

Love Emotions between Laura and Petrarch – an Approach by Mathematics and System Dynamics

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Laura, a very beautiful but also mysterious lady, inspired the famous poet Petrarch for poems, which express ecstatic love as well as deep despair.



F. J. Jones – a scientist for literary work – recognized in these changes between love and despair an oscillating behaviour – from 1328 to 1350 – which he called Petrarch's emotional cycle.



The mathematician S. Rinaldi investigated this cycle and established a mathematical model based on ordinary differential equation: two coupled nonlinear ODEs, reflecting Laura's and Petrarch's emotions for each other, drive an inspiration variable, which coincides with Petrarch's emotional cycle.

These ODEs were the starting point for the investigations in two directions: mapping the mathematical model to a suitable modelling concept, and trying to extend the model for love dynamics in modern times (F. Breitenecker et al.).

This contribution introduces and investigates a modelling approach for love dynamics and inspiration by means of System Dynamics, for Laura's and Petrarch's emotions as well as for a modern couple in love. In principle, emotions and inspiration emerge from a source and are fading into a sink. But the controlling parameters for increase and decrease of emotion create a broad variety of emotional behaviour and of degree of inspiration, because of the nonlinearities.

Experiments including an implementation of this model approach and selected simulations provide interesting case studies for different kinds of love dynamics - attraction, rejection and neglect – stable equilibria and chaotic cycles.

Keywords: love dynamics, limit cycles, dynamical systems, system dynamics

1. Introduction

Dynamic phenomena in physics, biology, economics, and all other sciences have been extensively studied with differential equations, since Newton introduced the differential calculus. But the dynamics of love, perhaps the most important phenomenon concerning our lives, has been tackled only very rarely by this calculus. In literature, two special contributions can be found:

- *Love Affairs and Differential Equations* by S. H. Strogatz ([1]), – harmonic oscillators making reference to Romeo and Juliet, and
- *Laura and Petrarch: an Intriguing Case of Cyclical Love Dynamics* by S. Rinaldi ([2]) – presenting a nonlinear ODE with cyclic solutions.

Based on these papers, recent work has been done, and is done, by embedding the mathematical concepts of Rinaldi into modelling concepts like Transfer Functions (Breitenecker et al., [3]) and System Dynamics (this paper), and by generalizing the model for modern times.

Section 2 presents the classification of Petrarch's poems by F. Jones, introducing Petrarch's emotional cycle. Section 3 summarises the ODE model approach for the Laura-Petrarch model

by S. Rinaldi. Section 4 introduces System Dynamics as appropriate modelling approach for the underlying dynamics, giving insight into the principle model structure and preparing a basis for a model extension. Section 5 shows identification results for the Laura-Petrarch model based on a MATLAB implementation with a GUI (MATLAB graphical user interface).

Section 6 presents analytical analysis for the Laura-Petrarch model as well as numerical analysis and experiments with this model.

Section 7 extends *Laura-Petrarch Model* to a general model for the dynamic love cycle – the *Woman-Man Model*, perhaps also applicable for modern times and modern love affairs, Section 8 concludes with experiments with the *Woman-Man model*, showing interesting case studies with changing appeal, etc.

2. Classification of Petrarch's poems

Francis Petrarch (1304–1374), is the author of the *Canzoniere*, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals). In Avignon, at the age of 23, he met Laura, a beautiful, but married lady. He immediately fell in love with her and, although his love was not reciprocated, he addressed more than 200 poems to her over the next 21 years. The poems express bouts of ardour and despair, snubs and reconciliations, making Petrarch the most lovesick poet of all time.

Unfortunately, only a few lyrics of the *Canzoniere* are dated. The knowledge of the correct chronological order of the poems is a prerequisite for studying the lyrical, psychological, and stylistical development of Petrarch and



Figure 1. Portraits of Laura and Petrarch, from Biblioteca Medicea Laurenziana, ms. Plut. cc. VIIIv-IX, Florence, Italy (courtesy of Ministero per i Beni Culturali e Ambientali), from [2].



Figure 2. Portraits of Laura and Petrarch, from Internet resources.

his work. For this reason, identification of the chronological order of the poems in the *Canzoniere* has been for centuries a problem of major concern for scholars.

In 1995, in his book *The Structure of Petrarch's Canzoniere* ([4]), Frederic Jones presented an interesting approach and solution to the chronological ordering problem of Petrarch's poems.

Jones concentrated on Petrarch's poems written at lifetime of Laura (the first, sonnet X, was written in 1330 and the last, sonnet CCXII, in 1347). First, he analysed 23 poems with fairly secure date. After a careful linguistic and lyrical analysis, he assigned grades to the poems, ranging from -1 to +1, establishing *Petrarch's emotional cycle*: the maximum grade (+1) stands for ecstatic love, while very negative grades correspond to deep despair. The following examples (in quotations, the English version is taken from an English translation of the *Canzoniere* by Frederic Jones) illustrate some of these grades:

- Sonnet LXXVI, great love, grade +0.6:

*Amor con sue promesse lusingando,
mi ricondusse alla prigione antica
[Love's promises so softly flattering me
have led me back to my old prison's thrall.]*

- Sonnet LXXIX, great despair, grade -0.6:

*Così mancando vo di giorno in giorno,
si chiusamente, ch'i' sol me ne accorgo
et quella che guardando il cor mi strugge.
[Therefore my strength is ebbing day by day,*

*which I alone can secretly survey,
and she whose very glance will
scourge my heart.]*

- Sonnet CLXXVI, melancholy, grade -0.4:

*Parme d'udir la, udendo i rami et l'ore
et le frondi, et gli augei lagnarsi, et l'acque
mormorando fuggir per l'erba verde.
[Her I seem to hear, hearing bough and
wind's caress, as birds and leaves lament, as
murmuring flees the streamlet coursing
through the grasses green.]*

Displaying the grades over time (Figure 3), F. Jones detected an oscillating behaviour, which he called Petrarch's emotional cycle $E(t)$, with a period of about four years.

In the second step, F. Jones analysed and 'graded' all the other poems of unknown date and checked, into which part of the cycle they could fit. Taking into account additional historical information, he could date these poems by locating them in Petrarch's emotional cycle $E(t)$, see Figure 3.

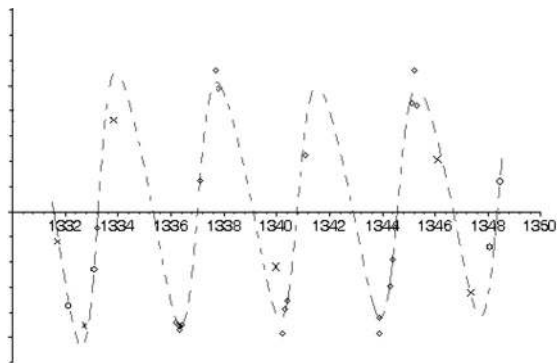


Figure 3. Petrarch's emotional cycle $E(t)$ - dashed line, with 'graded' poems (crosses for securely dated poems, and circles for poems dated based on the emotional cycle).

3. Mathematical Modelling Approach

The big challenge for S. Rinaldi was to set up an ODE model for the cyclic love dynamics and emotional dynamics of Petrarch and Laura, which would fit the experimentally founded emotional cycle of Petrarch.

