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Lotka-Volterra Systems and Polymatrix Replicators

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Resumo

Na década de 1970 John M. Smith e George R. Price [22] começaram a usar a teoria de jogos estratégicos desenvolvida por John von Neumann e Oskar Morgenstern [42] nos anos 1940 para investigar os processos dinâmicos de populações, dando assim origem à *Teoria de Jogos Evolutivos* (TJE).

Algumas classes de equações diferenciais ordinárias (e.d.o.s) que têm um papel central na TJE são os *sistemas Lotka-Volterra* (LV), a *equação do replicador*, o *replicador bimatricial* e o *replicador polimatricial*.

Muitas propriedades dos sistemas LV podem ser expressas geometricamente em termos do seu grafo associado, construido a partir da matriz de interacção do sistema. Para a classe dos sistemas LV estavelmente dissipativos provamos que a característica da sua matriz de interação, que é a dimensão da folheação invariante associada, é completamente determinada pelo grafo do sistema.

Nesta tese estudamos também fluxos analíticos definidos em politopos. Apresentamos uma teoria que nos permite analisar a dinâmica assintótica do fluxo ao longo da rede heteroclínica formada pelas arestas e vértices do politopo onde os fluxos estão definidos. Neste contexto, dado um fluxo definido num politopo, damos condições suficientes para a existência de variedades normalmente hiperbólicas estáveis e instáveis para ciclos heteroclínicos.

Nos jogos polimatriciais a população é dividida num número finito de grupos, cada um com um número finito de estratégias. As interacções entre indivíduos de quaisquer dois grupos podem ocorrer, inclusive do mesmo grupo. A equação diferencial associada a um jogo polimatricial, que designamos por *replicador polimatricial*, está definida num politopo dado por um produto finito de simplexos.

Karl Sigmund e Josef Hofbauer [16] e Wolfgang Jansen [18] apresentam condições suficientes para a *permanência* nos replicadores usuais. Nesta tese generalizamos esses resultados para os jogos polimatriciais.

Também para os replicadores polimatriciais estendemos o conceito de estabilidade dissipativa desenvolvido por Ray Redheffer *et al.* [25–29]. Neste contexto generalizamos um teorema de Waldyr Oliva *et al.* [6] sobre a natureza Hamiltoniana da dinâmica limite em replicadores polimatriciais "estavelmente dissipativos".

Apresentamos ainda alguns exemplos para ilustrar resultados e conceitos fundamentais desenvolvidos ao longo da tese.

Palavras-chave: Sistemas Lotka-Volterra, equação do replicador, jogo polimatricial, sistemas estavelmente dissipativos, ciclos heteroclinicos.

Abstract

In the 1970's John M. Smith and George R. Price [22] applied the theory of strategic games developed by John von Neumann and Oskar Morgenstern [42] in the 1940's to investigate the dynamical processes of biological populations, giving rise to the field of the *Evolutionary Game Theory* (EGT).

Some classes of ordinary differential equations (o.d.e.s) which plays a central role in EGT are the *Lotka-Volterra systems* (LV), the *replicator equation*, the *bimatrix replicator* and the *polymatrix replicator*.

Many properties of the LV systems can be geometrically expressed in terms of its associated graph, constructed from the system's interaction matrix. For the class of stably dissipative LV systems we prove that the rank of its defining matrix, which is the dimension of the associated invariant foliation, is completely determined by the system's graph.

In this thesis we also study analytic flows defined on polytopes. We present a theory that allows us to analyze the asymptotic dynamics of the flow along the heteroclinic network composed by the flowing-edges and the vertices of the polytope where the flow is defined. In this context, given a flow defined on a polytope, we give sufficient conditions for the existence of normally hyperbolic stable and unstable manifolds for heteroclinic cycles.

In polymatrix games population is divided in a finite number of groups, each one with a finite number of strategies. Interactions between individuals of any two groups are allowed, including the same group. The differential equation associated to a polymatrix game, that we designate as *polymatrix replicator*, is defined in a polytope given by a finite product of simplices.

Karl Sigmund and Josef Hofbauer [16] and Wolfgang Jansen [18] give sufficient conditions for *permanence* in the usual replicators. We generalize these results for polymatrix replicators.

Also for polymatrix replicators we extend the concept of stably dissipativeness developed by Ray Redheffer *et al.* [25–29]. In this context we generalize a theorem of Waldyr Oliva *et al.* [6] about the Hamiltonian nature of the limit dynamics in "stably dissipative" polymatrix replicators.

We present also some examples to illustrate fundamental results and concepts developed along the thesis.

Key-words: Lotka-Volterra systems, replicator equation, polymatrix game, stably dissipative systems, heteroclinic cycles.

Resumo Alargado

O livro "Theory of Games and Economic Behavior" [42], publicado em 1944 pelo matemático John von Neumann e o economista Oskar Morgenstern, marca o início do estudo matemático da tomada de decisões estratégicas, dando assim origem à area científica conhecida como *Teoria de Jogos*.

Passados cerca de 30 anos John Maynard Smith e George R. Price [22] usaram a teoria de jogos estratégicos desenvolvida por Neumann e Morgenstern para estudar os processos dinâmicos de populações, dando assim origem à *Teoria de Jogos Evolutivos* (TJE).

Existem algumas classes de equações diferenciais ordinárias (e.d.o.s) que têm um papel muito importante na TJE, nomeadamente os *sistemas Lotka-Volterra* (LV), a *equação do replicador*, o *replicador bimatricial* e o *replicador polimatricial*.

Nos anos 1920 os sistemas LV começaram a ser desenvolvidos por Alfred J. Lotka [21] e Vito Volterra [41], de forma independente um do outro, para modelar o processo evolutivo de sistemas químicos e de populações, respectivamente. Estes sistemas foram então designados por sistemas *Lotka-Volterra* em sua homenagem.

Normalmente os sistemas LV são classificados em termos das propriedades algébricas da sua matriz de interacção, propriedades estas que nos permitem uma análise qualitativa da dinâmica do sistema. Associado a um sistema LV dado, e a partir da sua matriz de interacção, podemos construir um grafo que representa as interacções entre os vários elementos da população. As propriedades deste grafo são muito úteis no estudo da dinâmica do sistema associado.

Ainda que possamos analisar completamente a dinâmica dos sistemas LV em baixa dimensão, para sistemas em dimensões superiores, o estudo da sua dinâmica está longe de ser completamente determinado, embora algumas classes particulares destes sistemas tenham sido amplamente estudadas.

Nas suas investigações V. Volterra [41] deu especial atenção aos sistemas predador-presa e à sua generalização para sistemas de cadeias alimentares para n espécies, que se enquadram na classe dos sistemas LV conservativos e dissipativos. O sistema LV para n populações está definido no subconjunto de $\mathbb{R}^n$ onde todas as coordenadas são não-negativas, i.e., $\mathbb{R}^n_+$.

A classe dos sistemas conservativos foi inicialmente estudada por V. Volterra de tal forma que ele fez uma caracterização Hamiltoniana no caso em que a matriz de interacção do sistema é anti-simétrica. Em 1998 Waldyr M. Oliva *et al.* [6] fizeram uma reinterpretação para o carácter Hamiltoniano da dinâmica dos sistemas conservativos.

Relativamente à classe dos sistemas LV dissipativos, nos anos 1980, Raymond M. Redheffer *et al.* [25–29] desenvolveram a teoria dos sistemas estavelmente dissipativos, i.e., sistemas que são dissipativos e que sofrendo pequenas perturbações permanecem dissipativos.

Outra classe de sistemas que tem um papel muito importante na TJE é a classe dos sistemas definidos pela denominada *equação do replicador*. Em 1978 Peter D. Taylor e Leo B. Jonker [39] iniciaram o estudo destas equações diferenciais no sentido de investigar a evolução das estratégias comportamentais.

A equação do replicador tem uma relação muito importante com os sistemas LV. Em 1981 Josef Hofbauer [15] provou que a equação do replicador para n estratégias é equivalente ao sistema LV para $n-1$ populações. A equação do replicador para n estratégias está definida no simplexo Δ^{n-1}.

No contexto da Teoria de Jogos, a equação do replicador tem sido estudada essencialmente no sentido de investigar a dinâmica deste tipo de sistemas.

O *replicador bimatricial*, associado aos jogos bimatriciais, também designados por *assimétricos*, foi inicialmente estudado por Peter Schuster e Karl Sigmund [32] e por P. Schuster, K. Sigmund, J. Hofbauer, e Robert Wolff [34]. Neste tipo de sistemas a "população" divide-se em dois grupos, por exemplo, machos e fêmeas, e cada grupo tem disponível um conjunto diferente de estratégias, digamos n estratégias para o primeiro grupo e m estratégias para o segundo. Um estado deste sistema é um par de vectores de probabilidade no prisma $(n+m-2)$-dimensional $\Gamma_{n,m} := \Delta^{n-1} \times \Delta^{m-1}$. Neste contexto as interacções apenas ocorrem entre indivíduos de grupos diferentes.

O estudo dos equilíbrios em jogos de n-jogadores foi iniciado nos anos 1950 por John Nash [23]. Uma classe de jogos de n-jogadores, designados por *jogos polimatriciais*, onde o payoff de cada jogador é a soma dos payoffs correspondentes a confrontos simultâneos com os adversários, foi estudada na década de 1970 por Joseph Howson [17] com o objectivo de investigar a existência de pontos de equilíbrio para este tipo de jogos. J. Howson [17] atribui o conceito de jogo polimatricial a E. Yanovskaya [43] em 1968.

Nos *jogos polimatriciais*, a população está dividida num número finito de grupos, cada um com um número finito de estratégias. São permitidas interacções entre indivíduos de quaisquer dois grupos, inclusive do mesmo grupo. No entanto, ocorre competição dentro de cada grupo, i.e., o sucesso relativo de cada estratégia é avaliado dentro do grupo correspondente.

Em [3] os autores introduzem a classe de e.d.o.s, designada por *replicador polimatricial*, que generaliza para os jogos polimatriciais a equação do replicador associada aos jogos simétricos e assimétricos.

O *replicador polimatricial* está definido num politopo dado por um produto finito de simplexos. A equação do replicador em dimensão n é o caso do replicador polimatricial definido no simplexo Δ^{n-1}. O jogo assimétrico para duas "populações", uma com n estratégias e a outra com m, é o caso do replicador polimatricial definido no prisma $\Gamma_{n,m}$, onde as submatrizes correspondentes às interacções dentro de cada grupo são nulas.

Damos em seguida uma breve visão geral dos nossos reultados. Para os sistemas LV em geral, apresentamos um resultado sobre a existência de folheações invariantes pelo campo vectorial associado, e neste contexto, para os sistemas LV dissipativos, apresentamos um resultado que estabelece uma relação entre estas folheações e o conjunto dos pontos de equilíbrio do sistema [7].

Muitas propriedades dos sistemas LV podem ser expressas geometricamente em termos do seu grafo associado, construido a partir da matriz de interacção do sistema. Para a classe dos sistemas LV estavelmente dissipativos provamos que a característica da sua matriz de interação, que é a dimensão da folheação invariante associada, é completamente determinada pelo grafo do sistema [7].

Nesta tese estudamos também fluxos analíticos definidos em politopos. Apresentamos uma teoria introduzida por Pedro Duarte [5] que nos permite analisar a dinâmica assintótica do fluxo ao longo da rede heteroclínica formada pelas arestas e vértices do politopo onde os fluxos estão definidos. Neste contexto, dado um fluxo definido num politopo, apresentamos um resultado que dá condições suficientes para a existência de variedades normalmente hiperbólicas estáveis e instáveis para ciclos heteroclínicos [2].

Um caso particular de fluxos analíticos definidos em politopos, são os fluxos associados aos replicadores polimatriciais. K. Sigmund e J. Hofbauer [16] e W. Jansen [18] apresentam condições suficientes para a *permanência* nos replicadores usuais. Neste contexto, generalizamos esses resultados para os replicadores polimatriciais [2]. Também para os replicadores polimatriciais estendemos o conceito de estabilidade dissipativa desenvolvido por Redheffer *et al.* [25–29]. Neste contexto generalizamos um teorema de W. Oliva *et al.* [6] sobre a natureza Hamiltoniana da dinâmica limite em replicadores polimatriciais "estavelmente dissipativos" [1].

Alguns exemplos são apresentados para ilustrar resultados e conceitos fundamentais desenvolvidos ao longo da tese.

Contents

Introduction **1**

1 Lotka-Volterra Systems **5**
1.1 Definitions and Properties 7
1.2 Conservative Systems 9
1.3 Dissipative Systems 13
1.4 Invariant Foliations 15
1.5 Replicator Equation 16

2 Stably Dissipative Lotka-Volterra Systems **21**
2.1 Stably Dissipative Matrices 22
2.2 Redheffer Reduction Algorithm 25
2.3 Stably Dissipative Graphs 30
2.4 The Rank of the Graph 31
2.5 Simplified Reduction Algorithm 32
2.6 Trimming Effect on Dynamics 37

3 Vector Fields on Polytopes **41**
3.1 Polytopes 42
3.2 Vector Fields on Polytopes 44
3.3 Dual Cone of a Polytope 44
3.4 Skeleton Vector Fields 47
3.5 Asymptotic Poincaré Maps 51
3.6 Invariant Manifolds 54

4 Polymatrix Replicators **59**
4.1 Definitions and Properties 60
4.2 Polymatrix Skeleton 63
4.3 Permanence 66
4.4 Conservative Polymatrix 70
4.5 Dissipative Polymatrix 73
4.6 Examples and Applications 87
4.6.1 Reduction Algorithm 87

4.6.2 Dissipative Polymatrix 92
4.6.3 Conservative Polymatrix 99

Bibliography **106**

Index **109**

Introduction

The book “Theory of Games and Economic Behavior” [42], published in 1944 by the mathematician John von Neumann and the economist Oskar Morgenstern, is considered the beginning of the study of strategic decision making, giving rise to the field designated as *Game Theory*. Thirty years later John Maynard Smith and George R. Price [22] applied the theory of strategic games developed by Neumann and Morgenstern to investigate the dynamical processes of biological populations, giving rise to the field of the *Evolutionary Game Theory* (EGT).

Some classes of ordinary differential equations (o.d.e.s) which plays a central role in EGT are the *Lotka-Volterra (LV) systems*, the *replicator equation*, the *bimatrix replicator*, and the *polymatrix replicator*.

In the 1920s Lotka-Volterra systems were independently introduced by A. J. Lotka [21] and V. Volterra [41] to model the evolution of chemical and biological ecosystems, respectively. The LV systems have become a mathematical model widely used by many scientific fields such as physics, chemistry, biology, and economy.

In 1978 Peter Taylor and Leo Jonker [39] introduced the replicator equation which is now central to EGT. These systems have been developed essentially on the context of game theory in terms of studying their dynamics.

There is an important relation between the replicator equation and the LV systems. In 1981 Josef Hofbauer [15] shows that the replicator equation corresponds - up to a change in velocity - to the generalized LV equation.

There are other types of replicator-like evolutionary systems, of which we single out the *bimatrix replicator*, and more generally the *polymatrix replicator*.

The *bimatrix replicator* was firstly introduced by Peter Schuster and K. Sigmund [32] and P. Schuster, K. Sigmund, J. Hofbauer, and Robert Wolff [34] to study the dynamics of bimatrix games. In these games, also called *asymmetric games*, two groups of individuals within a population (e.g. males and females), interact using different sets of strategies, say n strategies for the first group and m strategies for the second. In bimatrix games there are no interactions within each group.

In polymatrix games, the population is divided in a finite number of groups, each one with a finite number of strategies. Interactions between individuals of any two groups are allowed, including the same group.

In [3] the authors introduce a class of o.d.e.s, referred as *polymatrix replicator*, that generalizes to polymatrix games the replicator equations associated to symmetric and asymmetric games.

The polymatrix replicator, is defined in a polytope given by a finite product of simplexes. The replicator equation in dimension n is the case of the polymatrix replicator defined in the simplex Δ^{n-1}. The asymmetric games for two "populations", one with n strategies and the other with m, is the case of the polymatrix replicator defined on the prism $\Delta^{n-1} \times \Delta^{m-1}$, where the submatrices corresponding to interactions within each group are null.

I provide now a short overview of the organization of this thesis. In chapter 1 we present the definitions and basic properties of the LV systems giving particular attention to the classes of conservative and dissipative LV systems. For LV systems in general, we present a result about the existence of invariant foliations for the associated vector field. In this context, for dissipative LV systems, we state a relation between these foliations and the set of equilibria. We present also in this chapter a brief recall of the basic properties of the replicator equation, its relation with the LV systems and the notion of permanence.

Chapter 2 addresses essentially a particular class of dissipative LV systems, the *stably dissipative* LV systems. Namely, the *Redheffer reduction algorithm* that runs on the associated graph of the interaction matrix of the system. The main result of this chapter is the proof that the rank of the defining matrix of a stably dissipative LV system, which is the dimension of the associated invariant foliation, is completely determined by the system's graph.

In chapter 3 we address the study of analytic flows on polytopes, i.e., flows that leave invariant all faces of the polytope where they are defined. We pretend to analyze the asymptotic dynamics of the flow along the heteroclinic network formed by the flowing-edges and the vertices of the polytope. In this context, given a flow defined on a polytope, we give sufficient conditions for the existence of normally hyperbolic stable and unstable manifolds for heteroclinic cycles. This chapter gives the necessary theoretical background for the study of polymatrix games that are the main focus of chapter 4.

In the last chapter, 4, we give the definitions and basic properties of the *polymatrix replicators* associated to *polymatrix games*. Next we describe the skeleton character of the vector field defined by the polymatrix replicator equation. K. Sigmund and J. Hofbauer [16] and W. Jansen [18] give sufficient conditions for *permanence* in the usual replicators. We generalize these results to polymatrix replicators. We define also the classes of the *conservative* and *dissipative polymatrix games*. For the dissipative polymatrix games we extend the concept of stably dissipativeness introduced by R. Redheffer *et al.* [25–29]. In this context we generalize a theorem of W. Oliva *et al.* [6] about the Hamiltonian nature of the limit dynamics in “stably dissipative” polymatrix replicators. Finally we present some examples to illustrate fundamental results and concepts developed along the thesis.

All computations, formulas and pictures presented in sections 2.6, 4.6.1, 4.6.2, and 4.6.3 were done with *Wolfram Mathematica* software.

Chapter 1

Lotka-Volterra Systems

By the 1920s, A. J. Lotka [21] and V. Volterra [41], independently of each other, began to publish their studies in different scientific fields, respectively in autocatalytic reactions and in the evolution of biological populations, using the same differential equations. These systems were designated as Lotka-Volterra (LV) systems in their honour.

The LV systems have become a mathematical model widely used by many scientific fields such as physics, chemistry, biology, economy as well as other social sciences. In particular, these systems play an important role in the study of neural networks, biochemical reactions, cell change, resource management, epidemiology or evolution game theory, for instance.

Although we can fully analyse the dynamics of the LV systems in low dimension, for systems in higher dimensions, the study of the dynamics is far from fully understood, although some special classes have been widely studied.

Usually the LV systems are classified in terms of algebraic properties of its interaction matrix, such as *cooperative* (or *competitive*), *conservative* and *dissipative*. Cooperative and competitive systems have been widely studied by many authors like Smale [37], Hirsch [10–12], Zeeman [46, 47], Van Den Driessche *et al.* [40], Hofbauer *et al.* [14], Smith [38] and Karakostas *et al.* [19]. Typically these systems have a global attractor consisting of fixed points and connections between them.

In his work "Leçons sur la Théorie Mathématique de la Lutte pour la Vie" [41] Volterra gave special attention to predator-prey systems and their generalizations to food chain systems in n species, which fall in the class of dissipative and conservative LV systems.

Although dissipative systems have been addressed in the pioneering work of Volterra, this class has been the least studied. Volterra defined this systems looking for a generalization of the predator-prey model. Key references in this field are the book of Volterra [41] and the work of Redheffer and collaborators [25–29]. Especially in [25] Redheffer establishes the conditions for a matrix to be dissipative, and in [27] Redheffer *et al.* make a description of the attractor of these systems. Guo *et al.* [8] studied the necessary and sufficient conditions for which the real matrices of order 3 are dissipative. Rocha Filho *et al.* [30] have a numerical algorithm, and a package for Maple to obtain the positive diagonal matrix D, designated as *Volterra multiplier*, such that $Q_{AD} \leq 0$.

Redheffer *et al.* developed further the theory of dissipative LV systems, introducing and studying the class of stably dissipative systems [27–29]. Notice that we call *dissipative* and *stably dissipative* to systems that Redheffer *et al.* designated as *admissible* and *stably admissible*, respectively.

Conservative systems were first studied by Volterra in such a way that he made a Hamiltonian characterization in the case where the interaction matrix is skew-symmetric. Volterra proved that the dynamics of any n species conservative LV system can be embedded in a Hamiltonian system of dimension $2n$.

In 1998 Waldyr M. Oliva *et al.* [6] give a re-interpretation for the Hamiltonian character of the dynamics of any conservative LV system in terms of the existence of a Poisson structure on $\mathbb{R}^n_+$ which makes the system Hamiltonian. Another interesting fact from [6], which emphasizes the importance of studying Hamiltonian LV systems, is that the limit dynamics of any stably dissipative LV system is described by a conservative LV system.

To study the evolution of behavioural strategies, the so-called *replicator equation* introduced in 1978 by Taylor and Jonker [39], among others has been studied by Hofbauer *et al.* [13], Schuster *et al.* [32], and Zeeman [44, 45].

In 1981 J. Hofbauer [15] proved that the replicator equation for n strategies corresponds to the LV equation for $n-1$ populations.

This chapter is organized as follows. In section 1.1, we give the basic definitions and properties of the LV systems. In section 1.2, we characterize the conservative LV systems and in section 1.3 the dissipative ones. In section 1.4, for LV systems in general, we present a result about the existence of invariant foliations for the associated vector field. In this context, for dissipative LV systems, we state a relation between these foliations and the set of equilibria. Finally, in section 1.5, we recall some basic properties of the replicator equation, its relation with the LV systems and the notion of permanence.

1.1 Definitions and Properties

We call LV equation to the following system of differential equations

$$\frac{dx_i}{dt}(t) = x_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right), \quad i = 1, \dots, n, \tag{1.1}$$

where $x_i(t) \geq 0$ represents the density of population i in time t and r_i its intrinsic rate of decay or growth. Each coefficient a_{ij} represents the effect of population j over population i. If $a_{ij} > 0$ it means that population j benefits population i. The square matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ is said to be the interaction matrix of system (1.1).

The set

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \, : \, x_i \geq 0, \;\; i = 1, \dots, n\}.$$

is the phase space of (1.1).

There is a close relation between the study of the dynamical properties of a LV system and the algebraic properties of its interaction matrix.

Given a matrix $A \in M_n(\mathbb{R})$, we have the quadratic form $Q_A : \mathbb{R}^n \to \mathbb{R}$ defined by $Q_A(x) := x^T A\, x$. For simplicity, when its clear from the context, we simply write $Q_A \leq 0$, meaning that $Q_A(x) \leq 0$ for all $x \in \mathbb{R}^n$.

Usually the LV systems (1.1) are classified in terms of its interaction matrix.

Definition 1.1.1. We say that a LV system (1.1) with interaction matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ is:

a) *conservative* if there exists a positive diagonal matrix D such that AD is skew-symmetric;

b) *dissipative* if there is a positive diagonal matrix D such that $Q_{AD} \leq 0$.

Many properties of the LV systems can be geometrically expressed in terms of its associated graph, constructed from the system's interaction matrix.

Definition 1.1.2. Given a matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ of a LV system, we define its *associated graph* $G(A)$ to have vertex set $\{1, \dots, n\}$, and to contain an edge connecting vertex i to vertex j if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

For example, given the matrix

$$A = \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ 0 & * & 0 & * \\ 0 & * & * & 0 \end{bmatrix},$$

where $*$ represents non-zero real numbers, its associated graph is represented in figure 1.1.

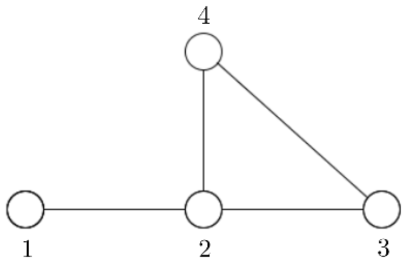


Figure 1.1: The associated graph $G(A)$ of a LV system with interaction matrix A.

Conservative systems can be characterized in terms of its associated graph as we can see by the next proposition due to Volterra [41].

Proposition 1.1.3. *A LV system with interaction matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ is conservative if and only if*

$$a_{ii} = 0, \quad \textit{for all } i \in \{1, \dots, n\},$$

$$a_{ij} \neq 0 \Rightarrow a_{ij}a_{ji} < 0, \quad \textit{for all } i \neq j,$$

and

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_s i_1} = (-1)^s a_{i_s i_{s-1}} \cdots a_{i_2 i_1} a_{i_1 i_s}, \tag{1.2}$$

for all finite sequence $(i_1, \dots, i_s)$, with $i_r \in \{1, \dots, n\}$ for $r = 1, \dots, s$.

We can observe that condition (1.2) means that for each closed path in the graph with an even (resp. odd) number of vertices, the product of the corresponding coefficients to each edge when we follow the path in one way is equal to the product (resp. minus the product) of these coefficients when we follow the path on the other way.

For example, a system whose associated graph is the represented in figure 1.1 is conservative if and only if

$$a_{23}a_{34}a_{42} = -a_{24}a_{43}a_{32}\,,$$

and conditions $a_{ii} = 0$ and $a_{ij} \neq 0 \Rightarrow a_{ij}a_{ji} < 0$ are satisfied for every $i, j \in \{1, \dots, 4\}$.

Equilibrium points $q \in \mathbb{R}^n_+$ of system (1.1) are the solutions of the linear system

$$r_i + \sum_{j=1}^{n} a_{ij} q_j = 0, \quad i = 1, \dots, n. \tag{1.3}$$

The existence of an equilibrium point in $\mathbb{R}^n_+$ is related with the orbit's behaviour in $\mathbb{R}^n_+$, as we can see by the following proposition (see [16] or [6], for instance).

Proposition 1.1.4. *System* (1.1) *admits an interior equilibrium point if and only if* $\operatorname{int}(\mathbb{R}^n_+)$ *contains* α *or* ω*-limit points.*

We can then observe that the limit behaviour of the orbits is related with the existence of equilibrium points. On the other hand, the following result shows that the time average of the orbits is related to the values that the coordinate functions take on the equilibrium points. In fact, if there exists a unique interior equilibrium point and if the solution do not converge to the boundary neither to infinity, then its time average converges to the equilibrium point.

Proposition 1.1.5. *Suppose that* $x(t)$ *is an orbit in* $\mathbb{R}^n_+$ *of the system* (1.1) *such that* $0 < m \leq x_i(t) \leq L$*, for all* $i \in \{1, \ldots, n\}$*. Then, there exists a sequence* $(T_k)_{k\in\mathbb{N}}$ *such that* $T_k \to +\infty$ *and an equilibrium point* $q \in \mathbb{R}^n_+$ *such that*

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x(t)\, dt = q.$$

Moreover, if system (1.1) *has only one equilibrium point* $q \in \mathbb{R}^n_+$*, then*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^{T} x(t)\, dt = q.$$

Proof. A proof of this proposition can be seen in [6]. □

Observe that in cases where the interaction matrix of the system is singular, the equilibrium points for which the time average of the orbits converges depend of the initial condition.

1.2 Conservative Systems

One of the main goals of Volterra in studying the class of conservative systems was the "mechanization" of biology. Looking for a variational principle of the system, Volterra developed a Hamiltonian formulation in the case where the interaction matrix is skew-symmetric, although this procedure has the cost of doubling the dimension of the system.

Conservative systems of classical mechanics can be seen in the context of Hamiltonian formulation, whose abstract version is based on the concept of *simplectic structure*, i.e., a closed nondegenerate differential 2-form.

A simplectic structure ω defined in an open set $M \subseteq \mathbb{R}^n$ can be represented in the form

$$\omega = \sum_{i,j=1}^{n} a_{ij}(x)\, dx_i \wedge dx_j\,, \tag{1.4}$$

satisfying

(i) function $x \mapsto A(x) = (a_{ij}(x))$ is smooth with values in the space of matrices;

(ii) $A(x)^T = -A(x)$;

(iii) $A(x)$ is nonsingular (ω is nondegenerate);

(iv) $\frac{\partial a_{ij}}{\partial x_k} + \frac{\partial a_{jk}}{\partial x_i} + \frac{\partial a_{ki}}{\partial x_j} = 0$, for all $i < j < k$ (ω is closed, $d\omega = 0$).

Given $u, v \in \mathbb{R}^n$ and $x \in M$, the 2-form (1.4) induces the bilinear skew-symmetric form

$$\omega_x(u, v) = u^T A(x)\, v\,.$$

From the nondegeneracy of ω, we can state the following proposition.

Proposition 1.2.1. *Given a function $H : M \to \mathbb{R}$, there exists a unique vector field X on M such that*

$$\omega_x\left(X(x), v\right) = DH_x(v)\,,$$

for all $x \in M$ and $v \in \mathbb{R}^n$.

In these conditions $X_H := X$ is said to be the *simplectic gradient* of H or the *Hamiltonian vector field*.

Observe that for the simplectic structure (1.4)

$$\begin{array}{rcl} X_H(x)^T A(x)\, v & = & \nabla H(x)\, v \\ \Leftrightarrow\ \ \nabla H(x) & = & -A(x) X_H(x) \\ \Leftrightarrow\ \ X_H(x) & = & -A^{-1}(x)\, \nabla H(x)\,. \end{array}$$

In these conditions, the next proposition follows.

Proposition 1.2.2. *H is constant along the orbits of X_H.*

Proof. Since $A(x)$ is skew-symmetric, follows that

$$\begin{array}{rcl} DH_x\left(X_H(x)\right) & = & \omega_x\left(X_H(x), X_H(x)\right) \\ & = & X_H(x)^T A(x)\, X_H(x)\,. \\ & = & 0\,. \end{array}$$

□

Note that if matrix $A(x)$ is constant and nonsingular (and skew-symmetric), then

$$\omega_x(u,v) = u^T A\, v \tag{1.5}$$

is a simplectic structure in $\mathbb{R}^n$.

For example, if $A = \left[\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right] \in M_{2n}(\mathbb{R})$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, then the Hamiltonian vector field of the simplectic structure (1.5) associated to $H(x,y)$ is

$$\begin{aligned} X_H(x,y) &= \left(-\frac{\partial H}{\partial y}(x,y), \frac{\partial H}{\partial x}(x,y)\right) \\ &= \left(-\frac{\partial H}{\partial y_1}, \ldots, -\frac{\partial H}{\partial y_n}, \frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right)\Big|_{(x,y)}, \end{aligned}$$

whence we get the classical Hamiltonian system

$$\begin{cases} \frac{dx}{dt} &= -\frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= \frac{\partial H}{\partial x} \end{cases}.$$

A generalization of the Hamiltonian system's theory is based on the notion of Poisson brackets.

Definition 1.2.3. A *Poisson bracket* in a smooth manifold M is given by a bilinear application $\{.,.\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ in the space of smooth functions that satisfies:

i) $\{f_1, f_2\} = -\{f_2, f_1\}$ (skew-symmetry),

ii) $\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2$ (Leibnitz identity),

iii) $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$ (Jacobi identity).

Definition 1.2.4. Given a function $h \in C^\infty(M)$ we define the vector field X_h by

$$X_h(f) = \{f, h\}, \quad \text{for all} \quad f \in C^\infty(M). \tag{1.6}$$

In that conditions X_h is said to be the *Poisson gradient* of h.

We call *Hamiltonian system* defined in a Poisson manifold M to the flow associated to the equation

$$\frac{dx}{dt} = X_h(x).$$

For example, if A is a skew-symmetric (constant) matrix,

$$\{f, g\} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

defines a Poisson structure in $\mathbb{R}^n$.

Suppose ω is a simplectic structure in M. Defining

$$\{f,g\}(x) = \omega_x(X_f(x), X_g(x)) \, ,$$

with $X_f(x)$ and $X_g(x)$ simplectic gradients, we have that $\{.\,,.\}$ is a Poisson structure. Moreover, the simplectic gradient coincides with the Poisson gradient since

$$\{f,h\}(x) = \omega_x(X_f(x), X_h(x)) = Df(X_h) = X_h(f) \, .$$

In this sense, the concept of Poisson structure generalizes the concept of simplectic structure.

If $\omega_x(u,v) = u^T A(x)\, v$, then

$$\{f,g\}(x) = \sum_{i,j=1}^{n} a_{ij}^{-1}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \, ,$$

where $a_{ij}^{-1}(x)$ is the (i,j) entry of the inverse matrix of $A(x)$.

Given a LV system (1.1) with an equilibrium point $q \in \int(\mathbb{R}^n_+)$ and interaction matrix $A \in M_n(\mathbb{R})$, we can write the LV system (1.1) as

$$\frac{dx}{dt} = X_{A,q}(x) \, , \tag{1.7}$$

where $X_{A,q}(x) = x * A\,(x-q)$ and $*$ denotes point-wise multiplication of vectors in $\mathbb{R}^n$. We designate by $X_{A,q}$ the LV vector field defined by the o.d.e. (1.7).

Theorem 1.2.5. *Let $A \in M_n(\mathbb{R})$ be a skew-symmetric nonsingular matrix. Then A^{-1} is skew-symmetric,*

$$\omega = \sum_{i,j=1}^{n} -\frac{a_{ij}^{-1}}{x_i x_j}\, dx_i \wedge dx_j$$

defines a simplectic structure in $\int(\mathbb{R}^n_+)$ and the simplectic gradient of

$$h = \sum_{i=1}^{n} (x_i - q_i \, \log x_i)$$

is the LV vector field $X_{A,q}$.

Proof. A proof of this theorem can be seen in [24]. □

Theorem 1.2.6. *Let $A \in M_n(\mathbb{R})$ be a skew-symmetric matrix. Then*

$$\{f, g\}(x) = \sum_{i,j=1}^{n} a_{ij}\, x_i x_j \, \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \,,$$

defines a Poisson structure in $\mathbb{R}^n_+$ and the Poisson gradient of

$$h = \sum_{i=1}^{n} (x_i - q_i \log x_i)$$

is the LV vector field $X_{A,q}$.

Proof. A proof of this theorem can be seen in [6]. □

Observe that if A is nonsingular, the Poisson structure of Theorem 1.2.6 corresponds to the simplectic structure of Theorem 1.2.5 .

Definition 1.2.7. We say that a graph G is a forest if $G = G_1 \cup \cdots \cup G_r$ (disjoint union), where each subgraph G_i is a tree, i.e, has no cycles.

Combining these concepts with Volterra criteria for a system to be conservative (see Proposition 1.1.3), Oliva *et al.* [6] state the following corollary.

Corollary 1.2.8. *Suppose LV system* (1.7) *has an equilibrium in $\int(\mathbb{R}^n_+)$. If the system's interaction matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ satisfies*

$$a_{ii} = 0, \qquad \textit{for all } i \in \{1, \ldots, n\},$$

$$a_{ij} \neq 0 \;\Rightarrow\; a_{ij} a_{ji} < 0, \qquad \textit{for all } i \neq j \,,$$

and its associated graph is a forest, i.e., has no cycles, then the system is conservative.

1.3 Dissipative Systems

Definition 1.3.1. We say that the LV system (1.7), the matrix A, or the vector field $X_{A,q}$, is *dissipative* if and only if there exists a positive diagonal matrix D such that $Q_{AD}(x) = x^T AD\, x \leq 0$ for every $x \in \mathbb{R}^n$.

Remark 1.3.2. *Notice that $Q_{AD}(x) = x^T AD\, x \leq 0$ for every $x \in \mathbb{R}^n$ is equivalent to*

$$Q_{D^{-1}A}(x) = x^T D^{-1} A\, x \leq 0 \,,$$

because

$$Q_{D^{-1}A}(x) = x^T D^{-1} A\, x = (D^{-1}x)^T AD\,(D^{-1}x) = Q_{AD}(D^{-1}x) \leq 0 \;.$$

Proposition 1.3.3. *When $X_{A,q}$ is dissipative, for any positive diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ such that $Q_{AD} \leq 0$, (1.7) admits the Lyapunov function $h : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ defined by*

$$h(x) = \sum_{i=1}^{n} \frac{x_i - q_i \log x_i}{d_i} \,, \tag{1.8}$$

which decreases along orbits of $X_{A,q}$.

