unfavorable permutations starting with A is the same as above, because one can, by a simple rule, make a one-to-one correspondence with the permutations formed with a letters A and $b - 1$ letters B. This rule is composed of two parts:
>
> (1) Given an unfavorable permutation starting with A, one removes the first occurrence of B that violates the law of the problem [i.e., causes the number of B's to equal the number of A's], then one exchanges the two groups separated by this letter: one obtains thus a permutation, uniquely determined, of a letters A and $b - 1$ letters B. Consider, for example, the unfavorable permutation $AABBABAA$, of five letters A and three letters B; by removing the first B that violates the law, one separates two groups AAB, $ABAA$; by exchanging these groups, one obtains the permutation $ABAAAAB$, formed of five letters A and two letters B.
>
> (2) Given an arbitrary permutation of a letters A and $b - 1$ letters B, one traverses it from right to left until one obtains a group where the number of A's exceeds [by one] the number of B's; one considers this group and that which

the letters placed at its left form; one exchanges these two groups, while placing between them a letter B: one thus forms an unfavorable permutation starting with A that is uniquely given. Consider, for example, the permutation $ABAAAAB$; while operating as described, one divides it in two groups $ABAA$, AAB; by exchanging these groups and placing the letter B between them, one forms the unfavorable permutation $AABBABAA$.

It results from all the above that the total number of unfavorable outcomes is twice the number of permutations one can form with a letters A and $b-1$ letters B . . . .

And so André finds that the number of good ballot permutations is

$$\binom{a+b}{a} - 2\binom{a+b-1}{a} = \frac{a-b}{a+b}\binom{a+b}{a}.$$

5. SOLVING THE GENERALIZED BALLOT PROBLEM. We now apply André's method to the generalized ballot problem. This solution makes crucial use of the sets $\mathcal{B}_i$, first considered in [**10**].

The Generalized Ballot Problem. Given positive integers a, b, k with $a > kb$, find the number of lattice paths starting at the origin and consisting of a upsteps $(1, 1)$ and b downsteps $(1, -k)$ such that no step ends on or below the x-axis.

Solution. For $0 \le i \le k$ let $\mathcal{B}_i$ denote the set of bad paths whose first bad step ends i units below the x-axis (observe that the paths in $\mathcal{B}_k$ necessarily start with a downstep). Clearly these $k+1$ sets are disjoint and their union is the set of all bad paths. Let $\mathcal{A}$ be the set of all paths consisting of a upsteps and $b-1$ downsteps, without regard to location in the plane; $|\mathcal{A}| = \binom{a+b-1}{a}$. We prove that $|\mathcal{B}_i| = |\mathcal{A}|$ for each i in the range $0 \le i \le k$ by providing a one-to-one correspondence between $\mathcal{B}_i$ and $\mathcal{A}$.

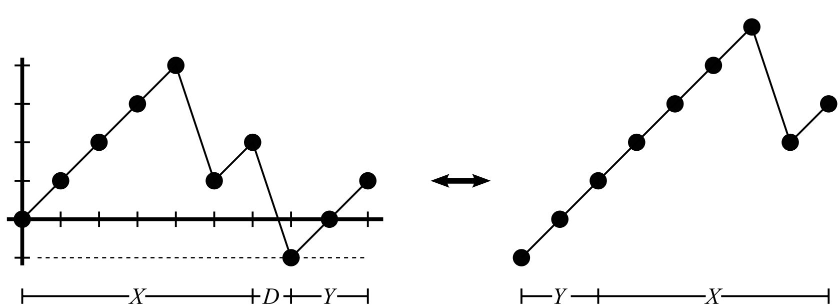


Figure 1. Example with $k = 3$. $XDY \in \mathcal{B}_1$ and $YX \in \mathcal{A}$.

Given a path $P \in \mathcal{B}_i$ we can write $P = XDY$ where D is the first downstep that ends on or below the x-axis (note that X is empty if $i = k$). The path YX is then uniquely determined and is an element of $\mathcal{A}$.

