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LOVE'S CIRCULAR PATCH PROBLEM REVISITED: CLOSED FORM SOLUTIONS FOR TRANSVERSE ISOTROPY AND SHEAR LOADING

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Abstract. This paper considers the title problem of uniform pressure or shear traction applied over a circular area on the surface of an elastic half space. The half space is transversely isotropic, where the planes of isotropy are parallel to the surface. A potential function method is adopted where the elastic field is written in terms of three harmonic functions. The known point force potential functions are used to find the solution for uniform pressure or shear traction over a circular area by quadrature. Using methods developed by Love (1929) and Fabrikant (1988), the elastic displacement and stress fields for normal and shear loading are evaluated in terms of closed form expressions containing complete elliptic integrals of the first, second, and third kinds. The solution for uniform normal pressure on an isotropic half space was previously found by Love (1929). The present results for transverse isotropy including shear loading are new. During the course of this research, a new relation has been discovered between different forms of the complete elliptic integral of the third kind. This has allowed the present solution to be put in a more convenient form than that used by Love. Following a limiting procedure allows the isotropic solution to be obtained. It is shown that for normal loading the present results agree with Love's solution, while the results for shear loading of an isotropic half space are also apparently new. Special consideration is also given to derive the limiting form of the elastic field on the z-axis and the surface ($z = 0$).

1. Introduction. Loading applied to the surface of an elastic half space is an old problem within the field of elasticity. The origins of this problem can be traced back to the early investigations of Boussinesq (1885). He gave a formal solution for the elastic field in terms of derivatives of a potential function. This logarithmic potential function was in the form of a double integral taken over the pressed area. It was noted by Love (1929) that, "The difficulty of evaluating the integral has been a serious obstacle to the development of the formal solution in special cases."

Love also noted an alternative method, developed by Lamb (1902) and Terazawa (1916), where the components of displacements and stresses were expressed in terms

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of single integrals containing Bessel functions. It was pointed out that this solution procedure was applicable only to circular boundaries. This method, now known as the Hankel transform method, was used by Terazawa (1916) to evaluate the elastic field in an isotropic half space when a circular region was acted upon by a uniform pressure (see Sneddon, 1951). Sneddon outlines the Hankel transform method and obtains Terazawa's solution as a special case of uniform pressure. Sneddon also noted that carrying out the inverse transform integrals cannot be done in a simple way and refers the reader to Terazawa's paper. The Hankel transform method was extended to shear loading by Muki (1960). He gave a general formulation for an arbitrary loading over a circular area on an isotropic half space. He considered uniform shear loading as a special case. He noted that the inverse Hankel transform integrals could be evaluated by the results in Eason et al. (1955). However, he did not include these results in this article (although it is apparent that he did work out the details since figures for subsurface stress fields were given). Some of the details of these integral evaluations may be in his previous work referenced in that paper.

It was apparently Love (1929) who first successfully applied Boussinesq's *potential method* to the solution of a special case. Concerned with the safety of foundations, Love investigated the stresses produced in an elastic, isotropic half space when a uniform pressure was applied to part of the surface. His choice of pressure distribution was motivated by his comment. "The law of distribution of pressure on the bases of walls and pillars is not known, but it would seem to be reasonable to assume that it is often not very far from being uniform." He used rectangular and circular patches of uniform pressure and evaluated the subsurface stress fields for each case. For the rectangular patch solution he was able to perform the double integrals of the derivatives of the logarithmic potential function directly in terms of elementary functions. For the circular area, he transformed the double integrals for the derivatives of the potential function over the circle into a line integral around the circle's circumference, evaluating them in terms of elliptic integrals.

Some pertinent details to his analysis for the circular patch are now discussed. Love defined the double integral of the logarithmic potential function as χ. He also used the symbol V, which is the derivative of χ with respect to z. Love evaluated all derivatives of χ and V needed for the stress field. Although he was apparently not concerned with the displacement field, the derivatives he evaluated were sufficient to also determine the radial displacement. He did not evaluate the function V, needed to obtain the normal displacement w. It is apparent from his analysis that the method he used to evaluate radial derivatives cannot be applied to z derivatives. However, he determined the second derivative of V with respect to z, needed for the stress field, through using the radial derivatives and the harmonic property of V. Similarly he found the first z derivative of V using the radial derivatives and the harmonic property of χ. The integral evaluations, and hence the elastic stress field, were given in terms of complete and incomplete elliptic integrals of the first and second kind.

In the following, Boussinesq's potential function method, now called the Green's function method, is again used. The present potential function formulation for transversely isotropic materials (where the isotropic planes are parallel to the free surface) was developed by Elliot (1948). It was put in its present form by Fabrikant (1989). It turns out that the point force Green's function potentials are identical to the corresponding isotropic one, apart from multiplicative material constants. Integration of these point normal and shear force potentials provides the solution for distributed loading. Love's potential χ is presently denoted as ψ while χ is now used for the integral of ψ with respect to z ($\chi = z\psi - \Phi$, where Φ is a function needed for the shear loading case not considered by Love). Love's analysis is followed to evaluate the radial derivatives of ψ (needed for normal loading) and also for χ (needed for shear loading). However, present analysis evaluates the derivatives in terms of complete elliptic integrals of the first, second, and third kinds in contrast to Love's use of incomplete elliptic integrals of the first and second kind. Additionally, the derivative of ψ with respect to z (Love's V) is obtained here to give the complete field of displacements.

This derivative is evaluated by a new method, developed by Fabrikant (1988), which utilizes an integral representation for the reciprocal of the distance between points in three-dimensional space. The complete elliptic integral of the third kind appears in a different, more convenient form than in the evaluations using Love's method for the radial derivatives. This has led to the discovery of an important transformation relation for this elliptic integral, similar to those already known for complete elliptic integrals of the first and second kind, albeit slightly more complicated. Nevertheless, it allows the elastic field for normal and shear loading to be put in a much more convenient form than that given by Love.