Proof. Let $D = \operatorname{diag}(d_1, \dots, d_n)$ be a positive diagonal matrix such that $Q_{AD} \leq 0$. The derivative of h along orbits of $X_{A,q}$ is given by

$$\frac{dh}{dt} = \sum_{i,j=1}^{n} \frac{a_{ij}}{d_i}(x_i - q_i)(x_j - q_j) = (x - q)^T D^{-1} A\,(x - q) \leq 0.$$

The last inequality follows by Remark 1.3.2. $\square$

Given a matrix A, we can define the *symmetric* and *skew-symmetric* parts of A by

$$A^{\text{sym}} = \frac{A + A^T}{2} \quad \text{and} \quad A^{\text{skew}} = \frac{A - A^T}{2} \,,$$

and the following decompositions hold,

$$A = A^{\text{sym}} + A^{\text{skew}} \quad \text{and} \quad A^T = A^{\text{sym}} - A^{\text{skew}} \,.$$

The following theorem is a characterization of dissipative matrices.

Theorem 1.3.4. *Let $A \in M_n(\mathbb{R})$ be a dissipative matrix and D a positive diagonal matrix such that $Q_{AD} \leq 0$. Suppose $a_{ii} = 0$ for all $i \in \{1, \dots, k\}$ and $a_{ii} < 0$ for all $i \in \{k+1, \dots, n\}$. Then there exists matrices $R \in M_k(\mathbb{R})$ skew-symmetric and $U \in M_{n-k}(\mathbb{R})$ such that $Q_U \leq 0$ and*

$$AD = \left[\begin{array}{c|c} R & S \\ \hline -S^T & U \end{array}\right],$$

Notice that

$$(AD)^{sim} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & U^{sim} \end{array}\right].$$

Proof. A proof of this theorem can be seen in [24]. $\square$

Observe that the reciprocal of this Theorem 1.3.4 is also valid. If there exists a positive diagonal matrix D such that

$$AD = \left[\begin{array}{c|c} R & S \\ \hline -S^T & U \end{array}\right],$$

with R and U in the conditions of the theorem, then A is dissipative.

1.4 Invariant Foliations

Consider a LV field $X_{A,q}$ with $rank(A) = k$, for some $1 \leq k \leq n$. Let $W \in M_{(n-k)\times n}(\mathbb{R})$ be a $(n-k) \times n$ matrix whose rows form a basis of

$$\mathrm{Ker}\left(A^T\right) = \{\, x \in \mathbb{R}^n \,:\, x^T A = 0 \,\}\,.$$

Define the map $g : \mathbb{R}^n_+ \to \mathbb{R}^{n-k}$, $g(x) = W\,\log x$, where

$$\log x = (\log x_1, \ldots, \log x_n)^T\,.$$

We say that a map $g : \mathbb{R}^n_+ \to \mathbb{R}^{n-k}$ is a *submersion* if for all $x \in \mathbb{R}^n_+$ its derivative at x is surjective.

Proposition 1.4.1. *The map $g : \mathbb{R}^n_+ \to \mathbb{R}^{n-k}$ is a submersion.*

Proof. The jacobian matrix of g is $Dg_x = W\,D_x^{-1}$, where the matrix $D_x^{-1} = \mathrm{diag}\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$. By the definition of W, Dg_x has maximal rank $n-k$. Hence $Dg_x : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is surjective. $\square$

We denote by $\mathcal{F}_A$ the pre-image foliation, whose leaves are the pre-images $g^{-1}(c)$ of g. By a classical theorem on Differential Geometry, each non-empty pre-image $g^{-1}(c)$ is a submanifold of dimension k. Recall that the dimension of a foliation is the common dimension of its leaves.

Definition 1.4.2. A foliation $\mathcal{F}$ is said to be *invariant* under a vector field X, and we say that $\mathcal{F}$ is X*-invariant*, if $X(x) \in T_x\mathcal{F}$ for every x, where $T_x\mathcal{F}$ denotes the tangent space at x to the unique leaf of $\mathcal{F}$ through x.

This definition is equivalent to say that the flow of X preserves the leaves of $\mathcal{F}$.

Proposition 1.4.3. *The foliation $\mathcal{F}_A$ is $X_{A,q}$-invariant with*

$$\dim(\mathcal{F}_A) = rank(A)\,.$$

Proof. We have

$$\begin{aligned} Dg_x\left(X_{A,q}(x)\right) &= Dg_x\left(D_x A(x-q)\right) \\ &= W D_x^{-1} D_x A(x-q) \\ &= W A(x-q) = 0\,, \end{aligned}$$

and the last equality follows by the definition of W. Hence $X_{A,q}(x) \in T_x\mathcal{F}_A$ and $\mathcal{F}_A$ is $X_{A,q}$-invariant. $\square$

The following proposition is a simple but key observation.

Proposition 1.4.4. *If $A \in M_{n\times n}(\mathbb{R})$ is dissipative and D is a positive diagonal matrix such that $Q_{AD} \leq 0$ then* $\operatorname{Ker}(A) = D\operatorname{Ker}(A^T)$.

Proof. Assume first that $Q_A \leq 0$ on $\mathbb{R}^n$ and consider the decomposition $A = M + N$ with $M = (A + A^T)/2$ and $N = (A - A^T)/2$. Clearly $\operatorname{Ker}(M) \cap \operatorname{Ker}(N) \subseteq \operatorname{Ker}(A)$. On the other hand, if $v \in \operatorname{Ker}(A)$ then $v^T M v = v^T A v = 0$. Because $Q_M = Q_A \leq 0$ this implies that $M v = 0$, i.e., $v \in \operatorname{Ker}(M)$. Finally, since $N = A - M$, $v \in \operatorname{Ker}(N)$. This proves that $\operatorname{Ker}(A) = \operatorname{Ker}(M) \cap \operatorname{Ker}(N)$. Similarly, one proves that $\operatorname{Ker}(A^T) = \operatorname{Ker}(M) \cap \operatorname{Ker}(N)$. Thus $\operatorname{Ker}(A) = \operatorname{Ker}(A^T)$.

In general, if $Q_{AD} \leq 0$, we have $\operatorname{Ker}((AD)^T) = \operatorname{Ker}(DA^T) = \operatorname{Ker}(A^T)$, and $\operatorname{Ker}(AD) = D^{-1}\operatorname{Ker}(A)$. Thus, from the previous case applied to AD we get $D^{-1}\operatorname{Ker}(A) = \operatorname{Ker}(A^T)$. $\square$

Considering a LV system (1.7), we see that the set of equilibrium points of $X_{A,q}$ is the affine space

$$E_{A,q} := \{\, x \in \mathbb{R}^n_+ \;:\; A(x - q) = 0 \,\}\,. \tag{1.9}$$

Theorem 1.4.5. *Given a dissipative LV system $X_{A,q}$, each leaf of $\mathcal{F}_A$ intersects transversely $E_{A,q}$ in a single point.*

Proof. A proof of this theorem can be seen in [24]. $\square$

1.5 Replicator Equation

In 1978 Taylor and Jonker [39] introduced a system of differential equations that in 1983 Schuster and Sigmund [33] designated as the *replicator equation*. These systems have been studied essentially in the context of EGT.

This equation models the frequency evolution of certain strategical behaviours within a biological population. In fact, the replicator equation says that the logarithmic growth of the usage frequency of each behavioural strategy is directly proportional to how well that strategy fares within the population.

In 1981 Hofbauer [15] stated an important relation between the LV systems and the replicator equation. In fact, he proved the equivalence of both systems (see Theorem 1.5.4 below).

In this section we present some elementary definitions and properties of the replicator equation that we will address later in chapter 4. For a more detailed introduction on the subject see [16] for instance.

Consider a population where individuals interact with each other according to a set of n possible strategies. The state of the population concerning this interaction is fully described by a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where x_i represents the frequency of individuals with strategy i, for $i = 1, \ldots, n$. Hence $x_i \geq 0$ for all i and $\sum_{i=1}^n x_i = 1$. The set of all population states is the simplex Δ^{n-1}.

If an individual using strategy i interacts with an individual using strategy j, a coefficient a_{ij} represents the average payoff for that interaction. Let $A = (a_{ij}) \in M_n(\mathbb{R}^n)$ be the matrix consisting of these a_{ij}'s. Assuming random encounters between individuals of that population,

$$(Ax)_i = \sum_{k=1}^n a_{ik} x_k$$

is the average payoff for strategy i and

$$x^T A\, x = \sum_{i=1}^n \sum_{k=1}^n a_{ik} x_i x_k$$

is the global average payoff of all population strategies. The growth rate $\frac{\frac{dx_i}{dt}}{x_i}$ of the frequency of strategy i are equal to the payoff difference $(Ax)_i - x^T A\, x$, which yields the *replicator equation*

$$\frac{dx_i}{dt} = x_i \left((Ax)_i - x^T A\, x \right), \quad i = 1, \ldots, n\,, \tag{1.10}$$

defined on the simplex

$$\Delta^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, \sum_{j=1}^n x_j = 1,\ x_i \geq 0\,,\ i = 1, \ldots, n\}\,.$$

Proposition 1.5.1. *The simplex Δ^{n-1} is invariant under* (1.10)*.*

Proof. The n-plane containing Δ^{n-1} given by $\sum_{i=1}^n x_i = 1$ is invariant because

$$\frac{d}{dt}\left(\sum_{i=1}^n x_i \right) = \sum_{i=1}^n \frac{dx_i}{dt} = \sum_{i=1}^n x_i (Ax)_i - \underbrace{\sum_{i=1}^n x_i}_{=1} \sum_{j=1}^n x_j (Ax)_j = 0\,.$$

Similarly, given any q-dimensional face σ of Δ^{n-1}, the q-plane containing σ is invariant. $\square$

Based on this result, from now on, we shall only consider the restriction of (1.10) to Δ^{n-1}.

Proposition 1.5.2. *For $x_j > 0$ we have the "replicator quotient rule"*

$$\frac{d}{dt}\left(\frac{x_i}{x_j}\right) = \frac{x_i}{x_j}\left((Ax)_i - (Ax)_j\right) .$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{x_i}{x_j}\right) &= \frac{\frac{dx_i}{dt}x_j - x_i\frac{dx_j}{dt}}{x_j^2} \\
&= \frac{x_i x_j\left((Ax)_i - x^T A\,x\right) - x_i x_j\left((Ax)_j - x^T A\,x\right)}{x_j^2} \\
&= \frac{x_i}{x_j}\left((Ax)_i - (Ax)_j\right) .
\end{aligned}$$

$\square$

Notice that the equilibrium points of (1.10) in $\mathrm{int}(\Delta^{n-1})$ are the solutions of $(Ax)_1 = \cdots = (Ax)_n$ and $\sum_{i=1}^n x_i = 1$ satisfying $x_i \geq 0$ for $i = 1, \ldots, n$.

Lemma 1.5.3. *The addition of a constant c_j to all entries in the j^{th}-column of $A \in M_n(\mathbb{R})$ does not change (1.10) on Δ^{n-1}.*

Proof. Consider the matrix $B = A + C$, where matrix C have zeros in all entries except in column j whose entries are all equal to a constant c_j. For each i we have

$$\begin{aligned}
\frac{dx_i}{dt} &= x_i\left((Bx)_i - x^T B\,x\right) \\
&= x_i\left((Ax)_i + c_j x_j - x^T A\,x - \sum_{k=1}^n c_j x_k x_j\right) \\
&= x_i\left((Ax)_i - x^T A\,x + c_j x_j(1 - x_j) - \sum_{\substack{k=1 \\ k\neq j}}^n c_j x_k x_j\right) \\
&= x_i\left((Ax)_i - x^T A\,x + c_j x_j\left(\sum_{\substack{k=1 \\ k\neq j}}^n x_k - \sum_{\substack{k=1 \\ k\neq j}}^n x_k\right)\right) \\
&= x_i\left((Ax)_i - x^T A\,x\right) .
\end{aligned}$$

$\square$

Considering a constant matrix $C \in M_n(\mathbb{R})$ with equal rows, the previous lemma says that adding $C \in M_n(\mathbb{R})$ to $A \in M_n(\mathbb{R})$ does not change the dynamics of system (1.10) on Δ^{n-1}.

We have that the replicator equation is a cubic equation on the compact set Δ^{n-1} while the LV equation is quadratic on $\mathbb{R}^n_+$. However, Hofbauer in [15] proved that the replicator equation in n variables $x_1, \ldots, x_n$ is equivalent to the LV equation in $n-1$ variables $y_1, \ldots, y_{n-1}$ (see also [16]).

Theorem 1.5.4. *There exists a differentiable invertible map from $\hat{S}_n = \{x \in \Delta^{n-1} : x_n > 0\}$ onto $\mathbb{R}^{n-1}_+$ mapping the orbits of the replicator equation*

$$\frac{dx_i}{dt} = x_i \left((Ax)_i - x^T A\, x \right), \quad i = 1, \ldots, n, \tag{1.11}$$

to time re-parametrization of the orbits of the LV equation

$$\frac{dy_i}{dt} = y_i \left(r_i + \sum_{j=1}^{n-1} a'_{ij} y_j \right), \quad i = 1, \ldots, n-1, \tag{1.12}$$

where $r_i = a_{in} - a_{nn}$ and $a'_{ij} = a_{ij} - a_{nj}$.

In Proposition 1.1.4 we have seen that a LV system admits an ω-limit point in $\mathrm{int}(\mathbb{R}^n_+)$ if and only if it has an equilibrium point in $\mathrm{int}(\mathbb{R}^n_+)$. Hence, from Theorem 1.5.4 we have that

Proposition 1.5.5. *If the replicator equation* (1.11) *has no equilibrium point in* $\mathrm{int}(\Delta^{n-1})$*, then every solution converges to the boundary of* Δ^{n-1}*.*

We have also a natural generalization of Theorem 1.1.5 in LV systems to the replicator equation.

Theorem 1.5.6. *If the replicator equation* (1.11) *admits a unique equilibrium point $q \in \mathrm{int}(\Delta^{n-1})$, and if the ω-limit of the orbit of $x(t)$ is in* $\mathrm{int}(\Delta^{n-1})$*, then*

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T x(t)\, dt = q\,.$$

Now we introduce the concept of *permanence* that is a stability notion introduced by Schuster *et al.* in [31].

Definition 1.5.7. A replicator equation (1.11) defined on Δ^{n-1} is said to be *permanent* if there exists $\delta > 0$ such that, for all $x \in \mathrm{int}(\Delta^{n-1})$,

$$\liminf_{t\to\infty} d\left(\varphi^t(x), \partial\Delta^{n-1}\right) > \delta\,,$$

where φ^t denotes the flow determined by system (1.11).

A system to be permanent means that sufficiently small perturbations cannot lead any species to extinction.

The following theorem due to Jansen [18] is also valid to LV systems.

Theorem 1.5.8. *If there is a point* $p \in \mathrm{int}(\Delta^{n-1})$ *such that for all boundary equilibria* $x \in \partial\Delta^{n-1}$,

$$p^T A\, x > x^T A\, x\,, \tag{1.13}$$

then X *is permanent.*

This Theorem 1.5.8 is a corollary of the following theorem which gives sufficient conditions for a system to be permanent. This result is stated and proved by Sigmund and Hofbauer in [16, Theorem 12.2.1].

Theorem 1.5.9. *Let* $P : \Delta^{n-1} \longrightarrow \mathbb{R}$ *be a smooth function such that* $P = 0$ *on* $\partial\Delta^{n-1}$ *and* $P > 0$ *on* $\mathrm{int}(\Delta^{n-1})$. *Assume there is a continuous function* $\Psi : \Delta^{n-1} \longrightarrow \mathbb{R}$ *such that*

(1) for any orbit $x(t)$ *in* $\mathrm{int}(\Delta^{n-1})$, $\frac{d}{dt}\log P(x(t)) = \Psi(x(t))$,

(2) for any orbit $x(t)$ *in* $\partial\Delta^{n-1}$, $\int_0^T \Psi(x(t))\, dt > 0$ *for some* $T > 0$.

Then the vector field X *is permanent.*

Chapter 2

Stably Dissipative Lotka-Volterra Systems

In most cases in real world, when we are modelling a phenomenon, we may not know exactly the interaction matrix of the system. So, its important to study systems that maintain their properties even when they slightly changed, i.e., systems that persist to small perturbations.

We study perturbations that do not change the system's associated graph. These systems are designated as *stably dissipative* systems.

The notion of *stably dissipativeness* is due to Redheffer *et al.* whom in a set of papers [25–29] studied the asymptotic stability of this class of systems, under the name of *stably admissible* LV systems. Redheffer and his collaborators designated by *admissible* a class of matrices that Volterra firstly classified as *dissipative* (see [41]).

In 2010, Zhao and Luo[48] presented a classification of stably dissipative systems in dimension five and studied their possible different dynamics.

In [7] we have proved that for the class of stably dissipative LV systems the associated graph completely determines the rank of its defining matrix. Moreover, the rank of its defining matrix is the dimension of the associated invariant foliation.

This chapter is organized as follows. In section 2.1, we introduce the formal definition of stably dissipative systems, i.e., systems that are dissipative and that maintain their properties for small enough perturbations. In section 2.2, we present the Redheffer reduction algorithm and explain how it runs on the graph associated to an interaction matrix of a LV system. In section 2.3, we define stably dissipative graphs and we will see how to characterize stably dissipative matrices that share the same graph. In section 2.4, we prove that the rank of the defining matrix of a stably dissipative LV system is completely determined by the system's graph. In section 2.5, based on the properties of the stably dissipative LV systems in terms of the rank of its associated graph, we state a simplified reduction algorithm that allows us to derive properties on stably dissipative graphs. Finally, in section 2.6, we define the trimming operation on a stably dissipative graph, which preserves its stably dissipativeness.

2.1 Stably Dissipative Matrices

Given a matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ we call *admissible perturbation* of A to any other matrix $\tilde{A} = (\tilde{a}_{ij}) \in M_n(\mathbb{R})$ such that

$$\tilde{a}_{ij} \approx a_{ij} \quad \text{and} \quad \tilde{a}_{ij} = 0 \Leftrightarrow a_{ij} = 0.$$

By definition, admissible perturbations $\tilde{A}$ of A are perturbations that maintain the same graph, i.e., such that $G(A) = G(\tilde{A})$.

Definition 2.1.1. A matrix $A \in M_n(\mathbb{R})$ is said to be *stably dissipative* if any small enough admissible perturbation $\tilde{A}$ of A is dissipative, i.e., if there exists $\varepsilon > 0$ such that for any admissible perturbation $\tilde{A}$ of A,

$$\max_{1 \le i,j \le n} |a_{ij} - \tilde{a}_{ij}| < \varepsilon \Rightarrow \tilde{A} \text{ is dissipative.}$$

A LV system (1.7) is said to be *stably dissipative* if its interaction matrix is stably dissipative.

Given a dissipative LV system (1.7) with interaction matrix A, we can choose a positive diagonal matrix D such that $Q_{AD} \le 0$. However, for stably dissipative systems we have an important additional property due to Redheffer and Zhou [29]. This lemma plays a key role in the theory of stably dissipative systems.

Lemma 2.1.2. *Let $A \in M_n(\mathbb{R})$ be a stably dissipative matrix. Then for any choice of a positive diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that $Q_{AD} \le 0$, the following condition holds*

$$\sum_{i,j=1}^{n} d_j a_{ij} w_i w_j = 0 \Rightarrow a_{ii} w_i = 0, \quad \textit{for all } \; i = 1, \ldots, n\,.$$

Given a stably dissipative matrix $A \in M_n(\mathbb{R})$, we have that $a_{ii} \le 0$ for all i. To study necessary and sufficient conditions for a matrix to be stably dissipative we consider two different cases: $a_{ii} < 0$ for all i, or else there are some $a_{ii} = 0$.

These two cases can be seen in more detail in [48]. However we present some theorems, together with some useful lemmas, that characterize stably dissipative matrices, as follows.

Theorem 2.1.3. *If $a_{ii} < 0$ for all $i \in \{1, \dots, n\}$, then $A \in M_n(\mathbb{R})$ is stably dissipative if and only if there exists a positive diagonal matrix D such that $Q_{AD} < 0$.*

Proof. A proof of this theorem can be seen in [7]. $\square$

Lemma 2.1.4. *Let i and j be adjacent vertices of $G(A)$ with $A \in M_n(\mathbb{R})$ stably dissipative. Then $a_{ii}a_{jj} > a_{ij}a_{ji}$.*

We denote by (i, j) the edge that connects vertices i and j.

Lemma 2.1.5. *Let $A \in M_n(\mathbb{R})$ be a stably dissipative matrix. Then, every cycle in $G(A)$ as at least an edge (i, j) such that $a_{ii} < 0$ and $a_{jj} < 0$, for some $i, j \in \{1, \dots, n\}$.*

Proofs of Lemma 2.1.4 and Lemma 2.1.5 can be seen in [27].

Theorem 2.1.6. *Let $A \in M_n(\mathbb{R})$ be a matrix such that $a_{ii} = 0$ for all $i \in \{1, \dots, n\}$ or there exists only one $k \in \{1, \dots, n\}$ such that $a_{kk} < 0$ and $a_{ii} = 0$ for all $i \neq k$. Then, A is stably dissipative if and only if (i) $G(A)$ has no cycles and (ii) $a_{ij} \neq 0 \Rightarrow a_{ij}a_{ji} < 0$, for all $i \neq j$.*

Proof. Given $A \in M_n(\mathbb{R})$ stably dissipative, by Lemma 2.1.4 and Lemma 2.1.5, (i) and (ii) hold because there is at most one $a_{ii} < 0$.

Suppose now that (i) and (ii) are satisfied. Since $G(A)$ has no cycles, at most is has $n - 1$ edges. Thus, by (ii), we can choose a positive diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ such that $d_i a_{ij} + d_j a_{ji} = 0$ for all $a_{ij} \neq 0$. Hence $\sum_{i,j=1}^{n} d_i a_{ij} w_i w_j = d_k a_{kk} w_k^2 \leq 0$ and A is dissipative. We can easily see that an arbitrary small enough perturbation $\tilde{A}$ of A also satisfies (i) and (ii). Therefore, $\tilde{A}$ is dissipative, and so we can conclude that A is stably dissipative. $\square$

In the case where there exists more than one i such that $a_{ii} < 0$ we have also a theorem that characterizes stably dissipative matrices (see Theorem 2.1.10 below). We present first two useful lemmas.

Given a graph G, we can denote it by $G = (V, E)$, where V is the set of all of its vertices and E the set of all of its edges.

Lemma 2.1.7. *Let $G = (V, E)$ be a graph. If G has no cycles, then there exists a partition $\{V_0, \dots, V_\ell\}$ of V satisfying:*

(i) *V_0 is a set of vertices such that for each connected component of the graph, V_0 contains exactly one vertex which is an endpoint of a single edge of the connected component;*

(ii) *for all $j \in \{1, \dots, \ell\}$ and for all $i \in V_j$, there exists one and only one $i' \in V_{j-1}$ such that $(i, i') \in E$.*

Proof. If G is a graph with no cycles, then the graph is a forest. So, on each connected component of the graph, consider one vertex that is an endpoint of a single edge in E, and define V_0 to be the set of all those vertices. Note that for each connected component of the graph there is more then one vertex that is an endpoint of a single edge in E but we consider only one in each connected component of the graph.

Recursively, for $j \in \{1, \dots, \ell\}$, we consider the set V_j of all vertices i such that there exists an edge that connects it to a vertex i' in the set V_{j-1}.

This recursive procedure of defining the sets V_j must finish because the vertices are finite.

Since V_0 has only one vertex in each connected component of the graph, by definition V_1 has also only one vertex in the same connected component (if not, the vertex in V_0 couldn't be an endpoint of a single edge in E). Naturally, the vertices in V_2, for each connected component of the graph, are linked with the only one vertex in V_1. Suppose now that for some $j \in \{3, \dots, \ell\}$, and some vertex i in V_j there exists more then one vertex i' in V_{j-1} such that $(i, i') \in E$. This would imply that the graph G had a cycle. $\square$

Definition 2.1.8. Given a matrix $A \in M_n(\mathbb{R})$ and a subset $I \subseteq \{1, \dots, n\}$, we denote by $A_I = (a_{ij})_{(i,j)\in I\times I}$ the *submatrix* $I \times I$ of A.

Lemma 2.1.9. *Let $A \in M_n(\mathbb{R})$ be a stably dissipative matrix. Then, for all $I \subseteq \{1, \dots, n\}$, the submatrix A_I is stably dissipative.*

Proof. Let I be a subset of $\{1, \dots, n\}$. Let $\tilde{A}_I$ be a perturbation of A_I. Consider $\tilde{A}$ the matrix whose entries $(i, j) \in I \times I$ are the corresponding entries of $\tilde{A}_I$ and for $(i, j) \notin I \times I$, $\tilde{a}_{ij} = a_{ij}$. Clearly, $\tilde{A}$ is a perturbation of A. So, there exists a positive diagonal matrix D such that $Q_{\tilde{A}D} \leq 0$. Considering now D_I the submatrix $I \times I$ of D, we have that $Q_{\tilde{A}_I D_I} \leq 0$, which concludes the proof. $\square$

Theorem 2.1.10. *Let $A \in M_n(\mathbb{R})$ be a matrix such that $a_{ii} = 0$ for all $i \leq k$ and $a_{ii} < 0$ for all $i > k$, for some $k \in \{1, \dots, n\}$. Let $M = (a_{ij})$ be the submatrix of A corresponding to $k + 1 \leq i, j \leq n$ and let $\tilde{G}(A)$ be the graph obtained from $G(A)$ removing all the edges (i, j) such that $i, j > k$. Then, A is stably dissipative if and only if (i) $\tilde{G}(A)$ has no cycles and (ii) there exists a positive diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ such that $Q_{MD_0} < 0$ and $d_j a_{ij} + d_i a_{ji} = 0$ for $i \leq k$ or $j \leq k$, where $D_0 = \operatorname{diag}(d_{k+1}, \dots, d_n)$.*

Proof. Given $A \in M_n(\mathbb{R})$ stably dissipative, by Lemma 2.1.9, M is stably dissipative. Then, by Theorem 2.1.3, Lemma 2.1.4 and Lemma 2.1.5 we obtain (i) and (ii).

Suppose now that (i) and (ii) are satisfied. By (ii), A is obviously dissipative. Since $\tilde{G}(A)$ has no cycles, we can consider, as in Lemma 2.1.7, $\{V_0, \dots, V_\ell\}$ partition of V, where V is the set of all vertices of $\tilde{G}(A)$.

Let $\tilde{A} = (\tilde{a}_{ij})$ be a small enough perturbation of $A = (a_{ij})$ and consider the positive diagonal matrix $\tilde{D} = \mathrm{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$ recursively defined by

$$\tilde{d}_i = \begin{cases} d_i & \text{if} \quad i \in V_0 \\ -\tilde{d}_{i'} \frac{\tilde{a}_{ii'}}{\tilde{a}_{i'i}} & \text{if} \quad i \in V_j \,, j \neq 0 \end{cases} ,$$

for $j \in \{1, \ldots, \ell\}$, where $i' \in V_{j-1}$ is the unique vertex of $\tilde{G}(A)$ such that $(i, i') \in E$.

Since $\tilde{D}$ is a perturbation of D, we have

$$Q_{\left[(\tilde{A}\tilde{D})_{ij}\right]_{k<i,j\leq n}} < 0$$

and

$$\tilde{d}_i \tilde{a}_{i'i} + \tilde{d}_{i'} \tilde{a}_{ii'} = 0 \,, \quad \text{for all} \;\; i \leq k \;\text{ or }\; j \leq k \,.$$

Hence $Q_{\tilde{A}\tilde{D}} \leq 0$. □

Finally we present a useful property for the stably dissipative systems.

Lemma 2.1.11. *Let D be a positive diagonal matrix. If A is a stably dissipative matrix, then AD and $D^{-1}A$ are also stably dissipative.*

Proof. Since A is dissipative, there exists a positive diagonal matrix D_1 such that $Q_{AD_1} \leq 0$, which is equivalent to $Q_{(AD)(D^{-1}D_1)} \leq 0$. Hence AD is dissipative. Analogously, since there exists a positive diagonal matrix D_1 such that $Q_{AD_1} \leq 0$, by Remark 1.3.2 we have $Q_{D_1^{-1}A} \leq 0$, which is equivalent to $Q_{(D_1^{-1}D)(D^{-1}A)} \leq 0$, and again by Remark 1.3.2 we have $Q_{(D^{-1}A)(D_1^{-1}D)^{-1}} \leq 0$, which shows that $D^{-1}A$ is dissipative.

Let B be a small enough admissible perturbation of AD. Then there exists admissible perturbations $\tilde{A}$ and $\tilde{D}$ of A and D, respectively, such that $B = \tilde{A}\tilde{D}$. Since A is stably dissipative, we have that $\tilde{A}$ is dissipative. Then, there exists a positive diagonal matrix D_2 such that $Q_{\tilde{A}D_2} \leq 0$, which iimplies $Q_{(\tilde{A}\tilde{D})(\tilde{D}^{-1}D_2)} \leq 0$. Hence $\tilde{A}\tilde{D}$ is dissipative.

A similar argument proves that $D^{-1}A$ is stably dissipative. □

2.2 Redheffer Reduction Algorithm

As stated before in Proposition 1.3.3, when A is dissipative, the LV system (1.7) admits the Lyapunov function

$$h(x) = \sum_{i=1}^{n} \frac{x_i - q_i \log x_i}{d_i} \,, \tag{2.1}$$

which decreases along the orbits of $X_{A,q}$.

Lemma 2.2.1. *If $q \in \mathrm{int}(\mathbb{R}^n_+)$, then h is a proper function.*

Proof. For each $i = 1, \ldots, n$ consider $h_i : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$h_i(x_i) = \frac{1}{d_i} \left(x_i - q_i \log x_i \right) .$$

We have that

$$\lim_{x_i \to 0^+} h_i(x_i) = +\infty , \qquad \lim_{x_i \to +\infty} h_i(x_i) = +\infty ,$$

and h_i reaches its minimum at $x_i = q_i$. Hence h_i is a proper function.

Consider now $\tilde{h} : \mathbb{R}^n_+ \to \mathbb{R}$ defined by

$$\tilde{h}(x) = \sum_{i=1}^{n} \left(h_i(x_i) - h_i(q_i) \right) .$$

We have that $\tilde{h}(x) \geq 0$ for all $x \in \mathbb{R}^n_+$. Hence, given any non-negative constant c,

$$\tilde{h}^{-1} \left( ]-\infty, c] \right) \subseteq \prod_{i=1}^{n} h_i^{-1} \left([0, c] \right) .$$

Since each h_i is proper, $h_i^{-1}([0,c])$ is compact. Hence $\prod_{i=1}^n h_i^{-1}([0,c])$ is compact. Since $\tilde{h}^{-1}(]-\infty, c])$ is closed, we have that $\tilde{h}$ is a proper function. Hence $h = \tilde{h} + \sum_{i=1}^n h_i(q_i)$ is a proper function. □

From now on, in this section, we will assume that $q \in \mathrm{int}(\mathbb{R}^n_+)$.

Since h is a proper function, $X_{A,q}$ determines a complete semi-flow

$$\phi^t_{A,q} : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n_+ ,$$

defined for all $t \geq 0$.

Definition 2.2.2. We call *attractor* of the LV system (1.7) to the following topological closure

$$\Lambda_{A,q} := \overline{\cup_{x \in \mathbb{R}^n_+} \omega(x)} ,$$

where $\omega(x)$ is the ω-limit of x by the semi-flow $\{\phi^t_{A,q} : \mathbb{R}^n_+ \to \mathbb{R}^n_+\}_{t \geq 0}$.

We need the following classical theorem (see [20, Theorem 2]).

Theorem 2.2.3 (LaSalle)**.** *Given a vector field $f(x)$ on a manifold M, consider the autonomous o.d.e. on M,*

$$x' = f(x). \tag{2.2}$$

Let $h : M \to \mathbb{R}$ be a smooth function such that

1. *h is a Lyapunov function, i.e., the derivative of h along the flow satisfies $\dot{h}(x) := Dh_x f(x) \leq 0$ for all $x \in M$.*

2. *h is bounded from below.*

3. *h is a proper function, i.e. $\{h \leq a\}$ is compact for all $a \in \mathbb{R}$.*

Then (2.2) *induces a complete semi-flow on M such that the topological closure of all its ω-limits is contained in the region where the derivative of h along the flow vanishes, i.e.,*

$$\overline{\cup_{x \in M} \omega(x)} \subseteq \{x \in M \colon \dot{h}(x) = 0\}.$$

Since h is a Lyapunov function for a dissipative LV system (1.7), from Theorem 2.2.3 we can deduce the following result about the attractor.

Proposition 2.2.4. *Given a dissipative LV system with interaction matrix $A \in M_n(\mathbb{R})$, an equilibrium $q \in \operatorname{int}(\mathbb{R}^n_+)$, and a positive diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ such that $Q_{AD} \leq 0$, we have*

$$\Lambda_{A,q} \subseteq \left\{ x \in \mathbb{R}^n_+ \,:\, Q_{AD}\left(D^{-1}(x-q)\right) = 0 \right\} .$$

Proof. By Theorem 2.2.3 the attractor $\Lambda_{A,q}$ is contained in the set where $\dot{h}(x) = 0$. The conclusion follows by the proof of Proposition 1.3.3 and Remark 1.3.2. □

Redheffer *et al.* [25–29] have characterized the class of stably dissipative systems and its attractor $\Lambda_{A,q}$ in terms of the graph $G(A)$. In particular, they described a simple reduction algorithm, running on the graph $G(A)$, that "deduces" every restriction of the form $\Lambda_{A,q} \subseteq \{x : x_i = q_i\}$, $1 \leq i \leq n$, that holds for every stably dissipative system with the same associated graph $G(A)$. To start this algorithm they use Lemma 2.1.2, which plays a key role in the theory of stably dissipative systems.

An immediate consequence of Proposition 2.2.4 and Lemma 2.1.2 is that

$$\Lambda_{A,q} \subseteq \{x \in \mathbb{R}^n_+ \,:\, x_i = q_i\} \tag{2.3}$$

for every $i = 1, \ldots, n$ such that $a_{ii} < 0$.

A species i is said to be of type $\bullet$ to state that $\Lambda_{A,q} \subseteq \{x \,:\, x_i = q_i\}$ holds. Similarly, a species i is said to be of type $\oplus$ to state that $\Lambda_{A,q} \subseteq \{x \,:\, X^i_{A,q}(x) = 0\}$, which means $\{x_i = \text{const.}\}$ is an invariant foliation under $\phi^t_{A,q} : \Lambda_{A,q} \to \Lambda_{A,q}$. Otherwise, a species i is said to be of type $\circ$, meaning that we don't know nothing about species i (at that moment).

This should be interpreted as a collection of statements about the attractor $\Lambda_{A,q}$.

Proposition 2.2.5. *Given neighbour vertices j, ℓ in the graph $G(A)$,*

(a) If j is of type $\bullet$ or $\oplus$ and all of its neighbours are of type $\bullet$, except for ℓ, then ℓ is of type $\bullet$;

(b) If j is of type $\bullet$ or $\oplus$ and all of its neighbours are of type $\bullet$ or $\oplus$, except for ℓ, then ℓ is of type $\oplus$;

(c) If j is of type $\circ$ and all of its neighbours are of type $\bullet$ or $\oplus$, then j is of type $\oplus$.

Proof. The proof involves the manipulation of algebraic relations holding on the attractor. To simplify the terminology we will say that some algebraic relation holds to mean that it holds on the attractor.

Observe that if j is of type $\bullet$ then $x_j = q_j$, and if j is of type $\oplus$ then $a_{jj} = 0$.

If j is of type $\bullet$ or $\oplus$ we have that $\frac{d}{dt}x_j = 0$. Then, by (1.7), we obtain

$$a_{j1}(x_1 - q_1) + \cdots + a_{jn}(x_n - q_n) = 0\,. \tag{2.4}$$

Let j, ℓ be neighbour vertices in the graph $G(A)$.

Let us prove (a). If j is of type $\bullet$ or $\oplus$ and all of its neighbours are of type $\bullet$, except for ℓ, then from 2.4 we obtain

$$a_{j\ell}(x_\ell - q_\ell) = 0\,,$$

from which follows that $x_\ell = q_\ell$ because $a_{j\ell} \neq 0$, which proves (a).

Let us prove (b). If j is of type $\bullet$ or $\oplus$ and all of its neighbours are of type $\bullet$ or $\oplus$, except for ℓ, then from 2.4 we obtain

$$a_{j\ell}(x_\ell - q_\ell) = C\,,$$

for some constant C. Hence x_ℓ is constant, which proves (b).

Let us prove (c). Suppose j is of type $\circ$ and all of its neighbours are of type $\bullet$ or $\oplus$. By (1.7) we have that

$$\frac{dx_j}{dt} = x_j \sum_{k=1}^{n} a_{jk}(x_k - q_k)\,.$$

Since all neighbours of j are of type $\bullet$ or $\oplus$ we obtain

$$\frac{dx_j}{dt} = x_j\, C\,,$$

for some constant C. Hence

$$x_j = B_0\, e^{Ct}\,,$$

where $B_0 = x_j(0)$. Since the system is dissipative we have that the constant C must be 0. Hence x_j is constant, which proves (c). $\square$

Based on these facts, Redheffer *et al.* introduced a reduction algorithm on the graph $G(A)$ to derive information on the specie's types of a stably dissipative LV system (1.7).