The basis for modelling is a predator-prey – type dynamics between the variables $L(t)$ and $P(t)$. $L(t)$ represents Laura's love for the poet at time t ; positive and high values of L mean warm

friendship, while negative values should be associated with coldness and antagonism. $P(t)$ describes Petrarch's love for Laura, whereby high values of P indicate ecstatic love, while negative values stand for despair. The general *Laura-Petrarch* model is given by:

$$\begin{aligned} \frac{dL(t)}{dt} &= -\alpha_L L(t) + R_L(P(t)) + \beta_L A_P \\ \frac{dP(t)}{dt} &= -\alpha_P P(t) + R_P(L(t)) + \beta_P A_L. \end{aligned}$$

$R_L(P)$ and $R_P(L)$ are reaction functions to be specified later, and A_P and A_L is the appeal (physical, as well as social and intellectual) of Petrarch and Laura to each other.

Next, it is necessary to model in more detail the complex personality Petrarch. A second variable has to be used to describe his poetic inspiration $I_P(t)$, expressing his productivity for poems (in general modelling the inspiration for work related to the love), and influencing the appeal A_L :

$$\begin{aligned} \frac{dL(t)}{dt} &= -\alpha_L L(t) + R_L(P(t)) + \beta_L A_P \\ \frac{dP(t)}{dt} &= -\alpha_P P(t) + R_P(L(t)) + \beta_P \frac{A_L}{1 + \delta_P I_P(t)} \\ \frac{dI_P(t)}{dt} &= -\alpha_{IP} I_P(t) + \beta_{IP} P(t). \end{aligned}$$

The rate of change of the love of Laura $L(t)$ is the sum of three terms. The first describes the forgetting process characterizing each individual. The second term $R_L(P)$ is the reaction of Laura to the love of Petrarch, and the third is her response to his appeal. The rate of change of the love of Petrarch $P(t)$ is of similar structure, with an extension: the response of Petrarch to the appeal of Laura depends also upon his inspiration $I_P(t)$. This takes into account the well-established fact that high moral tensions, like those associated with artistic inspiration, attenuate the role of the most basic instincts.

And there is no doubt that the tensions between Petrarch and Laura are of a passionate nature, as can be read in literature.

- In Sonett XXII, Petrarch writes:

*Con lei foss'io da che si parte il sole,
et non ci vedess' altri che le stelle,
sol una nocte, et mai non fosse l'alba.*

[Would I were with her when first sets the sun, and no one else could see us but the stars, one night alone, and it were never dawn.]

- And in his *Posteritati*, Petrarch confesses:

Libidem me prorsus expertem dicere posse optarem quidem, sed si dicat mentiar. [I would truly like to say absolutely that I was without libidinousness, but if I said so I would be lying].

The equation for the inspiration $I_P(t)$ says that the love of Petrarch sustains his inspiration which, otherwise, would exponentially decay. The reaction functions $R_L(P)$ and $R_P(L)$ are partly nonlinear. A linear approach would simply say that individuals love to be loved and hate to be hated.

The linearity of $R_P(L)$ is more or less obvious, since in his poems the poet has very intense reactions to the most relevant signs of antagonism from Laura: $R_P(L(t)) = \beta_P \cdot L(t)$.

A linear reaction function is not appropriate for Laura. Only around $R_L(P) = 0$ it can be assumed to be linear, thus interpreting the natural inclination of a beautiful high-society lady to stimulate harmless flirtations. But Laura never goes too far beyond gestures of pure courtesy: she smiles and glances.

However, when Petrarch becomes more demanding and puts pressure on her, even indirectly when his poems are sung in public, she reacts very promptly and rebuffs him, as described explicitly in a number of poems.

- In Sonnet XXI, Petrarch claims:

Mille fiate, o dolce mia guerrera, per aver co' begli occhi vostri pace v'aggio proferto il cor; ma voi non piace mirar si basso colla mente altera. [A thousand times, o my sweet enemy, to come to terms with your enchanting eyes I've offered you my heart, yet you despise aiming so low with mind both proud and free.]

Consequently, the reaction function $R_L(P)$ should, for $P > 0$, first increase, and then decrease. But the behaviour of Laura's reaction is also nonlinear for negative values of P . In fact, when $P < 0$ (when the poet despairs), Laura feels very sorry for him. Following her genuine Catholic ethic

she arrives at the point of overcoming her antagonism by strong feelings of pity, thus reversing her reaction to the passion of the poet. This behavioural characteristic of Laura is repeatedly described in the *Canzoniere*.

- In Sonnet LXIII Petrarch writes:

Volgendo gli occhi al mio novo colore che fa di morte rimembrar la gente, pietà vi mosse; onde, benignamente salutando, teneste in vita il core.

[Casting your eyes upon my pallor new, which thoughts of death recalls to all mankind,

pity in you I've stirred; whence, by your kind greetings, my heart to life's kept true.]

A good choice for Laura's reaction function $R_L(P)$ would be a cubic function of the following type, displayed in Figure 4.

$$R_L(P) = \beta_L P \cdot \left(1 - \left(\frac{P}{\gamma_L} \right)^2 \right).$$

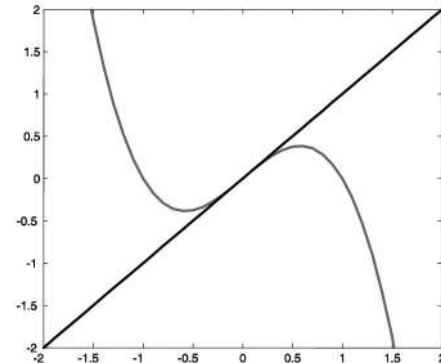


Figure 4. Nonlinear reaction function $R_L(P)$ of Laura – grey line, and linear reaction function $R_P(L)$ of Petrarch – straight black line.

With these approaches, the full Laura-Petrarch model is given by the following equations:

$$\frac{dL(t)}{dt} = -\alpha_L L(t) + \beta_L P \left(1 - \left(\frac{P}{\gamma_L} \right)^2 \right) + \beta_L A_P$$

$$\frac{dP(t)}{dt} = -\alpha_P P(t) + \beta_P L(t) + \beta_P \frac{A_L}{1 + \delta_P I_P(t)}$$

$$\frac{dI_P(t)}{dt} = -\alpha_{IP} I_P(t) + \beta_{IP} P(t).$$

4. Model Approach by System Dynamics

System Dynamics (SD) is a well known modelling approach, introduced by J. Forrester. SD is a methodology for studying and managing complex feedback systems, such as one finds in business and other social systems. In fact it has been used to address practically every sort of feedback system. Feedback refers to the situation of X affecting Y and Y in turn affecting X, perhaps through a chain of causes and effects. One cannot study the link between X and Y and, independently, the link between Y and X, and predict how the system will behave. Only the study of the whole system as a feedback system will lead to correct results.

For modelling, SD starts at qualitative level with a causal loop diagram. A *causal loop diagram* (CLD) is a diagram that aids in visualizing how interrelated variables affect one another.