2. Potential functions for transverse isotropy. The transversely isotropic half space is taken as the region $z > 0$ where the surface $z = 0$ is parallel to the plane of isotropy; see Fig. 1. A potential function formulation was first given by Elliot (1948). The notation of Fabrikant (1989) is presently adopted. The stress-strain relations in Cartesian components are

$$\begin{aligned}
\sigma_{xx} &= A_{11}\frac{\partial u}{\partial x} + (A_{11} - 2A_{66})\frac{\partial v}{\partial y} + A_{13}\frac{\partial w}{\partial z},\\
\sigma_{yy} &= (A_{11} - 2A_{66})\frac{\partial u}{\partial x} + A_{11}\frac{\partial v}{\partial y} + A_{13}\frac{\partial w}{\partial z},\\
\sigma_{zz} &= A_{13}\frac{\partial u}{\partial x} + A_{13}\frac{\partial v}{\partial y} + A_{33}\frac{\partial w}{\partial z},\\
\tau_{xy} &= A_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\\
\tau_{xz} &= A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),\\
\tau_{yz} &= A_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),
\end{aligned} \tag{1}$$

where u, v, and w are the displacements in the x, y, and z directions. Here A_{11}, A_{13}, A_{33}, A_{44}, and A_{66} are the elastic constants.

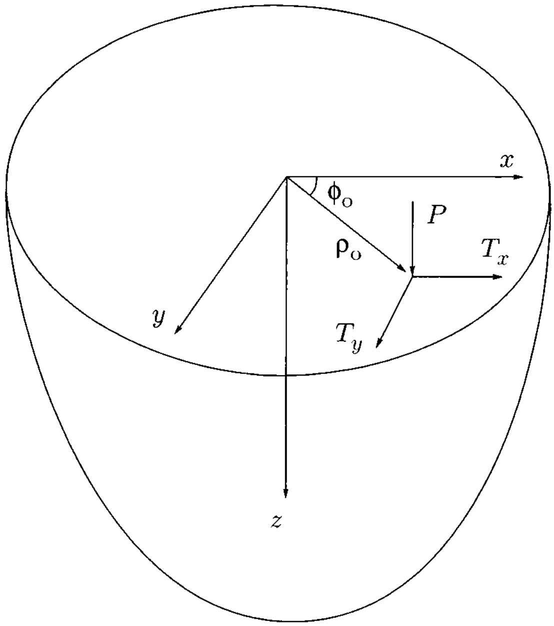

FIG. 1. Geometry and coordinate system for point loading

The solution of the equilibrium equations in terms of three potential functions F_1, F_2, and F_3 is given by Fabrikant (1989) in the form

$$u^c = \Lambda(F_1 + F_2 + iF_3), \qquad w = m_1 \frac{\partial F_1}{\partial z} + m_2 \frac{\partial F_2}{\partial z}, \tag{2}$$

with i being the complex number, $i = \sqrt{(-1)}$, m_1 and m_2 are constants defined below, and u^c is the complex displacement $u^c = u + iv$. The operator Λ and the operator Δ used subsequently are given as

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3}$$

The functions F_j satisfy the relations

$$\Delta F_j + \gamma_j^2 \frac{\partial^2 F_j}{\partial z^2} = 0, \quad j = 1, 2, 3, \tag{4}$$

where γ_j are also constants. The constant γ_3 is given as $\gamma_3^2 = A_{44}/A_{66}$ while $\gamma_j^2 = n_j$, $j = 1, 2$, and n_j are the two (real or complex conjugate) roots of the quadratic equation

$$A_{11}A_{44}n_j^2 + [A_{13}(A_{13} + 2A_{44}) - A_{11}A_{33}]n_j + A_{33}A_{44} = 0. \tag{5}$$

The constants m_j are related to γ_j as

$$m_j = \frac{A_{11}\gamma_j^2 - A_{44}}{A_{13} + A_{44}} = \frac{(A_{13} + A_{44})\gamma_j^2}{A_{33} - \gamma_j^2 A_{44}}, \quad j = 1, 2. \tag{6}$$

Using the stress combinations $\sigma_1 = \sigma_{xx} + \sigma_{yy} = \sigma_{\rho\rho} + \sigma_{\phi\phi}$, $\sigma_2 = \sigma_{xx} - \sigma_{yy} + 2i\tau_{xy} = e^{2i\phi}(\sigma_{\rho\rho} - \sigma_{\phi\phi} + 2i\tau_{\rho\phi})$, and $\tau_z = \tau_{xz} + i\tau_{yz} = e^{i\phi}(\tau_{\rho z} + i\tau_{\phi z})$, the stress field can be written in the following form:

$$\begin{aligned}
\sigma_1 &= 2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2 - (1+m_1)\gamma_3^2]F_1 + [\gamma_2^2 - (1+m_2)\gamma_3^2]F_2\},\\
\sigma_2 &= 2A_{66}\Lambda^2[F_1 + F_2 + iF_3],\\
\sigma_{zz} &= A_{44}\frac{\partial^2}{\partial z^2}[\gamma_1^2(1+m_1)F_1 + \gamma_2^2(1+m_2)F_2],\\
\tau_z &= A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1 + (1+m_2)F_2 + iF_3].
\end{aligned} \tag{7}$$

3. Point force Green's functions. The solution for uniform loading over a circular area will be obtained by integration of the point force Green's functions. The geometry and coordinate system are shown in Fig. 1. Using cylindrical coordinates (ρ, ϕ, z), a point force is applied on the surface at ρ_0, ϕ_0 with components T_x, T_y, and P in the x, y, and z directions, respectively.

The potential functions for these fundamental point force solutions were put in a very convenient form by Fabrikant (1989). For the point normal force P, the potentials are

$$\begin{aligned}
F_1(\rho,\phi,z;\rho_0,\phi_0) &= \frac{H\gamma_1}{(m_1-1)}P\ln[R_1 + z_1],\\
F_2(\rho,\phi,z;\rho_0,\phi_0) &= \frac{H\gamma_2}{(m_2-1)}P\ln[R_2 + z_2],\\
F_3(\rho,\phi,z;\rho_0,\phi_0) &= 0
\end{aligned} \tag{8}$$

with

$$R_j^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi-\phi_0) + z_j^2, \quad z_j = \frac{z}{\gamma_j}, \ j = 1,2,3, \tag{9}$$

and the constant H is defined as

$$H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33} - A_{13}^2)}. \tag{10}$$