Since $a_{ii} \leq 0$ for all i, before starting the reduction algorithm, there is an initial step that consists in colouring the vertices of the graph $G(A)$ according to the following rule.

Rule 1. *Colour in black, $\bullet$, every vertex $i \in \{1, \ldots, n\}$ such that $a_{ii} < 0$, and in white, $\circ$, all other vertices, i.e., every vertex $i \in \{1, \ldots, n\}$ such that $a_{ii} = 0$.*

The reduction procedure consists of the following rules, corresponding to valid inference rules:

Rule 2. *If j is a $\bullet$ or $\oplus$-vertex and all of its neighbours are $\bullet$, except for one vertex ℓ, then colour ℓ as $\bullet$.*

Rule 3. *If j is a $\bullet$ or $\oplus$-vertex and all of its neighbours are $\bullet$ or $\oplus$, except for one vertex ℓ, then draw $\oplus$ at the vertex ℓ.*

Rule 4. *If j is a $\circ$-vertex and all of is neighbours are $\bullet$ or $\oplus$, then draw $\oplus$ at the vertex j.*

Definition 2.2.6. Redheffer *et al.* define the *reduced graph* of the system, $\mathcal{R}(A)$, as the graph obtained from $G(A)$ by successive applications of the reduction Rules 2, 3 and 4 until they can no longer be applied. We designate this procedure as the *Redheffer Reduction Algorithm* (RRA).

In [27] Redheffer and Walter proved the following result, which in a sense states that the RRA on $G(A)$ can not be improved.

Theorem 2.2.7. *Given a stably dissipative matrix A,*

(a) If $\mathcal{R}(A)$ has only $\bullet$-vertices then A is nonsingular, the equilibrium point q is unique and every solution of (1.7) *converges to q as $t \to \infty$.*

(b) If $\mathcal{R}(A)$ has only $\bullet$ and $\oplus$-vertices, but not all $\bullet$, then A is singular, the equilibrium point q is not unique, and every solution of (1.7) *has a limit, as $t \to \infty$, that depends on the initial condition.*

(c) If $\mathcal{R}(A)$ has at least one $\circ$-vertex then there exists a stably dissipative matrix $\widetilde{A}$, with $G(\widetilde{A}) = G(A)$, such that the system (1.7) *associated with $\widetilde{A}$ has a nonconstant periodic solution.*

Based on these ideas Oliva *et al.* [6, Theorem 4.5] proved that the dynamics on the attractor of a stably dissipative LV system can be defined by a conservative lower dimensional LV system whose associated graph is a forest.

Theorem 2.2.8. *Consider a LV system* (1.7) *restricted to the set* $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$. *Suppose that the system is stably dissipative and has an equilibrium point* $q \in \mathrm{int}(\mathbb{R}^n_+)$. *Then the limit dynamics of* (1.7) *on the attractor* $\Lambda_{A,q}$ *is described by a lower dimensional Hamiltonian LV system.*

2.3 Stably Dissipative Graphs

Definition 2.3.1. We designate by *black and white graph* (BW graph) the triple $G = (V, E, \{V_\bullet, V_\circ\})$, where
(i) (V, E) is a graph,
(ii) $V = V_\bullet \cup V_\circ$ and $V_\bullet \cap V_\circ = \emptyset$.

Given a dissipative matrix $A \in M_n(\mathbb{R})$ we associate it the BW graph

$$(\mathcal{V}, \mathcal{E}, \{\mathcal{V}_\bullet, \mathcal{V}_\circ\}) \ , \tag{2.5}$$

where $G(A) = (\mathcal{V}_A, \mathcal{E}_A)$ and

$$\mathcal{V}_\bullet = \{\, 1 \leq i \leq n \ : \ a_{ii} < 0 \,\} \quad \text{and} \quad \mathcal{V}_\circ = \{\, 1 \leq i \leq n \ : \ a_{ii} = 0 \,\} \,.$$

This structure corresponds to the colouring procedure defined by Rule 1 of the RRA.

From now on in this chapter $G(A)$ denotes the BW graph (2.5).

Definition 2.3.2. We say that the graph $G(A)$ has a *strong link* ($\bullet\!\!-\!\!\bullet$) if there is an edge (i, j) between vertices i, j such that $a_{ii} < 0$ and $a_{jj} < 0$.

In [27] Redheffer and Walter gave the following property of stably dissipative matrices in terms of their associated graph (reformulation of Lemma 2.1.5).

Lemma 2.3.3. *If A is a stably dissipative matrix, then every cycle of $G(A)$ has at least one strong link* ($\bullet\!\!-\!\!\bullet$).

Definition 2.3.4. We say that a BW graph G is *stably dissipative* if and only if every cycle of G contains at least a strong link ($\bullet\!\!-\!\!\bullet$).

The name "stably dissipative" stems from the use we shall make of this class of graphs to characterize stably dissipative matrices. See Proposition 2.3.7 below.

Proposition 2.3.5. *Given a dissipative matrix $A \in M_n(\mathbb{R})$, there is a positive diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that $a_{ij}\, d_j = -a_{ji}\, d_i$ whenever $a_{ii} = 0$ or $a_{jj} = 0$, and for every $w \in \mathbb{R}^n$ and $k \in \mathcal{V}_\bullet$, $\sum_{i,j \in \mathcal{V}_\bullet} a_{ij} d_i w_i w_j \leq 0$.*

Proof. Let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a positive diagonal matrix such that for all $w \in \mathbb{R}^n$, $Q_{DA}(w) = \sum_{i,j} a_{ij} d_i w_i w_j \leq 0$. Assuming $a_{ii} = 0$, choose a vector $w \in \mathbb{R}^n$ with $w_i = 1$ and $w_k = 0$ for every $k \neq i, j$. Then

$$(a_{ij} d_j + a_{ji} d_i)\, x_j + a_{jj} d_j\, x_j^2 = Q_{DA}(x) \leq 0 \;,$$

which implies that $a_{ij} d_j + a_{ji} d_i = 0$, and everything else follows. $\square$

Definition 2.3.6. We say that a dissipative matrix $A \in M_n(\mathbb{R})$ is *almost skew-symmetric* if and only if $a_{ij} = -a_{ji}$ whenever $a_{ii} = 0$ or $a_{jj} = 0$, and the quadratic form $Q(x_k)_{k \in \mathcal{V}_\bullet} = \sum_{i,j \in \mathcal{V}_\bullet} a_{ij}\, x_i\, x_j$ is negative definite.

In this context, the following proposition is a reformulation of Theorem 2.1.10.

Proposition 2.3.7. *The matrix $A \in M_n(\mathbb{R})$ is stably dissipative if and only if $G(A)$ is a stably dissipative graph and there exists a positive diagonal matrix D such that AD is almost skew-symmetric.*

Proof. Assuming A is stably dissipative, by Lemma 2.3.3, $G(A)$ is stably dissipative. Take a diagonal matrix $D > 0$ according to Lemma 2.1.2, which implies that $Q(x_k)_{k \in \mathcal{V}_\bullet} = \sum_{i,j \in \mathcal{V}_\bullet} a_{ij} d_i x_i x_j$ is negative definite. By Proposition 2.3.5, AD is almost skew-symmetric.

Conversely, assume $G(A)$ is stably dissipative, assume there is a positive diagonal matrix D such that AD is almost skew-symmetric, and take $\tilde{A} = (\tilde{a}_{ij})$ some close enough perturbation of A. Let $\tilde{G}(A)$ be the partial graph of $G(A)$ obtained by removing every strong link ($\bullet\!-\!\bullet$). Because $G(A)$ is stably dissipative, the graph $\tilde{G}(A)$ has no cycles. Hence, since AD is almost skew-symmetric, the result follows by Theorem 2.1.10. $\square$

2.4 The Rank of the Graph

Definition 2.4.1. Given a stably dissipative graph G, we denote by $\mathrm{SD}(G)$ the set of all stably dissipative matrices A with $G(A) = G$.

This section's main theorem whose proof we present below is the following.

Theorem 2.4.2. *Let G be a stably dissipative graph. Then every matrix $A \in \mathrm{SD}(G)$ has the same rank.*

By this theorem we can define the *rank* of a stably dissipative graph G, denoted hereafter by $\mathrm{rank}(G)$, as the rank of any matrix in $\mathrm{SD}(G)$. Together with Proposition 1.4.3, we have the following result.

Corollary 2.4.3. *Let G be a stably dissipative graph. Then, for every matrix $A \in \mathrm{SD}(G)$, any stably dissipative LV system with matrix A has an invariant foliation of dimension* $\mathrm{rank}(G)$.

Definition 2.4.4. We shall say that a graph G has *constant rank* if and only if every matrix $A \in \mathrm{SD}(G)$ has the same rank.

With this terminology, Theorem 2.4.2 just states that every stably dissipative graph has constant rank.

2.5 Simplified Reduction Algorithm

Since the RRA (see Definition 2.2.6) runs on the graph $G(A)$, the conclusions drawn from the reduction procedure hold for all stably dissipative systems that share the same graph $G(A)$.

The following proposition is a slight improvement on item (*b*) of Theorem 2.2.7.

Proposition 2.5.1. *If $\mathcal{R}(A)$ has only $\bullet$ and $\oplus$-vertices then the system has an invariant foliation with a single globally attractive equilibrium point in each leaf.*

Proof. Combine Theorem 2.2.7 (*b*) with Theorem 1.4.5. $\square$

Remark 2.5.2. *In* [28] *Redheffer and Zhi Ming make the following statement:*

> *"Let A be stably dissipative and let every vertex $\circ$ in $G(A)$ be replaced arbitrarily by $\oplus$. Then A is nonsingular if and only if, by algebraic manipulations, every vertex can then be replaced by $\bullet$."*

We shall explain this statement in terms of a simpler reduction algorithm. Let us say that a species $i \in \{1, \ldots, n\}$ is a *restriction on the equilibria* of $X_{A,q}$ whenever $E_{A,q} \subset \{\, x \in \mathbb{R}^n_+ \,:\, x_i = q_i \,\}$, where $E_{A,q}$ is the set of all equilibria of (1.7) as defined in (1.9). Notice that every species of type $\bullet$ is also a restriction on the equilibria of $X_{A,q}$. Think of colouring i as black as the statement that i is a restriction to the equilibria of $X_{A,q}$. Notice that at the beginning of the reduction algorithm, described in the introduction of this section, the weaker interpretation that all black vertices correspond to restrictions on the equilibria is also valid. If we simply do not write $\oplus$-vertices, but consider every $\circ$-vertex as a $\oplus$-vertex, then the reduction Rules 3 and 4 can be discarded, while the Rule 2 becomes

(R) If all neighbours of a vertex j are $\bullet$-vertices, except for one vertex k, then we can colour k as a $\bullet$-vertex.

The idea implicit in Remark 2.5.2 is that (R) is a valid inference rule for the weaker interpretation of the colouring statements above. Assuming that every $\circ$-vertex is a $\oplus$-vertex amounts to looking for restrictions on the equilibria set $E_{A,q}$ instead of the attractor $\Lambda_{A,q}$. Let us still call *reduced graph* to the graph, denoted by $\mathcal{R}_*(G)$, obtained from G by successively applying rule (R) alone until it can no longer be applied. The previous considerations show that

Proposition 2.5.3. *Given a stably dissipative matrix A, every $\bullet$-vertex of $\mathcal{R}_*(G(A))$ is a restriction to the equilibria of $X_{A,q}$.*

We shall write $\mathcal{R}_*(G) = \{\bullet\}$ to express that all vertices of $\mathcal{R}_*(G)$ are $\bullet$-vertices.

Corollary 2.5.4. *If G is a stably dissipative graph such that $\mathcal{R}_*(G) = \{\bullet\}$ then every matrix $A \in \mathrm{SD}(G)$ is nonsingular. In particular G has constant rank.*

Proof. Given $A \in \mathrm{SD}(G)$, by Proposition 2.5.3 we have $E_{A,q} = \{q\}$, which automatically implies that A is nonsingular. $\square$

In fact, the converse statement of this corollary holds by Remark 2.5.2.

Proposition 2.5.5. *Let $A \in M_n(\mathbb{R})$ be a stably dissipative matrix. If A is nonsingular then $\mathcal{R}_*(G(A)) = \{\bullet\}$.*

Proof. Let $A \in M_n(\mathbb{R})$ be a stably dissipative matrix. By Proposition 2.2.4 we have that

$$A(x - q) = 0\,,$$

on $\Lambda_{A,q}$. Since A is nonsingular, the result follows. $\square$

We call any extreme $\circ$-vertex of G a $\circ$-*endpoint* of G.

Lemma 2.5.6. *Let G be a stably dissipative graph. If G has no $\circ$-endpoints, then $\mathcal{R}_*(G) = \{\bullet\}$.*

Proof. Let G be a stably dissipative graph with no $\circ$-endpoints. Assume, by contradiction, that $\mathcal{R}_*(G) \neq \{\bullet\}$. We shall construct a cycle in $\mathcal{R}_*(G)$ with no $\bullet\!\!-\!\!\bullet$ edges. Since every $\circ$-vertex of $\mathcal{R}_*(G)$ is also a $\circ$-vertex of G, this will contradict the assumption that G is stably dissipative.

In the following construction we always refer to the vertex colouring of $\mathcal{R}_*(G)$. Take j_0 to be any $\circ$-vertex. Then, given j_k take a neighbouring vertex j_{k+1} to be another $\circ$-vertex, if possible, or a $\bullet$-vertex otherwise. While the path is simple (no vertex repetitions) it can not end at some ∘-endpoint, and it can not contain any •–• edge because whenever we arrive to a •-vertex from a ∘-one we can always escape to another ∘-vertex. In fact, no •-vertex can be linked to a single ∘-vertex since otherwise we could reduce it to a •-vertex by applying rule (R). By finiteness this recursively defined path must eventually close, hence producing a cycle with no •–• edges. □

Given a stably dissipative graph G and some ∘-endpoint $i \in \mathcal{V}_\circ$, we define the *trimmed* graph $T_i(G)$ as follows: Let $i' \in \mathcal{V}$ be the unique vertex connected to i by some edge of G. Then $T_i(G)$ is the partial graph obtained from G by removing every edge incident with i' except with i. See an example in figure 2.1.

The trimming operation preserves the stable dissipativeness of the graph, i.e.,

Proposition 2.5.7. *$T_i(G)$ is stably dissipative whenever G is.*

Proof. The proof follows by Definition 2.3.4 because $T_i(G)$ is obtained by removing some edges from G. □

Figure 2.1: A graph G and it's trimmed graph $T_i(G)$.

Similarly we define the *trimmed matrix* $T_i(A)$ as follows: annihilate every entry of row i', except for $a_{i'i}$ and $a_{i'i'}$, and annihilate every entry of column i', except for $a_{ii'}$ and $a_{i'i'}$. See the example below, where matrix $A \in M_5(\mathbb{R})$ is the associated matrix to the graph G in figure 2.1.

$$A = \begin{bmatrix} \cdot & 0 & \cdot & * & \cdot \\ 0 & 0 & 0 & a_{ii'} & \cdot \\ \cdot & 0 & \cdot & * & \cdot \\ * & a_{i'i} & * & a_{i'i'} & * \\ \cdot & 0 & \cdot & * & \cdot \end{bmatrix} \qquad T_i(A) = \begin{bmatrix} \cdot & 0 & \cdot & 0 & \cdot \\ 0 & 0 & 0 & a_{ii'} & \cdot \\ \cdot & 0 & \cdot & 0 & \cdot \\ 0 & a_{i'i} & 0 & a_{i'i'} & 0 \\ \cdot & 0 & \cdot & 0 & \cdot \end{bmatrix}$$

The "$*$" above represent entries of A that are annihilated in $T_i(A)$, and "$\cdot$" are nonzero constants.

Lemma 2.5.8. *Let $i \in \mathcal{V}_\circ$ be some $\circ$-endpoint of a stably dissipative graph G. If $A \in \mathrm{SD}(G)$ then*

$$T_i(A) \in \mathrm{SD}\,(T_i(G)) \qquad \textit{and} \qquad \mathrm{rank}\,(T_i(A)) = \mathrm{rank}(A)\,.$$

Proof. Take $A \in \mathrm{SD}(G)$ and let $A' = T_i(A)$, where i is some $\circ$-endpoint. Denote, respectively, by col_j and row_j the j^{th} column and the j^{th} row of A, and denote by col'_j and row'_j the j^{th} column and the j^{th} row of the trimmed matrix. Since i is a $\circ$-endpoint, $a_{ii'}$ is the only nonzero entry in row_i, and $a_{i'i}$ is the only nonzero entry in col_i. Then the trimmed matrix A' is obtained from A by applying the following Gauss elimination rules, either simultaneously or in some arbitrary order

$$\begin{aligned} \mathrm{row}'_j &:= \mathrm{row}_j - \frac{a_{ji'}}{a_{ii'}}\,\mathrm{row}_i \qquad j \neq i'\ , \\ \mathrm{col}'_j &:= \mathrm{col}_j - \frac{a_{i'j}}{a_{i'i}}\,\mathrm{col}_i \qquad j \neq i'\ . \end{aligned}$$

Because Gauss elimination preserves the matrix rank we have $\mathrm{rank}(A') = \mathrm{rank}(A)$. To finish the proof, it is enough to see now that A' is stably dissipative. We use Proposition 2.3.7 for this purpose. First, $G(A') = T_i(G)$ is stably dissipative as observed above. Let D be a positive diagonal matrix such that AD is almost skew-symmetric. In view of Proposition 2.3.7, we only need to prove that $A'\,D$ is also almost skew-symmetric. Notice that G and $T_i(G)$ share the same black and white vertices. If $a'_{kk} = 0$ or $a'_{jj} = 0$ then also $a_{kk} = 0$ or $a_{jj} = 0$. Hence, because AD is almost skew-symmetric, $a_{kj}\,d_j = -a_{jk}\,d_k$. Looking at the Gauss elimination rules above, we have $a'_{kj} = a_{kj}$ and $a'_{jk} = a_{jk}$, or else $a'_{kj} = a'_{jk} = 0$. In either case we have $a'_{kj}\,d_j = -a'_{jk}\,d_k$. Finally, we need to see that $Q'(x_\ell)_{\ell\in\mathcal{V}_\bullet} = \sum_{k,j\in\mathcal{V}_\bullet} a'_{kj} d_j x_k x_j$ is a negative definite quadratic form. If i' is a $\circ$-vertex then $Q'(x_\ell)_{\ell\in\mathcal{V}_\bullet} = \sum_{k,j\in\mathcal{V}_\bullet} a_{kj} d_j\, x_k x_j$ is negative definite because AD is almost skew-symmetric. Otherwise, if i' is a $\bullet$-vertex, given a nonzero vector $(x_\ell)_{\ell\in\mathcal{V}_\bullet}$ we define $(x'_\ell)_{\ell\in\mathcal{V}_\bullet}$ letting $x'_\ell = x_\ell$ for $\ell \neq i'$, while $x'_{i'} = 0$. Then

$$Q'(x_\ell)_{\ell\in\mathcal{V}_\bullet} = \underbrace{a_{i'i'}}_{<0} d_{i'} x_{i'}^2 + \underbrace{\sum_{k,j\in\mathcal{V}_\bullet} a_{kj} d_j x'_k x'_j}_{=Q(x'_\ell)_{\ell\in\mathcal{V}_\bullet}\leq 0} < 0\ ,$$

since $(x_\ell)_{\ell\in\mathcal{V}_\bullet} \neq 0$ implies that either $x_{i'} \neq 0$ or else $(x'_\ell)_{\ell\in\mathcal{V}_\bullet} \neq 0$. This proves that Q' is negative definite. $\square$

As a simple corollary of the previous lemma we obtain

Lemma 2.5.9 (Trimming lemma)**.** *Let $i \in \mathcal{V}_\circ$ be some $\circ$-endpoint of a stably dissipative graph G. If $T_i(G)$ has constant rank then so has G, and* $\mathrm{rank}(G) = \mathrm{rank}(T_i(G))$.

We can now prove our main Theorem 2.4.2.

Proof of Theorem 2.4.2. Define recursively a sequence of graphs $G_0, G_1, \ldots, G_m$, with $G_0 = G$, and where $G_{i+1} = T_{j_i}(G_i)$ for some $\circ$-endpoint j_i of G_i. This sequence will end at some graph G_m with no $\circ$-endpoint. By Lemma 2.5.6 we have $\mathcal{R}_*(G_m) = \{\bullet\}$. The connected components of G_m are either reducible to $\bullet$-vertices by iteration of rule (R), or else composed by $\circ$-vertices alone. Since the $\circ$-components can not be trimmed anymore, they must be either formed of a single $\circ$-vertex, or else a single $\circ\!\!-\!\!\circ$ edge. By Corollary 2.5.4, G_m has constant rank. Finally, applying inductively Lemma 2.5.9 we see that all graphs G_i have constant rank. Hence G, in particular, has constant rank. $\square$

The previous proof gives a simple recipe to compute the rank of a graph. Trim G while possible. In the end, discard the single $\circ$-vertex components and count the remaining vertices. We can see some examples in table 2.1.

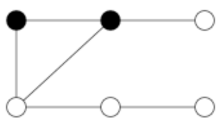
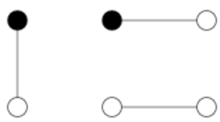
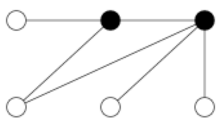
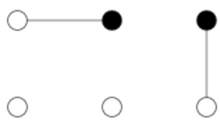
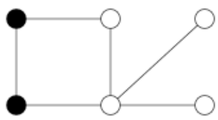
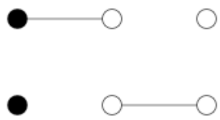
Original graph	Trimmed graph	Graph rank
		6
		4
		5

Table 2.1: Some graph trimming examples.

2.6 Trimming Effect on Dynamics

In this last section we use an example to describe the effect of trimming a stably dissipative matrix on the underlying dynamics.

Consider the stably dissipative LV system with interaction matrix

$$A = \left[\begin{array}{ccc|ccc|c} 0 & -2 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 & -1 \end{array}\right]$$

and equilibrium point $\mathbb{1} \in \mathbb{R}^7$ with all coordinates equal to 1. The associated graph $G(A)$ is represented in figure 2.2.

Figure 2.2: Associated graph of matrix A, $G(A)$.

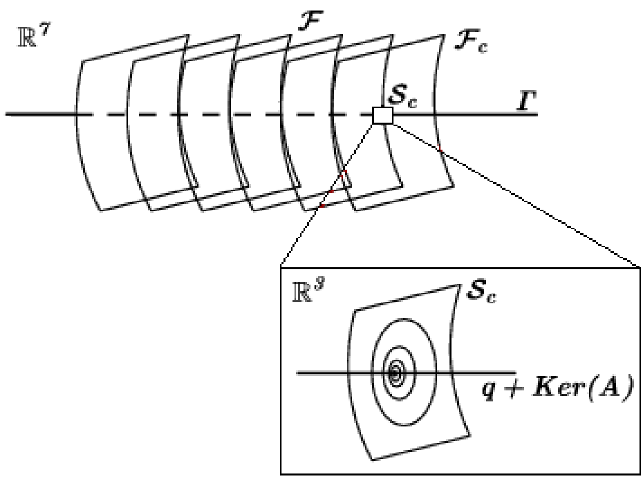


Figure 2.3: Phase portrait of a system E.

The null space, $\mathrm{Ker}(A^T)$, is generated by the vector $(1, 0, 2, 1, 0, 2, 0)$. Hence the foliation $\mathcal{F}$, with leaves $\mathcal{F}_c$ given by

$$\mathcal{F}_c = \{x \in \mathbb{R}^7 \,:\, \log x_1 + 2\log x_3 + \log x_4 + 2\log x_6 = c\},$$

is an invariant foliation with dimension $\operatorname{rank}(G(A)) = 6$ in $\mathbb{R}^7$. The system's phase portrait is represented in figure 2.3, being the attractor a $3D$-plan transversal to $\mathcal{F}$ given by

$$\Gamma = \{x \in \mathbb{R}^7 \,:\, x_1 = x_4,\, x_2 = x_5,\, x_3 = x_6,\, x_7 = 1\}\,.$$

The intersection of each leaf $\mathcal{F}_c$ with Γ is a surface $\mathcal{S}_c$ given by

$$\mathcal{S}_c = \mathcal{F}_c \cap \Gamma = \{(x_1, x_2, x_3, x_1, x_2, x_3, 1) \,:\, \log x_1 + 2\log x_3 = \frac{c}{2}\},$$

which is foliated into invariant curves by the level sets of h, defined in (2.1). Notice that $\mathcal{S}_c$ corresponds to an invariant leaf of the conservative system with graph $\circ\!\!-\!\!\!-\, \circ \,-\!\!\!-\!\!\circ$.

With the first trim on G we get the graph $T_6(G)$ represented in figure 2.4.

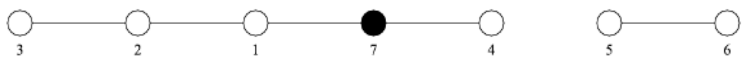


Figure 2.4: The trimmed graph of G, $T_6(G)$.

This corresponds to annihilate the entries $(4, 5)$ and $(5, 4)$ of the original matrix A. Notice that the components x_5 and x_6 of the system are independent of the rest. Hence the dynamics of this new system is the product of two independent LV systems represented in figure 2.5.

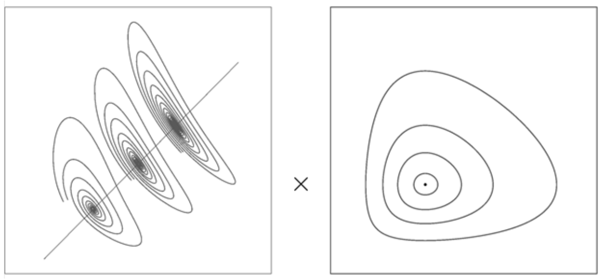


Figure 2.5: Representation of the dynamic of the system E_1.

The five dimensional system on the left of figure 2.5 has a straight line of equilibria. Moreover it leaves invariant a foliation of dimension four with a single globally attractive fixed point on each leaf. The right-hand side system is a typical conservative predator-prey.

Now we have two different possibilities of trimming the graph $T_6(G)$: we can either choose the $\circ$-endpoint 3 or else 4. In the first case we get the graph $T_3(T_6(G))$ represented in figure 2.6, whose dynamics is illustrated in figure 2.7.

Figure 2.6: The trimmed graph $T_3(T_6(G))$ of $T_6(G)$.

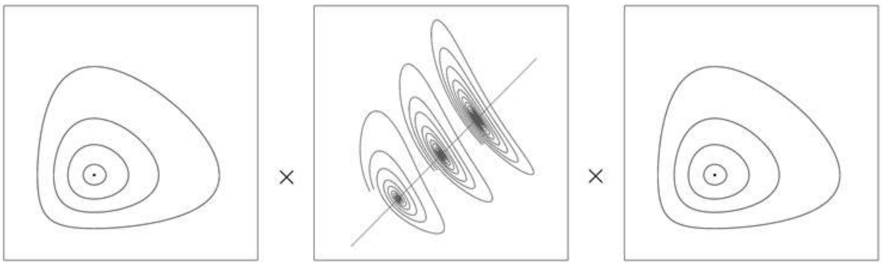


Figure 2.7: Representation of the system's dynamics associated to the graph $T_3(T_6(G))$.

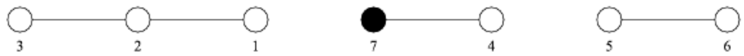


Figure 2.8: The trimmed graph $T_4(T_6(G))$ of $T_6(G)$.

Figure 2.9: Representation of the system's dynamics associated to the graph $T_4(T_6(G))$.

The three dimensional system in the middle of figure 2.7 has a straight line of equilibria. Moreover it leaves invariant a foliation of dimension two with a single globally attractive fixed point on each leaf. The left and right-hand side systems are typical conservative predator-preys.

In the second case we get the graph $T_4(T_6(G))$ represented in figure 2.8, whose dynamics is depicted in figure 2.9.

Here, the left-hand side three dimensional system is conservative, leaving invariant a foliation of dimension two transversal to a straight line of equilibria. The middle and right-hand side systems are typical predator-prey, respectively, dissipative and conservative.

Trimming $T_4(T_6(G))$ choosing the $\circ$-endpoint 1 we get the graph $T_1\left(T_4\left(T_6(G)\right)\right)$ represented in figure 2.10, whose dynamics is a product of three predator-prey systems, illustrated in figure 2.11, with a one dimensional system consisting of equilibria.

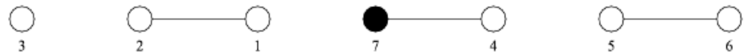


Figure 2.10: The trimmed graph $T_1(T_4(T_6(G)))$ of $T_4(T_6(G))$.

Figure 2.11: Representation of the system's dynamics associated to the graph $T_1(T_4(T_6(G)))$.

Notice that by trimming $T_3(T_6(G))$ we obtain an isomorphic graph to the one in figure 2.10.

Chapter 3

Vector Fields on Polytopes

In this chapter we address the study of analytic flows defined on polytopes. We present a theory that allows us to analyze the asymptotic dynamics of the flow along the heteroclinic network composed by the flowing-edges and the vertices of the polytope where the flow is defined.

Consider a flow φ_X^t defined on a polytope Γ^d associated to a vector field X. In [5] the author introduces a new method to encapsulate the asymptotic dynamics of φ_X^t along the heteroclinic network of Γ^d. This asymptotic behaviour is completely determined by local data obtained from X at each vertex singularity. We designate this local data as the *skeleton character*. From this data we construct a piecewise constant vector field χ defined in a geometric space $\mathcal{C}^*(\Gamma^d)$ designated as the *dual cone* of the polytope Γ^d. In some sense the orbits of χ give us information about the asymptotic behaviour of the flow φ_X^t along the heteroclinic network.

Looking to the heteroclinic network composed by the flowing-edges of Γ^d, we consider sets S, called *structural sets*, consisting of flowing-edges such that every cycle of the heteroclinic network of X contains at least one edge in S. Given a structural set S, we denote by $\Sigma \subset \Gamma^d$ a union of cross-sections to X, one at each flowing-edge in S. The flow φ_X^t induces a Poincaré return map P_S to Σ, designated as the *S-Poincaré map* of X.

At the level of the dual cone, the flow of χ also induces a return map on the union Π_S of the corresponding cross-sections in $\mathcal{C}^*(\Gamma^d)$, designated as the S-Poincaré map of χ, denoted by π_S. This map is piecewise linear and can be computed from the vector field's skeleton character. π_S carries the asymptotic behaviour of P_S along the flowing-edges in the sense that after a rescaling change of coordinates Ψ_ε, depending on a blow-up parameter ε, π_S is the C^∞ limit of $\Psi_\varepsilon \circ P_S \circ (\Psi_\varepsilon)^{-1}$ as ε tends to 0^+.

Because the Poincaré map π_S is computable, we can run algorithms to find the π_S-invariant linear algebra structures. If these structures are invariant under small non-linear perturbations, they will persist as invariant geometric structures for P_S, and hence for the flow φ_X^t.

We make use of this *stability principle* to prove, in Theorem 3.6.6, the existence of normally hyperbolic stable and unstable manifolds for heteroclinic cycles satisfying some appropriate assumptions.

We use the letters Σ and P for cross-sections on the polytope, and for the associated Poincaré maps between these cross-sections, respectively, while we will use the letters Π and π for the corresponding cross-sections on the dual cone, and for the associated Poincaré maps, respectively. Moreover, the interior and boundary of each of these cross-sections, e.g., $\mathrm{int}(\Pi)$, $\partial\Pi$, will refer to the topological interior and boundary with respect to the affine subspace generated by the respective cross-sections.

In this chapter we introduce the necessary theoretical background as well the corresponding main results (whose proofs can be seen in [2]) that will be required in chapter 4.

This chapter is organized as follows. In section 3.1, we define polytopes and all their associated notation, terminology and concepts. In section 3.2, we introduce the class of vector fields on polytopes to which our results apply. In section 3.3, we define the dual cone of a polytope, where the asymptotic dynamics along the heteroclinic network takes place. In section 3.4, we introduce the class of skeleton vector fields (piecewise constant vector fields) on the dual cone, whose dynamics encapsulate the asymptotic behaviour of the given flow. In section 3.5, we define the concept of structural set (of the heteroclinic network) and the associated Poincaré return map. Finally, in section 3.6, we give sufficient conditions (Theorem 3.6.6) for the existence of normally hyperbolic stable and unstable manifolds for heteroclinic cycles.

3.1 Polytopes

In this section we provide preliminary definitions and notations toward the definition of polytope. Given a compact convex set $K \subseteq \mathbb{R}^N$, we call *affine support* of K to the affine subspace spanned by K. The *dimension* of K is by definition the dimension of its affine support. We can now define d-dimensional simple polytopes.

Definition 3.1.1. A set $\Gamma^d \subset \mathbb{R}^N$ is called a *d-dimensional simple polytope* if it is a compact convex subset of dimension d, with affine support $E^d \subset \mathbb{R}^N$, for which there exists a family of affine functions $\{f_i : E^d \to \mathbb{R}\}_{i\in I}$ such that

(a) $\Gamma^d = \cap_{i\in I} f_i^{-1}([0,+\infty[)$.
(b) $\Gamma^d \cap f_i^{-1}(0) \neq \emptyset\,, \quad \forall i \in I$.
(c) Given $J \subseteq I$ such that $\Gamma^d \cap (\cap_{j\in J} f_j^{-1}(0)) \neq \emptyset$, the linear 1-forms $\mathrm{d}f_j$ are linearly independent at every point $p \in \cap_{j\in J} f_j^{-1}(0)$.

Given $J \subseteq I$ such that $\rho_J := \Gamma^d \cap (\cap_{j\in J} f_j^{-1}(0)) \neq \emptyset$ the subset ρ_J is an r-dimensional face of Γ^d, where $r = d - |J|$. We denote by $K^r(\Gamma^d)$ the set of all r-dimensional faces of Γ^d. Specially, we define

- the set of *vertices* $V := K^0(\Gamma^d)$, and denote its elements by letters like v, v', v_i, etc.
- the set of *edges* $E := K^1(\Gamma^d)$, and denote its elements by letters like γ, γ', γ_i, etc.
- the set of *faces* $F := K^{d-1}(\Gamma^d)$, and denote its elements by letters like σ, σ', σ_i, etc.
- the set of *corners* $C := \{\, (v, \gamma, \sigma) \in V \times E \times F :\ \gamma \cap \sigma = \{v\}\,\}$.

The second and third conditions of Definition 3.1.1 assert that every f_i defines a face $\sigma_i = \Gamma^d \cap f_i^{-1}(0)$. Thus, from now on we assume that the family of functions defining Γ^d is indexed in F instead of I, so that $\sigma = \Gamma^d \cap f_\sigma^{-1}(0)$ for all $\sigma \in F$.

A corner is given by a triple (v, γ, σ) where $v \in V$ is a vertex, $\gamma \in E$ is an edge, and $\sigma \in F$ is a face. However, any pair of these three elements in a corner determines uniquely the third one. Therefore, we will sometimes refer to the corner (v, γ, σ) shortly as (v, γ) or (v, σ). An edge γ with endpoints v_1, v_2 determines two corners (v_1, γ, σ_1) and (v_2, γ, σ_2), referred as end corners of γ. The faces σ_1, σ_2 will be referred as opposite faces of γ.

Definition 3.1.2. Given a vertex v, we denote by F_v and E_v the set of all faces, respectively edges, which contain v.

Since Γ^d is a simple polytope, both sets F_v and E_v have d elements. Condition (c) of Definition 3.1.1 guarantees that for every $v \in V$ the covectors $\{(\mathrm{d}f_\sigma)_v : \sigma \in F_v\}$ are linearly independent. This implies that, inside a neighbourhood U_v of the vertex v, the functions $\{f_\sigma : \sigma \in F_v\}$ can be used to define a coordinate system for Γ^d.

Given a vertex v, we define $\psi_v : U_v \to \mathbb{R}^{F_v} \equiv \mathbb{R}^d$ by

$$z \mapsto (\psi_v^\sigma(z))_{\sigma\in F_v} := (f_\sigma(z))_{\sigma\in F_v}.$$

It is clear that the restriction of ψ_v to $\tilde{N}_v := U_v \cap \Gamma^d$ is a local coordinate system for Γ^d sending the vertex v to the origin and every face σ to the hyperplane $x_\sigma = 0$. This restriction, still denoted by ψ_v, is called the *local v-coordinate system* of Γ^d. Shrinking the neighbourhoods $\tilde{N}_v$, if necessary, one can make them disjoint. Furthermore, we will assume that for every vertex v we have $[0, 1]^d \subset \psi_v(\tilde{N}_v)$. This can be achieved multiplying each defining function of Γ^d by a suitable positive constant.