The CLD consists of a set of nodes representing the variables connected together. The relationships between these variables, represented by arrows, can be labelled as positive or negative. There are two kinds of causal links, positive and negative. Positive causal links mean that the two nodes move in the same direction, i.e. if the node in which the link starts decreases, the other node also decreases. Similarly, if the node in which the link starts increases, the other node increases. Negative causal links are links in which the nodes change in opposite directions (an increase causes a decrease in another node, or a decrease causes an increase in another node).

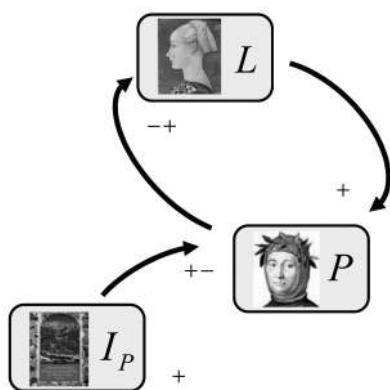


Figure 5. Causal loop diagram between Laura's and Petrarch's emotions and Petrarch's inspiration.

In case of Laura's and Petrarch's emotions $L(t)$ and $P(t)$, and Petrarch's inspiration $I_P(t)$, the causal relation is evident (Figure 5), but it is not possible to identify clearly positive and negative links.

SD continues the modelling process now at the quantitative level by a *stock and flow diagram* – (SFD), sometimes also called *level and rate diagram*. A stock variable is measured at one specific time. It represents a quantity existing at a given point in time, which may have been accumulated in the past. A *flow* variable is measured over an interval of time. Therefore a flow would be measured *per unit of time*. The variables in the CLD must be identified either as stock (level) or flow (rate) – or as auxiliary, and each stock (level) is connected in the SFD with its inflow – coming from a source – and by its outflow to a sink; flows are represented by double arrows and flow-controlling valves. The causal links from the CLP are found in the SFD as characterizing influences from stocks to flows (or from parameters and auxiliaries to flows).

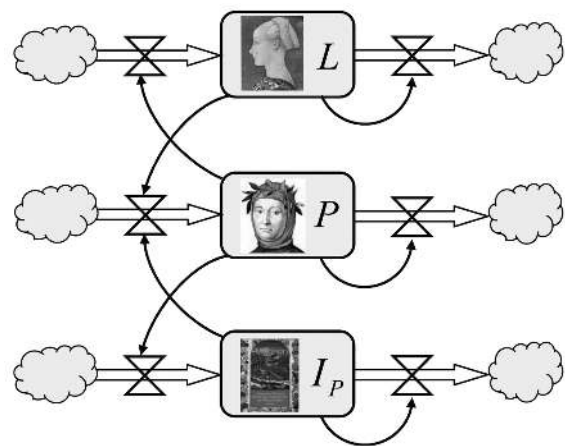


Figure 6. Qualitative stock and flow diagram for Laura's and Petrarch's emotions and for Petrarch's inspiration.

For the dynamics of emotion and inspiration under investigation, all $L(t)$, $P(t)$, and $I_P(t)$, are considered as stocks, whereby the adjoining flows are the changes (derivatives). A first simple SFD (Figure 6) for these dynamics shows similar structures for all three variables $L(t)$, $P(t)$, and $I_P(t)$. They are emerging from a source flow, influenced by other stocks, and they are fading to a sink (outflow) controlled by the stock itself. The stocks' feedback to the outflow let them 'converge' to zero (stabilizing

feedback), if no inflow is driving emotions and intuition.

SD’s modelling procedures now quantifies the SFD by introducing parameters and auxiliaries for the causal links and for the influences for the flows. Laura’s and Petrarch’s emotions and Petrarch’s intuition are fading with certain celerity, characterised by the parameters α_L , α_P , and α_{IP} in the direct feedbacks. And, in principle, also the feedback from Laura’s emotion $L(t)$ to Petrarch’s emotion $P(t)$ and vice versa is given by the parameters β_P and β_L , and the feedback from Petrarch’s emotion $P(t)$ to Petrarch’s inspiration $I_P(t)$ by the parameter β_{IP} . The appeal parameters A_L and A_P influence directly the inflow for the stocks $P(t)$ and $L(t)$ – indicated by additional control inputs for the respective inflows (summarized with the feedbacks from other stocks).

The SFD for the dynamics of emotion and inspiration in Figure 7 shows all basic feedbacks and direct inputs for the flows, weighted with parameters, but neglecting the nonlinear reaction of Laura, and the inspiration’s reciprocal feedback to Petrarch’s emotion (only ‘approximated’ by a classic feedback with parameter nature γ_L , which may be seen as linearization of the reciprocal feedback).

The model presented in the SFD in Figure 7 is still a simplified (linear) model, but the SFD can directly generate the system governing ODEs, which might be by used by any simulator.

This automatic generation of the ODE model is the last step in the SD modelling procedure. In principle, the SFD balances stocks and flows, so that for the simplified model the following ODEs are generated:

$$\begin{aligned} \frac{dL(t)}{dt} &= -\alpha_L L(t) + \beta_L P(t) + \beta_L A_P \\ \frac{dP(t)}{dt} &= -\alpha_P P(t) + \beta_P L(t) + \gamma_P I_P(t) \\ \frac{dI_P(t)}{dt} &= -\alpha_{IP} I_P(t) + \beta_{IP} P(t). \end{aligned}$$

Of course, the above equations are linear ODEs, and finally the nonlinearities must be modelled in the FSD. For this purpose, SD makes use of auxiliaries, which define the nonlinear nature of a feedback. Auxiliaries may have more than one input, and the output may be fed back into

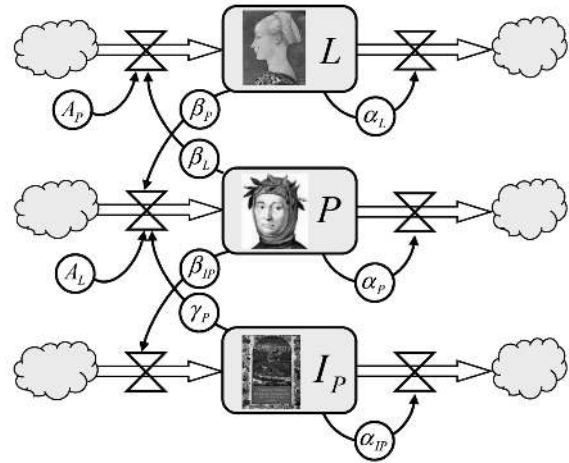


Figure 7. Stock and flow diagram for Laura’s and Petrarch’s emotions and for Petrarch’s inspiration with linear influences.

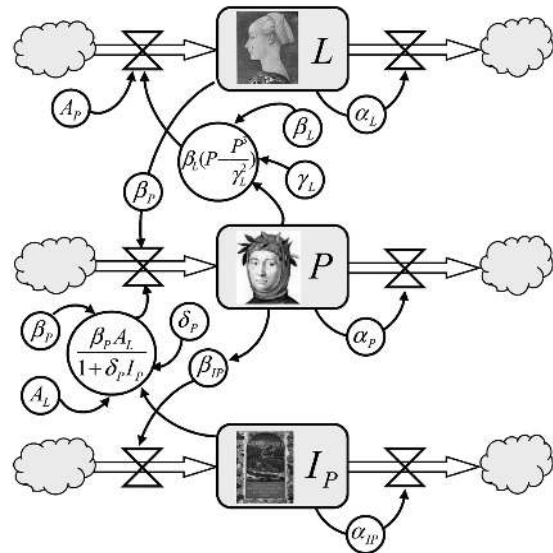


Figure 8. Stock and flow diagram for Laura’s and Petrarch’s emotions and for Petrarch’s inspiration with nonlinear reactions.

many different flows, or – if necessary, into further auxiliaries. Figure 8 presents the full SD model with all nonlinear reactions. The nonlinearities are of different quality. Choosing in the nonlinear cubic-like gain for Laura’s reaction $R_L(P)$ a big value for the parameter γ_L , the nonlinear auxiliary becomes almost linear (the nominator is bounded, usually less than 1). The nonlinear auxiliary for Laura’s appeal A_L becomes linear if the parameter δ_P is set to zero, letting the influence of Petrarch’s poetic inspiration vanish.