The potentials for point shear loading are given as

$$\begin{aligned}
F_1(\rho,\phi,z;\rho_0,\phi_0) &= \frac{H\gamma_1}{(m_1-1)}\frac{\gamma_2}{2}(T\overline{\Lambda} + \overline{T}\Lambda)\chi(z_1),\\
F_2(\rho,\phi,z;\rho_0,\phi_0) &= \frac{H\gamma_2}{(m_2-1)}\frac{\gamma_1}{2}(T\overline{\Lambda} + \overline{T}\Lambda)\chi(z_2),\\
F_3(\rho,\phi,z;\rho_0,\phi_0) &= i\frac{\gamma_3}{4\pi A_{44}}(T\overline{\Lambda} - \overline{T}\Lambda)\chi(z_3),
\end{aligned} \tag{11}$$

where $T = T_x + iT_y$, an overbar indicates complex conjugation, and the function $\chi(z_j)$ is

$$\chi(z_j) = z_j\ln[R_j + z_j] - R_j, \quad j = 1,2,3. \tag{12}$$

4. Mathematical preliminaries. Before deriving the solution for the title problem, a discussion of some mathematical details is necessary. The solution is evaluated in terms of complete elliptic integrals of the first, second, and third kinds. These are denoted as $\mathbf{F}(k), \mathbf{E}(k)$, and $\Pi(n,k)$ respectively. They are given in integral form as (Gradshteyn and Ryzhik, 1980)

$$\mathbf{F}(k) = \int_0^1 \frac{dx}{(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} = \int_0^{\pi/2} \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}, \tag{13}$$

$$\mathbf{E}(k) = \int_0^1 \frac{(1-k^2x^2)^{1/2}dx}{(1-x^2)^{1/2}} = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}d\theta, \tag{14}$$

$$\begin{aligned}\Pi(n,k) &= \int_0^1 \frac{dx}{(1-nx^2)(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} \\ &= \int_0^{\pi/2} \frac{d\theta}{(1-n\sin^2\theta)(1-k^2\sin^2\theta)^{1/2}},\end{aligned} \tag{15}$$

where k is called the modulus and $k' = (1-k^2)^{1/2}$ is the complementary modulus $(0 < k, k', n < 1)$. The solution will be explicitly written in terms of the coordinates (ρ, ϕ, z), the radius of loading $\rho = a$, and the two parameters $l_1(a)$, $l_2(a)$ given as

$$\begin{aligned}l_1(a) &= \tfrac{1}{2}\{[(\rho+a)^2+z^2]^{1/2} - [(\rho-a)^2+z^2]^{1/2}\}, \\ l_2(a) &= \tfrac{1}{2}\{[(\rho+a)^2+z^2]^{1/2} + [(\rho-a)^2+z^2]^{1/2}\}.\end{aligned} \tag{16}$$

These parameters, first introduced by Fabrikant (1989), allow the three-dimensional distance between the point $(a, 0, 0)$ on the surface and the interior point (ρ, ϕ, z) to be written in two-dimensional form, that is,

$$\rho^2 + a^2 + z^2 - 2a\rho\cos(\phi) = l_1^2(a) + l_2^2(a) - 2l_1(a)l_2(a)\cos(\phi) \tag{17}$$

where

$$l_1^2(a) + l_2^2(a) = \rho^2 + a^2 + z^2, \qquad l_1(a)l_2(a) = a\rho. \tag{18}$$

Note that it is easy to see $l_1(a) < l_2(a)$, while it can also be shown that $l_1(a) < \rho$. In the solution derived subsequently, the parameters k and n in Eqs. (13–15) will be given as

$$k = \frac{l_1(a)}{l_2(a)}, \qquad n = \frac{l_1^2(a)}{\rho^2}. \tag{19}$$

The solution for uniform normal pressure on an isotropic half space was derived in terms of elliptic integrals previously by Love (1929). He introduced two positive parameters r_1 and r_2 defined as

$$r_1^2 = (\rho-a)^2 + z^2, \qquad r_2^2 = (\rho+a)^2 + z^2. \tag{20}$$

Love used the modulus $k^L = r_1/r_2$ and the complementary modulus $k^{L'} = (1-k^{L^2})^{1/2}$, where a superscript L is used to denote Love's moduli. His solution was written in terms of complete elliptic integrals of the complementary modulus $k^{L'}$ as $\mathbf{E}(k^{L'}) = \mathbf{E}'$ and $\mathbf{F}(k^{L'}) = \mathbf{F}'$ (where Love used $\mathbf{K}$ instead of $\mathbf{F}$) and incomplete elliptic integrals as well. In order to compare the present solution with that from Love, the following relations are easily derived:

$$r_1 = l_2(a) - l_1(a), \qquad r_2 = l_2(a) + l_1(a),$$
$$k^L = \frac{l_2(a) - l_1(a)}{l_2(a) + l_1(a)} = \frac{1-k}{1+k}, \qquad k^{L'} = \left\{ \frac{4l_1(a)l_2(a)}{[l_2(a)+l_1(a)]^2} \right\}^{1/2} = \frac{2\sqrt{k}}{1+k}. \tag{21}$$

In addition to Eq. (21) the following conversion formulas are also useful (Gradshteyn and Ryzhik, 1980):

$$\mathbf{F}(k^{L'}) = \mathbf{F}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\mathbf{F}(k), \qquad \mathbf{E}(k^{L'}) = \mathbf{E}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{2\mathbf{E}(k) - (1-k^2)\mathbf{F}(k)}{1+k}. \tag{22}$$

It is important to note that some of the derivatives of the potential function needed for the elastic field can be evaluated in terms of complete elliptic integrals of the first and second kinds. Other derivatives require all three complete elliptic integrals. Love did not utilize the complete elliptic integral of the third kind in his solution. In its place he used a combination of complete and incomplete elliptic integrals of the first and second kind. Since the complete elliptic integral of the third kind is used presently, it is desirable to have a transformation formula for it, similar to Eq. (22) for the other two complete elliptic integrals. However, such a formula could not be found in the standard tables. The necessary formula is derived later in the paper, but we will have a need for it at an earlier point; so it is given without derivation presently as $(z > 0)$

$$\begin{aligned}\Pi\left(p, \frac{2\sqrt{k}}{1+k}\right) &= \frac{\pi l_2(a)(1+k)(a+\rho)}{4z\rho^2(a-\rho)} \left\{ 2\rho^2 - |a^2-\rho^2| + \frac{a^4-\rho^4}{|a^2-\rho^2|} \right\} \\ &\quad + \frac{(1+k)(a+\rho)}{(a-\rho)} \{\mathbf{F}(k) - 2\Pi(n,k)\}, \qquad p = \frac{4a\rho}{(a+\rho)^2}.\end{aligned} \tag{23}$$