Definition 3.1.3. Setting $N_v := \psi_v^{-1}([0, 1]^d)$, the pairwise disjoint local coordinate systems $\{(N_v, \psi_v) : v \in V\}$ will be referred as the *vertex coordinates* of Γ^d.

3.2 Vector Fields on Polytopes

Throughout the rest of the chapter, Γ^d will denote a d-dimensional simple polytope. For a d-dimensional polytope Γ^d, we denote by $\mathcal{C}^\omega(\Gamma^d)$ the space of functions defined on Γ^d that can be extended analytically to a neighbourhood of Γ^d, and denote by $\mathfrak{X}^\omega(\Gamma^d)$ the space of vector fields X, defined on Γ^d, that have an analytic extension to a neighbourhood of Γ^d and such that X is tangent to every r-dimensional face of Γ^d, for all $0 \leq r \leq d$. If $X \in \mathfrak{X}^\omega(\Gamma^d)$, then for every face $\sigma \in F$, $\mathrm{d}f_\sigma(X) = 0$ along $\sigma = \{z \in \Gamma^d : \, f_\sigma(z) = 0\}$. This implies that either $\mathrm{d}f_\sigma(X)$ is identically zero or else there exists a non-identically zero function $H_\sigma \in \mathcal{C}^\omega(\Gamma^d)$ such that

$$\mathrm{d}f_\sigma(X) = f_\sigma H_\sigma \, . \tag{3.1}$$

Definition 3.2.1. We say that the vector field X is *nondegenerate* if for all $\sigma \in F$ the function H_σ in (3.1) is such that $H_\sigma \not\equiv 0$ on σ.

Given a vertex v of Γ^d, denote by $T_v\Gamma^d$ the linear space of tangent vectors to Γ^d at v. For every corner (v, γ, σ), there is a unique vector $e_{(v,\sigma)}$ parallel to γ and such that $(\mathrm{d}f_\sigma)_v(e_{(v,\sigma)}) = 1$. Hence $\{\mathrm{e}_{(v,\sigma)}\}_{\sigma\in F_v}$ is the dual basis on $T_v\Gamma^d$ of the 1-form basis $\{(\mathrm{d}f_\sigma)_v\}_{\sigma\in F_v}$ on $(T_v\Gamma^d)^*$. For any $X \in \mathfrak{X}^\omega(\Gamma^d)$ the vectors $e_{(v,\sigma)}$ are eigenvectors of the derivative $(DX)_v$. Then

$$H_\sigma(v) = (\mathrm{d}f_\sigma)_v(D\,X)_v(e_{(v,\sigma)}) \, ,$$

is the eigenvalue associated to $e_{(v,\sigma)}$.

Definition 3.2.2. The *skeleton character* of $X \in \mathfrak{X}^\omega(\Gamma^d)$ is defined to be the family $\chi := (\chi^v_\sigma)_{(v,\sigma)\in V\times F}$ where

$$\chi^v_\sigma := \begin{cases} -H_\sigma(v) & \text{if } \sigma \in F_v \\ 0 & \text{, otherwise} \end{cases} \, ,$$

while the *skeleton character* at $v \in V$ is $\chi^v := (\chi^v_\sigma)_{\sigma\in F}$.

Definition 3.2.3. Given a vertex v, we define the sector at v

$$\Pi_v := \{\, (u_\sigma)_{\sigma\in F} \in \mathbb{R}^F : \, u_\sigma = 0 \quad \forall \sigma \notin F_v, \quad u_{\sigma'} \geq 0 \quad \forall \sigma' \in F_v \,\} \, .$$

3.3 Dual Cone of a Polytope

In this section we introduce the concept of a polytope's dual cone. Let Γ^d be a simple polytope with a defining family of affine functions $\{f_\sigma\}_{\sigma\in F}$. The *dual cone* of Γ^d is a subset of $\mathbb{R}^F_+$.

Definition 3.3.1. For every vertex $v \in V$ consider the set Π_v. The *dual cone* of Γ^d is defined to be

$$\mathcal{C}^*(\Gamma^d) := \bigcup_{v \in V} \Pi_v \ .$$

Dual cones of polytopes have a simplicial structure.

Definition 3.3.2. Given $0 \leq r \leq d$, for every $\rho \in K^{d-r}(\Gamma^d)$ the set

$$\Pi_\rho := \{ \, (x_\sigma)_{\sigma \in F} \in \mathbb{R}_+^F : \ x_\sigma = 0 \text{ if } \rho \not\subseteq \sigma \, \}$$

is called a *r-dimensional face* of $\mathcal{C}_r^*(\Gamma^d)$, and the union

$$\mathcal{C}_r^*(\Gamma^d) := \bigcup \{ \, \Pi_\rho : \ \rho \in K^{d-r}(\Gamma^d) \, \}$$

will be referred as *the r-dimensional skeleton* of the dual cone.

To justify the used “duality” terminology notice that

Remark 3.3.3. *Given faces ρ, ρ_1 and ρ_2 of Γ^d,*

- *If ρ has dimension $d - r$ then Π_ρ is a r-dimensional convex cone.*
- *If the face ρ is the convex hull of $\rho_1 \cup \rho_2$ then*

$$\Pi_\rho = \Pi_{\rho_1} \cap \Pi_{\rho_2} \ .$$

- *In particular, if γ is an edge with endpoints v_1 and v_2 then*

$$\Pi_\gamma = \Pi_{v_1} \cap \Pi_{v_2} \ .$$

In definition 3.1.3, for a given vertex v in Γ^d, we define the vertex coordinates of Γ^d. We can generalize this definition to every r-dimensional face of Γ^d.

For every $\sigma \in F$ set

$$N_\sigma := \{ \, z \in \Gamma^d : \ f_\sigma(z) \leq 1 \, \} \, . \tag{3.2}$$

More generally for every face $\rho \in K^{d-r}(\Gamma^d)$ define

$$N_\rho := \bigcap_{\substack{\sigma \in F \\ \sigma \supset \rho}} N_\sigma \ ,$$

which is a neighbourhood of ρ in Γ^d. Multiplying the functions f_σ, if needed, by some large constants, we can assume that

$$N_\rho \cap N_{\rho'} = \emptyset \quad \text{whenever} \quad \rho \cap \rho' = \emptyset \ ,$$

for all faces $\rho \in K^r(\Gamma^d)$ and $\rho' \in K^{r'}(\Gamma^d)$.

Definition 3.3.4. Given a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$, a vertex v and $\varepsilon > 0$, we define the rescaling v-coordinate $\Psi^X_{v,\varepsilon} : N_v \backslash \partial\Gamma^d \to \Pi_v$ by

$$z \mapsto y := \begin{cases} -\varepsilon^2 \log(\psi_v^\sigma(z)) & \text{if } \sigma \in F_v \\ 0 & \text{if } \sigma \notin F_v \end{cases}, \tag{3.3}$$

where $\psi_v : N_v \to [0,1]^d$ is the v-coordinate system in Definition 3.1.3.

Notice that there are natural identifications $\Pi_v \equiv \mathbb{R}_+^{F_d} \equiv \mathbb{R}_+^d$.

To shorten the convergence statements in the upcoming results, we introduce some terminology.

Definition 3.3.5. Suppose we are given a family of functions, or mappings, F_ε with varying domains $\mathcal{U}_\varepsilon$. Let F be another function with domain $\mathcal{U}$. Suppose that all these functions have the same target and source spaces, which are assumed to be linear spaces. We will say that $\lim_{\varepsilon\to 0^+} F_\varepsilon = F$ *in the C^∞ topology*, to mean that:

- *domain convergence*: for every compact subset $K \subseteq \mathcal{U}$, we have $K \subseteq \mathcal{U}_\varepsilon$ for all small enough $\varepsilon > 0$, and
- *derivative uniform convergence on compacts*: for every $k \in \mathbb{N}$

$$\lim_{\varepsilon\to 0^+} \sup_{u\in K\ 0\leq i\leq k} \left| D^i \left[F_\varepsilon(u) - F(u)\right] \right| = 0 .$$

If in a statement F_ε is written as a composition of two or more mappings then its domain should be understood as the composition domain.

For a given vertex $v \in V$ we define

$$\Pi_v(\varepsilon) := \{\, y \in \Pi_v :\ y_\sigma \geq \varepsilon \quad \text{for all} \quad \sigma \in F_v \,\} . \tag{3.4}$$

Lemma 3.3.6. *Consider the functions H_σ defined in* (3.1)*, and the rescaling v-coordinate $\Psi^X_{v,\varepsilon}$ specified in Definition 3.3.4. Then the push-forward of X by $\Psi^X_{v,\varepsilon}$ is*

$$(\Psi^X_{v,\varepsilon})_* X = \varepsilon^2 \tilde{X}^\varepsilon_v,$$

where $\tilde{X}^\varepsilon_v := \left(-H_\sigma\left((\Psi^X_{v,\varepsilon})^{-1}(y)\right)\right)_{\sigma\in F_v}$. Furthermore, the following limit holds in the C^∞ topology

$$\lim_{\varepsilon\to 0} (\tilde{X}^\varepsilon_v)_{|\Pi_v(\varepsilon)} = \chi^v .$$

Proof. A proof of this lemma can be seen in [2]. ∎

Definition 3.3.7. We define the *skeleton coordinate map* on Γ^d by

$$\psi : \Gamma^d \to W, \quad \psi(z) := (\psi_\sigma(z))_{\sigma\in F} , \quad \psi_\sigma(z) := \min\{1, f_\sigma(z)\} ,$$

where $W := \{\, x \in \mathbb{R}^F : \ 0 \le x_\sigma \le 1 \ \text{ for all } \ \sigma \in F \,\}$.

Notice also that the local coordinate (N_v, ψ_v) (see Definition 3.1.3) is simply the composition of the restriction of ψ to N_v with the orthogonal projection $\mathbb{R}^F \to \Pi_v \simeq \mathbb{R}^d$.

Next we introduce a family of rescaling change of coordinates from the polytope Γ^d to its dual cone $\mathcal{C}^*(\Gamma^d)$.

Definition 3.3.8. Given a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ and $\varepsilon > 0$, *the* ε*-rescaling* Γ^d*-coordinate for* X is the mapping $\Psi_\varepsilon^X : \Gamma^d \backslash \partial\Gamma^d \to \mathrm{C}^*(\Gamma^d)$ defined by

$$z \mapsto \Psi_\varepsilon^X(z) := \left(-\varepsilon^2 \log(\psi_\sigma(z))\right)_{\sigma\in F} ,$$

where ψ is the skeleton coordinate map above.

Notice that for every vertex v the restriction of Ψ_ε^X to $N_v \backslash \partial\Gamma^d$ is one-to-one and onto Π_v.

3.4 Skeleton Vector Fields

In this section we define and characterize skeleton vector fields on the dual cone. Every vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ yields a skeleton vector field on the dual cone of Γ^d.

Definition 3.4.1. *A skeleton vector field* on $\mathcal{C}^*(\Gamma^d)$ is a family $\chi = (\chi^v)_{v\in V}$ of vectors in $\mathbb{R}^F$ such that χ^v is tangent to Π_v for all $v \in F$. Alternatively, a skeleton vector field χ is a family $\chi = (\chi_\sigma^v)_{(v,\sigma)\in V\times F}$ such that $\chi_\sigma^v = 0$ for $\sigma \notin F_v$.

In this section, we study the skeleton vector fields associated to vector fields defined on polytopes.

Definition 3.4.2. Given a nondegenerate $X \in \mathfrak{X}^\omega(\Gamma^d)$, the skeleton character of X, $\chi = (\chi_\sigma^v)_{(v,\sigma)\in V\times F}$, (see Definition 3.2.2) is a skeleton vector field that we refer as the *skeleton vector field* of X.

We want to study the piecewise linear flows generated by skeleton vector fields on $\mathcal{C}^*(\Gamma^d)$.

Definition 3.4.3. Given a skeleton vector field χ, a vertex $v \in V$ is called

- χ*-attractive* if $\chi^v \in \Pi_v$,

- χ-*repelling* if $-\chi^v \in \Pi_v$,
- of *saddle type* otherwise.

The edges of Γ^d are also classified with respect to χ.

Definition 3.4.4. Let $\gamma \in E$ be an edge with end corners (v_i, σ_i) and (v_j, σ_j). The edge γ is called χ-*defined* if either $\chi_{\sigma_i}^{v_i} \chi_{\sigma_j}^{v_j} \neq 0$ or else $\chi_{\sigma_i}^{v_i} = \chi_{\sigma_j}^{v_j} = 0$. Moreover, we say that γ is

- a *flowing-edge* if $\chi_{\sigma_i}^{v_i} \chi_{\sigma_j}^{v_j} < 0$,
- a *neutral edge* if $\chi_{\sigma_i}^{v_i} = \chi_{\sigma_i}^{v_j} = 0$,
- an *attracting edge* if $\chi_{\sigma_i}^{v_i} < 0$ and $\chi_{\sigma_j}^{v_j} < 0$,
- a *repelling edge* if $\chi_{\sigma_i}^{v_i} > 0$ and $\chi_{\sigma_j}^{v_j} > 0$,
- a χ-*undefined* if non of the above happens.

Moreover, for flowing-edges we write $v_i \xrightarrow{\gamma} v_j$ whenever $\chi_{\sigma_i}^{v_i} < 0$ and $\chi_{\sigma_j}^{v_j} > 0$. The vertices v_i and v_j are called, respectively, the *source* of γ, denoted by $s(\gamma)$, and the *target* of γ, denoted by $t(\gamma)$.

Definition 3.4.5. The skeleton χ is said to be *regular* if it has no χ-undefined edges.

Definition 3.4.6. We denote by G_χ the directed graph $G_\chi = (V, E_\chi)$ where V is the vertex set of Γ^d, and E_χ is the set of all flowing-edges.

From now on, we will only consider regular skeleton vector fields.

Definition 3.4.7. We call *orbit* of χ to any continuous piecewise affine function $c : I \to \mathcal{C}^*(\Gamma^d)$, defined on some interval $I \subset \mathbb{R}$, such that

- $\frac{dc}{dt}(t) = \chi^v$ whenever $c(t)$ is interior to some sector Π_v, with $v \in V$,
- the set $\{\, t \in I :\ c(t) \in \mathcal{C}^*_{d-1}(\Gamma^d) \,\}$ is finite or countable.

Definition 3.4.8. Given a vertex $v \in V$, two flowing-edges $\gamma, \gamma' \in E_\chi$ and a face $\sigma' \in F$ such that $(v, \gamma', \sigma') \in C$ and $t(\gamma) = s(\gamma') = v$, we define the sector

$$\Pi_{\gamma,\gamma'} := \left\{\, x \in \mathrm{int}(\Pi_\gamma) :\ x_\sigma - \frac{\chi_\sigma^v}{\chi_{\sigma'}^v}\, x_{\sigma'} > 0, \quad \sigma \in F_v,\ \sigma \neq \sigma' \,\right\}.$$

Moreover, for each $v \in \Pi_{\gamma,\gamma'}$, we set

$$L_{\gamma,\gamma'}^{\chi}(x) = L_{\gamma,\gamma'}(x) := \left(x_\sigma - \frac{\chi_\sigma^v}{\chi_{\sigma'}^v}\, x_{\sigma'} \right)_{\sigma \in F}. \tag{3.5}$$

Next proposition relates the previous definition with the orbits of χ.

Proposition 3.4.9. *Given $v \in V$, $\gamma, \gamma' \in E_\chi$ and $\sigma' \in F$ such that $(v, \gamma', \sigma') \in C$ and $t(\gamma) = s(\gamma') = v$, the sector $\Pi_{\gamma,\gamma'}$ is the set of points $x \in \operatorname{int}(\Pi_\gamma)$ which are connected by the orbit segment $c(t) = x + t\chi^v$ $(t \geq 0)$ in Π_v to the point $x' = L^\chi_{\gamma,\gamma'}(x)$ in $\operatorname{int}(\Pi_{\gamma'})$.*

Proof. A proof of this proposition can be seen in [2]. □

Remark 3.4.10. *Given two corners (v, γ', σ') and (v, γ'', σ'') around the same vertex v, if v is χ-attractive or χ-repelling then it is not possible to connect any point of $\Pi_{\gamma'}$ to any point of $\Pi_{\gamma''}$ through a line parallel to the constant vector χ^v.*

Notice that the points in the boundary of $\Pi_{\gamma'}$ are in the intersection of more than two sectors Π_v with $v \in V$. Hence, if an orbit ends up in one of these points it might not be possible to continue it in a unique way. In the sequel we disregard this type of orbits.

We will extract information about the flow of the vector field X from the behaviour of the flow of the skeleton vector field χ. Instead of dealing with the flow of χ we introduce an associated Poincaré map π^χ.

Definition 3.4.11. Let $\Pi^\chi \subset \mathcal{C}^*_{d-1}(\Gamma^d)$ be the disjoint union

$$\Pi^\chi := \bigcup \left\{ \Pi_{\gamma,\gamma'} : \ t(\gamma) = s(\gamma'), \ (\gamma, \gamma') \in E_\chi \times E_\chi \right\} .$$

We define the *skeleton Poincaré map* $\pi^\chi : \Pi^\chi \subset \mathcal{C}^*_{d-1}(\Gamma^d) \to \mathcal{C}^*_{d-1}(\Gamma^d)$ by

$$\pi^\chi(x) := L_{\gamma,\gamma'}(x) \quad \text{whenever } x \in \Pi_{\gamma,\gamma'} .$$

Proposition 3.4.12. *Given $v \in V$, $\gamma, \gamma' \in E_\chi$ and $\sigma' \in F$ such that $(v, \gamma', \sigma') \in C$ and $t(\gamma) = s(\gamma') = v$, the linear map $L_{\gamma,\gamma'}$ is represented by the following matrix*

$$M_{\gamma,\gamma'} = \left(\delta_{\sigma,\sigma''} - \frac{\chi^v_\sigma}{\chi^v_{\sigma'}} \delta_{\sigma',\sigma''} \right)_{\sigma,\sigma'' \in F} ,$$

where $\delta_{\sigma,\sigma''}$ is the Kronecker delta symbol.

Proof. The proof follows by the definition of $L_{\gamma,\gamma'}$ in (3.5). □

Definition 3.4.13. A sequence of edges $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ is called a χ-*path* if ξ is a path of the graph G_χ, i.e., if

- $\gamma_j \in E_\chi$, for all $j = 0, 1, \ldots, m$,
- $t(\gamma_{j-1}) = s(\gamma_j)$, for all $j = 1, \ldots, m$.

The χ-path ξ is called a *cycle* when $\gamma_0 = \gamma_m$. The integer m is called the *length* of the path.

From now on we will write $\pi = \pi^\chi$ whenever the skeleton vector field χ is implicit from the context.

Definition 3.4.14. An *orbit segment* of $\pi = \pi^\chi$ is any finite sequence $\underline{x} = (x_0, x_1, \ldots, x_m)$ where $x_j \in \Pi^\chi$ and $x_j = \pi(x_{j-1})$ for all $j = 1, \ldots, m$. The unique χ-path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ such that for all $j = 1, \ldots, m$, $x_j \in \Pi_{\gamma_{j-1},\gamma_j}$, is called the *itinerary of* $\underline{x}$.

By definition, all edges in the itinerary of some orbit segment must be flowing-edges.

Definition 3.4.15. Given a χ-path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$, we define the *skeleton Poincaré map of* χ *along* ξ to be the mapping $\pi_\xi : \Pi_\xi \to \Pi_{\gamma_m}$

$$\pi_\xi := L_{\gamma_{m-1},\gamma_m} \circ \ldots \circ L_{\gamma_0,\gamma_1} \ ,$$

where

$$\begin{aligned} \Pi_\xi &:= \left\{ x \in \operatorname{int}(\Pi_{\gamma_0}) : \ \pi^j(x) \in \operatorname{int}(\Pi_{\gamma_j}) \ \text{ for all } \ j = 1, \ldots, m \right\} \\ &= \operatorname{int}(\Pi_{\gamma_0}) \cap \bigcap_{j=1}^{m} (L_{\gamma_{j-1},\gamma_j} \circ \ldots \circ L_{\gamma_0,\gamma_1})^{-1} \operatorname{int}(\Pi_{\gamma_j}) \ . \end{aligned}$$

Given a path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$, by Proposition 3.4.12 the matrix corresponding to the Poincaré map π_ξ is given by

$$M_\xi := M_{\gamma_{m-1},\gamma_m} \cdots M_{\gamma_0,\gamma_1} \ . \tag{3.6}$$

Any two paths ξ' and ξ'' where the end edge of ξ' is equal to the initial edge of ξ'' can be concatenated to form a new path ξ such that $\pi_\xi = \pi_{\xi''} \circ \pi_{\xi'}$.

We finish this section by introducing the concept of *structural set*, and the associated skeleton Poincaré map.

Definition 3.4.16. A non-empty set of edges $S \subset E_\chi$ is said to be a *structural set* for χ if

(i) any χ-cycle $\xi = (\gamma_0, \ldots, \gamma_m)$ contains an edge in S,

(ii) S is minimal w.r.t. (i).

Notice that the structural set S is in general not unique. The concept of structural set can be defined for general directed graphs. It corresponds to the homonym notion introduced by L. Bunimovich and B. Webb [4], but applied to the line graph[1].

[1]The line graph of a directed graph G, denoted by $L(G)$, is the graph whose vertices are the edges of G, and where $(\gamma, \gamma') \in E \times E$ is an edge of $L(G)$ if $t(\gamma) = s(\gamma')$.

Definition 3.4.17. We say that a χ-path $\xi = (\gamma_0, \ldots, \gamma_m)$ is a *branch* of S, or shortly an S-branch, if

(i) $\gamma_0, \gamma_m \in S$,

(ii) $\gamma_j \notin S$ for all $j = 1, \ldots, m-1$.

We denote by $\mathcal{B}_S(\chi)$ the set of all S-branches of G_χ.

Let $\Pi_S := \cup_{\xi \in \mathcal{B}_S(\chi)} \Pi_\xi$ where Π_ξ is defined inside Definition 3.4.15.

Definition 3.4.18. We define the S-Poincaré map to be $\pi_S : \Pi_S \to \Pi_S$ where $\pi_S(u) := \pi_\xi(u)$ for all $u \in \Pi_\xi$.

Proposition 3.4.19. *Given $X \in \mathfrak{X}^\omega(\Gamma^d)$, with a skeleton vector field χ, and a structural set $S \subset E_\chi$, if*

(i) χ is regular,

(ii) χ has no attracting or repelling edges,

(iii) all vertices are of saddle type,

then the following equality holds, up to a union of linear subspaces of dimension $d-2$,

$$\Pi_S = \bigcup_{\gamma \in S} \Pi_\gamma \ .$$

Proof. A proof of this proposition can be seen in [2]. □

3.5 Asymptotic Poincaré Maps

In this section, we study the asymptotic behaviour of the flow of a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ along the edges of Γ^d. The limit flow is described in terms of the skeleton Poincaré map π^χ associated with the skeleton vector field χ of X.

Definition 3.5.1. We say that a vector field $X \in \mathfrak{X}^\omega(\Gamma^d)$ is *regular* when it is nondegenerate and its skeleton vector field χ is regular (see Definition 3.4.5).

Throughout this section $X \in \mathfrak{X}^\omega(\Gamma^d)$ will denote a regular vector field and χ its skeleton vector field.

Given a corner $(v, \gamma, \sigma) \in C$, let

$$\Sigma_{v,\gamma} := (\Psi^X_{v,\varepsilon})^{-1}(\Pi_\gamma) \ .$$

This is a cross-section, transversal to X which intersects γ at a single point $z_{v,\gamma}$.

Given two flowing-edges $\gamma, \gamma' \in E_\chi$ such that $t(\gamma) = s(\gamma')$, let $D_{\gamma,\gamma'}$ be the set of points $x \in \mathrm{int}(\Sigma_{v,\gamma})$ such that the orbit $\{\varphi_X^t(x) \colon t \geq 0\}$ has a first transversal intersection with $\Sigma_{v,\gamma'}$. Then, the partial Poincaré map

$$P_{\gamma,\gamma'} : D_{\gamma,\gamma'} \subset \mathrm{int}(\Sigma_{v,\gamma}) \to \mathrm{int}(\Sigma_{v,\gamma'})$$

is defined by $P_{\gamma,\gamma'}(x) := \varphi_X^{\tau(x)}(x)$, where

$$\tau(x) := \min\{\, t > 0 : \ \varphi_X^t(x) \in \Sigma_{v,\gamma'} \,\} \, .$$

A flowing-edge $v \xrightarrow{\gamma} v'$ of a regular vector field X is a heteroclinic orbit with α-limit v and ω-limit v'. Given an edge $v \xrightarrow{\gamma} v'$, we denote by P_γ the Poincaré map from a small enough neighbourhood of $z_{v,\gamma}$ in $\Sigma_{v,\gamma}$ into $\Sigma_{v',\gamma}$.

Definition 3.5.2. Given a χ-path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$, the composition

$$P_\xi := (P_{\gamma_m} \circ P_{\gamma_{m-1},\gamma_m}) \circ \ldots \circ (P_{\gamma_1} \circ P_{\gamma_0,\gamma_1})$$

is referred as the *Poincaré map* of the vector field X along ξ. The domain of this composition is denoted by D_ξ.

Given a path ξ, the asymptotic behaviour of the Poincaré map P_ξ along ξ is given by the corresponding Poincaré map π_ξ of the skeleton vector field χ. More precisely we have

Proposition 3.5.3. *Given a χ-path $\xi = (\gamma_0, \ldots, \gamma_m)$ with $v_0 = s(\gamma_0)$ and $v_m = s(\gamma_m)$, let $\mathcal{U}_\xi^\varepsilon$ be the domain of the map $F_\xi^\varepsilon := \Psi_{v_m,\varepsilon}^X \circ P_\xi \circ (\Psi_{v_0,\varepsilon}^X)^{-1}$ from $\Pi_{\gamma_0}(\varepsilon)$ into $\Pi_{\gamma_m}(\varepsilon)$. Then*

$$\lim_{\varepsilon \to 0^+} \left(F_\xi^\varepsilon\right)_{|\mathcal{U}_\xi^\varepsilon} = \pi_\xi,$$

in the sense of Definition 3.3.5.

Proof. A proof of this proposition can be seen in [2]. $\square$

To encode the semi-global dynamics of the flow φ_X^t along the edges we will use Poincaré return maps to a system of cross-sections placed at the edges of a structural set (see Definition 3.4.16). Any orbit of the flow φ_X^t that shadows some heteroclinic circuit must intersect this cross-section system in a recurrent way.

Definition 3.5.4. Let $X \in \mathfrak{X}^\omega(\Gamma^d)$ be a regular vector field with a structural set, $S \subset E$. We define the S-Poincaré map $P_S : D_S \subset \Sigma_S \to \Sigma_S$ setting $\Sigma_S := \cup_{\gamma \in S} \Sigma_\gamma$, $D_S := \cup_{\xi \in \mathcal{B}_S(\chi)} D_\xi$ and $P_S(p) := P_\xi(p)$ for all $p \in D_\xi$. Notice that the domains D_ξ and $D_{\xi'}$ are disjoint for $\xi \neq \xi'$ in $\mathcal{B}_S(\chi)$.

By construction, the suspension of the S-Poincaré map $P_S : D_S \subset \Sigma_S \to \Sigma_S$ embeds (up to a time re-parametrization) in the flow of the vector field X. In this sense the dynamics of the map P_S encapsulates the qualitative behaviour of the flow φ_X^t of X along the edges of Γ^d. The following theorem is a corollary of Proposition 3.5.3.

Theorem 3.5.5. *Let $X \in \mathfrak{X}^\omega(\Gamma^d)$ be a regular vector field with skeleton vector field χ and a structural set $S \subset E_\chi$. Then*

$$\lim_{\varepsilon \to 0^+} \Psi_\varepsilon^X \circ P_S \circ (\Psi_\varepsilon^X)^{-1} = \pi_S$$

in the sense of Definition 3.3.5.

Proof. A proof of this theorem can be seen in [2]. □

Consider now a function $h : \Gamma^d \to \mathbb{R}$,

$$h(x) = -\sum_{j=1}^{N} c_j \, \log x_j \, , \tag{3.7}$$

to be a first integral of the flow of X, where c_j are constants.

Definition 3.5.6. We call *skeleton of the first integral h* to the function $\eta : \mathcal{C}^*(\Gamma^d) \to \mathbb{R}$ defined by

$$\eta(u) := \sum_{j=1}^{N} c_j \, u_j \, .$$

Proposition 3.5.7. *In these conditions the following limit holds*

$$\lim_{\varepsilon \to 0^+} \varepsilon^2 h \left(\Psi_\varepsilon^X\right)^{-1} = \eta \, .$$

Proof. Observing that $\left(\Psi_\varepsilon^X\right)^{-1}(z) = \left(e^{-\frac{\psi_\sigma(z)}{\varepsilon^2}}\right)_{\sigma \in F}$, the equality

$$\varepsilon^2 h \left(\Psi_\varepsilon^X\right)^{-1} = \eta$$

follows. □

Proposition 3.5.8. *If (3.7) is a first integral of the flow of X, the Poincaré map π_S preserves the function η, i.e.,*

$$\eta \circ \pi_S = \eta \, .$$

Proof. Since the flow is h-invariant, by Theorem 3.5.5 and Proposition 3.5.7,

$$\eta \circ \pi_S = \lim_{\varepsilon \to 0^+} \varepsilon^2 h \left(\Psi_\varepsilon^X\right)^{-1} \circ \Psi_\varepsilon^X \circ P_S \circ (\Psi_\varepsilon^X)^{-1} = \lim_{\varepsilon \to 0^+} \varepsilon^2 h \left(\Psi_\varepsilon^X\right)^{-1} = \eta \,.$$

□

3.6 Invariant Manifolds

Let $X \in \mathfrak{X}^\omega(\Gamma^d)$ be a regular vector field with skeleton χ, and consider a fixed χ-structural set S (see Definition 3.4.16).

As before, $\mathbb{1} \in \mathbb{R}^F_+$ stands for the vector with all coordinates equal to 1. We will write

$$\overline{u} := \mathbb{1} \cdot u = \sum_{\sigma \in F} u_\sigma \,.$$

Let $G = G_\chi$ denote the directed graph of χ (see Definition 3.4.6) and recall that $\mathcal{B}_S(\chi)$ represents the set of all S-branches of G (see Definition 3.4.17). Given a χ-path $\xi = (\gamma_1, \ldots, \gamma_m)$ such that the cone Π_ξ has non-empty interior, we define the $(d-2)$-simplex

$$\Delta_\xi^\chi := \{\, u \in \mathrm{int}(\Pi_\xi) : \ \overline{u} = 1 \,\}$$

and set $\Delta_S^\chi := \cup_{\xi \in \mathcal{B}_S(G)} \Delta_\xi^\chi$. Analogously, for each edge $\gamma \in E_\chi$ we define $\Delta_\gamma := \{\, u \in \mathrm{int}(\Pi_\gamma) : \ \overline{u} = 1 \,\}$ and set $\Delta_S := \cup_{\gamma \in S} \Delta_\gamma$.

Definition 3.6.1. Given a χ-path $\xi = (\gamma_1, \ldots, \gamma_m)$, the *projective Poincaré map along* ξ is the map $\hat{\pi}_\xi : \Delta_\xi^\chi \subset \Delta_{\gamma_1} \to \Delta_{\gamma_m}$ defined by

$$\hat{\pi}_\xi(u) := \pi_\xi(u) / \overline{\pi_\xi(u)} \,.$$

The *projective S-Poincaré map* is the application $\hat{\pi}_S : \Delta_S^\chi \subset \Delta_S \to \Delta_S$ defined by $\hat{\pi}_S(u) := \hat{\pi}_\xi(u)$ for all $u \in \Delta_\xi^\chi$.

Definition 3.6.2. A *periodic point* of $\hat{\pi}_S$ is any point $u \in \Delta_S^\chi$ such that $u = (\hat{\pi}_S)^n(u)$, for some $n \geq 1$. If the *period* n is minimum, denoting by ξ_j the unique S-branch such that $(\hat{\pi}_S)^j(u) \in \Delta_{\xi_j}^\chi$ for all $j = 0, 1, \ldots, n-1$, and concatenating these branches, we obtain a cycle ξ such that $u = \hat{\pi}_\xi(u)$. We refer to this cycle ξ as the *itinerary* of the periodic point u.

Definition 3.6.3. Let u be a periodic point of $\hat{\pi}_S$ whose itinerary is the cycle ξ. Then we say that u is an *eigenvector* of π_ξ, i.e., $\pi_\xi(u) = \lambda\, u$, and the number $\lambda = \lambda(u) > 0$ will be referred as the *eigenvalue* of u. Define $\sigma(u)$ to be the maximum ratio $\lambda / |\lambda'|$ where λ' ranges over all eigenvalues of π_ξ different from λ.

Given a periodic point u of $\hat{\pi}_S$, with itinerary ξ, the eigenvalues of $D(\hat{\pi}_S)_u$ are the ratios $\lambda(u)/\lambda'$ where λ' ranges over the eigenvalues of π_ξ associated to eigen-directions different from u. Then next proposition follows easily.

Proposition 3.6.4. *Let u be a periodic point of $\hat{\pi}_S$ with itinerary ξ.*

(a) If $\sigma(u) < 1$ then u is an attracting periodic point of $\hat{\pi}_S$,

(b) If $\sigma(u) > 1$ then u is a repelling periodic point of $\hat{\pi}_S$.

Let M be a smooth Riemann manifold, $\varphi^t : M \to M$ a flow (discrete or continuous) of class C^s $(s \geq 1)$ and $V \subseteq M$ an invariant submanifold for φ^t.

Definition 3.6.5. We say that V is *s-normally hyperbolic* for φ^t if the tangent bundle of M over V, $T_V M$, has a $D\varphi^t$-invariant splitting

$$T_V M = E^u \oplus TV \oplus E^s \,,$$

and there exists $\sigma > 1$ and $c > 0$ such that for all $p \in V$, for all $0 \leq k \leq s$, and for all $t \geq 0$ ($t \in \mathbb{Z}$ or $t \in \mathbb{R}$) :

(i) $m(D\varphi^t_{|E^u_p}) > c\,\sigma^t\, \|D\varphi^t_{|T_pV}\|^k$,

(ii) $\|D\varphi^t_{|E^s_p}\| < c^{-1}\,\sigma^{-t}\, m(D\varphi^t_{|T_pV})^k$,

where $m(A) = \min_{\|v\|=1} \|Av\|$ denotes the minimum expansion of a linear map A, and $\|A\| = \max_{\|v\|=1} \|Av\|$ is the operator norm of A. In case φ^t is smooth, and s can be taken arbitrarily large, we say that V is ∞*-normally hyperbolic.*

Because the skeleton Poincaré map is computable, we can run algorithms to find the structures invariant for the S-Poincaré map. If these structures are stable under small non-linear perturbations, they will persist as invariant geometric structures for the P_S-Poincaré map, and hence for the flow. Based on this stability principle we prove in the following theorem on the existence of normally hyperbolic stable and unstable manifolds of heteroclinic cycles.