Given a path $Q \in \mathcal{A}$, scan the path from right to left until a vertex is found lying $k-i$ units below the terminal vertex of Q (note that this vertex is the terminal vertex of Q itself if $i = k$). Such a vertex must exist since the initial vertex of Q lies more than k units below the terminal vertex of Q. Write $Q = YX$, where Y and X are the paths joined at that vertex. Construct path $P = XDY$ by interchanging X and Y

and inserting a downstep D between them. Translating P to start at $(0, 0)$, we see X touches the x-axis only at the origin, and XD ends i units below the x-axis; hence $P \in \mathcal{B}_i$.

Therefore, the number of good paths is

$$\binom{a+b}{a} - (k+1)\binom{a+b-1}{a} = \frac{a-kb}{a+b}\binom{a+b}{a}. \qquad \blacksquare$$

6. ANDRÉ AND THE REFLECTION METHOD. When did the reflection method make its first appearance as a solution to the ballot problem? Bertrand's textbook on probability [**5**, p. 18], first published in 1888, solves the ballot problem exactly as André did. In the following years, evidently many other minor variations appeared (see [**8**] and the references therein).

In 1923 Aebly [**1**] conceives of ballot permutations as paths starting from a corner on a rectangular grid, and he observes a certain symmetry of bad paths across the diagonal. Mirimanoff [**13**] (the very next article in the same journal) builds on Aebly's observations and relates this symmetry to ballot permutations of A's and B's by applying the transposition (A, B) to an initial segment of a bad ballot permutation. Viewed geometrically, Mirimanoff's method is exactly the reflection method we know today.

When did authors start claiming that André actually used the reflection method in his solution? This is more difficult to ascertain. In 1947 Dvoretzky and Motzkin [**8**] introduce a new method of proof to the ballot problem and its generalization. They accurately describe André's contribution, making *no mention* of the reflection method, and write "André's proof or variations of it may be found in most of the classical treatises on the theory of probability." In 1957, citing this paper of Dvoretzky and Motzkin, Feller [**9**, p. 66] not quite accurately writes "As these authors point out, most of the formally different proofs in reality use the reflection principle . . . ," suggesting, perhaps, that the reflection method was the original method of solution. Feller, four pages later, states the reflection principle and makes the footnote, "The probability literature attributes this method to D. André, (1887)." The first edition of Feller's book (1950) mentions neither André nor the reflection principle.

The earliest source that this author could find linking André and the reflection method is J. L. Doob [**7**, p. 393] (1953), who writes while describing Brownian motion processes, " . . . similar exact evaluations are easily made . . . using what is known as the reflection principle of Désiré André." While these early sources do not explicitly state that André used the reflection method in his own proof, clearly this could be inferred, and other writers since then have naturally assumed that André did indeed utilize the reflection method.

Original articles and translations of Bertrand, Barbier, André, Aebly, and Mirimanoff can be found at `http://webspace.ship.edu/msrenault/ballotproblem/`. Many thanks to my colleague Paul Taylor for his assistance with the translations.

REFERENCES

1. J. Aebly, Démonstration du problème du scrutin par des considérations géométriques, *L'enseignement mathématique* **23** (1923) 185–186.
2. D. André, Solution directe du problème résolu par M. Bertrand, *Comptes Rendus de l'Académie des Sciences*, Paris, **105** (1887) 436–437.
3. É. Barbier, Généralisation du problème résolu par M. J. Bertrand, *Comptes Rendus de l'Académie des Sciences*, Paris, **105** (1887) p. 407.
4. J. Bertrand, Solution d'un problème, *Comptes Rendus de l'Académie des Sciences*, Paris, **105** (1887) p. 369.
5. ________, *Calcul des Probabilités*, 3rd ed., Chelsea Pub., Bronx, NY 1972.