As with the simplified model in Figure 7, the ODE system is generated automatically from the FSD in Figure 8, to be used by a simulator, to be used directly in the SD modelling and simulation environment.

It can be concluded, that System Dynamics is a very appropriate tool for modelling the emotional dynamics under investigation – again it is underlined, that SD offers big benefits for modelling social systems, because relations and influences can be modelled step by step, from qualitative causal links to quantitative stock and flow diagrams, which can be simulated directly.

Finally it should be remarked, that SFDs can be directly mapped to block diagrams with transfer functions: stock and output flow with direct feedback of stock correspond to a first-order transfer function, with input represented by the input flow (mapped on a sum block); parameters in feedback links are mapped to gain blocks, auxiliaries nonlinear blocks (for details on approach by transfer functions see [3]).

5. Identification of the Laura-Petrarch Model

The big challenge is to identify the model parameters in the nonlinear Laura-Petrarch model, with two appeal parameters, with three gains, with three time constants, and with two parameters for the nonlinearity – in sum ten parameters. A brute-force identification starting with arbitrary values for these parameters is not successful, especially as the appeals may also be negative.

Consequently, first the size of the parameters and relations between them should be qualitatively analysed, following S. Rinaldi ([2]). The time constants α_L , α_P , and α_{IP} describe the forgetting processes. For Laura and Petrarch, obviously $\alpha_L > \alpha_P$ holds, because Laura never appears to be strongly involved, while the poet definitely has a tenacious attachment, documented by poems:

- In sonnet XXXV Petrarch claims:

*Solo et pensoso i piu deserti campi
vo mesurando a passi tardi e lenti,*

.....

*Ma pur si aspre vie ne' si selvage
cercar non so ch' Amor non venga sempre
ragionando con meco, et io col' lui.*

[Alone and lost in thought, each lonely strand

I measure out with slow and laggard step,

.....

*Yet I cannot find such harsh and savage trails
where love does not pursue me as I go,
with me communing, as with him do I.]*

The inspiration of the poet wanes very slowly, because Petrarch continues to write (over one hundred poems) for more than ten years after the death of Laura. The main theme of these lyrics is not his passion for Laura, which has long since faded, but the memory for her and the invocation of death:

- In Sonett CCLXVIII, written about two years after Laura's demise, Petrarch remembers:

*Tempo e ben di morire,
et o tardato piu ch'i non vorrei.*

*Madonna e morta, et a
seco il mio core;
e volendol seguire,
interromper conven quest'anni rei,
perche mai veder lei
di qua non spero, et l'aspettar
m'e noia.*

*[It's time indeed to die,
and I have lingered more than I desire.
My lady's dead, and with her my heart lies;
and, keen with her to fly,
I now would from this wicked world retire,
since I can no more aspire
on earth to see her, and delay will me de-
stroy.]*

Consequently, between the time constants α_{IP} and α_P the relation $\alpha_{IP} < \alpha_P$ must hold.

As Petrarch's inspiration holds about ten years, whereas Laura forgets Petrarch in about four months, and Petrarch's passion fades in one year, suitable relations and values are

$$\alpha_L \sim 3 \cdot \alpha_P, \alpha_P \sim 10 \cdot \alpha_{IP}, \alpha_P \sim 1.$$

The gains or reaction parameters β_L , β_P , and β_{IP} also can be estimated qualitatively, with respect to the time constants:

$$\beta_L \sim \alpha_P, \beta_P \sim 5 \cdot \alpha_P, \beta_{IP} \sim 10 \cdot \alpha_P.$$

Here the assumption is that Laura's reaction equals the forgetting time of Petrarch, and Petrarch reacts five times stronger. For simplicity, the parameters γ_L and δ_P are normalized to one, since it is always possible to scale $P(t)$ and $I_P(t)$ suitably.

The choice of the appeal parameters A_L and A_P is crucial, because these parameters determine the qualitative behaviour of the love dynamics – cyclic nonlinear behaviour, or damped oscillation toward an equilibrium. In case of Laura and Petrarch, cyclic love dynamics are expected in order to meet the experimentally founded emotional cycle $E(t)$ of Petrarch.

Clearly, Petrarch loves Laura, so $A_L > 0$ must hold. By contrast, Petrarch is a cold scholar interested in history and letters. He is appointed a *cappellanus continuus commensalis* by Cardinal Giovanni Colonna, and this ecclesiastic appointment brings him frequently to Avignon, where Laura lives. Consequently, Petrarch’s appeal A_P is assumed to be negative. Appropriate choices for the appeals A_L and A_P are:

$$A_L \sim 2, A_P \sim -1.$$

The negativity of the appeal of Petrarch for Laura is somehow recognized by the poet himself:

- In sonnet XLV, while Petrarch is talking about Laura’s mirror, he says:

*Il mio adversario in cui veder solete
gli occhi vostri ch’Amore
e’l ciel honora, ...
[My rival in whose depths
you’re wont to see
your own dear eyes which
Love and heaven apprise, ...]*

The above estimated ten parameter values, together with zero initial values for the love dynamics and for the poetic inspiration, are a good choice for identification. The ODE model has been implemented in MATLAB (for comparison, the transfer function model has been implemented in Simulink); for identification, a least squares method can be used:

$$\sum (P(t_k) - E_k)^2 \rightarrow \min$$

There, it makes sense to use relations between parameters, so that the number of parameters to be identified is reduced. As the quality of data is relatively poor, also different sets of parameter values may be seen as good approximation. Figure 9 shows an identification result for $P(t)$, with data E_k (‘graded’ poems).

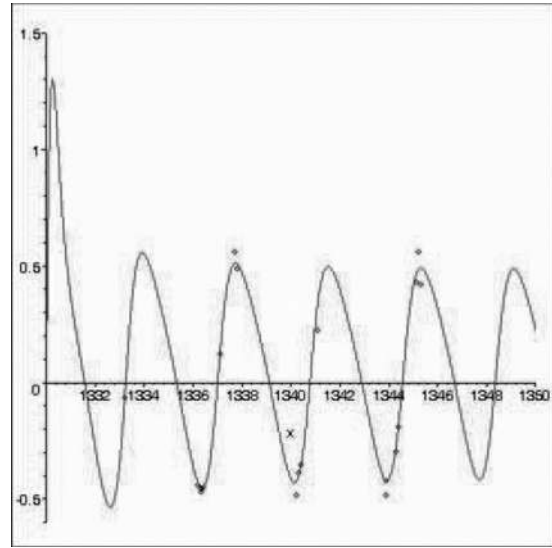


Figure 9. Result of model identification: love dynamics $P(t)$ for Petrarch coinciding with data from Petrarch’s emotional cycle $E(t_k)$, with data E_k (crosses and circles).