Finally, it will be necessary to differentiate the complete elliptic integrals of the first and second kind with respect to z and ρ, where k is a function of these coordinates through Eq. (19). The derivatives of the complete elliptic integrals with respect to k are found in Gradshteyn and Ryzhik (1980) as

$$\frac{\partial \mathbf{E}(k)}{\partial k} = \frac{\mathbf{E}(k) - \mathbf{F}(k)}{k}, \qquad \frac{\partial \mathbf{F}(k)}{\partial k} = -\frac{\mathbf{F}(k)}{k} + \frac{\mathbf{E}(k)}{k(1-k^2)}. \tag{24}$$

Additionally, one can verify the results

$$\frac{\partial k}{\partial z} = -\frac{2zk}{l_2^2(a)(1-k^2)}, \qquad \frac{\partial k}{\partial \rho} = \frac{a(a^2+z^2-\rho^2)}{l_2^4(a)(1-k^2)}, \tag{25}$$

where we have used the following differential relations provided by Fabrikant (1989):

$$\begin{aligned}\frac{\partial l_1(a)}{\partial z} &= -\frac{z l_1(a)}{l_2^2(a) - l_1^2(a)}, \qquad & \frac{\partial l_2(a)}{\partial z} &= \frac{z l_2(a)}{l_2^2(a) - l_1^2(a)}, \\ \frac{\partial l_1(a)}{\partial \rho} &= \frac{\rho[a^2 - l_1^2(a)]}{l_1(a)[l_2^2(a) - l_1^2(a)]}, \qquad & \frac{\partial l_2(a)}{\partial \rho} &= \frac{\rho[l_2^2(a) - a^2]}{l_2(a)[l_2^2(a) - l_1^2(a)]}.\end{aligned} \tag{26}$$

The needed derivatives of the complete elliptic integrals of the first and second kinds can be written directly using the chain rule and the above results.

5. Uniform normal pressure on a transversely isotropic half space. Consider the half space shown in Fig. 1 with the surface being free of shear stress and a uniform normal compressive stress being applied within the region $\rho < a$. The magnitude of the pressure is denoted as σ. The solution can be obtained by replacing the force P in Eq. (8) with $\sigma\rho_0\,d\rho_0\,d\phi_0$ and integrating the result over $0 < \rho_0 < a$, $0 < \phi_0 < 2\pi$. The potential functions become

$$\begin{aligned} F_1(\rho,\phi,z) &= \frac{H\gamma_1\sigma}{(m_1-1)}\psi(\rho,z_1), \\ F_2(\rho,\phi,z) &= \frac{H\gamma_2\sigma}{(m_2-1)}\psi(\rho,z_2), \\ F_3(\rho,\phi,z) &= 0, \end{aligned} \tag{27}$$

where

$$\psi(\rho,z) = \int_0^{2\pi}\int_0^a \ln[R+z]\rho_0\,d\rho_0\,d\phi_0, \qquad R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2, \tag{28}$$

where ψ corresponds to Love's function χ. Note that the integral is independent of the polar angle ϕ and is thus a function of ρ and z only.

To determine the elastic field, the derivatives of this potential function are needed. Since

$$\Lambda\psi(\rho,z_j) = e^{i\phi}\frac{\partial}{\partial\rho}\psi(\rho,z_j), \qquad \frac{\partial}{\partial z}\psi(\rho,z_j) = \frac{1}{\gamma_j}\frac{\partial}{\partial z_j}\psi(\rho,z_j), \tag{29}$$

it is necessary to evaluate $\frac{\partial}{\partial\rho}\psi(\rho,z)$ and $\frac{\partial}{\partial z}\psi(\rho,z)$ for the displacement field. To find the ρ derivative, Love's procedure is adopted from his paper, and it is easy to show that

$$\frac{\partial}{\partial\rho}\psi(\rho,z) = \frac{\pi}{2\rho}\{a^2+\rho^2-|a^2-\rho^2|\} + \frac{z}{4\rho}I_1 - \frac{z(a^2+\rho^2)}{2\rho}I_2 + \frac{z(a^2-\rho^2)^2}{4\rho}I_3, \tag{30}$$

where

$$\begin{aligned} I_1 &= \int_0^{2\pi}\frac{(a^2+\rho^2-2a\rho\cos\theta)\,d\theta}{\overline{R}}, \\ I_2 &= \int_0^{2\pi}\frac{d\theta}{\overline{R}}, \qquad \overline{R}^2 = \rho^2+a^2+z^2-2a\rho\cos\theta, \\ I_3 &= \int_0^{2\pi}\frac{d\theta}{(a^2+\rho^2-2a\rho\cos\theta)\overline{R}}. \end{aligned} \tag{31}$$

Considering I_2 first, Eq. (17), the transformation $\theta = \pi - 2\alpha$ and $\cos(2\alpha) = 1 - 2\sin^2\alpha$ yield

$$I_2 = \frac{4}{l_2(a)}\int_0^{\pi/2}\frac{d\alpha}{[(1+k)^2-4k\sin^2\alpha]^{1/2}} = \frac{4}{(1+k)l_2(a)}\mathbf{F}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{4}{l_2(a)}\mathbf{F}(k) \tag{32}$$

where Eq. (22) has been used. For I_1, it is rewritten and evaluated as

$$\begin{aligned} I_1 &= \int_0^{2\pi} \overline{R}\, d\theta - z^2 I_2 = 4(1+k) l_2(a) \mathbf{E}\left(\frac{2\sqrt{k}}{1+k}\right) - z^2 I_2 \\ &= 8 l_2(a) \mathbf{E}(k) - \frac{4}{l_2(a)} [l_2^2(a) - l_1^2(a) + z^2] \mathbf{F}(k). \end{aligned} \tag{33}$$