Theorem 3.6.6. *Given $X \in \mathfrak{X}^\omega(\Gamma^d)$ regular with skeleton χ, denote by φ^t_X the flow of X. Given a periodic point u_0 of $\hat{\pi}_S$ with itinerary ξ, let C_ξ be the heteroclinic cycle determined by ξ. Assume u_0 has eigenvalue $\lambda(u_0) \neq 1$ and the linear map π_ξ has no other eigenvalue of absolute value λ. Then there exists a manifold $W \subset \Gamma^d$ such that*

(a) W is of class C^∞,

(b) W is either forward or backward invariant under the flow of X,

(c) W is ∞-normally hyperbolic,

(d) If $\sigma(u) < 1$ then W is normally contractive,

(e) If $\sigma(u) > 1$ then W is normally repelling,

(f) If $\lambda(u) > 1$ then W is forward invariant and for every $p \in W$

$$\lim_{t\to+\infty} d(\varphi_X^t(p), C_\xi) = 0 \,.$$

(g) If $\lambda(u) < 1$ then W is backward invariant and for every $p \in W$

$$\lim_{t\to+\infty} d(\varphi_X^{-t}(p), C_\xi) = 0 \,.$$

Proof. Let γ_0 be the first edge of the cycle ξ, and identify the hyperplane that contains Π_{γ_0} with $\mathbb{R}^{d-1}$. Consider the $(d-1)\times(d-1)$ matrix M_ξ that represents the linear map $\pi_\xi : \Pi_\xi \subset \Pi_{\gamma_0} \to \Pi_{\gamma_0}$, whose eigenvalues we denote by $\lambda_0, \lambda_1, \dots, \lambda_k$ with $\lambda_0 = \lambda > 0$ and $|\lambda_j| \neq \lambda$ for $j = 1, \dots, k$. Consider the following system of coordinates $\Phi : \Pi_{\gamma_0} \equiv \mathbb{R}_+^{d-1} \to \mathbb{R} \times \Delta^{d-2}$, $u \mapsto (r,\theta)$ defined by the relations

$$\begin{cases} e^r &= \overline{u} \\ \theta &= u/\overline{u} \end{cases} \,. \tag{3.8}$$

In these coordinates the map $f_\xi = \pi_\xi : \Pi_\xi \subset \Pi_{\gamma_0} \to \Pi_{\gamma_0}$ is given by

$$f_\xi(r,\theta) := \big(\, r + \log \overline{M_\xi \theta},\ \hat{\pi}_\xi(\theta)\,\big) \,.$$

The point u_0 is a fixed point of $\hat{\pi}_\xi$, with coordinates $r = 0$ and $\theta = u_0$. The line $V = \{\, (r,\theta) \in \mathbb{R} \times \Delta^{d-2} \ :\ \theta = u_0 \,\}$ is f_ξ-invariant. Notice that $m(D(f_\xi)|_{T_pV}) = 1 = \big\| D(f_\xi)|_{T_pV} \big\|$. Since u_0 is a hyperbolic fixed point of $\hat{\pi}_\xi$ there is a $D(\hat{\pi}_\xi)_{u_0}$-invariant decomposition $T_{u_0}\Delta^{d-2} = E^s \oplus E^u$ such that for all $p = (r, u_0) \in V$ and all $k \in \mathbb{N}$,

$$\| D(f_\xi)|_{E^s}(p) \| < 1 = m(D(f_\xi)|_{T_pV})^k \,,$$

and similarly

$$\big\| D(f_\xi)|_{T_pV} \big\|^k = 1 < m(D(f_\xi)|_{E^u(p)}) \,.$$

Because k is arbitrary, V is s-normally hyperbolic for any $s \in \mathbb{N}$. Consider now the half-line $V_+ = \{\, (r,\theta) \in \mathbb{R} \times \Delta^{d-2} \,:\, r \geq 0,\ \theta = u_0 \,\}$.

If $\lambda < 1$ then $\log \lambda = \log \overline{M_\xi\, u_0} < 0$, V_+ is backward invariant and overflowing for the map f_ξ.

If $\lambda > 1$ then $\log \lambda = \log \overline{M_\xi\, u_0} > 0$, V_+ is forward invariant and overflowing for the inverse map f_ξ^{-1}.

Because they are analogous, we only address the case $\lambda < 1$. Consider the family of mappings

$$f_{\xi,\varepsilon}(r,\theta) := \Phi \circ \Psi_\varepsilon \circ P_\xi \circ (\Psi_\varepsilon)^{-1} \circ \Phi^{-1}(r,\theta\,) \,,$$

with $\Psi_\varepsilon := \Psi_{v,\varepsilon}^X$ and where v is the source vertex of the first flowing-edge in ξ. In a neighbourhood of V_+ this map converges uniformly (as $\varepsilon \to 0^+$) to $f(r,\theta) := \Phi \circ \pi_\xi \circ \Phi^{-1}(r,\theta)$. Then by [9, Theorem 4.1], ignoring for now the fact that V_+ is not compact, for every small enough $\varepsilon > 0$ there exists a unique $f_{\xi,\varepsilon}$-invariant normally hyperbolic manifold V_+^ε of class C^∞ close to V_+. Setting $W_0 := (\Psi_\varepsilon)^{-1}\Phi^{-1}(V_+^\varepsilon)$, this manifold is normally hyperbolic and over-flowing under the Poincaré map P_ξ. For each $x \in W_0$, consider the orbit segment of x by the flow φ_X^t from time 0 to the first return time $\tau(x)$ to W_0,

$$\varsigma_x^\tau := \{\varphi_X^t(x) \,:\, t \in [0, \tau(x))\}.$$

Consider $W := \bigcup_{x\in W_0} \varsigma_x^\tau$. As W_0 is P_ξ-invariant in the sense that $W_0 \subseteq P_\xi(W_0)$, we have W a normally hyperbolic manifold invariant under the flow (in the same sense as W_0) of class C^∞. Statements (d)-(g) of W w.r.t. the flow follow from the corresponding properties of W_0 w.r.t. P_ξ and V_+^ε w.r.t. $f_{\xi,\varepsilon}$.

We now briefly explain how the non compactness of V_+, in the application of [9, Theorem 4.1], can be skirted. The idea is to compactify the mappings f_ξ and $f_{\xi,\varepsilon}$, or, more precisely, their domain $[0,+\infty) \times \Delta^{d-2}$. Notice Δ^{d-2} is already compact. Take any diffeomorphism $h : [0,1) \to [0,+\infty)$, e.g., $h(x) = x/(1-x)$, and define $\tilde f_\xi : [0,1] \times \Delta^{d-2} \to [-1,1] \times \Delta^{d-2}$ by

$$\tilde f_\xi(s,\theta) := \left(h^{-1}(h(s) + \log \overline{M_\xi \theta}),\, \hat\pi_\xi(\theta)\right) \;.$$

The compactification $\tilde f_{\xi,\varepsilon}$ is defined analogously, so that $\tilde f_\xi = \lim_{\varepsilon\to 0} \tilde f_{\xi,\varepsilon}$. Then $\tilde V_+ = [0,1] \times \Delta^{d-2}$ is a compact normally hyperbolic manifold such that $\tilde f_\xi(\tilde V_+) \supseteq \tilde V_+$, and in fact ∞-normally hyperbolic. To see this take any $s \in \mathbb{N}$. For each $n \geq 1$ let $\tilde V_+^{(n)} := \tilde f_\xi^{-n}(\tilde V_+)$. Because $p = (1, u_0)$ is a parabolic fixed point of $f|_{\tilde V_+}$, the derivatives of $\tilde f_\xi^n|_{\tilde V_+^{(n)}} : \tilde V_+^{(n)} \to \tilde V_+$ and its inverse tend to 1 as $n \to \infty$. Thus, letting $M_n = \|D\tilde f_\xi^n|_{\tilde V_+^{(n)}}\|$ and $m_n = m(D\tilde f_\xi^n|_{\tilde V_+^{(n)}})$, we can choose n large enough so that none of the eigenvalues λ_j^n, of $(\hat\pi_\xi)^n$ at u_0, lies inside the interval $[m_n^s, M_n^s]$, which implies that $\tilde V_+$ is s-normally hyperbolic. The rest of the argument goes without change. $\square$

If (f) holds, the manifold W is called the stable manifold of C_ξ, and denoted by $W^s_{loc}(\xi)$. If (g) holds, the manifold W is called the unstable manifold of C_ξ, and denoted by $W^u_{loc}(\xi)$.

Chapter 4

Polymatrix Replicators

Evolutionary Game Theory (EGT) originated from the work of John Maynard Smith and George R. Price [22] who applied the theory of strategic games developed by John von Neumann and Oskar Morgenstern [42] to evolution problems in Biology. Unlike Game Theory, EGT investigates the dynamical processes of biological populations.

As stated before, LV systems and the replicator equation are classes of o.d.e.s which plays a central role in EGT.

Another fundamental class of models in EGT are the *bimatrix replicators*, associated to bimatrix games, where two groups of individuals within a population, e.g. males and females, interact using different sets of strategies, say n strategies for the first group and m strategies for the second. There are no interactions within each group. The state of this model is a pair of probability vectors in the $(n+m-2)$-dimensional prism $\Delta^{n-1} \times \Delta^{m-1}$. A more detailed study of bimatrix replicators can be found in [16], for instance.

The theory of equilibria for n-person games was initiated in the years 1950s by John Nash [23]. A subclass of n-person games, referred as *polymatrix games*, where the payoff of each player is the sum of the payoffs corresponding to simultaneous contests with the opponents, was studied in the years 1970s by J. Howson [17] who attributes the concept to E. Yanovskaya [43] in 1968.

In polymatrix games, a population is divided in a finite number of groups, each one with a finite number of strategies. Interactions between individuals of any two groups are allowed, including the same group.

The differential equation associated to a polymatrix game, introduced in [3] and designated as *polymatrix replicator*, is defined in a finite product of simplices.

Polymatrix replicator generalizes the symmetric and asymmetric replicator equations. The replicator equation in dimension n is the case of the polymatrix replicator with one group defined in the simplex Δ^{n-1}. The asymmetric games for two "populations", one with n strategies and the other with m, is the case of the polymatrix replicator with two groups defined on the prism $\Delta^{n-1} \times \Delta^{m-1}$, where the submatrices corresponding to interactions within each group are null.

In this chapter we address essentially the study of polymatrix replicators. In section 4.1, we state the basic definitions and properties of the polymatrix replicators. In section 4.2, we describe the skeleton character of the vector field defined by the polymatrix replicator. K. Sigmund and J. Hofbauer [16] and W. Jansen [18] give sufficient conditions for *permanence* in the usual replicators. In section 4.3 we generalize these results to polymatrix replicators. In sections 4.4 and 4.5, we define the classes of conservative and dissipative polymatrix games, and study their properties. In particular, we extend to polymatrix replicators the concept of stably dissipativeness introduced by Redheffer *et al.* [27–29]. In this context we generalize a theorem of Oliva *et al.* [6] about the Hamiltonian nature of the limit dynamics in "stably dissipative" polymatrix replicators. Finally, in section 4.6, we give examples in the scope of this work's applicability.

4.1 Definitions and Properties

In this section we introduce the evolutionary polymatrix games. This class of systems contains both the replicator models and the evolutionary bimatrix games.

Consider a population divided in p groups, labelled by an integer α ranging from 1 to p. Individuals of each group $\alpha = 1, \ldots, p$ have exactly n_α strategies to interact with other members of the population. The strategies of a group α are labelled by positive integers j in the range

$$n_1 + \ldots + n_{\alpha-1} < j \leq n_1 + \ldots + n_\alpha \, .$$

We will write $j \in \alpha$ to mean that j is a strategy of the group α. Hence the strategies of all population are labelled by the integers $j = 1, \ldots, n$, where $n = n_1 + \cdots + n_p$.

This context can be formalized in the following definition.

Definition 4.1.1. A *polymatrix game* is an ordered pair $(\underline{n}, A)$ where $\underline{n} = (n_1, \ldots, n_p)$ is a list of positive integers, called the game type, and $A \in M_n(\mathbb{R})$ a square matrix of dimension $n = n_1 + \ldots + n_p$.

The matrix A is the *payoff matrix.* Given strategies $i \in \alpha$ and $j \in \beta$, in the groups α and β respectively, the entry $a_{ij} = a_{ij}^{\alpha\beta}$ represents an average payoff for an individual using the first strategy in some interaction with an individual using the second. Thus, the payoff matrix A can be decomposed into $n_\alpha \times n_\beta$ block matrices $A^{\alpha\beta}$, with entries $a_{ij}^{\alpha\beta}$, where α and β range from 1 to p.

Definition 4.1.2. Two polymatrix games $(\underline{n}, A)$ and $(\underline{n}, B)$ with the same type are said to be equivalent, and we write $(\underline{n}, A) \sim (\underline{n}, B)$, when for $\alpha, \beta = 1, \ldots, p$, all the rows of the block matrix $A^{\alpha\beta} - B^{\alpha\beta}$ are equal.

The *state* of the population is described by a point $x = (x^\alpha)_\alpha$ in the *prism*

$$\Gamma_{\underline{n}} := \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1} \subset \mathbb{R}^n ,$$

where $\Delta^{n_\alpha-1} = \{x \in \mathbb{R}^{n_\alpha} : \sum_{i=1}^{n_\alpha} x_i = 1\}$, and the entry $x_j = x_j^\alpha$ represents the usage frequency of strategy j within the group α.

Definition 4.1.3. Considering $d = n - p$, we denote by $\Gamma_{\underline{n}}$ the d-dimensional simple polytope whose affine support is the d-dimensional space $E^d \subset \mathbb{R}^n$ defined by the p equations

$$\sum_{i\in\alpha} x_i^\alpha = 1, \quad 1 \le \alpha \le p .$$

Definition 4.1.4. A polymatrix game $(\underline{n}, A)$ determines the following o.d.e. on the prism $\Gamma_{\underline{n}}$

$$\frac{dx_i^\alpha}{dt} = x_i^\alpha \left((A\,x)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha\beta} x^\beta \right), \qquad i \in \alpha,\ \alpha = 1, \ldots, p , \tag{4.1}$$

called a *polymatrix replicator system.*

This equation says that the growth rate of each frequency x_i^α is the difference between its payoff $(Ax)_i = \sum_{j=1}^n a_{ij} x_j$ and the average payoff of all strategies in the group α. The underlying vector field on $\Gamma_{\underline{n}}$ will be denoted by $X_{\underline{n},A}$, or simply by X_A when the type $\underline{n}$ is clear from the context. The flow $\varphi^t_{\underline{n},A}$ associated to $X_{\underline{n},A}$ leaves the prism $\Gamma_{\underline{n}}$ invariant. Hence, by compactness of $\Gamma_{\underline{n}}$, this flow is complete.

In the case $p = 1$, we have $\Gamma_{\underline{n}} = \Delta^{n-1}$ and (4.1) is the usual replicator equation associated to the payoff matrix A.

When $p = 2$, and $A^{11} = A^{22} = 0$, $\Gamma_{\underline{n}} = \Delta^{n_1-1} \times \Delta^{n_2-1}$ and (4.1) becomes the bimatrix replicator equation associated to the pair of payoff matrices (A^{12}, A^{21}).

The polytope $\Gamma_{\underline{n}}$ is parallel to the affine subspace

$$H_{\underline{n}} := \left\{ x \in \mathbb{R}^n : \ \sum_{j\in\alpha} x_j = 0, \ \text{ for } \ \alpha = 1, \ldots, p \right\} . \tag{4.2}$$

For each $\alpha = 1, \ldots, p$, we denote by $\pi_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ the projection

$$x \mapsto y, \quad y_i := \left\{ \begin{array}{lll} x_i & \text{if} & i \in \alpha \\ 0 & \text{if} & i \notin \alpha \end{array} \right. . \tag{4.3}$$

Lemma 4.1.5. *Given a matrix $C \in M_n(\mathbb{R})$, the following statements are equivalent:*

(a) $C^{\alpha\beta}$ has equal rows, for all $\alpha, \beta \in \{1, \ldots, p\}$,

(b) $Cx \in H_{\underline{n}}^\perp$, for all $x \in \mathbb{R}^n$.

Moreover, if any of these conditions holds then $X_{\underline{n},C} = 0$ on $\Gamma_{\underline{n}}$.

Proof. Assume (a). Since $H_{\underline{n}}^\perp$ is spanned by the vectors $\pi_\alpha(\mathbb{1})$ with $\alpha = 1, \ldots, p$, we have $v \in H_{\underline{n}}^\perp$ iff $v_i = v_j$ for all $i, j \in \alpha$. Because all rows of C in the group α are equal, we have $(Cx)_i = (Cx)_j$ for all $i, j \in \alpha$. Hence item (b) follows.

Next assume (b). For all $i \in \alpha$, with $\alpha \in \{1, \ldots, p\}$, $Ce_i \in H_{\underline{n}}^\perp$, which implies that $c_{i,k} = c_{j,k}$ for all $j \in \alpha$. This proves (a).

If (a) holds, then for any $\alpha \in \{1, \ldots, p\}$, $i, j \in \alpha$ and $k = 1, \ldots, n$, we have $c_{ik} = c_{jk}$. Hence for any $x \in \Gamma_{\underline{n}}$, and $i, j \in \alpha$ with $\alpha \in \{1, \ldots, p\}$, $(C\,x)_i = (C\,x)_j$, which implies that $X_{\underline{n},C} = 0$ on $\Gamma_{\underline{n}}$. □

Proposition 4.1.6. *Given two polymatrix games $(\underline{n}, A)$ and $(\underline{n}, B)$ with the same type $\underline{n}$, $X_{\underline{n},A} = X_{\underline{n},B}$ on $\Gamma_{\underline{n}}$ if $(\underline{n}, A) \sim (\underline{n}, B)$.*

Proof. Follows from Lemma 4.1.5 and the linearity of the correspondence $A \mapsto X_{\underline{n},A}$. □

Given a polymatrix game $(\underline{n}, A)$ the following proposition characterizes the equilibria of the associated polymatrix replicator.

Proposition 4.1.7. *Given a polymatrix game $(\underline{n}, A)$, a point $q \in \mathrm{int}(\Gamma_{\underline{n}})$ is an equilibrium of $X_{\underline{n},A}$ if and only if $(A\,q)_i = (A\,q)_j$ for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.*

Proof. Suppose that $q \in \mathrm{int}\,(\Gamma_{\underline{n}})$ is an equilibrium point of $X_{\underline{n},A}$. Then, for all $\alpha = 1 \ldots, p$ and every $i \in \alpha$,

$$q_i \left((A\,q)_i - \sum_{\beta=1}^{p} (q^\alpha)^T A^{\alpha\beta} q^\beta \right) = 0\,.$$

Since $q \in \mathrm{int}\,(\Gamma_{\underline{n}})$,

$$(A\,q)_i = \sum_{\beta=1}^{p} (q^\alpha)^T A^{\alpha\beta} q^\beta\,.$$

Hence $(Aq)_i = (Aq)_j$, for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.

Suppose now that $q \in \mathrm{int}\,(\Gamma_{\underline{n}})$ such that $(Aq)_i = (Aq)_j$, for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$. Then, for each $i \in \alpha$,

$$\begin{aligned} q_i \left((A\,q)_i - \sum_{\beta=1}^{p} (q^\alpha)^T A^{\alpha\beta} q^\beta \right) &= q_i \left((A\,q)_i - \sum_{j\in\alpha} q_j (A\,q)_j \right) \\ &= q_i \left((A\,q)_i - \underbrace{\sum_{j\in\alpha} q_j}_{=1} (A\,q)_i \right) \\ &= 0\,. \end{aligned}$$

□

4.2 Polymatrix Skeleton

Consider the polymatrix game $(\underline{n}, A)$ whose associated polymatrix replicator is defined on the d-dimensional prism $\Gamma_{\underline{n}}$. The *defining functions* of $\Gamma_{\underline{n}}$ are $f_i : E^d \to \mathbb{R}$, $f_i(x) := x_i$, with $i \in \alpha$ and $\alpha = 1, \ldots, p$.

We define

- $V_{\underline{n}} := V(\Gamma_{\underline{n}})$ the set of the vertices of $\Gamma_{\underline{n}}$;
- $E_{\underline{n}} := E(\Gamma_{\underline{n}})$ the set of the edges of $\Gamma_{\underline{n}}$;
- $F_{\underline{n}} := F(\Gamma_{\underline{n}})$ the set of the faces of $\Gamma_{\underline{n}}$;
- $F_{\underline{n},v} := F_v(\Gamma_{\underline{n}})$ the set of the faces of $\Gamma_{\underline{n}}$ containing the vertex v.

The polytope $\Gamma_{\underline{n}}$ has exactly $\prod_{\alpha=1}^{p} n_\alpha$ vertices

$$V_{\underline{n}} = \{\, e_{i_1} + \cdots + e_{i_p} :\ i_\alpha \in \alpha \ \text{ for } \ \alpha = 1, \ldots, p \,\}\,,$$

where the vectors e_{i_α} stand for the canonical basis of $\mathbb{R}^n$.

The polytope $\Gamma_{\underline{n}}$ has exactly n faces

$$F_{\underline{n}} = \{\, \sigma_j :\ j = 1, \ldots, n \,\}\,,$$

where $\sigma_j := \Gamma_{\underline{n}} \cap \{f_j = 0\}$.

For each vertex $v = e_{i_1} + \cdots + e_{i_p}$ the set $F_{\underline{n},v}$ of $d = n - p$ faces containing v is

$$F_{\underline{n},v} = \{\, \sigma_j :\ j \in \alpha,\ j \neq i_\alpha,\ \alpha = 1, \ldots, p \,\}\,.$$

A vertex v in $V_{\underline{n}}$ is given by $v = e_{i_1} + \cdots + e_{i_p}$. We can label the vertex v with $(i_1, \ldots i_p)$ where each $i_\alpha \in \alpha$ for $\alpha = 1, \ldots, p$. We define the set of vertex labels of $\Gamma_{\underline{n}}$ by

$$\mathbb{V}_{\underline{n}} := \{(i_1, \ldots, i_p) \,:\, i_\alpha \in \alpha \text{ for } \alpha = 1, \ldots, p\}.$$

Observing that the polytope has exactly n faces we can also label each face of $\Gamma_{\underline{n}}$ with $j = 1, \ldots, n$. We define the set of face labels of $\Gamma_{\underline{n}}$ by

$$\mathbb{F}_{\underline{n}} := \{1, \ldots, n\},$$

where a vertex v with label $(i_1, \ldots, i_p)$ belongs to the face j if and only if $j \neq i_\alpha$ for all $\alpha = \{1, \ldots, p\}$. So, given a vertex v with label $(i_1, \ldots, i_p)$, we define the set of the face labels of $\Gamma_{\underline{n}}$ that contain v as

$$\mathbb{F}_{\underline{n},v} := \{\, j \,:\, j \notin \{i_1, \ldots, i_p\}\,\}.$$

Beyond the incidence relations between the vertices and faces of $\Gamma_{\underline{n}}$ we have a natural bijection between vertices and its labels, as well between faces and its labels.

Proposition 4.2.1. *Given $\underline{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$, we have the following bijections:*

(1) $\Phi_V : \mathbb{V}_{\underline{n}} \to V_{\underline{n}}$, $(i_1, \ldots, i_p) \mapsto e_{i_1} + \cdots + e_{i_p}$*;*

(2) $\Phi_F : \mathbb{F}_{\underline{n}} \to F_{\underline{n}}$, $j \mapsto \sigma_j$ *, where* $\sigma_j = \{x \in \Gamma_{\underline{n}} \,:\, f_j(x) = 0\}$.

Moreover, for all $v \in V_{\underline{n}}$ and all $\sigma \in F_{\underline{n}}$ if

$$\Phi_V(i_1, \ldots, i_p) = v \quad \textit{and} \quad \Phi_F(j) = \sigma,$$

then

$$v \in \sigma \quad \Leftrightarrow \quad j \notin \{i_1, \ldots, i_p\}.$$

Proof. Observing that the polytope has exactly n faces we can identify each face σ of $\Gamma_{\underline{n}}$ with the unique $j \in \{1, \ldots, n\}$ such that

$$j \notin \bigcup_{(i_1,\ldots,i_p)\in\sigma} \{i_1, \ldots, i_p\}.$$

The proof of the statements follows from the corresponding definitions. $\square$

Given a vertex v we have the corresponding sector in $\mathbb{R}^F_+$ defined by

$$\Pi_v = \{(x_i)_{i\in F_{\underline{n}}} \in \mathbb{R}^F_+ \,:\, x_i = 0 \text{ if } i \notin \mathbb{F}_{\underline{n},v}\}.$$

Analogously, given an edge $\gamma = (v', v'')$, the corresponding sector in $\mathbb{R}_+^F$ is defined by

$$\Pi_\gamma = \{(x_i)_{i\in F_{\underline{n}}} \in \mathbb{R}_+^F \,:\, x_i = 0 \;\text{ if }\; i \notin \mathbb{F}_{\underline{n},v'} \cap \mathbb{F}_{\underline{n},v''}\} \,.$$

Let us fix a general face $i \in F_{\underline{n}}$. We can rewrite (4.1) as

$$\frac{dx_i}{dt} = x_i \left((A\,x)_i - \sum_{k\in\alpha} \sum_{j=1}^{n} a_{kj} x_k x_j \right), \qquad i \in \alpha,\; \alpha = 1, \ldots, p \,, \tag{4.4}$$

Considering the Taylor development of the right hand side of (4.4) in the variable x_i around zero, we have

$$\frac{dx_i}{dt} = A_1\, x_i + A_2\,(x_i)^2 + A_3\,(x_i)^3 \,,$$

where each coefficient A_ℓ is a polynomial in the remaining variables x_j with $j \neq i$. The first coefficient is

$$A_1 := \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} x_j - \sum_{\substack{k\in\alpha\\ k\neq i}} \sum_{\substack{j=1\\ j\neq i}}^{n} a_{kj} x_k x_j \,, \tag{4.5}$$

the second coefficient is

$$A_2 := a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} x_j - \sum_{\substack{k\in\alpha\\ k\neq i}} a_{ki} x_k \,, \tag{4.6}$$

and the third coefficient is $A_3 := -a_{ii}$.

Proposition 4.2.2. *Let X_A be the vector field associated to a polymatrix replicator* (4.4) *defined on $\Gamma_{\underline{n}}$. Then*

$$X_A \textit{ is regular } \Leftrightarrow \text{ (4.5) } \textit{does not vanish identically on } \Gamma_{\underline{n}} \,.$$

Furthermore, if X_A is regular, for every vertex $v \in V_{\underline{n}}$ with label $(j_1, \ldots, j_p)$, the skeleton character of X_A is the family $\chi = (\chi_i^v)_{(v,i)\in V_{\underline{n}}\times \mathbb{F}_{\underline{n}}}$ where

$$\chi_i^v = \begin{cases} \sum_{\beta=1}^{p} (a_{j_\alpha j_\beta} - a_{i j_\beta}) & \textit{if} \;\; i \in \mathbb{F}_{\underline{n},v} \\ 0 & , \;\; \textit{otherwise} \,. \end{cases}$$

Proof. The proof follows from Definition 3.2.2 and Definition 3.4.5. ∎

4.3 Permanence

In this section we generalize to polymatrix replicators the definition and some properties of *permanence* stated in the context of LV and replicator systems.

If an orbit in the interior of the state space converges to the boundary, this corresponds to extinction. Despite we give a formal definition of permanence in polymatrix replicators (see Definition 4.3.3), as we saw in the context of the LV systems and the replicator equation, we say that a system is *permanent* if there exists a compact set K in the interior of the state space such that all orbits starting in the interior of the state space end up in K. This means that the boundary of the state space is a repellor.

Consider a polymatrix game $(\underline{n}, A)$. Throughout the rest of the section, X will denote the associated vector field on the d-dimensional prism $\Gamma_{\underline{n}}$.

Proposition 4.3.1. *If the flow of X has no interior fixed point then it admits a strict Lyapunov function on* $\operatorname{int}(\Gamma_{\underline{n}})$*. In particular the system has no periodic orbits, and no α or ω-limits inside* $\operatorname{int}(\Gamma_{\underline{n}})$*.*

Proof. Consider the convex set

$$K = \{Ax \,:\, x \in \operatorname{int}(\Gamma_{\underline{n}})\},$$

the open convex set

$$\tilde{K} = \{ty \,:\, y \in K,\, t > 0\},$$

and the linear space $L \subset \mathbb{R}^n$ generated by the vectors $\{\pi_\alpha(\mathbb{1}) : \alpha = 1, \ldots, p\}$. If $\tilde{K} \cap L$ is non-empty there exists a point $q \in \operatorname{int}(\Gamma_{\underline{n}})$ and a number $t > 0$ such that $t\,(A\,q) \in L$, which implies that q is an equilibrium point of X. Thus, under the assumptions of the proposition, by Minkowski's Separating Theorem, there exists a linear hyperplane $H \subset \mathbb{R}^n$ that contains L and doesn't intersect $\tilde{K}$.

Let c be the unit normal to H that points to the half-space bounded by H and not containing $\tilde{K}$. Then $\langle c, A\,q\rangle < 0$ for all $q \in \operatorname{int}(\Gamma_{\underline{n}})$.

Consider now

$$V(x) = \sum_{\alpha=1}^{p} \sum_{i\in\alpha} c_i^\alpha \,\log x_i^\alpha\,.$$

Since $L \subset H = c^\perp$, for each α we have that $\sum_{i\in\alpha} c_i^\alpha = 0$. Therefore, differentiating V along the flow in $\operatorname{int}(\Gamma_{\underline{n}})$,

$$\frac{dV}{dt} = \sum_{\alpha=1}^{p} \sum_{i\in\alpha} c_i^\alpha \left((A\,x)_i - \pi_\alpha(x)^T A\,x\right) = \sum_{\alpha=1}^{p} \sum_{i\in\alpha} \langle c_i^\alpha, (A\,x)_i\rangle < 0\,,$$

which proves that V is a strict Lyapunov function. $\square$

The following result is a generalization of the average principle in LV systems (see Theorem 1.1.5) and replicator equation (see Theorem 1.5.6) to the polymatrix replicator systems.

Proposition 4.3.2 (Average Principle)**.** *Let $x(t) \in \operatorname{int}(\Gamma_{\underline{n}})$ be an interior orbit of the vector field X such that for some $\varepsilon > 0$ and some time sequence $T_k \to +\infty$, as $k \to +\infty$, one has*

(1) $d\left(x(T_k), \partial\Gamma_{\underline{n}}\right) \geq \varepsilon$ *for all* $k \geq 0$,

(2) $\displaystyle\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x(t)\, dt = q,$

(3) $\displaystyle\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \pi_\alpha\left(x(t)\right)^T A\, x(t)\, dt = a_\alpha \quad$ *for all* $\alpha \in \{1, \ldots, p\}$.

Then q is an equilibrium of X and $a_\alpha = \pi_\alpha(q)^T A\, q$, for all $\alpha \in \{1, \ldots, p\}$. Moreover,

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x(t)^T A\, x(t)\, dt = q^T A\, q\,.$$

Proof. Let $\alpha \in \{1, \ldots, p\}$ and $i, j \in \alpha$. Observe that from (2) we obtain

$$\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} (Ax)_i\, dt = (Aq)_i\,.$$

By (1) we have for all k, $\varepsilon < x_i^\alpha(T_k) < 1 - \varepsilon$. Hence

$$\begin{aligned}
(Aq)_i - (Aq)_j &= e_i^T Aq - e_j^T Aq \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \left(e_i^T Ax - e_j^T Ax\right)\, dt \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \left(\log \frac{x_i^\alpha(T_k)}{x_j^\alpha(T_k)} - \log \frac{x_i^\alpha(0)}{x_j^\alpha(0)}\right) = 0\,.
\end{aligned}$$

It follows that q is an equilibrium of X, and for all $i, j \in \alpha$, $\alpha = 1, \ldots, p$, $(Aq)_i = (Aq)_j = \pi_\alpha(q)^T A\, q$.

Finally, using (1)-(3),

$$
\begin{aligned}
0 &= \lim_{k\to+\infty} \frac{1}{T_k} \left(\log x_i^\alpha(T_k) - \log x_i^\alpha(0)\right) \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \frac{\frac{dx_i^\alpha}{dt}(t)}{x_i^\alpha(t)}\, dt \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \left((Ax)_i - \sum_{\beta=1}^{p} (x^\alpha)^T A^{\alpha\beta} x^\beta \right) dt \\
&= \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \big((Ax)_i - \pi_\alpha(x)^T A\, x \big)\, dt \\
&= (Aq)_i - \lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} \pi_\alpha(x)^T A\, x\, dt = (Aq)_i - a_\alpha\,,
\end{aligned}
$$

which implies that $a_\alpha = \pi_\alpha(q)^T A\, q$, and hence

$$
\lim_{k\to+\infty} \frac{1}{T_k} \int_0^{T_k} x^T A\, x\, dt = q^T A\, q\,.
$$

□

In the context of the polymatrix replicator systems we have a natural generalization of the definition of permanence in the replicator equation (see Definition 1.5.7).

Definition 4.3.3. Given a vector field X defined in $\Gamma_{\underline{n}}$, we say that the associated flow φ_X^t is *permanent* if there exists $\delta > 0$ such that $x \in \operatorname{int}(\Gamma_{\underline{n}})$ implies

$$
\liminf_{t\to+\infty} \; d\left(\varphi_X^t(x), \partial\Gamma_{\underline{n}}\right) \geq \delta\,.
$$

The following theorem generalizes Theorem 1.5.9 for polymatrix replicators.

Theorem 4.3.4. *Let $\Phi : \Gamma_{\underline{n}} \to \mathbb{R}$ be a smooth function such that $\Phi = 0$ on $\partial\Gamma_{\underline{n}}$ and $\Phi > 0$ on $\operatorname{int}(\Gamma_{\underline{n}})$. Assume there is a continuous function $\Psi : \Gamma_{\underline{n}} \to \mathbb{R}$ such that*

(1) for any orbit $x(t)$ in $\operatorname{int}(\Gamma_{\underline{n}})$, $\frac{d}{dt}\log\Phi(x(t)) = \Psi(x(t))$,

(2) for any orbit $x(t)$ in $\partial\Gamma_{\underline{n}}$, $\int_0^T \Psi(x(t))\, dt > 0$ for some $T > 0$.

Then the vector field X is permanent.

Proof. In the proof of Theorem 1.5.9, Sigmund and Hofbauer [16, Theorem 12.2.1] use an argument that is abstract and applicable to a much wider class of systems, including polymatrix replicator systems. □

Remark 4.3.5. *Sigmund and Hofbauer in* [16, Theorem 12.2.2] *prove that for the conclusion in Theorem 4.3.4 it is enough to check (2) for all ω-limit orbits in $\partial\Gamma_{\underline{n}}$. Thus, defining*

(2') for any ω-limit orbit $x(t)$ in $\partial\Gamma_{\underline{n}}$, $\int_0^T \Psi(x(t))\, dt > 0$ for some $T > 0$,

we have that condition (2') implies (2).

The k-dimensional face skeleton of $\Gamma_{\underline{n}}$, denoted by $\partial_k\Gamma_{\underline{n}}$, is the union of all j-dimensional faces of $\Gamma_{\underline{n}}$ with $j \leq k$. In particular, the edge skeleton of $\Gamma_{\underline{n}}$ is the union $\partial_1\Gamma_{\underline{n}}$ of all vertices and edges of $\Gamma_{\underline{n}}$.

The following theorem generalizes Jansen's Theorem [18] (see Theorem 1.5.8) for polymatrix replicators.

Theorem 4.3.6. *If there is a point $q \in \mathrm{int}\,(\Gamma_{\underline{n}})$ such that for all boundary equilibria $x \in \partial\Gamma_{\underline{n}}$,*

$$q^T A\, x > x^T A\, x\,, \tag{4.7}$$

then X is permanent.

Proof. The proof we present here is essentially an adaptation of the argument used in the proof of Theorem 13.6.1 in [16].

Take the given point $q \in \mathrm{int}\,(\Gamma_{\underline{n}})$ and consider $\Phi : \Gamma_{\underline{n}} \to \mathbb{R}$,

$$\Phi(x) := \prod_{i=1}^{n} (x_i)^{q_i} \;\; .$$

We can easily see that $\Phi = 0$ on $\partial\Gamma_{\underline{n}}$ and $\Phi > 0$ on $\mathrm{int}\,(\Gamma_{\underline{n}})$.

Consider now the continuous function $\Psi : \Gamma_{\underline{n}} \to \mathbb{R}$,

$$\Psi(x) := q^T A\, x - x^T A\, x\;.$$

We have that

$$\frac{d}{dt} \log \Phi(x(t)) = \Psi(x(t))\,.$$

It remains to show that for any orbit $x(t)$ in $\partial\Gamma_{\underline{n}}$, there is a $T > 0$ such that

$$\int_0^T \Psi\left(x(t)\right)\, dt > 0\,. \tag{4.8}$$

We will prove by induction in $k \in \mathbb{N}$ that if $x(t) \in \partial_k\Gamma_{\underline{n}}$ then (4.8) holds for some $T > 0$.

If $x(t) \in \partial_0\Gamma_{\underline{n}}$ then $x(t) \equiv q'$ for some vertex q' of $\Gamma_{\underline{n}}$. Since by (4.7) $\Psi(q') > 0$, (4.8) follows. Hence the induction step is true for $k = 0$.

Assume now that conclusion (4.8) holds for every orbit $x(t) \in \partial_{m-1}\Gamma_{\underline{n}}$, and consider an orbit $x(t) \in \partial_m\Gamma_{\underline{n}}$. Then there is an m-dimensional face $\sigma \in K^m(\Gamma_{\underline{n}})$ that contains $x(t)$. We consider two cases:

(i) If $x(t)$ converges to $\partial\sigma$, i.e., $\lim_{t\to+\infty} d\,(x(t), \partial\sigma) = 0$ then the ω-limit of $x(t)$, $\omega(x)$, is contained in $\partial\sigma$. By induction hypothesis, (4.8) holds for all orbits inside $\omega(x)$, and consequently, by Remark 4.3.5 the same is true about $x(t)$.