Of interest are, of course, also the love dynamics $L(t)$ for Laura and Petrarch’s poetic inspiration $I_P(t)$. All variables are presented in a MATLAB GUI, which drives experiments with the Laura-Petrarch model. The GUI (Figure 10) offers parameter input (sliders) and displays time

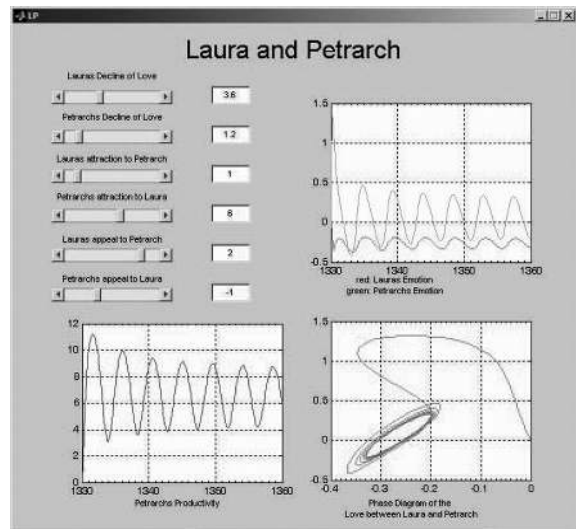


Figure 10. MATLAB GUI for experimenting with the Laura-Petrarch model:
 i) above left: sliders for gain β_L , time constant α_L , gain β_P , time constant α_P , and appeals A_L and A_P ;
 ii) upper right: love dynamics for Laura – $L(t)$, green – and Petrarch – $P(t)$, red – over time period 1130 – 1360;
 iii) lower left: poetic inspiration of Petrarch – $I_P(t)$, blue – over time period 1130 – 1360;
 iv) lower right: phase portrait $P(L)$ of love dynamics of Petrarch and Laura – P over L .

courses for $P(t)$ and $L(t)$ (together), the time course for $I_P(t)$, and a phase portrait $P(L)$.

Figure 10 shows all results for the identified parameters. The results of the numerical solution are qualitatively in full agreement with the *Canzoniere* and with the analysis of Frederic Jones. After a first high peak, Petrarch's love $P(t)$ tends toward a regular cycle characterized by alternate positive and negative peaks. Also, Laura's love $L(t)$ and Petrarch's poetic inspiration $I_P(t)$ tend towards a cyclic pattern.

At the beginning, Petrarch's inspiration $I_P(t)$ rises much more slowly than his love and then remains positive during the entire period. This might explain why Petrarch writes his first poem more than three years after he has met Laura, but then continues to produce lyrics without any significant interruption.

By contrast, Laura's love is always negative. This is in perfect agreement with the *Canzoniere*, where Laura is repeatedly described as adverse.

- In sonnet XXI, Petrarch calls Laura

dolce mia guerrera
[my sweet enemy].

- But in sonnet XLIV Petrarch says:

ne lagrima pero discese anchora
da' be' vostr'occhi, ma disdegno et ira.
[and still no tears your lovely eyes assail,
nothing as yet, but anger and disdain.]

The fit between $P(t)$ for Petrarch's love and $E(t_k)$ for Petrarch's emotional cycle is actually very good. It is of similar quality which is usually obtained when calibrating models of electrical and mechanical systems. Moreover, the fit could be further improved by slightly modifying the parameter values and by loosening some parameter relations.

But, improvement might be skipped, citing and agreeing with Rinaldi: *I do not want to give the impression that I believe that Petrarch had been producing his lyrics like a rigid, deterministic machine.* Nevertheless, one can conclude that the *Laura-Petrarch model* strongly supports Frederic Jones's conjecture on Petrarch's emotional cycle.

6. Experiments and Analysis with the Laura-Petrarch model

Experiments with the parameters show that the cyclic love dynamics may change to a damped oscillation converging to equilibrium. It is difficult to find out which parameter quality causes a cyclic behaviour, and which the damped oscillations. Starting with the classic Laura-Petrarch parameters, for instance, an increase of only one parameter α_L by a factor of 2.5 changes the qualitative behaviour essentially (Figure 11) – this parameter change means that Laura forgets Petrarch in about half time than before.

In this case all variables for $P(t)$, $L(t)$, and $I_P(t)$, show strongly damped oscillations, reaching equilibrium almost in ten years. The phase portrait $P(L)$ underlines this convergence in relatively fast time to an equilibrium of about $P = 0.05$ and $L = -0.0105$. This equilibrium still has a negative value for Laura's love emotion, but it is very small – almost as small as the positive value for Petrarch's emotion (see Figure 11).

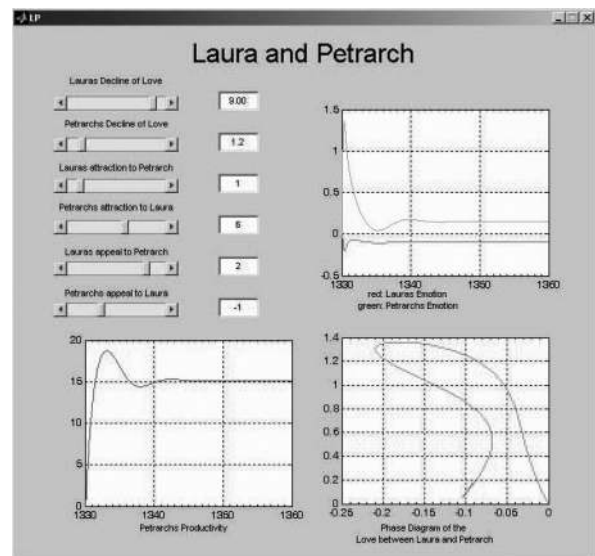


Figure 11. Experiment with Laura-Petrarch parameters and changed parameter α_L by a factor of 2.5: strongly damped variables converging to equilibrium with very small positive and negative values for P and L , resp.

In principle, the existence of an equilibrium is tied to the existence of steady state solutions for the ODE. Setting all derivatives to zero, and substituting I_P by the corresponding steady state for P , i.e.

$$I_P(t \rightarrow \infty) = \frac{\beta_{IP}}{\alpha_{IP}} P(t \rightarrow \infty),$$

results in two coupled nonlinear algebraic equations for P and L at steady state:

$$\begin{aligned} 0 &= -\alpha_L L + \beta_L P - \beta_L P^3 + \beta_L A_P \\ 0 &= -\alpha_P(\alpha_{IP} + \beta_{IP} P)P \\ &\quad + \beta_P(\alpha_{IP} + \beta_{IP} P)L + \alpha_{IP}\beta_P A_L. \end{aligned}$$

Rinaldi investigated in detail, by means of analytical methods, the case $A_L < 0$ and $A_P > 0$. First he tested the robustness of the Laura – Petrarch cyclic love dynamics, with respect to perturbations of the parameters.

For this purpose, the package LOCBIF, a professional software package for the analysis of the bifurcations of continuous-time dynamical systems, has been used. By varying only one parameter at a time, Rinaldi detected a supercritical Hopf bifurcation, by which the cycle eventually disappears.