In a similar manner I_3 becomes

$$I_3 = \frac{4}{(1+k) l_2(a) (a+\rho)^2} \Pi\left(p, \frac{2\sqrt{k}}{1+k}\right), \qquad p = \frac{4a\rho}{(a+\rho)^2}. \tag{34}$$

To evaluate this last integral, Love introduced Jacobian elliptic functions and evaluated it through a very complicated analysis as a combination of complete and incomplete elliptic integrals of the first and second kinds. Combining the above results leads to the present evaluation as

$$\begin{aligned} \frac{\partial}{\partial \rho} \psi(\rho, z) &= \frac{\pi}{2\rho}\{a^2 + \rho^2 - |a^2 - \rho^2|\} + \frac{2 z l_2(a)}{\rho} \mathbf{E}(k) \\ &- \frac{z}{\rho l_2(a)} [2 l_2^2(a) + a^2 + \rho^2] \mathbf{F}(k) + \frac{z (a^2 - \rho^2)^2}{\rho (1+k) l_2(a) (a+\rho)^2} \Pi\left(p, \frac{2\sqrt{k}}{1+k}\right). \end{aligned} \tag{35}$$

Note that the first term above in brackets is independent of z and has the value of $2\rho^2$ for $\rho < a$ and $2a^2$ for $\rho > a$. Thus, it is continuous at $\rho = a$, but its ρ derivative is discontinuous. This function must have continuous derivatives at $\rho = a$ based on physical considerations. It must then be concluded that the last term above also has a discontinuous ρ derivative to cancel the effect of the first term. Without a proof, it can be conjectured that this is true since $\rho \to a$ results in $p \to 1$ and the complete elliptic integral of the third kind tends to infinity by Eq. (15). Since this term is multiplied by $(a^2 - \rho^2)^2$, the product of the two must be bounded with a discontinuous derivative. Obviously, the above form of the expression resulting from this last term is not very convenient for a numerical evaluation in the subsurface region $\rho \to a$ since very careful numerical treatment is necessary to obtain the correct limiting value. A much better form can be obtained using Eq. (23). This provides the result

$$\frac{\partial}{\partial \rho} \psi(\rho, z) = \frac{\pi a^2}{\rho} + \frac{2 z l_2(a)}{\rho} \mathbf{E}(k) - \frac{2z}{\rho l_2(a)} [\rho^2 + l_2^2(a)] \mathbf{F}(k) - \frac{2z(a^2 - \rho^2)}{\rho l_2(a)} \Pi(n, k). \tag{36}$$

The function $\Pi(n, k)$ is continuous and bounded at every point inside the half space. Similarly to Eq. (35), Eq. (36) cannot be evaluated numerically along the z-axis since $\rho = 0$ there. The limits for the elastic field are evaluated analytically at $\rho = 0$ in a later section. It is also shown subsequently that for $z \to 0$ with $\rho < a$, $n \to 1$ and thus $\Pi(n, k) \to \infty$. However, in a later section it is also shown that $z\Pi(n, k)$ approaches a finite limit and the surface values of the elastic field are analytically obtained.

Moving back to the derivatives of the potential function, the z derivative of $\psi(\rho, z)$ is evaluated in Appendix A with the result

$$\frac{\partial}{\partial z}\psi(\rho, z) = \frac{4}{l_2(a)}\{l_2^2(a)\mathbf{E}(k) - [\rho^2 - l_1^2(a)]\mathbf{F}(k) - z^2\Pi(n, k)\}. \tag{37}$$

Using the above derivatives the displacement field can be given as

$$\begin{aligned} u^c = \frac{H\sigma}{\rho}e^{i\phi}\sum_{j=1}^{2}\frac{\gamma_j}{(m_j - 1)}\Bigg\{\pi a^2 + 2z_j l_{2j}(a)\mathbf{E}(k_j) - \frac{2z_j}{l_{2j}(a)}[\rho^2 + l_{2j}^2(a)]\mathbf{F}(k_j) \\ - \frac{2z_j(a^2 - \rho^2)}{l_{2j}(a)}\Pi(n_j, k_j)\Bigg\}, \end{aligned} \tag{38}$$

$$w = 4H\sigma\sum_{j=1}^{2}\frac{m_j}{(m_j - 1)}\frac{1}{l_{2j}(a)}\{l_{2j}^2(a)\mathbf{E}(k_j) - [\rho^2 - l_{1j}^2(a)]\mathbf{F}(k_j) - z_j^2\Pi(n_j, k_j)\}. \tag{39}$$

In the above two equations, the following definitions have been employed:

$$\begin{aligned} l_{1j}(a) &= \tfrac{1}{2}\{[(\rho + a)^2 + z_j^2]^{1/2} - [(\rho - a)^2 + z_j^2]^{1/2}\}, \\ l_{2j}(a) &= \tfrac{1}{2}\{[(\rho + a)^2 + z_j^2]^{1/2} + [(\rho - a)^2 + z_j^2]^{1/2}\}, \end{aligned} \tag{40}$$

$$k_j = \frac{l_{1j}(a)}{l_{2j}(a)}, \qquad n_j = \frac{l_{1j}^2(a)}{\rho^2}$$

where z_j is defined in Eq. (9).

For the stress field, the following derivatives are required:

$$\begin{aligned} \frac{\partial}{\partial z^2}\psi(\rho, z_j) = \frac{1}{\gamma_j^2}\frac{\partial^2}{\partial z_j^2}\psi(\rho, z_j), \qquad \Lambda\frac{\partial}{\partial z}\psi(\rho, z_j) = \frac{e^{i\phi}}{\gamma_j}\frac{\partial^2}{\partial\rho\partial z_j}\psi(\rho, z_j), \\ \Lambda^2\psi(\rho, z_j) = \Lambda\Lambda\psi(\rho, z_j) = \left[e^{i\phi}\frac{\partial}{\partial\rho} + \frac{ie^{i\phi}}{\rho}\frac{\partial}{\partial\phi}\right]\left[e^{i\phi}\frac{\partial}{\partial\rho}\psi(\rho, z_j)\right] \\ = e^{i2\phi}\left[\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right]\psi(\rho, z_j). \end{aligned} \tag{41}$$