6. L. Comtet, *Advanced Combinatorics, the Art of Finite and Infinite Expansions*, D. Reidel, Dordrecht and Boston, 1974.
7. J. L. Doob, *Stochastic Processes*, John Wiley, New York 1953.
8. A. Dvoretzky and Th. Motzkin, A problem of arrangements, *Duke Math. J.* **14** (1947) 305–313.
9. W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed., John Wiley, New York 1957.
10. I. P. Goulden and L. G. Serrano, Maintaining the spirit of the reflection principle when the boundary line has arbitrary integer slope, *J. Combin. Theory Ser. A* **104** (2003) 317–326.
11. P. Hilton, D. Holton, and J. Pedersen, *Mathematical Vistas From a Room with Many Windows*, Springer-Verlag, New York, 2002.
12. P. Hilton and J. Pedersen, Catalan numbers, their generalizations, and their uses, *Math. Intelligencer* **13** (1991) 64–75.
13. D. Mirimanoff, A propos de l'interprétation géométrique du problème du scrutin, *L'enseignement mathématique* **23** (1923) 187–189.
14. J. V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill, New York and London, 1937.

Mathematics Department, Shippensburg University, 1871 Old Main Drive, Shippensburg, PA 17257
msrenault@ship.edu

On Sums of Positive Integers That Are Not of the Form $ax + by$

Amitabha Tripathi

If a and b are coprime and positive integers, then the set $a\mathbb{Z} + b\mathbb{Z} = \{ax + by : x, y \in \mathbb{Z}\} = \mathbb{Z}$. The subset $\Gamma(a, b) := \{ax + by : x, y \geq 0\}$ of $a\mathbb{Z} + b\mathbb{Z}$ is only slightly more difficult to describe. It is obvious that $\Gamma(a, b)$ is an infinite set, but not so obvious that $\Gamma^c(a, b) = \mathbb{N} \setminus \Gamma(a, b)$ is finite. Indeed, it was known to Sylvester [**2**] that the largest integer in $\Gamma^c(a, b)$ is $ab - a - b = (a - 1)(b - 1) - 1$. It is not hard to show this, nor even that the number of elements in $\Gamma^c(a, b)$ is $(a - 1)(b - 1)/2$. However, there appears to be no easy way to sum the integers in $\Gamma^c(a, b)$; for instance, a generating function is used to do this in [**1**]. The purpose of this note is to show that the sum $s(a, b)$ can be determined directly, and quite easily. In fact, we do a bit more.

Given k positive integers $a_1, a_2, \dots, a_k$ with $\gcd(a_1, a_2, \dots, a_k) = 1$, we consider the complement of the set $\Gamma(a_1, a_2, \dots, a_k) := \{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_i \geq 0\}$ in $\mathbb{N}$. Observe that if $n \in \Gamma(a_1, a_2, \dots, a_k)$, then $n + ma_1 \in \Gamma(a_1, a_2, \dots, a_k)$ for every nonnegative integer m. As a consequence of the well-ordering of $\mathbb{N}$, there is a *least* positive integer in $\Gamma(a_1, a_2, \dots, a_k)$ among those congruent to i modulo a_1 for each i with $1 \leq i \leq a_1 - 1$. We denote this minimum by m_i, and note that $\Gamma^c(a_1, a_2, \dots, a_k)$ can be expressed as the union of $a_1 - 1$ arithmetic progressions, one for each i between 1 and $a_1 - 1$. The i th arithmetic progressions has first term i, last term $m_i - a_1$ and common difference a_1. This makes it easy to express the sum $s(a_1, a_2, \dots, a_k)$ of the integers in $\Gamma^c(a_1, a_2, \dots, a_k)$ in terms of these minima.

By the definition of the m_i's, $m_i - a_1$ is the *largest* positive integer in $\Gamma^c(a_1, a_2, \dots, a_k)$ among those congruent to i modulo a_1. Hence the sum of elements in $\Gamma^c(a_1, a_2, \dots, a_k)$ congruent to i modulo a_1 is easily seen to be $(m_i - i)(m_i + i - a_1)/2a_1$. It follows that