(ii) If $x(t)$ does not converge to $\partial\sigma$, there exists $\varepsilon > 0$ and a sequence $T_k \to +\infty$ such that $d\,(x(T_k), \partial\sigma) \geq \varepsilon$ for all $k \geq 0$. Let us write

$$\bar{x}(T) = \frac{1}{T}\int_0^T x\,dt \quad \text{and} \quad a_\alpha(T) = \frac{1}{T}\int_0^T \pi_\alpha(x)^T A\,x\,dt$$

for all $\alpha = 1, \ldots, p$. Since the sequences $\bar{x}(T_k)$ and $a_\alpha(T_k)$ are bounded, there is a subsequence of T_k, that we will keep denoting by T_k, such that $\bar{x}(T_k)$ and $a_\alpha(T_k)$ converge, say to q' and a_α, respectively, for all $\alpha = 1, \ldots, p$. By Proposition 4.3.2, q' is an equilibrium point in σ and $a_\alpha = \pi_\alpha(q')^T A\,q'$. Therefore

$$\frac{1}{T_k}\int_0^{T_k} \Psi\,(x(t))\;dt$$

converges to $q^T A\,q' - q'^T A\,q'$, which by (4.7) is positive. This implies (4.8) and hence proves the permanence of X. $\square$

4.4 Conservative Polymatrix

Definition 4.4.1. We say that any vector $q \in \mathbb{R}^n$ is a *formal equilibrium* of a polymatrix game $(\underline{n}, A)$ if

(a) $(Aq)_i = (Aq)_j$ for all $i, j \in \alpha$, and all $\alpha = 1, \ldots, p$,

(b) $\sum_{j\in\alpha} q_j = 1$ for all $\alpha = 1, \ldots, p$.

Observe that a formal equilibrium of a polymatrix game $(\underline{n}, A)$ is an equilibrium of the natural extension of $X_{\underline{n},A}$ to the affine subspace spanned by $\Gamma_{\underline{n}}$.

The matrix A induces a quadratic form $Q_A : H_{\underline{n}} \to \mathbb{R}$ defined by $Q_A(w) := w^T A\,w$, where $H_{\underline{n}}$ is defined in (4.2).

Definition 4.4.2. We call diagonal matrix of type $\underline{n}$ to any diagonal matrix $D = \mathrm{diag}(d_i)$ such that $d_i = d_j$ for all $i, j \in \alpha$ and $\alpha = 1, \ldots, p$.

Definition 4.4.3. A polymatrix game $(\underline{n}, A)$ is called *conservative* if it has a formal equilibrium q, and there exists a positive diagonal matrix D of type $\underline{n}$ such that $Q_{A\,D} = 0$ on $H_{\underline{n}}$.

Let $\{e_1,\ldots,e_n\}$ denote the canonical basis in $\mathbb{R}^n$, and $V_{\underline{n}}$ be the set of vertices of $\Gamma_{\underline{n}}$. Each vertex $v \in V_{\underline{n}}$ can be written as $v = e_{i_1} + \cdots + e_{i_p}$, with $i_\alpha \in \alpha$, $\alpha = 1,\ldots,p$, and it determines the set

$$\mathcal{V}_v := \{\, (i, i_\alpha) : \ i \in \alpha,\ i \neq i_\alpha,\ \alpha = 1,\ldots,p \,\}$$

of cardinal $n - p = \dim(H_{\underline{n}})$. Notice that $(i,j) \in \mathcal{V}_v$ if and only if $i \neq j$ are in the same group, and $v_j = 1$. Hence there is a natural identification

$$\mathcal{V}_v \equiv \{\, i \in \{1,\ldots,n\} : \ v_i = 0 \,\} .$$

For every vertex v, the family $\mathcal{B}_v := \{\, e_i - e_j : \ (i,j) \in \mathcal{V}_v \,\}$ is a basis of $H_{\underline{n}}$.

Lemma 4.4.4. *Given a vertex v of $\Gamma_{\underline{n}}$ and $x, q \in \Gamma_{\underline{n}}$, we have*

$$x - q = \sum_{(i,j)\in\mathcal{V}_v} (x_i - q_i)\,(e_i - e_j) .$$

Proof. Let v be a vertex of $\Gamma_{\underline{n}}$. Notice that for all $\alpha = 1,\ldots,p$,

$$-(x_{i_\alpha} - q_{i_\alpha}) = \sum_{\substack{i\neq i_\alpha \\ i\in\alpha}} (x_i - q_i) .$$

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{V}_v} (x_i - q_i)\,(e_i - e_j) &= \sum_{\alpha=1}^{p} \sum_{\substack{i\neq i_\alpha \\ i\in\alpha}} (x_i - q_i)(e_i - e_{i_\alpha}) \\
&= \sum_{\alpha=1}^{p} \sum_{\substack{i\neq i_\alpha \\ i\in\alpha}} (x_i - q_i)e_i - \sum_{\alpha=1}^{p} \sum_{\substack{i\neq i_\alpha \\ i\in\alpha}} (x_i - q_i)e_{i_\alpha} \\
&= \sum_{\alpha=1}^{p} \sum_{\substack{i\neq i_\alpha \\ i\in\alpha}} (x_i - q_i)e_i + \sum_{\alpha=1}^{p} (x_{i_\alpha} - q_{i_\alpha})e_{i_\alpha} \\
&= \sum_{\alpha=1}^{p} \sum_{i\in\alpha} (x_i - q_i)e_i = x - q
\end{aligned}$$

□

Given ordered pairs of strategies in the same group $(i,j), (k,\ell)$, i.e., $i, j \in \alpha$ and $k, \ell \in \beta$ for some $\alpha, \beta \in \{1,\ldots,p\}$, we define

$$A_{(i,j),(k,\ell)} := a_{ik} + a_{j\ell} - a_{i\ell} - a_{jk} .$$

Proposition 4.4.5. *The coefficients $A_{(i,j),(k,\ell)}$ do not depend on the representative A of the polymatrix game $(\underline{n}, A)$.*

Proof. Consider the matrix $B = A - C$, where matrix C is constant by blocks, and each block $C^{\alpha\beta} = \left(c_{ij}^{\alpha\beta}\right)_{i\in\alpha, j\in\beta}$ have equal rows for all $\alpha, \beta = 1, \ldots, p$. Let $(i,j) \in \alpha$ and $(k,\ell) \in \beta$, for some $\alpha, \beta = 1, \ldots, p$, we have that

$$\begin{aligned} B_{(i,j),(k,\ell)} &= b_{ik} + b_{j\ell} - b_{i\ell} - b_{jk} \\ &= a_{ik} - c_k^{\alpha\beta} + a_{j\ell} - c_\ell^{\alpha\beta} - a_{i\ell} + c_\ell^{\alpha\beta} - a_{jk} + c_k^{\alpha\beta} \\ &= A_{(i,j),(k,\ell)}\,, \end{aligned}$$

where $c_k^{\alpha\beta}$ is the constant entry on the k^{th}-column of $C^{\alpha\beta}$. □

Definition 4.4.6. Given $v \in V_{\underline{n}}$, we define $A_v \in M_{n-p}(\mathbb{R})$ the matrix with entries $A_{(i,j),(k,\ell)}$, indexed in $\mathcal{V}_v \times \mathcal{V}_v$, and $G(A_v)$ its associated graph (see Definition 1.1.2).

Proposition 4.4.7. *This matrix represents the quadratic form $Q_A : H_{\underline{n}} \to \mathbb{R}$ in the basis $\mathcal{B}_v$.*

Proposition 4.4.8. *Let q be a formal equilibrium of the polymatrix game $(\underline{n}, A)$. The quadratic form $Q_A : H_{\underline{n}} \to \mathbb{R}$ is given by*

$$Q_A(x-q) = \sum_{(i,j),(k,\ell)\in\mathcal{V}_v} A_{(i,j),(k,\ell)}\,(x_i - q_i)\,(x_k - q_k)\,. \tag{4.9}$$

Proof. Using Lemma 4.4.4, we have

$$\begin{aligned} Q_A(x-q) &= \left(\sum_{(i,j)\in\mathcal{V}_v} (x_i - q_i)(e_i - e_j)\right)^T A \left(\sum_{(k,\ell)\in\mathcal{V}_v} (x_k - q_k)(e_k - e_\ell)\right) \\ &= \sum_{(i,j),(k,\ell)\in\mathcal{V}_v} (e_i - e_j)^T A (e_k - e_\ell)(x_i - q_i)(x_k - q_k) \\ &= \sum_{(i,j),(k,\ell)\in\mathcal{V}_v} A_{(i,j),(k,\ell)}(x_i - q_i)(x_k - q_k)\,, \end{aligned}$$

□

Remark 4.4.9. *For all $w \in H_{\underline{n}}$, $Q_{D^{-1}A}(w) = Q_{AD}(D^{-1}w)$. Hence, because $D\,H_{\underline{n}} = H_{\underline{n}}$ for any diagonal matrix D of type $\underline{n}$*

(1) $Q_{A\,D}(w) = 0 \quad \forall\, w \in H_{\underline{n}} \quad \Leftrightarrow \quad Q_{D^{-1}A}(w) = 0 \quad \forall\, w \in H_{\underline{n}}$.

(2) $Q_{A\,D}(w) \leq 0 \quad \forall\, w \in H_{\underline{n}} \quad \Leftrightarrow \quad Q_{D^{-1}A}(w) \leq 0 \quad \forall\, w \in H_{\underline{n}}$.

Lemma 4.4.10. *Given $A \in M_n(\mathbb{R})$, if q is a formal equilibrium of $X_{\underline{n},A}$, and $D = \mathrm{diag}(d_i)$ is a positive diagonal matrix of type $\underline{n}$, then the derivative of*

$$h(x) = -\sum_{i=1}^{n} \frac{q_i}{d_i} \log x_i \tag{4.10}$$

along the flow of $X_{\underline{n},A}$ satisfies

$$\frac{dh}{dt}(x) = Q_{D^{-1}A}(x-q)\,.$$

Proof.

$$\begin{aligned}
\frac{dh}{dt}(x) &= -\sum_{i=1}^{n} \frac{q_i}{d_i} \frac{\frac{dx_i}{dt}}{x_i} = -\sum_{i=1}^{n} \frac{q_i}{d_i}((Ax)_i - \sum_{\beta=1}^{p}(x^\alpha)^t A^{\alpha\beta} x^\beta) \\
&= -q^T D^{-1} A\, x + x^T D^{-1} A\, x = (x-q)^T D^{-1} A\, x \\
&= (x-q)^T D^{-1} A\, x - \underbrace{(x-q)^T D^{-1} A\, q}_{=0} \\
&= (x-q)^T D^{-1} A\,(x-q) = Q_{D^{-1}A}(x-q)\,.
\end{aligned}$$

To explain the vanishing term notice that for all $\alpha \in \{1,\ldots,p\}$ and $i,j \in \alpha$, $(A\,q)_i = (A\,q)_j$, $d_i = d_j$ and $\sum_{k\in\alpha}(x_k - q_k) = 0$. □

Proposition 4.4.11. *If $(\underline{n}, A)$ is conservative, q is a formal equilibrium of $X_{\underline{n},A}$, and $D = \mathrm{diag}(d_i)$ is a positive diagonal matrix of type $\underline{n}$ such that $Q_{A\,D} = 0$ on $H_{\underline{n}}$, then (4.10) is a first integral for the flow of $X_{\underline{n},A}$, i.e., $\frac{dh}{dt} = 0$ along the flow of $X_{\underline{n},A}$.*

Moreover, $X_{\underline{n},A}$ is Hamiltonian w.r.t. a stratified Poisson structure on the prism $\Gamma_{\underline{n}}$, having h as its Hamiltonian function.

Proof. The first part follows from Lemma 4.4.10 and Remark 4.4.9. The second follows from [3, theorem 3.20]. □

4.5 Dissipative Polymatrix

Definition 4.5.1. A polymatrix game $(\underline{n}, A)$ is called *dissipative* if it has a formal equilibrium q, and there exists a positive diagonal matrix D of type $\underline{n}$ such that $Q_{A\,D} \le 0$ on $H_{\underline{n}}$.

Proposition 4.5.2. *If $(\underline{n}, A)$ is dissipative, q is a formal equilibrium of $X_{\underline{n},A}$, and $D = \mathrm{diag}(d_i)$ is a positive diagonal matrix of type $\underline{n}$ such that $Q_{A\,D} \le 0$*

on $H_{\underline{n}}$, then

$$h(x) = -\sum_{i=1}^{n} \frac{q_i}{d_i} \log x_i$$

is a Lyapunov decreasing function for the flow of $X_{\underline{n},A}$, i.e., $\frac{dh}{dt} \leq 0$ along the flow of $X_{\underline{n},A}$.

Proof. Follows from Lemma 4.4.10 and Remark 4.4.9. □

Definition 4.5.3. A polymatrix game $(\underline{n}, A)$ is called *admissible* if $(\underline{n}, A)$ is dissipative and for some vertex $v \in \Gamma_{\underline{n}}$ the matrix A_v is stably dissipative (see Definition 2.1.1).

Proposition 4.5.4. *Let q be a formal equilibrium of the polymatrix game $(\underline{n}, A)$. Given $v \in V_{\underline{n}}$ and $(i,j) \in \mathcal{V}_v$, then we have the "polymatrix quotient rule"*

$$\frac{d}{dt}\left(\frac{x_i}{x_j}\right) = \frac{x_i}{x_j} \sum_{(k,\ell)\in\mathcal{V}_v} A_{(i,j),(k,\ell)}\,(x_k - q_k)\,. \tag{4.11}$$

Proof. Let v be a vertex of $\Gamma_{\underline{n}}$, $(i,j) \in \mathcal{V}_v$, and q be a formal equilibrium. Using Lemma 4.4.4, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{x_i}{x_j}\right) &= \frac{x_i}{x_j}\left((Ax)_i - (Ax)_j\right) \\
&= \frac{x_i}{x_j}\Big((A(x-q))_i - (A(x-q))_j\Big) \\
&= \frac{x_i}{x_j} \sum_{(k,\ell)\in\mathcal{V}_v} (e_i - e_j)^T A(e_k - e_\ell)(x_k - q_k) \\
&= \frac{x_i}{x_j} \sum_{(k,\ell)\in\mathcal{V}_v} A_{(i,j),(k,\ell)}(x_k - q_k)\,.
\end{aligned}$$

□

Proposition 4.5.5. *If the dissipative polymatrix replicator associated to $(\underline{n}, A)$ has an equilibrium $q \in \operatorname{int}(\Gamma_{\underline{n}})$, then for any state $x_0 \in \operatorname{int}(\Gamma_{\underline{n}})$ and any pair of strategies i, j in the same group, the solution $x(t)$ of (4.1) with initial condition $x(0) = x_0$ satisfies*

$$\frac{1}{c} \leq \frac{x_i(t)}{x_j(t)} \leq c\,, \quad \textit{for all} \quad t \geq 0\,,$$

where $c = c(x)$ is a constant depending on x.

Proof. Notice that the Lyapunov function h in Proposition 4.5.2 is a proper function because $q \in \operatorname{int}(\Gamma_{\underline{n}})$. Given $x_0 \in \operatorname{int}(\Gamma_{\underline{n}})$, $h(x_0) = a$ for some constant $a > 0$. By Proposition 4.5.2 the compact set $K = \{x \in \operatorname{int}(\Gamma_{\underline{n}})\colon h(x) \leq$

$a\}$ is forward invariant by the flow of $X_{\underline{n},A}$. In particular, the solution of the polymatrix replicator with initial condition $x(0) = x_0$ lies in K. Hence the quotient $\frac{x_i}{x_j}$ has a minimum and a maximum in K. $\square$

Proposition 4.5.6. *Given a dissipative polymatrix game* $(\underline{n}, A)$*, if* $X_{\underline{n},A}$ *admits an equilibrium* $q \in \operatorname{int}(\Gamma_{\underline{n}})$ *then there exists a* $X_{\underline{n},A}$*-invariant foliation* $\mathcal{F}$ *on* $\operatorname{int}(\Gamma_{\underline{n}})$ *such that every leaf of* $\mathcal{F}$ *contains exactly one equilibrium point.*

Proof. Fix some vertex $v \in V_{\underline{n}}$. Recall that the entries of A_v are indexed in the set $\mathcal{V}_v \equiv \{\, i \in \{1, \ldots, n\} : \, v_i = 0 \,\}$. Given a vector $w = (w_i)_{i \in \mathcal{V}_v} \in \mathbb{R}^{n-p}$, we denote by $\bar{w}$ the unique vector $\bar{w} \in H_{\underline{n}}$ such that $\bar{w}_i = w_i$ for all $i \in \mathcal{V}_v$.

Let $\mathcal{E} \subset \mathbb{R}^n$ be the affine subspace of all points $x \in \mathbb{R}^n$ such that for all $\alpha = 1, \ldots, p$ and all $i, j \in \alpha$, $(A\,x)_i = (A\,x)_j$ and $\sum_{j \in \alpha} x_j = 1$. By definition $\mathcal{E} \cap \operatorname{int}(\mathbb{R}^n)$ is the set of interior equilibria of $X_{\underline{n},A}$. We claim that $\mathcal{E} = \{q + \bar{w} \colon w \in \operatorname{Ker}(A_v)\}$. To see this it is enough to remark that $w \in \operatorname{Ker}(A_v)$ if and only if

$$(A\bar{w})_i - (A\bar{w})_j = (e_i - e_j)^T A \bar{w} = 0\,, \quad \forall\, (i,j) \in \mathcal{V}_v.$$

Given $b \in \operatorname{Ker}(A_v^T)$, consider the function $g_b : \operatorname{int}(\mathbb{R}^n_+) \to \mathbb{R}$ defined by $g_b(x) := \sum_{j=1}^n \bar{b}_j \, \log x_j$. The restriction of g_b to $\Gamma_{\underline{n}}$ is invariant by the flow of $X_{\underline{n},A}$. Note that we can write

$$g_b(x) = \sum_{l=1}^n \bar{b}_l \, \log x_l = \sum_{(i,j) \in \mathcal{V}_v} b_i \, \log\left(\frac{x_i}{x_j}\right),$$

and differentiating g_b along the flow of $X_{\underline{n},A}$, by Proposition 4.5.5 we get

$$\dot{g}_b(x) = b^T A_v (x_k - q_k)_{k \in \mathcal{V}_v} = 0 \quad \text{ for all } \; x \in \Gamma_{\underline{n}}.$$

Fix a basis $\{b_1, \ldots, b_k\}$ of $\operatorname{Ker}(A_v^T)$, and define $g : \operatorname{int}(\mathbb{R}^n_+) \to \mathbb{R}^k$ by $g(x) := (g_{b_1}(x), \ldots, g_{b_k}(x))$. This map is a submersion. For that consider the matrix $B \in M_{k \times n}(\mathbb{R})$ whose rows are the vectors $\bar{b}_j$, $j = 1, \ldots, k$. We can write $g(x) = B \, \log x$, where $\log x = (\log x_1, \ldots, \log x_n)$. Hence $Dg_x = B\, D_x^{-1}$, where $D_x = \operatorname{diag}(x_1, \ldots, x_n)$, and because B has maximal rank, $\operatorname{rank}(B) = k$, the map g is a submersion. Hence g determines the foliation $\mathcal{F}$ whose leaves are the pre-images $g^{-1}(c) = \{g \equiv c\}$ with $c \in \mathbb{R}^k$.

Let us now explain why each leaf of $\mathcal{F}$ contains exactly one point in $\mathcal{E}$. Consider the vector subspace parallel to $\mathcal{E}$, $E_0 := \{\bar{w} \colon w \in \operatorname{Ker}(A_v)\}$. Because $(\underline{n}, A)$ is dissipative, $A_v \in M_{n-p}(\mathbb{R})$ is also dissipative, and by Proposition 1.4.4, $\operatorname{Ker}(A_v)$ and $\operatorname{Ker}(A_v^T)$ have the same dimension. Therefore $\dim(E_0) = k$. Let $\{c_1, \ldots, c_{n-k}\}$ be a basis of $E_0^\perp \subset \mathbb{R}^n$ and consider the matrix $C \in M_{(n-k) \times n}(\mathbb{R})$ whose rows are the vectors c_j, $j = 1, \ldots, n-k$. The matrix C provides the following description $\mathcal{E} = \{x \in \mathbb{R}^n \colon C\,(x - q) = 0\}$.

Consider the matrix $U = \left[\begin{array}{c} B \\ \hline C \end{array}\right] \in M_{n\times n}(\mathbb{R})$, which is nonsingular because by Proposition 1.4.4, $\mathrm{Ker}(A_v) = D\,\mathrm{Ker}(A_v^T)$, for some positive diagonal matrix D.

The intersection $g^{-1}(c) \cap \mathcal{E}$ is described by the system

$$x \in g^{-1}(c) \cap \mathcal{E} \quad \Leftrightarrow \quad \left\{ \begin{array}{l} B \log x = c \\ C(x - q) = 0 \end{array} \right. .$$

Considering $u = \log x$, this system becomes

$$\left\{ \begin{array}{l} B\, u = c \\ C(e^u - q) = 0 \end{array} \right. .$$

It is now enough to see that

$$\left\{ \begin{array}{l} B\, u = c \\ C(e^u - q) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} B\, u' = c \\ C(e^{u'} - q) = 0 \end{array} \right.$$

imply $u = u'$. By the mean value theorem, for every $i \in \{1, \ldots, n\}$ there is some $\tilde{u}_i \in [u_i, u_i']$ such that

$$e^{u_i} - e^{u_i'} = e^{\tilde{u}_i}(u_i - u_i'),$$

which in vector notation is to say that

$$e^u - e^{u'} = D_{e^{\tilde{u}}}(u - u') = e^{\tilde{u}} * (u - u').$$

Hence

$$\begin{aligned} \left\{ \begin{array}{l} B\,(u - u') = 0 \\ C(e^u - e^{u'}) = 0 \end{array} \right. \quad &\Leftrightarrow \quad \left\{ \begin{array}{l} B\,(u - u') = 0 \\ C\, D_{e^{\tilde{u}}}(u - u') = 0 \end{array} \right. \\ &\Leftrightarrow \quad \left[\begin{array}{c} B \\ \hline C\, D_{e^{\tilde{u}}} \end{array}\right](u - u') = 0 \\ &\Leftrightarrow \quad U \left[\begin{array}{c|c} I & 0 \\ \hline 0 & D_{e^{\tilde{u}}} \end{array}\right](u - u') = 0\,. \end{aligned}$$

Therefore, because $\left[\begin{array}{c|c} I & 0 \\ \hline 0 & D_{e^{\tilde{u}}} \end{array}\right]$ is non-singular, we must have $u = u'$.

Restricting $\mathcal{F}$ to $\mathrm{int}(\Gamma_{\underline{n}})$ we obtain a $X_{\underline{n},A}$-invariant foliation on $\mathrm{int}(\Gamma_{\underline{n}})$. Notice that the restriction $g|_{\mathrm{int}(\mathbb{R}^n)} : \mathrm{int}(\mathbb{R}^n) \to \mathbb{R}^k$ is invariant by the flow of $X_{\underline{n},A}$ because all its components are.

Since all points in $\mathrm{int}(\Gamma_{\underline{n}}) \cap \mathcal{E}$ are equilibria, each leaf of the restricted foliation contains exactly one equilibrium point. $\square$

Definition 4.5.7. We call attractor of the polymatrix replicator (4.1) to the following topological closure

$$\Lambda_{\underline{n},A} := \overline{\cup_{x\in\Gamma_{\underline{n}}}\omega(x)} \ ,$$

where $\omega(x)$ is the ω-limit of x by the flow $\{\varphi^t_{\underline{n},A} : \Gamma_{\underline{n}} \to \Gamma_{\underline{n}}\}_{t\in\mathbb{R}}$.

Proposition 4.5.8. *Given a dissipative polymatrix replicator associated to $(\underline{n}, A)$ with an equilibrium $q \in \operatorname{int}(\Gamma_{\underline{n}})$, and $D = \operatorname{diag}(d_i)$ is a positive diagonal matrix of type $\underline{n}$, we have that*

$$\Lambda_{\underline{n},A} \subseteq \{\, x \in \Gamma_{\underline{n}} \, : \, Q_{D^{-1}A}(x - q) = 0 \,\} \ .$$

Proof. By Theorem 2.2.3 the attractor $\Lambda_{\underline{n},A}$ is contained in the region where $\frac{dh}{dt}(x) = 0$. The conclusion follows then by Lemma 4.4.10. □

Given an admissible polymatrix replicator associated to $(\underline{n}, A)$ with an equilibrium $q \in \operatorname{int}(\Gamma_{\underline{n}})$, we say that a strategy i is of type $\bullet$ to mean that the following inclusion holds $\Lambda_{\underline{n},A} \subseteq \{x \, : \, x_i = q_i\}$. Similarly, we say that a strategy i is of type $\oplus$ to state that $\Lambda_{\underline{n},A} \subseteq \{x \, : \, X^i_{\underline{n},A}(x) = 0\}$. Otherwise, a species i is said to be of type $\circ$, meaning that we don't know nothing about species i (at that moment). Given two strategies i and j in the same group, we say that i and j are *related* when the orbits of the attractor $\Lambda_{\underline{n},A}$ are tangent to the foliation $\{\, \frac{x_i}{x_j} = \text{const.} \,\}$.

For any $v \in V_{\underline{n}}$ we denote by a^v_{ij} the entries of the matrix A_v. Moreover, given a subset $J \subset \{1, \ldots, n\}$ we will denote by $A_{v,J}$ the submatrix $A_{v,J} := (a^v_{ij})_{i,j\notin J}$.

With this terminology we have

Proposition 4.5.9. *Given an admissible polymatrix game $(\underline{n}, A)$ with an equilibrium $q \in \operatorname{int}(\Gamma_{\underline{n}})$ the following statements hold:*

(1) For any graph $G(A_v)$:

- *(a) if $J \subset \{1, \ldots, n\}$ is a set of strategies of type $\bullet$, the submatrix $A_{v,J}$ is stably dissipative, and $a^v_{ii} < 0$, for some $i \notin J$, then i is of type $\bullet$;*
- *(b) if $J \subset \{1, \ldots, n\}$ is a set of strategies of type $\bullet$ or $\oplus$, the submatrix $A_{v,J}$ is stably dissipative and $a^v_{ii} < 0$, for some $i \notin J$, then i is of type $\oplus$;*

(2) For any graph $G(A_v)$, if j and the unique strategy j' in the same group as j, with $v_{j'} = 1$, are both of type $\bullet$ or $\oplus$ then:

- *(c) if all neighbours of j but (possibly) l in $G(A_v)$ are of type $\bullet$, then l is also of type $\bullet$;*

(d) if all neighbours of j but (possibly) l in $G(A_v)$ are of type $\bullet$ or $\oplus$, then l is also of type $\oplus$;

(3) For any graph $G(A_v)$ with $v \in V_{\underline{n}}$:

(e) if all neighbours of a strategy j in $G(A_v)$ are of type $\bullet$ or $\oplus$, then j is related to the unique strategy j', in the same group as j, such that $v_{j'} = 1$.

Proof. The proof involves the manipulation of algebraic relations holding on the attractor. To simplify the terminology we will say that some algebraic relation holds to mean that it holds on the attractor.

Choose a positive diagonal matrix D of type $\underline{n}$ such that $Q_{AD} \leq 0$ on $H_{\underline{n}}$, and set $\tilde{A} := D^{-1}A$. By Lemma 2.1.11, for any $v \in V_{\underline{n}}$, the matrices A_v and $\tilde{A}_v$ have the same dissipative and stably dissipative character.

In the proofs of items (a) and (b) we will use Lemma 4.5.17 and Corollary 4.5.15 that appears below.

Proof of (a) when $J = \emptyset$: Given $v \in V_{\underline{n},A}$ such that A_v is stably dissipative, for any solution $x(t)$ of the polymatrix replicator in the attractor, we have that $Q_{\tilde{A}_v}(x(t) - q) = 0$. Hence, since $\tilde{A}_v$ is stably dissipative and $a^v_{ii} < 0$, by Lemma 2.1.2 follows that $x_i(t) = q_i$ on the attractor, which proves i is of type $\bullet$.

Proof of (a) when $J \neq \emptyset$: Given a subset J of strategies of type $\bullet$, define $c : J \to (0,1)$ by $c(j) = q_j$. Then $\Lambda_{\underline{n},A} \subset \Gamma_{\underline{n}}(c)$ (defined in (4.14) below). By Corollary 4.5.15 the affine homeomorphism $\psi_{\underline{n},c} : \Gamma_{\underline{n}}(c) \to \Gamma_{\underline{n}(c)}$ embeds $\Lambda_{\underline{n},A}$ in the polymatrix replicator $X_{\underline{n}(c),A(c)}$ on $\Gamma_{\underline{n}(c)}$.

Let $A(c)$ be the matrix in Definition 4.5.13 and $\check{v} = (v_j)_{j \notin J}$. It follows by Lemma 4.5.17 that there exists a positive diagonal matrix $\check{D}$ of type $\underline{n}(c)$ such that $(A(c)\check{D})_{\check{v}}$ is a submatrix of $(AD)_v$ indexed in $\mathcal{V}_v \setminus J$. On the other hand, since $A_{v,J}$ is a submatrix of A_v, then $A_{v,J}D_{\check{v}}$ is a submatrix of $A_v D_v = (AD)_v$ indexed in $\mathcal{V}_v \setminus J$. Therefore $A(c)_{\check{v}}\check{D}_{\check{v}} = (A(c)\check{D})_{\check{v}} = A_{v,J}D_{\check{v}}$, because these are submatrices of $(AD)_v$ with the same set of indexes.

Since $A_{v,J}$ is stably dissipative, by Lemma 2.1.11, the matrix $A(c)_{\check{v}}$ is stably dissipative. We can now apply the case (a), with $J = \emptyset$, to the polymatrix game $(\underline{n}(c), A(c))$, vertex $\check{v}$ and strategy i. By this property, the attractor of the $X_{\underline{n}(c),A(c)}$ is contained in some affine hyperplane $\{x_i = q'_i\}$. Hence, using the embedding $\psi_{\underline{n},c}$, the same holds for $\Lambda_{\underline{n},A}$. This shows that i is of type $\bullet$.

Proof of (b): Consider a subset J of strategies of type $\bullet$ or $\oplus$. Given $x \in \mathrm{int}(\Gamma_{\underline{n}})$ there exists a function $c : J \to (0,1)$ (depending on x) such that $\omega(x) \subset \Gamma_{\underline{n}}(c)$. By Corollary 4.5.15 the affine homeomorphism $\psi_{\underline{n},c} : \Gamma_{\underline{n}}(c) \to \Gamma_{\underline{n}(c)}$ embeds $\omega(x)$ in the polymatrix replicator $X_{\underline{n}(c),A(c)}$ on $\Gamma_{\underline{n}(c)}$.

Arguing as in the previous item we obtain that $A(c)_{\check{v}} = A_{v,J}D_{\check{v}}(\check{D}_{\check{v}})^{-1}$. Hence, as before, $A(c)_{\check{v}}$ is stably dissipative.

We can now apply the case (a), with $J = \emptyset$, to the polymatrix game $(\underline{n}(c), A(c))$, vertex $\check{v}$ and strategy i. By this property, the $X_{\underline{n}(c),A(c)}$-invariant set $\psi_{\underline{n},c}(\omega(x))$ is contained in some affine hyperplane $\{x_i = q_i'\}$, where q_i' depends on x. Hence the same holds for $\omega(x)$. This shows that i is of type $\oplus$.

Let j and j', the unique strategy in the same group as j with $v_{j'} = 1$, be strategies of type $\bullet$ or $\oplus$. Then $\frac{d}{dt}\left(\frac{x_j}{x_{j'}}\right) = 0$, i.e., by Proposition 4.5.4,

$$\sum_{(k,k')\in\mathcal{V}_v} A_{(j,j'),(k,k')}\,(x_k - q_k) = 0\,. \tag{4.12}$$

Observe that if j is of type $\bullet$ then $x_j = q_j$. Otherwise $a^v_{jj} = A_{(j,j'),(j,j')} = 0$.
Let l be a neighbour of j in the graph $G(A_v)$.
Proof of (c): If all neighbours of j but l in $G(A_v)$ are of type $\bullet$, then by (4.12)

$$A_{(j,j'),(l,l')}\,(x_l - q_l) = 0\,,$$

from which follows that $x_l = q_l$, because $A_{(j,j'),(l,l')} \neq 0$. Hence l is of type $\bullet$.
Proof of (d): If all neighbours of j but l in $G(A_v)$ are of type $\bullet$ or $\oplus$, then by (4.12)

$$A_{(j,j'),(l,l')}\,(x_l - q_l) = C\,,$$

for some constant C. Then x_l is constant, because $A_{(j,j'),(l,l')} \neq 0$. Hence l is of type $\oplus$.
Proof of (e): Suppose all neighbours of a strategy j are of type $\bullet$ or $\oplus$ and take j' the unique strategy, in the same group as j, such that $v_{j'} = 1$. By the polymatrix quotient rule (see Proposition 4.5.4),

$$\frac{d}{dt}\left(\frac{x_j}{x_{j'}}\right) = \frac{x_j}{x_{j'}} \sum_{(k,k')\in\mathcal{V}_v} A_{(j,j'),(k,k')}\,(x_k - q_k)\,.$$

Since all neighbours of j are of type $\bullet$ or $\oplus$ we obtain

$$\frac{d}{dt}\left(\frac{x_j}{x_{j'}}\right) = \frac{x_j}{x_{j'}}\,C\,,$$

for some constant C. Hence

$$\frac{x_j}{x_{j'}} = B_0\, e^{Ct}\,,$$

where $B_0 = \frac{x_j(0)}{x_{j'}(0)}$. By Proposition 4.5.5 we have that the constant C must be 0. Hence there exists a constant $B_0 > 0$ such that $\frac{x_j}{x_{j'}} = B_0$, which proves (e). □

Proposition 4.5.10. *If in a group α all strategies are of type $\bullet$ (resp. of type $\bullet$ or $\oplus$) except possibly for one strategy i, then i is also of type $\bullet$ (resp. of type $\oplus$).*

Proof. Suppose that in a group α all strategies are of type $\bullet$ or $\oplus$ except for one strategy i. We have that $x_k = c_k$, for some constant c_k, for each $k \neq i$. Thus,

$$x_i = 1 - \sum_{\substack{j \in \alpha \\ j \neq i}} x_j = 1 - \sum_{j = \bullet} x_j - \sum_{k = \oplus} x_k = 1 - \sum_{j = \bullet} q_j - \sum_{k = \oplus} c_k \,.$$

Hence i is of type $\oplus$.

If in a group α all strategies are of type $\bullet$, the proof is analogous. □

Proposition 4.5.11. *Assume that in a group α with n strategies, $n - k$ of them, with $0 \leq k < n$, are of type $\bullet$ or $\oplus$, and denote by S the set of the remaining k strategies. If the graph with vertex set S, obtained by drawing an edge between every pair of related strategies in S, is connected, then all strategies in S are of type $\oplus$.*

Proof. Since all strategies in $\alpha \setminus S$ are of type $\bullet$ or $\oplus$, for the strategies in S we have that

$$\sum_{i \in S} x_i = 1 - C \,, \tag{4.13}$$

where $C = \sum_{j \in \alpha \setminus S} x_j$.

Let G_S be the graph with vertex set S obtained drawing an edge between every pair of related strategies in S. Since G_S is connected we have that it contains a tree. Considering the $k - 1$ relations between the strategies in S given by that tree, we have $k - 1$ linearly independent equations of the form $x_i = C_{ij} x_j$ for pairs of strategies i and j in S, where C_{ij} is a constant. Together with (4.13) we obtain k linear independent equations for the k strategies in S, which implies that $x_i =$ constant, for every $i \in S$. This concludes the proof. □

Based on these facts we introduce a reduction algorithm on the set of graphs $\{\, G(A_v) \;:\; v \in V_{\underline{n}} \,\}$ to derive information on the strategies of an admissible polymatrix game $(\underline{n}, A)$.

In each step, we also register the information obtained about each strategy in what we call the "information set", where all strategies of the polymatrix are represented.

The algorithm is about labelling (or colouring) strategies with the "colours" $\bullet$ and $\oplus$. The algorithm acts upon all graphs $G(A_v)$ with $v \in V_{\underline{n}}$ as well as on the information set. It is implicit that after each rule application, the new labels (or colours) are transferred between the graphs $G(A_v)$ and the information set, that is, if in a graph $G(A_v)$ a strategy i has been coloured $i = \bullet$, then in all other graphs containing the strategy i, we colour it $i = \bullet$, as well on the information set.

Rule 1. *Initially, for each graph $G(A_v)$ such that A_v is stably dissipative colour in black ($\bullet$) any strategy i such that $a^v_{ii} < 0$. Colour in white ($\circ$) all other strategies.*

This rule follows from item (a) of Proposition 4.5.9 with $J = \emptyset$.