Following Love one can obtain the result

$$\frac{\partial^2}{\partial\rho^2}\psi(\rho, z) = \frac{\pi}{2\rho^2}\left\{\rho^2 - a^2 + \frac{a^4 - \rho^4}{|a^2 - \rho^2|}\right\} - \frac{z}{4\rho^2}I_1 + \frac{za^2}{2\rho^2}I_2 - \frac{z(a^4 - \rho^4)}{4\rho^2}I_3, \tag{42}$$

which becomes

$$\begin{aligned} \frac{\partial^2}{\partial\rho^2}\psi(\rho, z) = \frac{\pi}{2\rho^2}\left\{\rho^2 - a^2 + \frac{a^4 - \rho^4}{|a^2 - \rho^2|}\right\} - \frac{2zl_2(a)}{\rho^2}\mathbf{E}(k) \\ + \frac{z}{\rho^2 l_2(a)}[a^2 - \rho^2 + 2l_2^2(a)]\mathbf{F}(k) - \frac{z(a^4 - \rho^4)}{\rho^2(1 + k)l_2(a)(a + \rho)^2}\Pi\left(p, \frac{2\sqrt{k}}{1 + k}\right). \end{aligned} \tag{43}$$

Using Eq. (23) this expression is rewritten as

$$\frac{\partial^2}{\partial\rho^2}\psi(\rho,z) = -\frac{\pi a^2}{\rho^2} - \frac{2zl_2(a)}{\rho^2}\mathbf{E}(k) + \frac{2z}{\rho^2 l_2(a)}[l_2^2(a) - \rho^2]\mathbf{F}(k) + \frac{2z(a^2+\rho^2)}{\rho^2 l_2(a)}\Pi(n,k). \quad (44)$$

Similarly one can derive

$$\frac{\partial^2}{\partial\rho\partial z}\psi(\rho,z) = \frac{4l_2(a)}{\rho}[\mathbf{E}(k) - \mathbf{F}(k)]. \quad (45)$$

Since $\psi(\rho,z)$ is a harmonic function, one can write

$$\frac{\partial^2}{\partial z^2}\psi(\rho,z) = -\frac{\partial^2}{\partial\rho^2}\psi(\rho,z) - \frac{1}{\rho}\frac{\partial}{\partial\rho}\psi(\rho,z) = \frac{4z}{l_2(a)}[\mathbf{F}(k) - \Pi(n,k)]. \quad (46)$$

Using the above results the components of the stress field can now be found. These are

$$\begin{aligned}\sigma_1 &= 2H\sigma A_{66}\sum_{j=1}^{2}\frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j(m_j-1)}\frac{\partial^2}{\partial z_j^2}\psi(\rho,z_j)\\ &= 8H\sigma A_{66}\sum_{j=1}^{2}\frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j(m_j-1)}\frac{z_j}{l_{2j}(a)}[\mathbf{F}(k_j)-\Pi(n_j,k_j)],\end{aligned} \quad (47)$$

$$\begin{aligned}\sigma_{zz} &= \frac{\sigma}{2\pi(\gamma_1-\gamma_2)}\sum_{j=1}^{2}\gamma_j(-1)^{j+1}\frac{\partial^2}{\partial z_j^2}\psi(\rho,z_j)\\ &= \frac{2\sigma}{\pi(\gamma_1-\gamma_2)}\sum_{j=1}^{2}\gamma_j(-1)^{j+1}\frac{z_j}{l_{2j}(a)}[\mathbf{F}(k_j)-\Pi(n_j,k_j)],\end{aligned} \quad (48)$$

$$\begin{aligned}\tau_z &= \frac{\sigma}{2\pi(\gamma_1-\gamma_2)}e^{i\phi}\sum_{j=1}^{2}(-1)^{j+1}\frac{\partial^2}{\partial\rho\partial z_j}\psi(\rho,z_j)\\ &= \frac{2\sigma}{\pi\rho(\gamma_1-\gamma_2)}e^{i\phi}\sum_{j=1}^{2}(-1)^j l_{2j}(a)[\mathbf{F}(k_j)-\mathbf{E}(k_j)],\end{aligned} \quad (49)$$

$$\begin{aligned}\sigma_2 &= 2H\sigma A_{66}\sum_{j=1}^{2}\frac{\gamma_j}{(m_j-1)}\Lambda^2\psi(\rho,z_j)\\ &= 4H\sigma A_{66}\frac{e^{i2\phi}}{\rho^2}\sum_{j=1}^{2}\frac{\gamma_j}{(m_j-1)}\left\{-\pi a^2 - 2z_j l_{2j}(a)\mathbf{E}(k_j)\right.\\ &\qquad\left. + 2z_j l_{2j}(a)\mathbf{F}(k_j) + \frac{2z_j a^2}{l_{2j}(a)}\Pi(n_j,k_j)\right\}.\end{aligned} \quad (50)$$

The identity

$$\frac{(m_j+1)}{(m_j-1)} = \frac{(-1)^{j+1}}{2\pi H A_{44}(\gamma_1-\gamma_2)} \quad (51)$$

was used to transform some of the expressions for the stress components. Equations (38, 39) for displacements and Eqs. (47–50) for stresses comprise a complete solution to the elastic field for this problem.

6. Uniform shear loading on a transversely isotropic half space. Now the half space shown in Fig. 1 is free of normal stress, and a uniform shear stress is applied within the region $\rho < a$. The magnitude of the shear stress is denoted as τ, $\tau = \tau_x + i\tau_y$, where τ_x and τ_y are uniform shear loadings in the x and y directions respectively. The solution can be obtained by replacing the complex force T in Eqs. (11) with $\tau\rho_0\, d\rho_0\, d\phi_0$ and integrating the result over $0 < \rho_0 < a$, $0 < \phi_0 < 2\pi$. The potential functions become

$$\begin{aligned} F_1(\rho,\phi,z) &= \frac{H\gamma_1}{(m_1-1)}\frac{\gamma_2}{2}(\tau\overline{\Lambda}+\overline{\tau}\Lambda)[z_1\psi(\rho,z_1)-\Phi(\rho,z_1)], \\ F_2(\rho,\phi,z) &= \frac{H\gamma_2}{(m_2-1)}\frac{\gamma_1}{2}(\tau\overline{\Lambda}+\overline{\tau}\Lambda)[z_2\psi(\rho,z_2)-\Phi(\rho,z_2)], \\ F_3(\rho,\phi,z) &= i\frac{\gamma_3}{4\pi A_{44}}(\tau\overline{\Lambda}-\overline{\tau}\Lambda)[z_3\psi(\rho,z_3)-\Phi(\rho,z_3)], \end{aligned} \tag{52}$$

where $\psi(\rho, z)$ is defined above and $\Phi(\rho, z)$ is given as

$$\Phi(\rho,z) = \int_0^{2\pi}\int_0^a R\rho_0\, d\rho_0\, d\phi_0, \qquad R^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z^2. \tag{53}$$