Rinaldi continued with a detailed analysis of the limit cycle using substitution of the variable $L(t)$, transformation of parameters, and singular perturbation for $P(t)$ and $I_P(t)$. Results derive that, indeed, a supercritical Hopf bifurcation causes a change of the qualitative behaviour. One detailed result is a stability chart (Figure 12) for new parameters ϵ and μ :

$$\epsilon = \alpha_{IP}, \quad \mu = \frac{\beta_{IP}}{\alpha_{IP}}, \quad \beta_P \beta_L > \alpha_P \alpha_L.$$

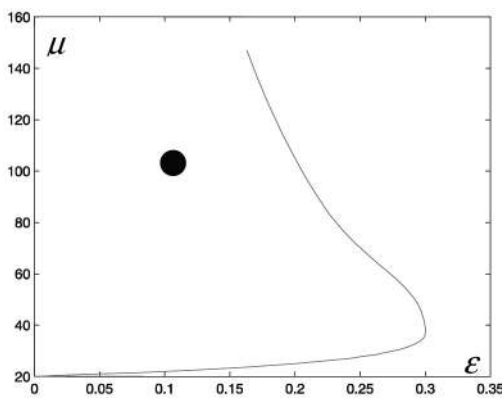


Figure 12. Stability chart $\mu(\epsilon)$ with border (red line; Hopf bifurcation) between cyclic solutions (left) and solutions converging to equilibrium (right); circle indicates the classical cyclic Laura-Petrarch solution.

In Figure 12, the border between cyclic solutions and solutions converging to equilibrium can be seen, represented as graph $\mu(\epsilon)$.

In general, cyclic solutions exist only in cases of nonsymmetric reactions, or in simple cases for appeal parameters with different signs. More details about these analytical investigations can be found in Rinaldi’s work ([2]).

Usually, if people fall in love, both are attractive for each other, which means that A_L and A_P must be positive. In this case, we meet only damped oscillations converging to a stable steady state:

$$A_L > 0, A_P > 0 \rightarrow \exists(P > 0, L > 0).$$

An interesting experiment is the case of an attractive Petrarch. Assuming e.g. that Petrarch is a young beautiful man, almost like Apollo, he may have the appeal $A_L \sim 6$ to Laura, three times the appeal of Laura to him (all other parameters unchanged).

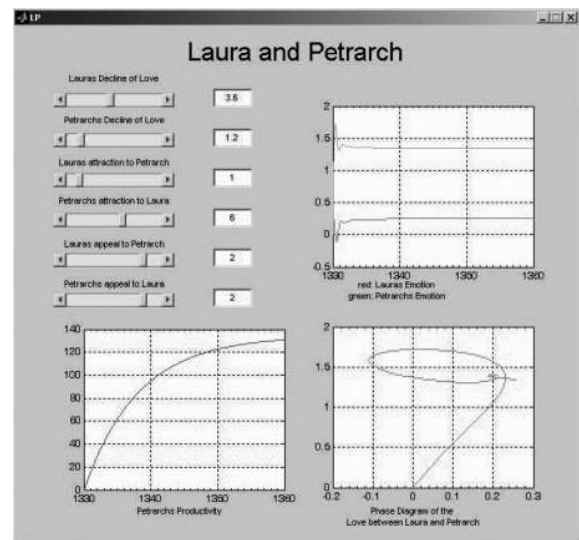


Figure 13. Experiment with Laura-Petrarch parameters and positive appeal $A_L \sim 6$ of Petrarch to Laura: very strongly damped variables converging to an equilibrium with positive steady states for \dot{P} and L , resp.

Figure 13 shows the MATLAB GUI with the results of this experiment. All variables $P(t)$, $L(t)$, and $I_P(t)$, are very strongly damped and converge to steady states with relative high positive values. The poetic inspiration $I_P(t)$ shows almost (negative) exponential behaviour. Nevertheless, $L(t)$ becomes negative for a short time period in the first three years. The phase diagram shows two crossings and it seems that the love dynamics first cycles two times, before it decides to converge to a stable positive feeling of both lovers.

The MATLAB GUI allows experiments with a broad variety of parameter sets. Of interest are, for instance, also cases which cycle more than hundred times before they decide to converge to an equilibrium. For 'extreme' parameter values the numerical solution may cause problems, or at least, take a long time; as ODE solver, a stiff solver has been chosen, having the best relative performance.

7. From Laura-Petrarch Model to Woman-Man Model

In times of gender equality women, as well as men, may play an active part in a love affair. Consequently, women also express their love by poems or other media, and they confess their love to public. By this, an additional stock with flow for the women's inspiration can be introduced easily. For Laura and Petrarch, this would mean that also Laura writes poems, that Petrarch's appeal is influenced by Laura's poetic inspiration, and that Petrarch shows more sensibility in his reactions to Laura. Consequently, the structure of the System Dynamics model (Figure 7) suggests a genuine and natural extension: symmetric stocks, flows, and feedbacks gains as well for 'Petrarch' and for 'Laura', which should now generally represent a man and a woman who fall in love.

The *Woman-Man Model* presented in Figure 14 describes the love dynamics $W(t)$ for a woman, and $M(t)$ for a man both falling in love with each other; love inspires both to communicate their love to public, in letters, in videos, with CDs and DVDs, etc. – represented by the inspiration variables $I_W(t)$ and $I_M(t)$.

A quantification of the qualitative SD model (Figure 14) with linear causal links and influences yields a SD – representation of *Man-Woman Model* with full symmetry (Figure 15).

As with the simplified Laura-Petrarch Model, the ODEs corresponding to the SFD in Figure 15 are linear, and they show the full symmetry between woman's and man's emotions and inspirations:

$$\begin{aligned} \frac{dW(t)}{dt} &= -\alpha_L W(t) + \beta_W M(t) + \gamma_W A_M \\ \frac{dI_W(t)}{dt} &= -\alpha_{IW} I_W(t) + \beta_{IW} W(t) \end{aligned}$$

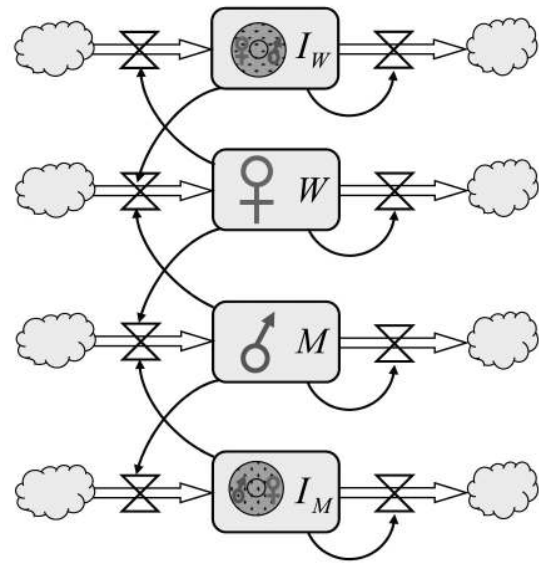


Figure 14. Qualitative System Dynamics Model for Woman's and Man's emotions and inspirations.