The reduction procedure consists in applying the following rules, corresponding to valid inferences rules. For each graph $G(A_v)$ such that for some set of strategies $J \subset \{1, \ldots, n\}$ of type $\bullet$ or $\oplus$ the submatrix $A_{v,J} = (a^v_{ij})_{i,j \notin J}$ is stably dissipative:

Rule 2. *If $J \subset \{1, \ldots, n\}$ is a set of strategies of type $\bullet$, the submatrix $A_{v,J}$ is stably dissipative and $a^v_{ii} < 0$, for some $i \notin J$, then colour $i = \bullet$.*

Rule 3. *If $J \subset \{1, \ldots, n\}$ is a set of strategies of type $\bullet$ or $\oplus$, the submatrix $A_{v,J}$ is stably dissipative and $a^v_{ii} < 0$, for some $i \notin J$, then colour $i = \oplus$.*

Rule 4. *If i has colour $\bullet$ or $\oplus$, the unique strategy i' in the same group as i with $v_{i'} = 1$ has also colour $\bullet$ or $\oplus$ and all neighbours of i in $G(A_{v,J})$ but j are $\bullet$, then colour $j = \bullet$.*

Rule 5. *If i has colour $\bullet$ or $\oplus$, the unique strategy i' in the same group as i with $v_{i'} = 1$ has also colour $\bullet$ or $\oplus$ and all neighbours of i but j in $G(A_v)$ are $\bullet$ or $\oplus$, then colour $j = \oplus$.*

For each graph $G(A_v)$ such that $v \in V_{\underline{n}}$:

Rule 6. *If i has colour $\circ$ and all neighbours of i in $G(A_v)$ are $\bullet$ or $\oplus$, then we put a link between strategies j and j' in the "information set", where j' is the unique strategy such that $v_{j'} = 1$ and j' is in the same group as j.*

The following rules can be applied to the set of all strategies of the polymatrix game.

Rule 7. *If in a group all strategies have colour $\bullet$ (respectively, $\bullet, \oplus$) except for one strategy i, then colour $i = \bullet$ (respectively, $i = \oplus$).*

Rule 8. *If in a group some strategies have colour $\bullet$ or $\oplus$, and the remaining strategies are related forming a connected graph, then colour with $\oplus$ all that remaining strategies.*

We define the *reduced information set* $\mathcal{R}(\underline{n}, A)$ as the $\{\bullet, \oplus, \circ\}$-coloring on the set of strategies $\{1, \ldots, n\}$, which is obtained by successive applications to the graphs $G(A_v)$, $v \in V_{\underline{n}}$, of the reduction rules 1-6, until they can no longer be applied.

Proposition 4.5.12. *Let* $(\underline{n}, A)$ *be an admissible polymatrix game, and consider the associated polymatrix replicator* (4.1) *with an interior equilibrium* $q \in \mathrm{int}(\Gamma_{\underline{n}})$.

1. *If all vertices of* $\mathcal{R}(\underline{n}, A)$ *are* $\bullet$ *then* q *is the unique globally attractive equilibrium.*

2. *If* $\mathcal{R}(\underline{n}, A)$ *has only* $\bullet$ *or* $\oplus$ *vertices then there exists an invariant foliation with a unique globally attractive equilibrium in each leaf.*

Proof. Item (1) is clear because if all strategies are of type $\bullet$ then for every orbit $x(t) = (x_1(t), \ldots, x_n(t))$ of (4.1), and every $i = 1, \ldots, n$, one has $\lim_{t\to+\infty} x_i(t) = q_i$.

Likewise, if $\mathcal{R}(\underline{n}, A)$ has only $\bullet$ or $\oplus$ vertices then every orbit of (4.1) converges to an equilibrium point, which depends on the initial condition. But by Proposition 4.5.6 there exists an invariant foliation $\mathcal{F}$ with a single equilibrium point in each leaf. Hence, the unique equilibrium point in each leaf of $\mathcal{F}$ must be globally attractive. $\square$

Given a type $\underline{n}$ with total length $n = n_1 + \ldots + n_p$ and a function $c : J \to (0, 1)$ on some subset $J \subset \{1, \ldots, n\}$, define

$$\Gamma_{\underline{n}}(c) := \Gamma_{\underline{n}} \cap \bigcap_{j\in J} \{x_j = c(j)\}. \tag{4.14}$$

The group partition in $\{1, \ldots, n\}$ induces a splitting in $\{1, \ldots, n\} \setminus J$, and we denote by $\underline{n}(c)$ the corresponding type of total length $n - |J|$. There is a natural affine homeomorphism $\psi_{\underline{n},c} : \Gamma_{\underline{n}}(c) \to \Gamma_{\underline{n}(c)}$ defined by $\psi_{\underline{n},c}(x) = (y_l)_{l\notin J}$, where for each group $\alpha = 1, \ldots, p$ and index $l \in \alpha$,

$$y_l = \left(1 - \sum_{j\in\alpha\cap J} c(j)\right)^{-1} x_l.$$

Next we define the *c-reduction* $(\underline{n}(c), A(c))$ of a polymatrix game $(\underline{n}, A)$ for any given *restriction data function* $c : J \to (0, 1)$.

Definition 4.5.13. Let c be a one-step restriction data function, i.e., $J = \{l\}$. If $l \in \alpha$ and $n_\alpha > 1$ we define $A(c) = (a_{ij}(c))$ to be the matrix indexed in $\{1, \ldots, n\} \setminus \{l\}$ with entries

$$a_{ij}(c) := \begin{cases} a_{ij} - a_{lj} & \text{if } j \notin \alpha \\ (a_{ij} - a_{lj})(1 - c(l)) + (a_{il} - a_{ll})c(l) & \text{if } j \in \alpha \setminus \{l\} \end{cases}. \tag{4.15}$$

while if $l \in \alpha$ and $n_\alpha = 1$ we set

$$a_{ij}(c) := a_{ij} + \frac{a_{il}}{p-1}. \tag{4.16}$$

When $J = \{l_1, \ldots, l_k\}$ with $l_1 < \ldots < l_k$ and $k \geq 2$ the matrix $A(c)$ indexed on $\{1, \ldots, n\} \setminus J$ is defined recursively through a sequence of one-step reductions, first in l_1, second in l_2 and so on.

Proposition 4.5.14. *Given a polymatrix game* $(\underline{n}, A)$ *and a restriction data function* $c : J \to (0,1)$*, if* $(x, X_{n,A}(x)) \in T(\Gamma_{\underline{n}}(c))$ *then*

$$X_{\underline{n}(c),A(c)}(\psi_{\underline{n},c}(x)) = (D\psi_{\underline{n},c})_x X_{n,A}(x) \; .$$

Proof. It is enough to address the case where $c : J \to (0,1)$ is a one-step restriction data function, i.e., $J = \{l\}$. We assume here that $l \in \alpha$ with $\alpha \in \{1, \ldots, p\}$ and $n_\alpha > 1$. The case $n_\alpha = 1$ follows from (4.16) in a simpler way. Because $(x, X_{\underline{n},A}(x)) \in T(\Gamma_{\underline{n}}(c))$ we have $x \in \Gamma_{\underline{n}} \cap \{x_l = c(l)\}$ and $X^l_{\underline{n},A}(x) = 0$.

Since $\sum_{\substack{j \in \alpha \\ j \neq l}} x_j = 1 - c(l)$, considering the change of variables $y = \psi_{\underline{n},c}(x)$

$$y_j = \begin{cases} \frac{x_j}{1-c(l)} & \text{if } \; j \in \alpha \setminus \{l\} \\ x_j & \text{if } \; j \notin \alpha \end{cases} \quad , \tag{4.17}$$

we have that $\sum_{j \in \alpha \setminus \{l\}} y_j = 1$.

By Proposition 4.1.6, we can assume $A = (a_{ij})$ has all entries equal to zero in row l, i.e., $a_{lj} = 0$ for all j. Thus we obtain

$$\frac{dx_l}{dt} = x_l \left(- \sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha\beta} x^\beta \right) \; .$$

Hence, making $x_l = c(l)$, the replicator equation (4.1) becomes

(i) if $i \in \alpha \setminus \{l\}$,

$$\frac{dx_i}{dt} = x_i \left(\sum_{\substack{j=1 \\ j \neq l}}^{n} a_{ij} x_j + a_{il} c(l) - \sum_{\substack{k \in \alpha \\ k \neq l}} \sum_{j=1}^{n} a_{kj} x_k x_j \right) \tag{4.18}$$

(ii) if $i \in \beta \neq \alpha$, the equation is essentially the same, with $x_l = c(l)$.

Observe that $\sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha\beta} x^\beta = 0$ because we are assuming that $x \in \Gamma_{\underline{n}} \cap \{x_l = c(l)\}$ and $X^l_{\underline{n},A}(x) = 0$.

Hence we can add

$$-\frac{c(l)}{1 - c(l)} \sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha\beta} x^\beta$$

to each equation for $\frac{dx_i}{dt}$, with $i \in \alpha \setminus \{l\}$, without changing the vector field $X_{\underline{n},A}$ at the points $x \in \Gamma_{\underline{n}} \cap \{x_l = c(l)\}$ where $X_{\underline{n},A}(x)$ is tangent to $\{x_l = c(l)\}$. So equation (4.18) becomes

$$\frac{dx_i}{dt} = x_i \left(\sum_{\substack{j=1 \\ j\neq l}}^{n} a_{ij}x_j + a_{il}c(l) - \frac{1}{1-c(l)} \sum_{\substack{k\in\alpha \\ k\neq l}} \sum_{j=1}^{n} a_{kj}x_k x_j \right) \tag{4.19}$$

Now, using the change of variables (4.17), equation (4.19) becomes

$$\frac{dy_i}{dt} = y_i \left(f_i - \sum_{\substack{k\in\alpha \\ k\neq l}} y_k f_k \right) \quad (i \in \alpha) , \tag{4.20}$$

where $f_i = \sum_{j\in\alpha\setminus\{l\}} a_{ij}(1-c(l))y_j + a_{il}c(l) + \sum_{j\notin\alpha} a_{ij}y_j$.

Let $\breve{\alpha} \equiv \alpha \setminus \{l\}$. Setting $a_{il}c(l) = a_{il}c(l)(\sum_{j\in\breve{\alpha}} y_j)$,

$$\frac{dy_i}{dt} = y_i \left(g_i - \sum_{k\in\beta} y_k g_k \right) , \quad i \in \beta,\ \beta \in \{1, \ldots, p\} , \tag{4.21}$$

where $g_i = \sum_{j\in\breve{\alpha}}(a_{ij}(1-c(l)) + a_{il}c(l))y_j + \sum_{j\notin\breve{\alpha}} a_{ij}y_j$, defines a new polymatrix game in dimension $n-1$. In fact, (4.21) is the replicator equation of the polymatrix game $(\underline{n}(c), A(c))$, where, since we have assumed that $a_{lj} = 0$ for all j, (4.15) becomes

$$a_{ij}(c) = \begin{cases} a_{ij} & \text{if } j \notin \breve{\alpha} \\ a_{ij}(1-c(l)) + a_{il}c(l) & \text{if } j \in \breve{\alpha} . \end{cases} \tag{4.22}$$

□

Corollary 4.5.15. *Let $(\underline{n}, A)$ be a polymatrix game, $c : J \to (0,1)$ a restriction data function and $\Lambda \subset \Gamma_{\underline{n}}(c)$ a $X_{\underline{n},A}$-invariant set. Then the affine homeomorphism $\psi_{\underline{n},c} : \Gamma_{\underline{n}}(c) \to \Gamma_{\underline{n}(c)}$ embeds Λ in the flow of $X_{\underline{n}(c),A(c)}$ on $\Gamma_{\underline{n}(c)}$. In other words, Λ lives on a lower dimension polymatrix replicator of type $\underline{n}(c)$.*

Lemma 4.5.16. *Given a polymatrix game $(\underline{n}, A)$ and a diagonal matrix D of type $\underline{n}$, we have*

$$(A\,D)_v = A_v\,D_v ,$$

where A_v is given in Definition 4.4.6 and D_v is the submatrix of D indexed in $\mathcal{V}_v = \{\, i \in \{1, \ldots, n\} :\ v_i = 0 \,\}$.

Proof. Given indices $i, k \in \mathcal{V}_v$, take j, resp. l, in the group of i, resp. k, such that $v_j = v_l = 1$.

Since D is of type $\underline{n}$ we have $d_k = d_l$. By Definition 4.4.6,

$$\begin{aligned} ((A\,D)_v)_{ik} = (A\,D)_{(i,j),(k,l)} &= a_{ik}d_k + a_{jl}d_l - a_{il}d_l - a_{jk}d_k \\ &= (a_{ik} + a_{jl} - a_{il} - a_{jk})\,d_k \\ &= A_{(i,j),(k,l)}\,d_k = (A_v\,D_v)_{ik}\,. \end{aligned}$$

□

Lemma 4.5.17. *Let $(\underline{n}, A)$ be a dissipative polymatrix game and D a positive diagonal matrix of type $\underline{n}$ such that $Q_{AD} \leq 0$ on $H_{\underline{n}}$. Given a restriction data function $c : J \to (0,1)$, and a vertex v of $\Gamma_{\underline{n}}$ such that $\mathcal{V}_v \supset J$, there exists a positive diagonal matrix $\check{D}$ of type $\underline{n}(c)$ such that $(A(c)\check{D})_{\check{v}}$ is the submatrix of $(AD)_v$ indexed in $\mathcal{V}_v \setminus J$ and $\check{v} = (v_j)_{j \notin J}$ is a vertex of $\Gamma_{\underline{n}(c)}$. In particular $(\underline{n}(c), A(c))$ is dissipative.*

Proof. It is enough to address the case where $c : J \to (0,1)$ is a one-step restriction data function, i.e., $J = \{l\}$.

We assume here that $l \in \alpha$ with $\alpha \in \{1, \ldots, p\}$ and $n_\alpha > 1$.

By Proposition 4.1.6, we can assume $A = (a_{ij})$ has all entries equal to zero in row l, i.e., $a_{lj} = 0$ for all j. The matrix $(AD)_v$ is indexed in $\mathcal{V}_v$. Since $l \in \mathcal{V}_v$ the vertex $\check{v}$ is determined by the exact same strategies as v.

As in the proof of Proposition 4.5.14 the matrix $A(c)$ is given by (4.22). Hence the matrix $A(c)_{\check{v}}$ has entries

$$a_{ij}^{\check{v}}(c) = \begin{cases} a_{ij}^v & \text{if } j \notin \check{\alpha} \\ a_{ij}^v(1 - c(l)) & \text{if } j \in \check{\alpha} \end{cases}.$$

Let us write $\tilde{d}_\beta = d_j$ for $j \in \beta$ and $\beta \in \{1, \ldots, p\}$ and denote by $\check{D}$ the positive diagonal matrix

$$\check{D} = \operatorname{diag}\left(\tilde{d}_1 I_1, \ldots, \frac{\tilde{d}_\alpha}{1 - c(l)} I_\alpha, \ldots, \tilde{d}_p I_p\right).$$

By Lemma 4.5.16, $(A(c)\check{D})_{\check{v}} = A(c)_{\check{v}}\,\check{D}_{\check{v}}$. All entries of this matrix are of the form $a_{ij}^v d_j$. Therefore $(A(c)\check{D})_{\check{v}}$ is the submatrix of $A_v D_v = (AD)_v$ indexed in $\mathcal{V}_v \setminus \{l\}$.

Let us prove now that $(\underline{n}(c), A(c))$ is dissipative. Since $Q_{AD} \leq 0$ on $H_{\underline{n}}$, by Proposition 4.4.8 we have that $w^T (AD)_v w \leq 0$ for all $w \in \mathbb{R}^{\mathcal{V}_v}$. Hence, because $(A(c)\check{D})_{\check{v}}$ is a submatrix of $(AD)_v$, we also have $w^T (A(c)\check{D})_{\check{v}} w \leq 0$ for all $w \in \mathbb{R}^{\mathcal{V}_{\check{v}}}$. This implies that $Q_{A(c)\check{D}} \leq 0$ on $H_{\underline{n}(c)}$.

By definition $(\underline{n}, A)$ has a formal equilibrium q. A simple calculation shows that $\psi_{\underline{n},c}(q)$ is a formal equilibrium of the the c-reduction $(\underline{n}(c), A(c))$. $\square$

Remark 4.5.18. *When $c : J \to (0,1)$ is a one-step restriction data function with $J = \{l\}$, and the group $\alpha \in \{1, \ldots, p\}$ that contains the strategy l satisfies $n_\alpha = 1$, the entries of the matrix $A(c)$ are given by* (4.16) *and $A(c)_{\check{v}}$ is a submatrix of A_v. In this case, although the assumptions of Lemma 4.5.17 are not satisfied (because $v_l = 1$) the conclusions of this lemma remain valid taking $\check{D} = \operatorname{diag}(d_j)_{j \neq l}$.*

Lemma 4.5.19. *Let $(\underline{n}, A)$ be a dissipative polymatrix game and $c : l \mapsto c(l)$ a one-step restriction data function. Given $v \in V_{\underline{n},A}$ such that A_v is stably dissipative and either (i) $l \in \mathcal{V}_v$, or else (ii) $l \notin \mathcal{V}_v = \emptyset$ and $\{l\}$ is a group of $(\underline{n}, A)$, then $A(c)_{\check{v}}$ is stably dissipative.*

Proof. By assumption $(\underline{n}, A)$ is dissipative. Hence there exists a positive diagonal matrix D of type $\underline{n}$ such that $Q_{AD} \leq 0$ on $\Gamma_{\underline{n}}$. By Lemma 4.5.17 and Remark 4.5.18, $(\underline{n}(c), A(c))$ is a dissipative polymatrix game.

Since A_v is stably dissipative, by Lemma 2.1.11 the matrix $(AD)_v = A_v D_v$ is also stably dissipative. Because of Lemma 4.5.17 (or Remark 4.5.18), $(A(c)\check{D})_{\check{v}}$ is a submatrix of $(AD)_v$. Hence Lemma 2.1.9 implies that $A(c)_{\check{v}}\check{D}_{\check{v}}$ is stably dissipative. Finally applying Lemma 2.1.11 again, we conclude that $A(c)_{\check{v}}$ is stably dissipative. $\square$

Proposition 4.5.14 and Lemma 4.5.19 allows us to generalize [6, Theorem 4.5], on the Hamiltonian nature of the limit dynamics, to the class of *admissible* polymatrix replicators.

Theorem 4.5.20. *Consider a polymatrix replicator $X_{\underline{n},A}$ on $\Gamma_{\underline{n}}$, and assume that $(\underline{n}, A)$ is admissible and has an equilibrium $q \in \operatorname{int}(\Gamma_{\underline{n}})$. Then the limit dynamics of $X_{\underline{n},A}$ on the attractor $\Lambda_{\underline{n},A}$ is embeddable in a Hamiltonian polymatrix replicator on some lower dimensional prism $\Gamma_{\underline{n}'}$.*

Proof. By definition there exists a vertex v of $\Gamma_{\underline{n}}$ such that $A_v = (a_{ij}^v)$ is stably dissipative. Applying Proposition 4.5.14 and Lemma 4.5.19 we obtain a new polymatrix replicator in lower dimension that is admissible.

We can iterate this process until the corresponding vertex $\check{v}$ in the polytope is such that, $a_{ii}^{\check{v}} = 0$ for all i with $\check{v}_i = 0$.

Let us denote the resulting polymatrix game by $(\underline{r}, A')$. By Proposition 2.3.7, for some positive diagonal matrix D' of type $\underline{r}$, $(A'\,D')_{\check{v}}$ is skew-symmetric. Hence $Q_{A'\,D'} = 0$ on $H_{\underline{r}}$, and by Definition 4.4.3 the polymatrix game $(\underline{r}, A')$ is conservative. Thus by Proposition 4.4.11 the vector field $X_{\underline{r},A'}$ is Hamiltonian. $\square$

4.6 Examples and Applications

In this section we present examples that illustrate some of the main results of this chapter as well as of chapter 3.

In example 4.6.1 we begin with an admissible polymatrix game to illustrate the reduction algorithm on the set of graphs $\{\, G(A_v) \ :\ v \in V_{\underline{n}} \,\}$ to derive information on the strategies of the polymatrix game. Since this polymatrix game is admissible, we also illustrate the procedure to obtain the Hamiltonian polymatrix replicator defined in a lower dimensional prism whose dynamics describes the limit dynamics of the admissible game.

In example 4.6.2 we present a dissipative polymatrix game that has a single attractive equilibrium in the interior of the phase space, and a heteroclinic cycle in the network formed by the flowing-edges of the polytope where it is defined. We analyse the global dynamics of the flow in the polytope, in particular the asymptotic dynamics along the heteroclinic cycle.

Finally, in example 4.6.3 we present a conservative polymatrix replicator. We will see that for all sufficiently large energy level this system dynamics is chaotic.

We will use the notation introduced in sections 4.1 and 4.2.

4.6.1 Reduction Algorithm

Consider the polymatrix replicator system associated to the polymatrix game $\mathcal{G} = ((3,2), A)$, where

$$A = \begin{bmatrix} -1 & 8 & -7 & 3 & -3 \\ -10 & -1 & 11 & 3 & -3 \\ 11 & -7 & -4 & -6 & 6 \\ -3 & -3 & 6 & 0 & 0 \\ 3 & 3 & -6 & 0 & 0 \end{bmatrix}.$$

We denote by $X_{\mathcal{G}}$ the vector field associated to this polymatrix replicator defined on the polytope $\Gamma_{(3,2)} = \Delta^2 \times \Delta^1$.

In this example we want to illustrate the reduction algorithm on the set of graphs $\{\, G(A_v) \ :\ v \in V_{(3,2)} \,\}$ to derive information on the strategies of the polymatrix game $\mathcal{G}$ as described in section 4.5. Moreover, we will see that this polymatrix game is admissible and that its limit dynamics on the attractor is described by a Hamiltonian polymatrix replicator defined in a lower dimensional prism.

In this game the strategies are divided in two groups, $\{1,2,3\}$ and $\{4,5\}$. The vertices of the phase space $\Gamma_{(3,2)}$ will be designated by pairs in $\{1,2,3\} \times \{4,5\}$, where the label (i,j) stands for the point $e_i + e_j \in \Gamma_{(3,2)}$. To simplify

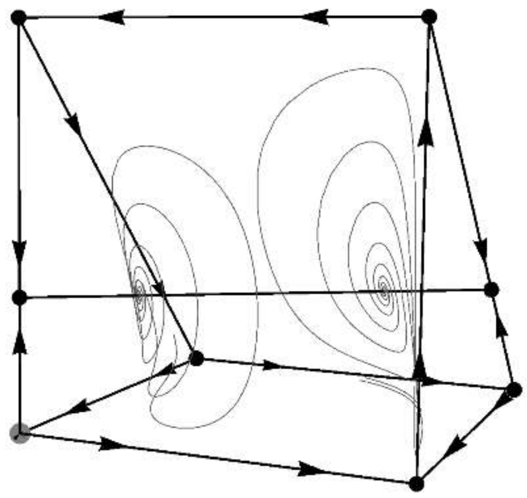


Figure 4.1: Four orbits in two different leafs of the polymatrix game $\mathcal{G}$.

$v_1 = (1,4)$	$v_2 = (1,5)$	$v_3 = (2,4)$	$v_4 = (2,5)$	$v_5 = (3,4)$	$v_6 = (3,5)$

Table 4.1: Vertex labels.

the notation we designate the prism vertices by the letters $v_1, \ldots, v_6$ according to table 4.1.

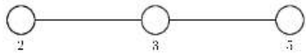
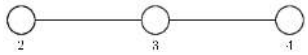
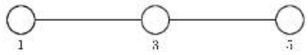
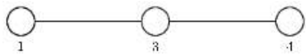
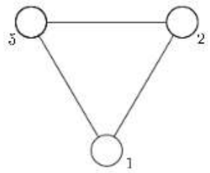
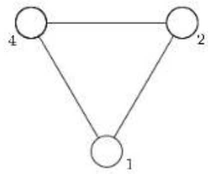
Vertex	A_v	$G(A_v)$
v_1	$\begin{bmatrix} 0 & 27 & 0 \\ -27 & -9 & 18 \\ 0 & -18 & 0 \end{bmatrix}$	
v_2	$\begin{bmatrix} 0 & 27 & 0 \\ -27 & -9 & -18 \\ 0 & 18 & 0 \end{bmatrix}$	
v_3	$\begin{bmatrix} 0 & -27 & 0 \\ 27 & -9 & 18 \\ 0 & -18 & 0 \end{bmatrix}$	
v_4	$\begin{bmatrix} 0 & -27 & 0 \\ 27 & -9 & -18 \\ 0 & 18 & 0 \end{bmatrix}$	
v_5	$\begin{bmatrix} -9 & 18 & -18 \\ -36 & -9 & -18 \\ 18 & 18 & 0 \end{bmatrix}$	
v_6	$\begin{bmatrix} -9 & 18 & 18 \\ -36 & -9 & 18 \\ -18 & -18 & 0 \end{bmatrix}$	

Table 4.2: Matrix A_v and its graph $G(A_v)$ for each vertex v.

The point $q \in \operatorname{int}\left(\Gamma_{(3,2)}\right)$ given by

$$q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right) ,$$

is an equilibrium of our polymatrix replicator $X_{\mathcal{G}}$. In particular it is also a formal equilibrium of $\mathcal{G}$ (see Definition 4.4.1).

The quadratic form $Q_A : H_{(3,2)} \to \mathbb{R}$ induced by matrix A is

$$Q_A(x) = -9\, x_3^2 \leq 0 ,$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in H_{(3,2)}$. By Definition 4.5.1, $\mathcal{G}$ is dissipative.

In table 4.2 we present for each vertex v in the prism the corresponding

Step	Rule	Vertex	Strategy	Group 1	Group 2
1	1	v_1, v_2, v_3, v_4	3	○ 1 ○ 2 ● 3	○ 4 ○ 5
2	6	v_4 (or v_5)	$4, 5$	○ 1 ○ 2 ● 3	○ 4 — ○ 5
3	8	–	$4, 5$	○ 1 ○ 2 ● 3	$\oplus$ 4 $\oplus$ 5
4	3	v_5	$1, 2$	$\oplus$ 1 $\oplus$ 2 ● 3	$\oplus$ 4 $\oplus$ 5

Table 4.3: Information set of all strategies (by group) of $\mathcal{G}$, where for each step, we mention the rule, the vertex (or vertices) and the strategy (or strategies) to which we apply the rule.

matrix A_v and graph $G(A_v)$.

Considering vertex $v_1 = (1, 4)$ for instance, by Proposition 2.3.7, we have that matrix A_{v_1} is stably dissipative. Hence, by Definition 4.5.3, $\mathcal{G}$ is admissible.

Table 4.3 represents the steps of the reduction procedure applied to $\mathcal{G}$. Let us describe it step by step:

(Step 1) Initially, considering the vertices v_1, v_2, v_3 and v_4 we apply *rule 1* to the corresponding graphs $G(A_{v_1}), G(A_{v_2}), G(A_{v_3})$ and $G(A_{v_4})$, and we colour in black ($\bullet$) strategy 3. We obtain the graphs depicted in column "Step 1" in table 4.4;

(Step 2) In this step we can consider vertex v_4 (or v_5) to apply *rule 6*. Hence, we put a link between strategies 4 and 5 in group 2;

(Step 3) In this step we apply *rule 8* to strategies 4 and 5, and we colour with $\oplus$ that strategies. We obtain the graphs depicted in column "Step 3" in table 4.4;

(Step 4) Finally, considering vertex v_5, we apply *rule 3* to the corresponding graph $G(A_{v_5})$, with $J = \{5\}$, and we colour with $\oplus$ strategy 1 and 2. We obtain the graphs depicted in column "Step 4" table 4.4.

Since $\mathcal{G}$ is admissible and has an equilibrium $q \in \operatorname{int}\left(\Gamma_{(3,2)}\right)$, by Theorem 4.5.20 we have that its limit dynamics on the attractor $\Lambda_{\mathcal{G}}$ is described

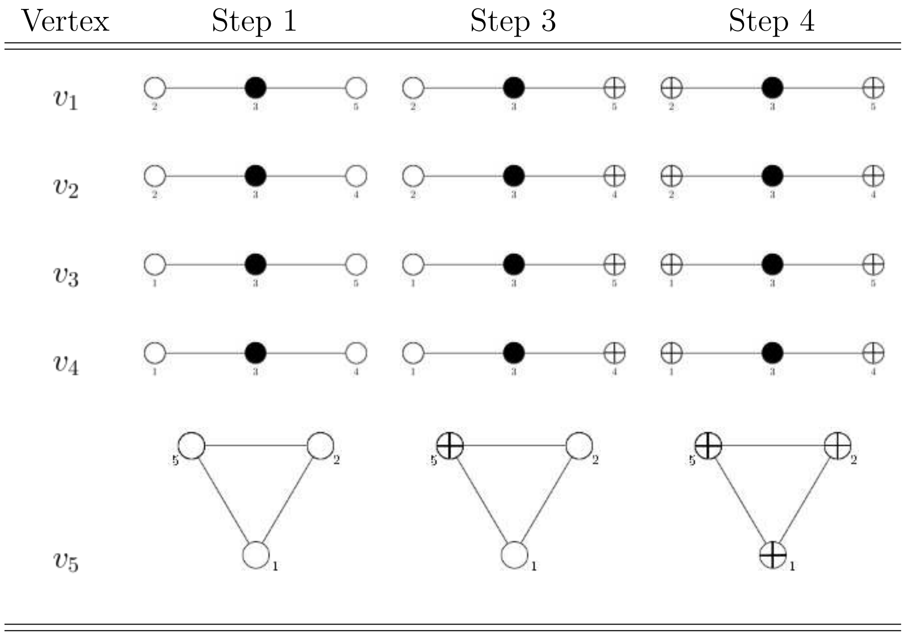


Table 4.4: The graphs obtained in each step of the reduction algorithm for $\mathcal{G}$.

by a Hamiltonian polymatrix replicator in a lower dimensional prism. Considering the strategy 3 in group 1, by Definition 4.5.13 we obtain the $(q, 3)$-reduction $((2, 2), A(3))$ where $\tilde{A} := A(3)$ is the matrix

$$\tilde{A} = \begin{bmatrix} -9 & 9 & 9 & -9 \\ -9 & 9 & 9 & -9 \\ -6 & 6 & 6 & -6 \\ -6 & 6 & 6 & -6 \end{bmatrix} .$$

Consider now the polymatrix replicator associated to the game $\tilde{\mathcal{G}} = \left((2, 2), \tilde{A}\right)$, which is equivalent to the trivial game $((2, 2), 0)$. Hence its replicator dynamics on the polytope $\Gamma_{(2,2)} = \Delta^1 \times \Delta^1$ is trivial, in the sense that all points are equilibria. In particular the associated vector field $X_{\tilde{\mathcal{G}}} = 0$ is Hamiltonian.

Since the reduced information set $\mathcal{R}(\mathcal{G})$ is of type $\{\bullet, \oplus\}$, by Proposition 4.5.12 the flow of $X_{\mathcal{G}}$ admits an invariant foliation with a single globally attractive equilibrium on each leaf, as we can see in figure 4.1.

Therefore, the attractor $\Lambda_{\mathcal{G}}$ is just a line segment of equilibria, which embeds in the Hamitonian flow of $X_{\tilde{\mathcal{G}}} = 0$, as asserted by Proposition 4.5.14.

4.6.2 Dissipative Polymatrix

Consider the polymatrix replicator system associated to the polymatrix game $\mathcal{G} = ((2,2,2), A)$, where

$$A = \begin{bmatrix} 0 & -102 & 0 & 79 & 0 & 18 \\ 102 & 0 & 0 & -79 & -18 & 9 \\ 0 & 0 & 0 & 0 & 9 & -18 \\ -51 & 51 & 0 & 0 & 0 & 0 \\ 0 & 102 & -79 & 0 & -18 & -9 \\ -102 & -51 & 158 & 0 & 9 & 0 \end{bmatrix}.$$

We denote by $X_{\mathcal{G}}$ the vector field associated to this polymatrix replicator defined on the polytope $\Gamma_{(2,2,2)} = \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3$.

We will prove that this system is dissipative, has a single attractive equilibrium in the interior of the phase space, and an heteroclinic cycle whose local unstable manifold extends to a global invariant surface containing the interior equilibrium.

The cube $[0,1]^3$ has six faces labelled by an index j ranging from 1 to 6, and designated by $\sigma_1, \ldots, \sigma_6$. The vertices of the phase space $[0,1]^3$ will be designated by tuples in $\{1,2\} \times \{3,4\} \times \{5,6\}$, where the label (i,j,k) stands for the point $e_i + e_j + e_k \in \Gamma_{(2,2,2)}$. To simplify the notation we designate the cube vertices by the letters $v_1, \ldots, v_8$ according to table 4.5.

$v_1 = (1,3,5)$	$v_2 = (1,3,6)$	$v_3 = (1,4,5)$	$v_4 = (1,4,6)$
$v_5 = (2,3,5)$	$v_6 = (2,3,6)$	$v_7 = (2,4,5)$	$v_8 = (2,4,6)$

Table 4.5: Vertex labels.

The skeleton character of $X_{\mathcal{G}}$ is displayed in table 4.6, whose entries are the components of the skeleton vector field χ of $X_{\mathcal{G}}$.

In this model all twelve edges of $[0,1]^3$ are flowing-edges and will be designated by $\gamma_1, \ldots, \gamma_{12}$, according to table 4.7, where we write $\gamma = (v_i, v_j)$ to mean that γ is a flowing-edge from v_i to v_j.

The graph of the skeleton vector field χ (see Definition 3.4.6) is represented in figure 4.2. Looking at the graph in figure 4.2, we can see that

$$S = \{\, \gamma_5 = (v_3, v_1),\ \gamma_8 = (v_6, v_8) \,\}$$

is a structural set for χ (see Definition 3.4.16), whose S-branches are displayed in table 4.8.

The S-Poincaré map $\pi_S : \Pi_S \to \Pi_S$ of χ (see Definition 3.4.18) is depicted in figure 4.3. Notice that $\Pi_S = \Pi_{\gamma_5} \cup \Pi_{\gamma_8}$, where $\Pi_{\gamma_5} = \Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Pi_{\xi_3}$ and

χ^v_σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
v_1	0	-84	0	60	0	-162
v_2	0	-93	0	33	144	0
v_3	0	74	-60	0	0	75
v_4	0	65	-33	0	-93	0
v_5	84	0	0	-42	0	-111
v_6	93	0	0	-69	93	0
v_7	-74	0	42	0	0	126
v_8	-65	0	69	0	-144	0

Table 4.6: The skeleton character of $X_{\mathcal{G}}$.

$\gamma_1 = (v_1, v_2)$	$\gamma_2 = (v_4, v_3)$	$\gamma_3 = (v_5, v_6)$	$\gamma_4 = (v_8, v_7)$
$\gamma_5 = (v_3, v_1)$	$\gamma_6 = (v_4, v_2)$	$\gamma_7 = (v_5, v_7)$	$\gamma_8 = (v_6, v_8)$
$\gamma_9 = (v_1, v_5)$	$\gamma_{10} = (v_2, v_6)$	$\gamma_{11} = (v_7, v_3)$	$\gamma_{12} = (v_8, v_4)$

Table 4.7: Edge labels.

$\Pi_{\gamma_8} = \Pi_{\xi_4} \cup \Pi_{\xi_5} \cup \Pi_{\xi_6}$. Because the remaining coordinates vanish, we consider the coordinates (u_2, u_6) on Π_{γ_5} and (u_1, u_5) on Π_{γ_8}. Table 4.9 gives the matrix representation and the corresponding defining conditions for all the branches of the S-Poincaré map π_S, regarding the fixed coordinates. In all domains Π_{ξ_j}, the inequalities $u_1 \geq 0$, $u_2 \geq 0$, $u_5 \geq 0$ and $u_6 \geq 0$ are implicit.