To find the elastic field, the derivatives of the potential functions above are needed. It is important to point out that $\chi(z)$ in Eq. (12) is the integral of $\psi(\rho, z)$ with respect to z. This gives the useful relations

$$\frac{\partial}{\partial z}\chi(z) = \frac{\partial}{\partial z}[z\psi(\rho,z)-\Phi(\rho,z)] = \psi(\rho,z), \qquad \frac{\partial^2}{\partial z^2}\chi(z) = \frac{\partial}{\partial z}\psi(\rho,z), \tag{54}$$

and any z derivatives of $\chi(z)$ can be expressed in terms of $\psi(\rho, z)$.

Another important property is now discussed. It is easily shown that $\chi(z)$ is a harmonic function. Thus one may write

$$\begin{aligned} \frac{\partial}{\partial z}\psi(\rho,z) &= \frac{\partial^2}{\partial z^2}\chi(z) = -\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]\chi(z) \\ &= -z\frac{\partial^2}{\partial\rho^2}\psi(\rho,z) - \frac{z}{\rho}\frac{\partial}{\partial\rho}\psi(\rho,z) + \frac{\partial^2}{\partial\rho^2}\Phi(\rho,z) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Phi(\rho,z). \end{aligned} \tag{55}$$

The ρ derivatives of $\Phi(\rho, z)$ are evaluated in Appendix B. They are

$$\frac{\partial}{\partial\rho}\Phi(\rho,z) = \frac{4l_2^3(a)(1+k^2)}{3\rho}\mathbf{E}(k) - \frac{4l_2^3(a)(1-k^2)}{3\rho}\mathbf{F}(k), \tag{56}$$

$$\begin{aligned} \frac{\partial^2}{\partial\rho^2}\Phi(\rho,z) &= \frac{4l_2(a)}{3\rho^2}[l_1^2(a)+z^2+a^2-2\rho^2]\mathbf{F}(k) \\ &\quad - \frac{4l_2(a)}{3\rho^2}[z^2+a^2-2\rho^2]\mathbf{E}(k). \end{aligned} \tag{57}$$

Substituting Eq. (37) into the left-hand side of Eq. (55) and substituting Eqs. (35, 43, 56, 57) into the right-hand side proves Eq. (23) after some algebra.

Now, returning to the elastic field, the displacement w can be written as

$$\begin{aligned} w &= \frac{H\gamma_1\gamma_2}{2}\sum_{j=1}^{2}\frac{m_j}{\gamma_j(m_j-1)}(\tau\overline{\Lambda}+\overline{\tau}\Lambda)\frac{\partial}{\partial z_j}\chi(\rho,z_j) \\ &= \frac{H\gamma_1\gamma_2}{\rho}(\cos\phi\tau_x+\sin\phi\tau_y)\sum_{j=1}^{2}\frac{m_j}{\gamma_j(m_j-1)}\left\{\pi a^2+2z_jl_{2j}(a)\mathbf{E}(k_j)\right. \\ &\qquad\left. -\frac{2z_j}{l_{2j}(a)}[\rho^2+l_{2j}^2(a)]\mathbf{F}(k_j)-\frac{2z_j(a^2-\rho^2)}{l_{2j}(a)}\Pi(n_j,k_j)\right\}, \end{aligned} \tag{58}$$

where Eq. (33) was utilized. The tangential displacements become

$$u^c = \frac{H\gamma_1\gamma_2}{2}\sum_{j=1}^{2}\frac{1}{(m_j-1)}(\tau\Delta+\overline{\tau}\Lambda^2)\chi(\rho,z_j)-\frac{\gamma_3}{4\pi A_{44}}(\tau\Delta-\overline{\tau}\Lambda^2)\chi(\rho,z_3). \tag{59}$$

Using the results

$$\begin{aligned} \Delta\chi(\rho,z_j) &= -\frac{\partial^2}{\partial z_j^2}\chi(\rho,z_j) = -\frac{\partial}{\partial z_j}\psi(\rho,z_j), \\ \Lambda^2\chi(\rho,z_j) &= e^{i2\phi}\left[\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]\chi(\rho,z_j) \\ &= -e^{i2\phi}\left[\frac{\partial}{\partial z_j}\psi(\rho,z_j)+\frac{2z_j}{\rho}\frac{\partial}{\partial\rho}\psi(\rho,z_j)-\frac{2}{\rho}\frac{\partial}{\partial\rho}\Phi(\rho,z_j)\right], \end{aligned} \tag{60}$$

where the harmonic property of $\chi(\rho,z)$ has been used in the last expression, the displacements become

$$\begin{aligned} u^c = &\frac{H\gamma_1\gamma_2}{2}\sum_{j=1}^{2}\frac{1}{(m_j-1)}\left\{-\tau\frac{4}{l_{2j}(a)}\{l_{2j}^2(a)\mathbf{E}(k_j)-[\rho^2-l_{1j}^2(a)]\mathbf{F}(k_j)-z_j^2\Pi(n_j,k_j)\}\right. \\ &+\overline{\tau}e^{i2\phi}\left[-\frac{2\pi a^2z_j}{\rho^2}+\frac{4l_{2j}(a)}{3\rho^2}(2a^2-\rho^2-z_j^2)\mathbf{E}(k_j)\right. \\ &\qquad\left.\left.+\frac{4l_{2j}(a)}{3\rho^2}[\rho^2-2a^2+z_j^2+l_{1j}^2(a)]\mathbf{F}(k_j)+\frac{4a^2z_j^2}{\rho^2l_{2j}(a)}\Pi(n_j,k_j)\right]\right\} \\ &-\frac{\gamma_3}{4\pi A_{44}}\left\{-\tau\frac{4}{l_{23}(a)}\{l_{23}^2(a)\mathbf{E}(k_3)-[\rho^2-l_{13}^2(a)]\mathbf{F}(k_3)-z_3^2\Pi(n_3,k_3)\}\right. \\ &-\overline{\tau}e^{i2\phi}\left[-\frac{2\pi a^2z_3}{\rho^2}+\frac{4l_{23}(a)}{3\rho^2}(2a^2-\rho^2-z_3^2)\mathbf{E}(k_3)\right. \\ &\qquad\left.\left.+\frac{4l_{23}(a)}{3\rho^2}[\rho^2-2a^2+z_3^2+l_{13}^2(a)]\mathbf{F}(k_3)+\frac{4a^2z_3^2}{\rho^2l_{23}(a)}\Pi(n_3,k_3)\right]\right\}. \end{aligned} \tag{61}$$