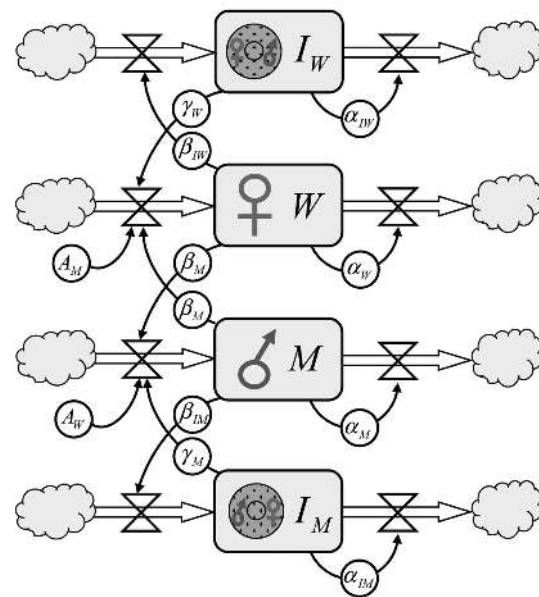


Figure 15. Simplified System Dynamics Model for Woman-Man Model with linear influences.

$$\begin{aligned} \frac{dM(t)}{dt} &= -\alpha_M M(t) + \beta_M M(t) + \gamma_M A_W \\ \frac{dI_M(t)}{dt} &= -\alpha_{IM} I_M(t) + \beta_{IM} M(t). \end{aligned}$$

Finally, the nonlinearities must be modelled in the FSD by auxiliaries, replacing the weighting parameters in the feedback. Because of the symmetry in emotions and inspirations, the model makes use of two nonlinear cubic-like reaction functions for woman's and man's re-

actions to each other, $R_W(M)$ and $R_M(W)$, and of two nonlinear relations between inspiration, appeal, and emotion. The nonlinearities for the reactions become almost linear for big values for the parameters γ_W and γ_M , resp; the nonlinearities for the inspiration-appeal relation become linear, if the parameters δ_W and δ_M tend towards zero. Figure 16 presents this complete nonlinear *Woman-Man Model* in SD notation (SFD).

Compared with the Laura-Petrarch Model, the Woman-Man Model must make use of an increased number of parameters: four fading parameters (instead of three), four (linear) weighting factors for the cross-feedbacks (instead of three), two appeal parameters (instead of one), and four parameters in the nonlinear functions (instead of two) – in sum 14 parameters. An analysis of *Man-Woman Model* is almost impossible, but numerical experiments may give interesting insight into love dynamics.

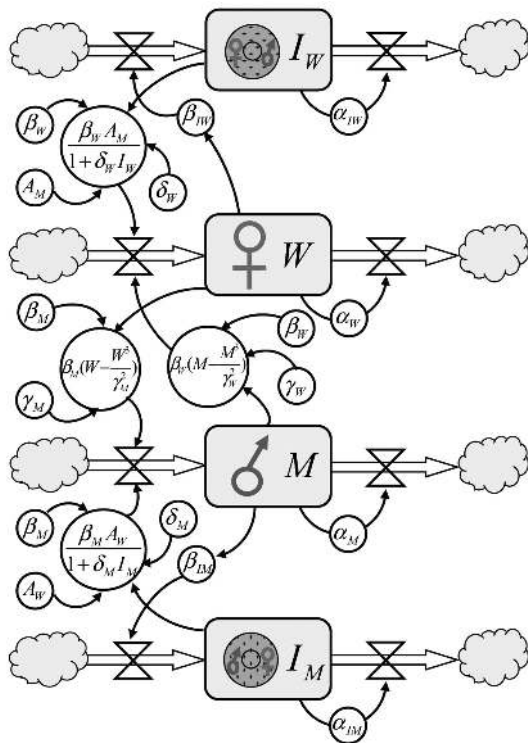


Figure 16. System Dynamics Model Stock and flow diagram for Woman-Man Model with nonlinear reactions.

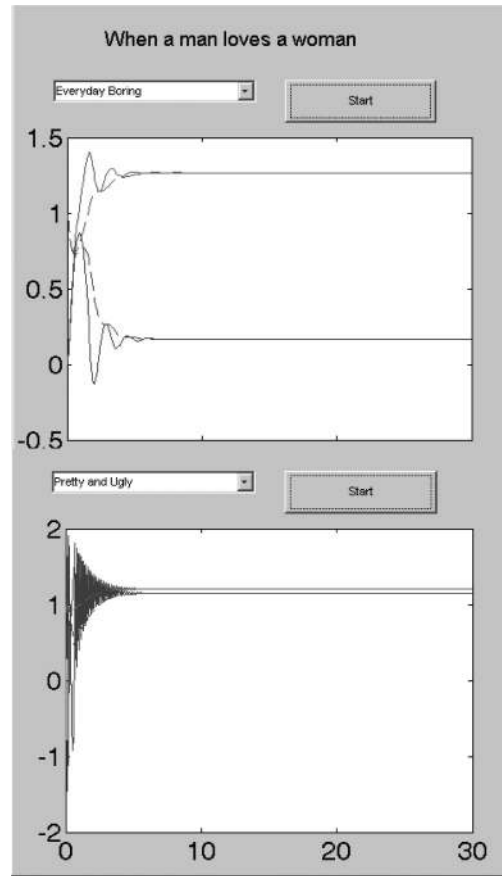


Figure 17. Experiment with Woman-Man Models: case studies 'Everyday Boring' and 'Pretty and Ugly' (red/blue – woman's/man's love emotion; red/blue dashed – woman's/man's love inspiration).

The ODE model equivalent to the SFD in Figure 16 shows now typical symmetric structure:

$$\begin{aligned} \frac{dW(t)}{dt} &= -\alpha_L W(t) + \beta_W M \left(1 - \left(\frac{M}{\gamma_W} \right)^2 \right) \\ &\quad + \beta_W \frac{A_M}{1 + \delta_W I_W(t)} \\ \frac{dI_W(t)}{dt} &= -\alpha_{IW} I_W(t) + \beta_{IW} W(t) \\ \frac{dM(t)}{dt} &= -\alpha_M M(t) + \beta_M W \left(1 - \left(\frac{W}{\gamma_M} \right)^2 \right) \\ &\quad + \beta_M \frac{A_W}{1 + \delta_M I_M(t)} \\ \frac{dI_M(t)}{dt} &= -\alpha_{IM} I_M(t) + \beta_{IM} M(t). \end{aligned}$$

As the Laura-Petrarch Model, the Woman-Man Model has been implemented in MATLAB (ODE model) and in Simulink (transfer functions model).

Experiments can be controlled by a simplified MATLAB GUI, as suggested by case studies. Figure 17 shows results of two case studies in a GUI, with predefined parameters characterizing typical parameter configurations ('Everyday Boring' and 'Pretty and Ugly').

8. Woman-Man Model with Dynamic Appeal Parameters

Does the Woman-Man model reflect reality? The model is able to mimicry different situations, but with one assumption: the appeal parameters A_M and A_W are constant. This assumption may not meet reality; the appeal for each other may change and may be controlled.