To represent the projective map $\hat{\pi}_S : \Delta^\chi_S \subset \Delta_S \to \Delta_S$ (see Definition 3.6.1), we identify $\Delta_{\gamma_5} = [0,1]$, $\Delta_{\gamma_8} = [1,2]$ and $\Delta_S = [0,2]$. With these identifications, the mapping $\varphi = \hat{\pi}_S : [0,2] \to [0,2]$ is given in (4.23), while its graph is shown in figure 4.4.

$$\varphi(x) = \begin{cases} \frac{3626x}{3626-1343x} & \text{if} \quad 0 \leq x \leq \frac{49}{236} \\ \frac{49-8746x}{784-8031x} & \text{if} \quad \frac{49}{236} \leq x \leq \frac{14}{41} \\ \frac{316-19459x}{1928-18002x} & \text{if} \quad \frac{14}{41} \leq x \leq 1 \\ \frac{677x+21488}{43653-21488x} & \text{if} \quad 1 \leq x \leq \frac{5153}{4438} \\ \frac{8374-18904x}{137199-129854x} & \text{if} \quad \frac{5153}{4438} \leq x \leq \frac{274}{209} \\ \frac{629x-222}{4924x-5242} & \text{if} \quad \frac{274}{209} \leq x \leq 2 \end{cases} \tag{4.23}$$

The points 0 and 1 are fixed points of this projective map. They correspond to the invariant boundary lines of the cones Π_{ξ_1} and Π_{ξ_6}, and they are both repelling fixed points. The map φ has also an attractive periodic orbit $\{p, p'\}$, of period two, corresponding to the eigenvectors of $\pi_{\xi_3} \circ \pi_{\xi_4}$

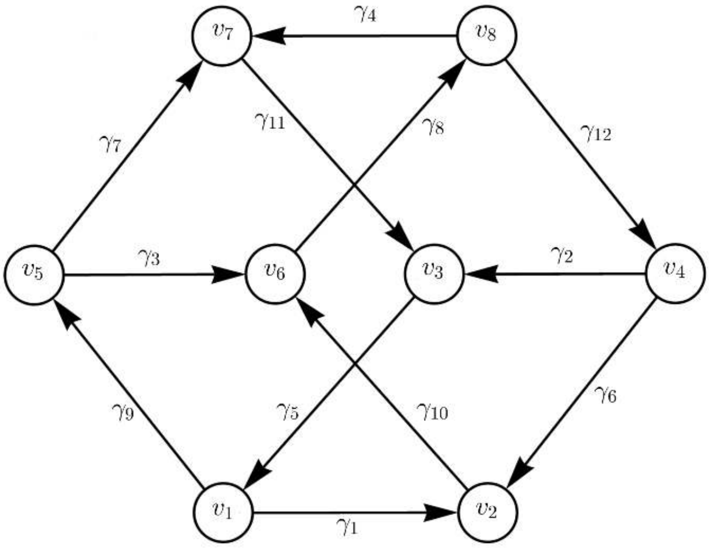


Figure 4.2: The oriented graph of χ, where the label i represents the edge γ_i.

From\To	γ_5	γ_8
γ_5	ξ_1	ξ_2, ξ_3
γ_8	ξ_4, ξ_5	ξ_6

$\xi_1 = (\gamma_5, \gamma_9, \gamma_7, \gamma_{11}, \gamma_5)$	$\xi_2 = (\gamma_5, \gamma_1, \gamma_{10}, \gamma_8)$	$\xi_3 = (\gamma_5, \gamma_9, \gamma_3, \gamma_8)$
$\xi_4 = (\gamma_8, \gamma_4, \gamma_{11}, \gamma_5)$	$\xi_5 = (\gamma_8, \gamma_{12}, \gamma_2, \gamma_5)$	$\xi_6 = (\gamma_8, \gamma_{12}, \gamma_6, \gamma_{10}, \gamma_8)$

Table 4.8: S-branches of χ.

and $\pi_{\xi_4} \circ \pi_{\xi_3}$ (see Definition 3.6.3). Notice that both intervals I_1 and I_6 are overflowing, i.e., $\varphi(I_1) \supset I_1$ and $\varphi(I_6) \supset I_6$. Hence, since φ is one-to-one, the complementary intervals are forward invariant, i.e., $\varphi(I_2 \cup I_3) \subset I_4 \cup I_5$ and $\varphi(I_4 \cup I_5) \subset I_2 \cup I_3$. We have that p and p' are the only fixed points of φ^2. Hence we have the following complete description of the projective dynamics: the ω-limit of any point in $]0,1[\cup]1,2]$ is the attractive periodic orbit $\{p, p'\}$. The α-limit of any point in $[0,2] \setminus \{p, p'\}$ is one of the two fixed points 0 or 1.

Regarding the periodic orbit $\{p, p'\}$, concatenating the paths ξ_3 and ξ_4 we obtain the cycle

$$\xi_{34} := (\gamma_5, \gamma_9, \gamma_3, \gamma_8, \gamma_4, \gamma_{11}, \gamma_5),$$

ξ	def. equations of Π_ξ	matrix of π_ξ
ξ_1	$49u_6 - 187u_2 \geq 0$	$\begin{bmatrix} 1 & 0 \\ -\frac{1343}{3626} & 1 \end{bmatrix}$
ξ_2	$14u_6 - 27u_2 \leq 0$	$\begin{bmatrix} \frac{11}{23} & \frac{52}{207} \\ \frac{1445}{713} & -\frac{1180}{2139} \end{bmatrix}$
ξ_3	$\begin{cases} 49u_6 - 187u_2 \leq 0 \\ 14u_6 - 27u_2 \geq 0 \end{cases}$	$\begin{bmatrix} \frac{2900}{5957} & \frac{210}{851} \\ \frac{11594}{5957} & -\frac{434}{851} \end{bmatrix}$
ξ_4	$65u_5 - 144u_1 \leq 0$	$\begin{bmatrix} \frac{7}{10} & \frac{11}{40} \\ \frac{357}{148} & -\frac{145}{296} \end{bmatrix}$
ξ_5	$\begin{cases} 65u_5 - 144u_1 \geq 0 \\ 715u_5 - 3723u_1 \leq 0 \end{cases}$	$\begin{bmatrix} \frac{14717}{20150} & \frac{81}{310} \\ \frac{3723}{1612} & -\frac{55}{124} \end{bmatrix}$
ξ_6	$715u_5 - 3723u_1 \geq 0$	$\begin{bmatrix} 1 & 0 \\ -\frac{21488}{22165} & 1 \end{bmatrix}$

Table 4.9: Branches of π_S.

which determines a heteroclinic cycle of $X_\mathcal{G}$ along the polytope's boundary. The periodic point p corresponds to a fixed point $\hat{\pi}_{\xi_{34}}(w_0) = w_0 \in \text{int}\,(\Pi_{\xi_{34}})$, i.e., w_0 is an eigenvector of $\pi_{\xi_{34}}$ (see Definition 3.6.3). The associated eigenvalue is

$$\lambda_1 \approx 0.946652\,.$$

The eigenvalue of the second eigenvector of $\pi_{\xi_{34}}$, outside $\Pi_{\xi_{34}}$, is

$$\lambda_2 \approx 0.774422\,.$$

Hence $\sigma(w) = \frac{\lambda_2}{\lambda_1} < 1$, and by Proposition 3.6.4 we have that w_0 is an attractive periodic point of $\varphi = \hat{\pi}_S$. Thus, by Theorem 3.6.6, there exists a normally contractive local unstable manifold $W^u_{loc}(\xi_{34})$ for the heteroclinic cycle associated to ξ_{34}.

Since the ω-limit of any point $x \in [0,2] \setminus \{0,1\}$ is the attractive periodic orbit $\{p, p'\}$, and the eigenvalue of the eigenvector w_0 is $\lambda_1 < 1$, it follows that for all $u \in \text{int}\,(\Pi_S)$,

$$\lim_{n\to+\infty} \pi_S^n(u) = 0\ .$$

This means that near the polytope's edge skeleton all orbits of the flow $\varphi^t_{X_\mathcal{G}}$ are attracted to $W^u_{loc}(\xi_{34})$, and at the same time pulled away from the heteroclinic cycle ξ_{34}.

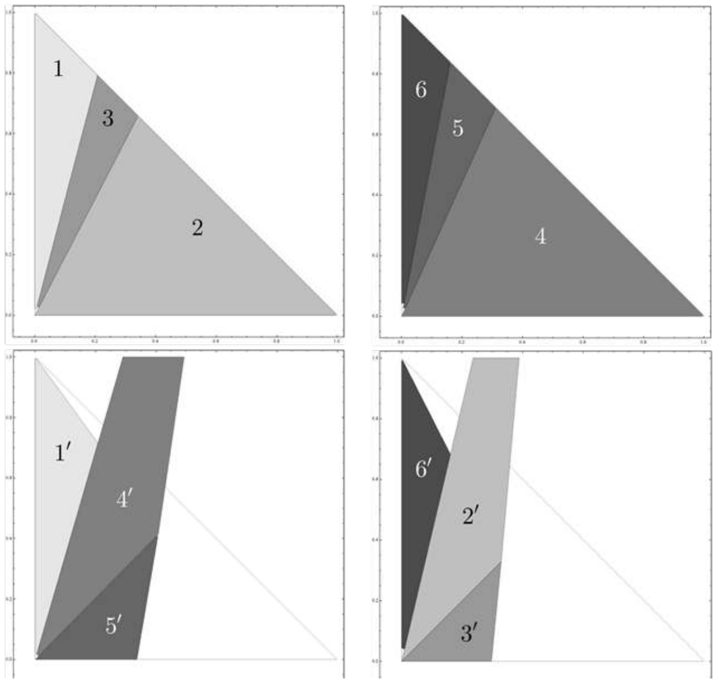


Figure 4.3: The S-Poincaré map $\pi_S : \Pi_S \to \Pi_S$. The pictures above represent the domains Π_{ξ_j} labelled from 1 to 6. The pictures below represent the images $\pi_{\xi_j}(\Pi_{\xi_j})$ labelled from $1'$ to $6'$.

Our polymatrix replicator only has one equilibrium in int $\left(\Gamma_{(2,2,2)}\right)$,

$$q = \left(\frac{1}{2}, \frac{1}{2}, \frac{71}{158}, \frac{87}{158}, \frac{2}{3}, \frac{1}{3}\right),$$

while it has 10 equilibria in $\partial\Gamma_{(2,2,2)}$, eight of them vertices, and the remaining two on different 2-faces,

$$q_1 = \left(\frac{7}{17}, \frac{10}{17}, \frac{37}{79}, \frac{42}{79}, 1, 0\right) \quad \text{and} \quad q_2 = \left(\frac{23}{34}, \frac{11}{34}, \frac{65}{158}, \frac{93}{158}, 0, 1\right).$$

The equilibrium $q \in$ int $\left(\Gamma_{(2,2,2)}\right)$ satisfies

(1) $(Aq)_1 = (Aq)_2 = -\frac{3}{2}$, $(Aq)_3 = (Aq)_4 = 0$ and $(Aq)_5 = (Aq)_6 = \frac{1}{2}$;

(2) $q_1 + q_2 = q_3 + q_4 = q_5 + q_6 = 1$.

By Definition 4.4.1, q is a formal equilibrium of $\mathcal{G}$.

The quadratic form $Q_{AD} : H_{(2,2,2)} \to \mathbb{R}$ induced by matrix A is

$$Q_{AD}(x) = -x_3^2 \leq 0,$$

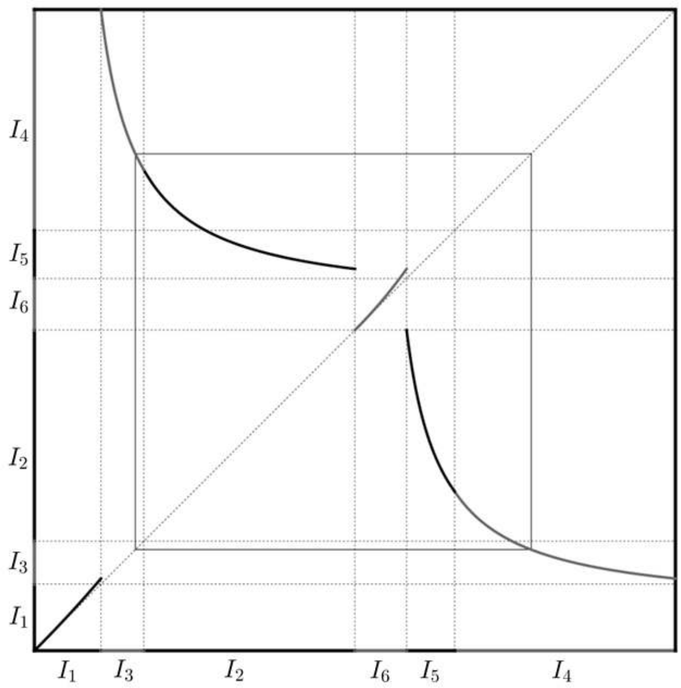


Figure 4.4: The projective map $\varphi : [0, 2] \to [0, 2]$ with $I_j = \Delta^{\chi}_{\xi_j}$ for $j = 1, \ldots, 6$.

where $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in H_{(2,2,2)}$ and D is the positive diagonal matrix of type $(2, 2, 2)$ given by

$$D = \begin{bmatrix} \frac{1}{51} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{51} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{79} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{79} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{bmatrix}.$$

By Definition 4.5.1, $\mathcal{G}$ is dissipative.

The equilibrium $q \in \operatorname{int}\left(\Gamma_{(2,2,2)}\right)$ is hyperbolic. The eigenvalues of q are

$$\zeta_\pm \approx -0.545809 \pm 37.0244\, i\ , \qquad \text{and} \qquad \zeta_1 \approx -2.90838\,.$$

The local manifold $W^u_{\text{loc}}(\xi_{34})$ extends to a global surface of class C^5 that contains the interior equilibrium q. Let us explain this.

At the equilibrium point q we have a DX_q-invariant decomposition $T_p\Gamma = E^{ss} \oplus E^s$, where E^s is the eigen-plane associated to the complex eigenvalues $\zeta_\pm$ and E^{ss} is the eigen-direction associated to the eigenvalue ζ_1.

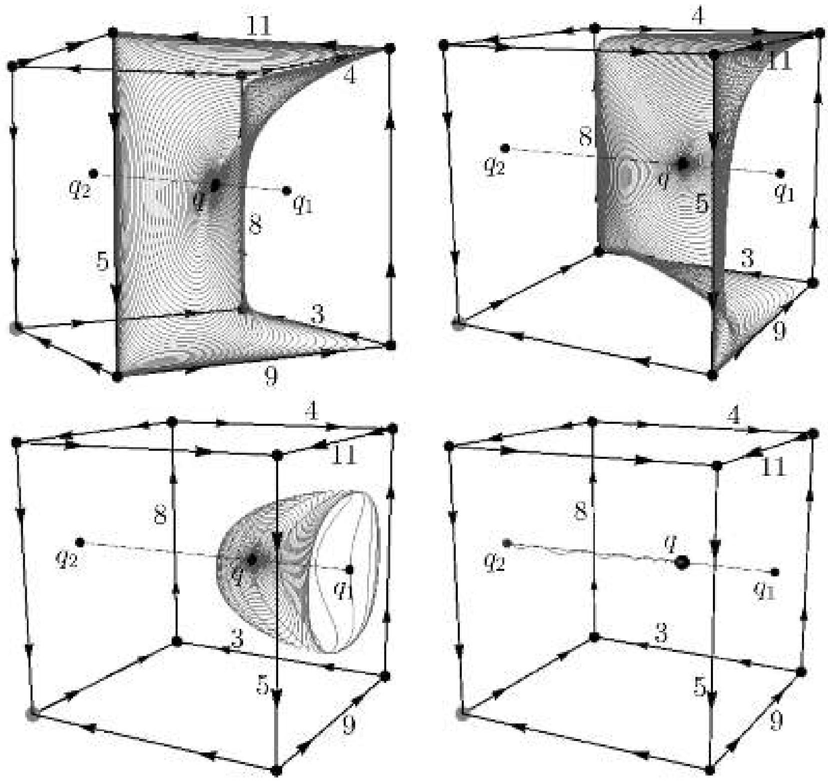


Figure 4.5: An approximation of the $X_{\mathcal{G}}$-invariant manifold on the polytope from two different perspectives (up) and two different orbits starting near the respective faces equilibrium (down).

Since

$$\zeta_1 < 5\,\mathrm{Re}(\zeta_\pm) < 0 \tag{4.24}$$

there exist central stable manifolds of class C^5 tangent to E^s at q (see [35], Theorem III.8). These central manifolds are not unique because of the contractive behaviour on the central direction.

The ω-limit of any interior point $x \in \operatorname{int}\left(\Gamma_{(2,2,2)}\right)$ is the equilibrium q. This follows by Proposition 4.5.8, because the system is dissipative and has a unique interior equilibrium. This system has a strict global Lyapunov function $h : \operatorname{int}\left(\Gamma_{(2,2,2)}\right) \to \mathbb{R}$ for $X_{\mathcal{G}}$. This function h has an absolute minimum at q and satisfies

$$\frac{dh}{dt}(x) = Dh_x\left(X_{\mathcal{G}}\right) < 0$$

for all $x \in \operatorname{int}\left(\Gamma_{(2,2,2)}\right)$ with $x \neq q$.

Consider the set

$$W^u(\xi_{34}) := \bigcup_{t>0} \varphi_X^t(W^u_{\text{loc}}(\xi_{34})) \ ,$$

that is a global unstable manifold. The equilibrium q is an accumulation point of $W^u(\xi_{34})$. In fact, $W = W^u(\xi_{34}) \cup \{q\}$ must be a smooth surface with at most a possible cusp singularity at q. Obviously this is not possible due to (4.24). Actually, W must be of class C^5 at q.

We finish with a rough description of the replicator interior dynamics, which can easily be proven from the previously established facts. The ω-limit of any point in int $\left(\Gamma_{(2,2,2)}\right)$ is always the equilibrium point q. The stable and strong stable invariant manifolds associated to the equilibrium q satisfy:

- $W^s(q) = W^u(\xi_{34}) \cup \{q\}$,
- $W^{ss}(q) = W^u(q_1) \cup \{q\} \cup W^u(q_2)$.

Regarding the α-limit of a point $x \in \text{int}\left(\Gamma_{(2,2,2)}\right)$ we have three possibilities:

- If $x \in W^s(q)$ then $\alpha(x)$ is the heteroclinic cycle ξ_{34}.
- If x is on the component of $\Gamma_{(2,2,2)} \setminus W^s(q)$ that contains q_1 then $\alpha(x)$ can be any one of the following alternatives:
  - the equilibrium point q_1,
  - one of the closed orbits around q_1 in the face σ_6,
  - the heteroclinic cycle ξ_1.
- If x is on the component of $\Gamma_{(2,2,2)} \setminus W^s(q)$ that contains q_2 then $\alpha(x)$ can be any one of the following alternatives:
  - the equilibrium point q_2,
  - one of the closed orbits around q_2 in the face σ_5,
  - the heteroclinic cycle ξ_6.

4.6.3 Conservative Polymatrix

Consider the polymatrix replicator system associated to the polymatrix game $\mathcal{G} = ((5), A)$, where

$$A = \begin{bmatrix} 0 & -2 & 2 & -2 & 2 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -3 & 0 \\ 2 & 0 & 3 & 0 & -2 \\ -2 & 0 & 0 & 2 & 0 \end{bmatrix} .$$

χ^v_σ	σ_1	σ_2	σ_3	σ_4	σ_5
v_1	0	-2	2	-2	2
v_2	2	0	-2	0	0
v_3	-2	2	0	-3	0
v_4	2	0	3	0	-2
v_5	-2	0	0	2	0

Table 4.10: The skeleton character of $X_{\mathcal{G}}$.

We denote by $X_{\mathcal{G}}$ the vector field associated to this polymatrix replicator defined on the polytope $\Gamma_{(5)} = \Delta^4$.

We will prove that this system is conservative and the dynamics in all sufficiently large energy levels is chaotic, i.e., the flow contains horse-shoes.

The point $q \in \operatorname{int}\left(\Gamma_{(5)}\right)$ given by

$$q = \left(\frac{1}{8}, \frac{5}{16}, \frac{1}{8}, \frac{1}{8}, \frac{5}{16}\right) ,$$

satisfies

(1) $(Aq)_1 = (Aq)_2 = (Aq)_3 = (Aq)_4 = (Aq)_5 = 0$;

(2) $q_1 + q_2 + q_3 + q_4 + q_5 = 1$.

By Definition 4.4.1, q is a formal equilibrium of the polymatrix game $\mathcal{G}$.

Since the matrix A is skew-symmetric, the quadratic form $Q_A : H_{(5)} \to \mathbb{R}$ induced by matrix A is zero. By Definition 4.4.3, the polymatrix game $\mathcal{G}$ is conservative.

The prism $\Gamma_{(5)}$ has five faces labelled by an index j ranging from 1 to 5, and designated by $\sigma_1, \ldots, \sigma_5$. The vertices of the phase space $\Gamma_{(5)}$ are also labelled by i in $\{1, \ldots, 5\}$, where the label i stands for the point $e_i \in \Gamma_{(5)}$. To simplify the notation we designate the prism vertices by $1, \ldots, 5$.

The skeleton character of $X_{\mathcal{G}}$ is displayed in table 4.10, whose entries are the components of the skeleton vector field χ of $X_{\mathcal{G}}$.

The edges of $\Gamma_{(5)}$ will be designated by $\gamma_1, \ldots, \gamma_{10}$, according to table 4.11, where we write $\gamma = (i, j)$ to mean that γ is a flowing edge from vertex i to vertex j. In this model not all edges of $\Gamma_{(5)}$ are flowing-edges. There are three neutral edges, $\gamma_6, \gamma_7, \gamma_9$, and seven flowing-edges, $\gamma_1, \ldots, \gamma_5, \gamma_8, \gamma_{10}$. See figure 4.6.

The graph of the skeleton vector field χ (see Definition 3.4.6) is represented in figure 4.6. Looking at the graph in figure 4.6, we can see that

$$S = \{\, \gamma_1 = (1, 2),\ \gamma_4 = (5, 1) \,\}$$

$\gamma_1 = (1,2)$	$\gamma_2 = (3,1)$	$\gamma_3 = (1,4)$	$\gamma_4 = (5,1)$	$\gamma_5 = (2,3)$
$\gamma_6 = (2,4)$	$\gamma_7 = (2,5)$	$\gamma_8 = (3,4)$	$\gamma_9 = (3,5)$	$\gamma_{10} = (4,5)$

Table 4.11: Edge labels.

is a structural set for χ (see Definition 3.4.16), whose S-branches denoted by $\xi_1, \dots, \xi_5$ are displayed in table 4.12.

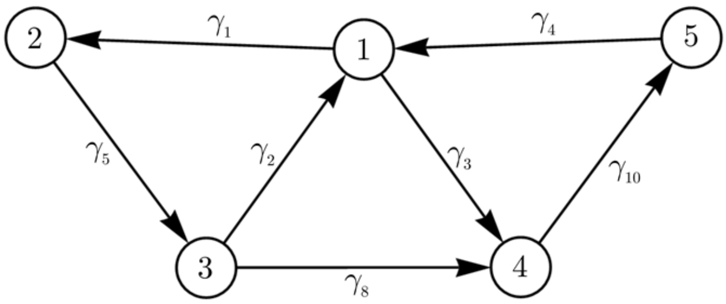


Figure 4.6: The oriented graph of χ.

From\To	γ_1	γ_4
γ_1	ξ_1	ξ_2, ξ_3
γ_4	ξ_4	ξ_5

$\xi_1 = (\gamma_1, \gamma_5, \gamma_2, \gamma_1)$	$\xi_2 = (\gamma_1, \gamma_5, \gamma_8, \gamma_{10}, \gamma_4)$	$\xi_3 = (\gamma_1, \gamma_5, \gamma_2, \gamma_3, \gamma_{10}, \gamma_4)$
$\xi_4 = (\gamma_4, \gamma_1)$	$\xi_5 = (\gamma_4, \gamma_3, \gamma_{10}, \gamma_4)$	

Table 4.12: S-branches of χ.

For this example the Hamiltonian is the function $h : \Gamma_{(5)} \to \mathbb{R}$ defined by

$$h(x) = -\left(\frac{1}{8}\log x_1 + \frac{5}{16}\log x_2 + \frac{1}{8}\log x_3 + \frac{1}{8}\log x_4 + \frac{5}{16}\log x_5\right).$$

By Definition 3.5.6 the skeleton of h is $\eta : \mathcal{C}^*(\Gamma_{(5)}) \to \mathbb{R}$,

$$\eta(u) = \frac{1}{8}u_1 + \frac{5}{16}u_2 + \frac{1}{8}u_3 + \frac{1}{8}u_4 + \frac{5}{16}u_5\,.$$

We define the sets

$$\Delta_\gamma = \Pi_\gamma \cap \eta^{-1}(1)\,, \quad \Delta_\xi = \Pi_\xi \cap \eta^{-1}(1)$$

and

$$\Delta_S = \bigcup_{i=1}^{5} \Delta_{\xi_i}\,,$$

where

$$\eta^{-1}(1) := \{u \in \mathcal{C}^*(\Gamma_{(5)}) \,:\, \eta(u) = 1\}\,.$$

Consider the S-Poincaré map $\pi_S : \Pi_S \to \Pi_S$ of χ (see Definition 3.4.18). Notice that $\Pi_S = \Pi_{\gamma_1} \cup \Pi_{\gamma_4}$, where $\Pi_{\gamma_1} = \Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Pi_{\xi_3}$ and $\Pi_{\gamma_4} = \Pi_{\xi_4} \cup \Pi_{\xi_5}$. Because the remaining coordinates vanish, we consider the coordinates (u_3, u_4, u_5) on Π_{γ_1} and (u_2, u_3, u_4) on Π_{γ_4}. Table 4.13 gives the matrix representation and the corresponding defining conditions for all the S-branches of the Poincaré map π_S, regarding the fixed coordinates. In all domains Π_{ξ_j}, the inequalities $u_2 \geq 0$, $u_3 \geq 0$, $u_4 \geq 0$ and $u_5 \geq 0$ are implicit.

ξ	def. equations of Π_ξ	matrix of π_ξ
ξ_1	$5u_3 - 2u_4 \leq 0$	$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
ξ_2	$3u_3 - 2u_4 \geq 0$	$\begin{bmatrix} 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{3}{2} \\ 1 & -\frac{2}{3} & 1 \end{bmatrix}$
ξ_3	$\begin{cases} 5u_3 - 2u_4 \geq 0 \\ 3u_3 - 2u_4 \leq 0 \end{cases}$	$\begin{bmatrix} \frac{5}{2} & -1 & 0 \\ -\frac{15}{4} & \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & 1 & 1 \end{bmatrix}$
ξ_4	$u_2 - u_4 \leq 0$	$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
ξ_5	$u_2 - u_4 \geq 0$	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$

Table 4.13: Branches of π_S.

Since the Poincaré map π_S preserves the function η (see Proposition 3.5.8) we can consider the restriction of π_S to Δ_S. In figure 4.7 we represent the Poincaré map $\pi_S : \Delta_S \to \Delta_S$. The sets $\Delta_{\xi_1}, \ldots, \Delta_{\xi_5}$ are represented in the top of figure 4.7, labelled from 1 to 5, and the π_S images of $\Delta_{\xi_1}, \ldots, \Delta_{\xi_5}$ are represented in the bottom of the same figure, labelled from $1'$ to $5'$.

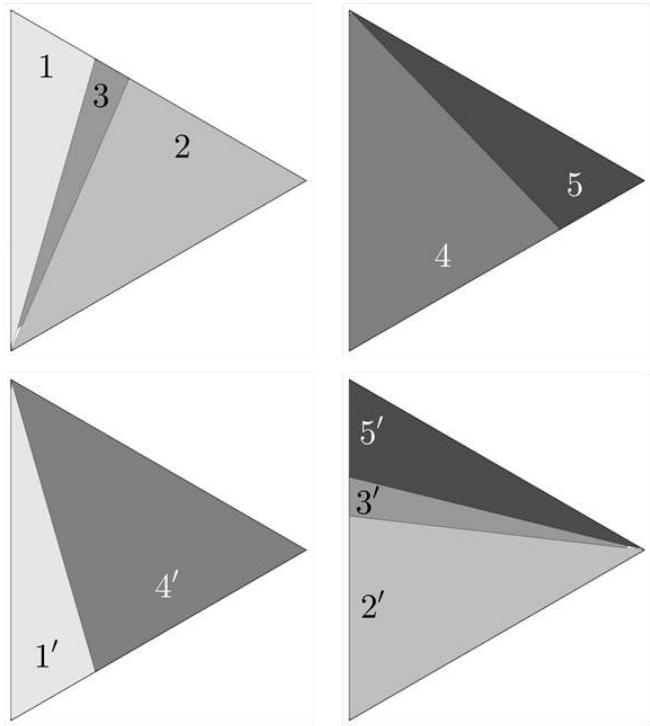


Figure 4.7: The S-Poincaré map $\pi_S : \Delta_S \to \Delta_S$. The pictures above represent the sets $\Delta_{\xi_1}, \ldots, \Delta_{\xi_5}$, labelled from 1 to 5. The pictures below represent the π_S iterates of $\Delta_{\xi_1}, \ldots, \Delta_{\xi_5}$, labelled from $1'$ to $5'$.

Figure 4.8 shows 2000 π_S-iterates (white dots) of a random point. Following the itinerary of that random point we consider the sequence of 25 S-branches

$$\begin{aligned} &\xi_1, \xi_1, \xi_3, \xi_4, \xi_2, \xi_4, \xi_2, \xi_4, \xi_2, \xi_4, \xi_2, \xi_4, \xi_2, \\ &\xi_4, \xi_2, \xi_5, \xi_5, \xi_5, \xi_5, \xi_5, \xi_4, \xi_2, \xi_4, \xi_1, \xi_1 \,. \end{aligned} \tag{4.25}$$

Let M be the matrix of the Poincaré map associated to the path (4.25), as defined in (3.6).

Numerically we have computed the eigenvalues of M,

$$\lambda_u = 948.618, \quad \lambda_s = 0.00105417, \quad \text{and} \quad \lambda = 1\,,$$

and the corresponding eigenvectors

$$w_u = (0.284213, -0.923978, 0.255906)\,,$$

$$w_s = (0.112816, 0.906168, -0.407593)\,,$$

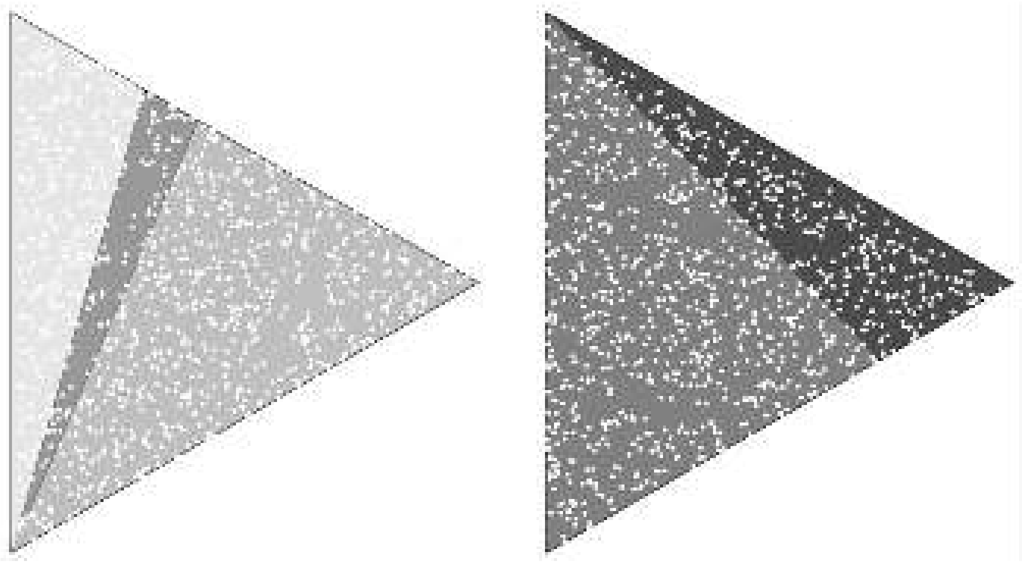


Figure 4.8: One orbit with 2000 iterates.

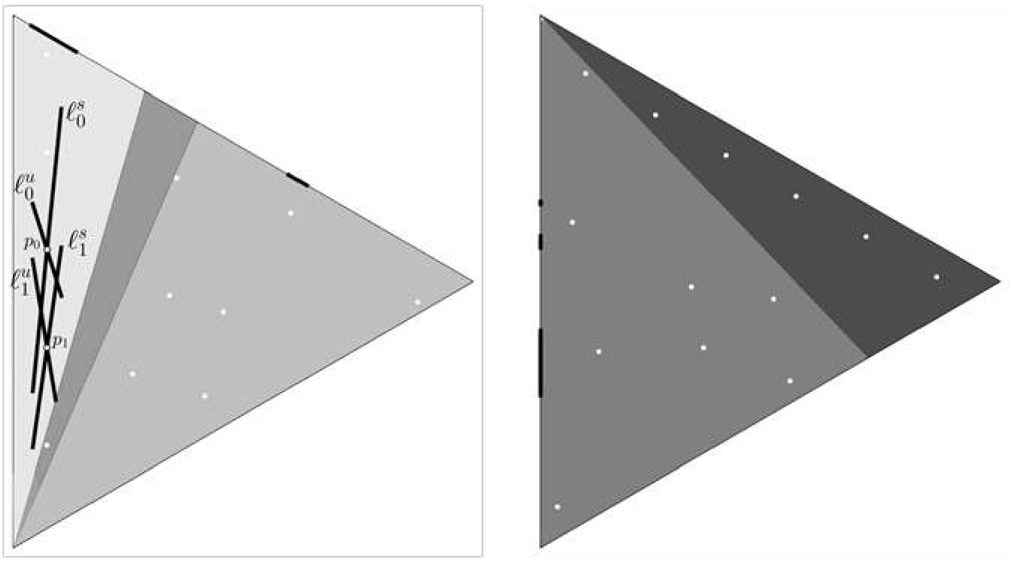


Figure 4.9: Stable and unstable manifolds at the periodic point p_0.

and

$$p_0 = (0.132969, 0.946979, 0.292489)\,.$$

By the defining equations given in table 4.13 we can see that $p_0 \in \Delta_{\xi_1}$. Hence p_0 is a periodic point of π_S with period 25. The iterates of p_0 are represented by the white dots in figure 4.9.

Let ℓ_0^u and ℓ_0^s be the small line segments through p_0, contained in Δ_{ξ_1}, respectively with the (eigen) directions of the expanding (associated to w_u) and the contracting (associated to w_s) eigen-spaces of the π_S periodic point p_0. We denote by ℓ_1^u and ℓ_1^s the π_S images of ℓ_0^u and ℓ_0^s, respectively, i.e.,

$$\ell_1^u = \pi_S(\ell_0^u) \quad \text{and} \quad \ell_1^s = \pi_S(\ell_0^s)\,.$$

Let $p_1 = \pi_S(p_0)$. Our numerical experiment shows that ℓ_0^u intersects ℓ_1^s transversally, and ℓ_0^s intersects ℓ_1^u transversally as well. These intersection points, respectively q_0 and q_1, belong to Δ_{ξ_1}.

Let Ω be the union of the π_S-orbits of p_0, q_0 and q_1. We have that Ω is a compact set contained in Δ_S. Moreover, Ω is a hyperbolic π_S-invariant set. In particular, this implies the existence of a horse-shoe for the S-Poincaré map $\pi_S : \Delta_S \to \Delta_S$ (see [36]).

Therefore, by Theorem 3.5.5 in each level set $h^{-1}(c)$, where h is the Hamiltonian defined in (4.10) and $c \approx \infty$ is a large energy level, the Poincaré map P_S (see Definition 3.5.4) has an invariant hyperbolic set conjugated to Ω. This implies transversal homoclinic points and hence the existence of a horse-shoe. This conclusion follows by the stability of hyperbolic invariant sets under perturbations (see [35, Theorem 7.8]).

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Index

admissible matrix, 21
admissible perturbation, 22
almost skew-symmetric matrix, 30
attracting edge, 48

black and white graph, 29

dual cone, 45

flowing-edge, 48
formal equilibrium, 71

Hamiltonian system, 11
Hamiltonian vector field, 10

invariant foliation, 15–16
itinerary, 54

Lotka-Volterra, 7–9
  attractor, 26
  conservative, 7, 9–13
  dissipative, 7, 13–14
  graph, 7
  stably dissipative, 22

manifold
  normally hyperbolic, 55

neutral edge, 48

orbit segment, 50

path, 49
  cycle, 50
  length, 50
period, 54
periodic point, 54
permanence
  polymatrix replicator, 69
  replicator equation, 19
Poincaré map, 52
Poisson bracket, 11
Poisson gradient, 11
polymatrix game, 61
  admissible, 74
  conservative, 71
  dissipative, 74
polymatrix replicator
  attractor, 75
polymatrix replicator system, 61
polytope, 42
  corner, 43
  edge, 43, 64
  face, 43, 64
  vertex, 43, 64
  vertex coordinates, 43
projective Poincaré map, 54
  eigenvalue, 54
  eigenvector, 54

quotient rule
  polymatrix, 75
  replicator, 17

rank, stably dissipative graph, 31
Redheffer Reduction Algorithm, 29
reduced graph, 28, 32
reduction algorithm
  polymatrix, 78
regular skeleton, 48
repelling edge, 48
replicator equation, 16–20
  differential equation, 17
  permanent, 19

simplectic gradient, 10

simplectic structure, 9
skeleton character, 44
skeleton Poincaré map, 49, 50
skeleton vector field, 47
source vertex, 48
stably admissible, 21
stably dissipative matrix, 22
strong link, 30
structural set, 50
  branch, 51
submatrix, 24

target vertex, 48
trimmed matrix, 34
trimming, graph, 33

vector field
  regular, 51
vertex
  χ-attractive, 48
  χ-repelling, 48
  saddle type, 48