A dynamic appeal can be easily modelled by time-dependent appeal variables $A_M(t)$ and $A_W(t)$, resulting in a small change in the ODE model:

$$\begin{aligned} \frac{dW(t)}{dt} &= -\alpha_L W(t) + \beta_W M \left(1 - \left(\frac{M}{\gamma_W} \right)^2 \right) \\ &\quad + \beta_W \frac{A_M(t)}{1 + \delta_W I_W(t)} \\ \frac{dI_W(t)}{dt} &= -\alpha_{IW} I_W(t) + \beta_{IW} W(t) \\ \frac{dM(t)}{dt} &= -\alpha_M M(t) + \beta_M W \left(1 - \left(\frac{W}{\gamma_M} \right)^2 \right) \\ &\quad + \beta_M \frac{A_W(t)}{1 + \delta_M I_M(t)} \\ \frac{dI_M(t)}{dt} &= -\alpha_{IM} I_M(t) + \beta_{IM} M(t). \end{aligned}$$

Case studies may become now very complicated, because not only 14 parameters have to be chosen appropriately, but also the function $A_M(t)$ and $A_W(t)$ have to be provided meaningfully. An extended version of the MATLAB GUI presented in Figure 17 allows additionally providing predefined appeal functions. Figure 18 and Figure 19 show results for perhaps interesting cases: the appeal decreases exponentially – should happen, and second, the appeal $A_M(t)$ is an increasing step function – could model a plastic surgery.

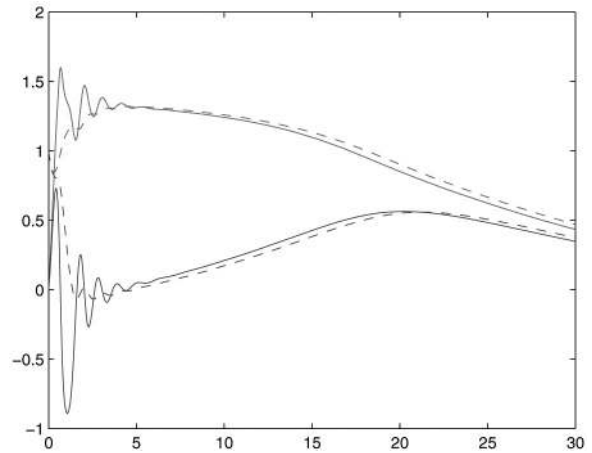


Figure 18. Experiment with Woman-Man Model: exponentially decreasing appeal functions (red/blue – woman's/man's love emotion; red/blue dashed – woman's/man's love inspiration).

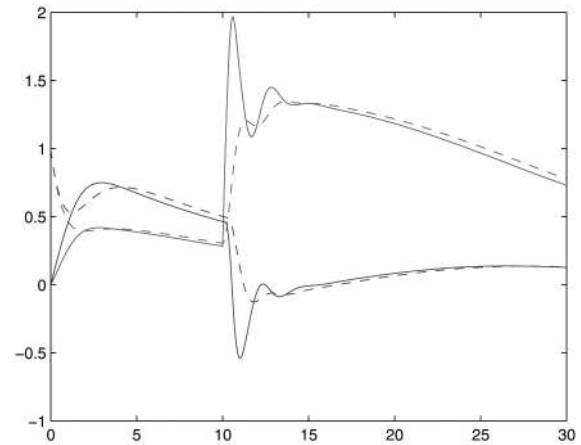


Figure 19. Experiment with Woman-Man Model – positive step in appeal at ten years (time courses as in Figure 18).

9. Conclusion

In principle, the contribution could show,

- that it is possible to model in some detail the love dynamics between two persons by System Dynamics, resulting in nonlinear ODE systems by ODEs,
- that, additionally, a poetic inspiration caused by love emotions can be modelled in System Dynamics, extending the ODE system for the love dynamics,
- and that the model can be validated with data from history, reflecting the emotions between Petrarch and Laura.

Of course, this contribution presents serious investigations. But is it possible to investigate the dynamics of love, perhaps the most important phenomenon concerning our lives, seriously, by methods of mathematics and engineering? One could also conclude, it might be better not to tackle the secrets of love, because described and controlled by formula, it is not love any longer. In this view, the contribution might be seen as reference to Petrarch and the most beautiful love poems the author has ever read.

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Received: June, 2008

Accepted: September, 2008

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In 2006 he initiated the LAURA Group at Vienna University of Technology, gathering people interested in modelling and simulation of emotions, starting the group with the Laura – Petrarch model. Felix Breiteneker is head of LAURA Group; he develops different model approaches, and acts as narrator in multimedia speeches on love emotions.

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Currently he is concentrating on E-learning, especially enhancing the mathematical education of engineering students through E-learning using MATLAB and Maple T.A by developing web courses for blended learning. Up to now, Florian Judex has published 10 papers in the area of modelling and simulation. He is member of ASIM, the German Simulation Society, and winner of the TU Vienna E-learning Prize 2008.

In LAURA Group, Florian Judex programs and implements the MATLAB and MAPLE models and GUIs, and presents them in multimedia speeches.

NIKOLAS POPPER has earned a degree in technical mathematics at the Vienna University of Technology. His thesis title is “Simulation of the Respiratory System – Compartment Modelling and Modelling of Perfusion”. After research stays abroad in Barcelona, Universidad Politècnica de Catalunya and Moscow, Idaho, University of Idaho, he has gained experience in industry projects as well as research and development knowledge. Currently he is working in the area of visualization in computer graphics, modelling and simulation of health economy and theory of modelling & simulation. Nikolas Popper is co-proprietor of the company “Die Drahtwarenhandlung” Simulation Services. The company also offers technical solutions (defect detection on pictures, modelling & simulation, . . .) as well as animations and films in the area of science journalism, and E-learning. Furthermore, he is doing a PhD thesis in the area of alternative and structural dynamic models at the Vienna University of Technology.

For LAURA Group, he is preparing multimedia content for multimedia speeches, based on historical research on Petrarch, Laura, and on research on emotions modelling.

KATHARINA BREITENECKER studied “Technical Physics” at the Vienna University of Technology with an emphasis on nuclear physics. During her last terms of her master’s she was also involved in modelling and simulation, doing some projects at the department of atomic physics. In 2005 she got her master degree with a master thesis about the analyses of trace elements by nuclear activation analysis in Greek pumice within an UNESCO project. She continued working on her PhD on the migration behaviour of transuranic elements in the soil by cellular automata modelling, finished in 2008. Currently she is working for the International Atomic Energy Agency in the Safeguards Analytical Laboratory.

She is an active member of the Austrian Society for Nuclear Techniques, of the German Society for Liquid Scintillation Counting, and of the German Simulation Society ASIM. She is also presenting E-learning courses organized by ARGESIM. She has made publications in both, nuclear analytics (radiometry, radiochemistry), and mathematical modelling.

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ANNA MATHE studied Media Design in Linz and now works as a freelancer for web design, copywriting and graphic design. Since 2003, she has managed organisational affairs for ARGESIM – ARGE Simulation News TU Vienna – in supporting organization of ARGESIM seminars, in co-ordination and supporting editing and production of SNE (Simulation News Europe). Since 2008, she is engaged in administration and in graphical design for ASIM, the German Simulation Society.

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In LAURA Group, Andreas Mathe takes care of quality management. In multimedia speeches, he co-ordinates voice, sound and multimedia presentations, and he presents sonnets in English.