Now the stress field is evaluated. To start with, σ_1 and σ_{zz} can be written as

$$\sigma_1 = HA_{66}\gamma_1\gamma_2\sum_{j=1}^{2}\frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j^2(m_j-1)}(\tau\overline{\Lambda}+\overline{\tau}\Lambda)\frac{\partial^2}{\partial z_j^2}\chi(\rho,z_j), \tag{62}$$

$$\sigma_{zz} = \frac{\gamma_1\gamma_2}{4\pi(\gamma_1-\gamma_2)}\sum_{j=1}^{2}(-1)^{j+1}(\tau\overline{\Lambda}+\overline{\tau}\Lambda)\frac{\partial^2}{\partial z_j^2}\chi(\rho,z_j). \tag{63}$$

Using Eqs. (29, 45, 54) these two stress components are

$$\sigma_1 = 8HA_{66}\gamma_1\gamma_2(\cos\phi\tau_x + \sin\phi\tau_y)\sum_{j=1}^{2}\frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j^2(m_j-1)}\frac{l_{2j}(a)}{\rho}[\mathbf{E}(k_j)-\mathbf{F}(k_j)], \tag{64}$$

$$\sigma_{zz} = \frac{2\gamma_1\gamma_2}{\pi(\gamma_1-\gamma_2)}(\cos\phi\tau_x + \sin\phi\tau_y)\sum_{j=1}^{2}(-1)^{j+1}\frac{l_{2j}(a)}{\rho}[\mathbf{E}(k_j)-\mathbf{F}(k_j)]. \tag{65}$$

The expression for τ_z becomes

$$\tau_z = \frac{\gamma_1\gamma_2}{4\pi(\gamma_1-\gamma_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}}{\gamma_j}(\tau\Delta+\overline{\tau}\Lambda^2)\frac{\partial}{\partial z_j}\chi(\rho,z_j) - \frac{1}{4\pi}(\tau\Delta-\overline{\tau}\Lambda^2)\frac{\partial}{\partial z_3}\chi(\rho,z_3). \tag{66}$$

Using Eqs. (36, 41, 44, 54) and Eq. (46) coupled with the fact that $\psi(\rho,z_j)$ is harmonic in its two variables, the shear stress expression is evaluated as

$$\begin{aligned}\tau_z = \frac{\gamma_1\gamma_2}{4\pi(\gamma_1-\gamma_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}}{\gamma_j}&\left\{\tau\left[-\frac{4z_j}{l_{2j}(a)}\mathbf{F}(k_j)+\frac{4z_j}{l_{2j}(a)}\Pi(n_j,k_j)\right]\right.\\ &\left.+\frac{\overline{\tau}e^{i2\phi}}{\rho^2}\left[-2\pi a^2-4z_jl_{2j}(a)\mathbf{E}(k_j)+4z_jl_{2j}(a)\mathbf{F}(k_j)+\frac{4a^2z_j}{l_{2j}(a)}\Pi(n_j,k_j)\right]\right\}\\ -\frac{1}{4\pi}&\left\{\tau\left[-\frac{4z_3}{l_{23}(a)}\mathbf{F}(k_3)+\frac{4z_3}{l_{23}(a)}\Pi(n_3,k_3)\right]\right.\\ &\left.-\frac{\overline{\tau}e^{i2\phi}}{\rho^2}\left[-2\pi a^2-4z_3l_{23}(a)\mathbf{E}(k_3)+4z_3l_{23}(a)\mathbf{F}(k_3)+\frac{4a^2z_3}{l_{23}(a)}\Pi(n_3,k_3)\right]\right\}.\end{aligned} \tag{67}$$

The last expression for σ_2 has the form

$$\sigma_2 = HA_{66}\gamma_1\gamma_2\sum_{j=1}^{2}\frac{1}{(m_j-1)}(\tau\Lambda\Delta+\overline{\tau}\Lambda^3)\chi(\rho,z_j) - \frac{1}{2\pi\gamma_3}(\tau\Lambda\Delta-\overline{\tau}\Lambda^3)\chi(\rho,z_3). \tag{68}$$

Using the results of Eq. (60) provides

$$\Lambda[\Delta\chi(\rho,z_j)] = \Lambda\left[-\frac{\partial}{\partial z_j}\psi(\rho,z_j)\right] = -e^{i\phi}\frac{\partial^2}{\partial\rho\partial z_j}\psi(\rho,z_j),$$

$$\begin{aligned}\Lambda^3\chi(\rho,z) &= \left[e^{i\phi}\frac{\partial}{\partial\rho}+\frac{ie^{i\phi}}{\rho}\frac{\partial}{\partial\phi}\right][\Lambda^2\chi(\rho,z)]\\ &= e^{i3\phi}\left[-\frac{\partial^2}{\partial\rho\partial z}\psi(\rho,z)+\frac{6z}{\rho^2}\frac{\partial}{\partial\rho}\psi(\rho,z)+\frac{2}{\rho}\frac{\partial}{\partial z}\psi(\rho,z)\right.\\ &\qquad\left.-\frac{2z}{\rho}\frac{\partial^2}{\partial\rho^2}\psi(\rho,z)-\frac{6}{\rho^2}\frac{\partial}{\partial\rho}\Phi(\rho,z)+\frac{2}{\rho}\frac{\partial^2}{\partial\rho^2}\Phi(\rho,z)\right].\end{aligned} \tag{69}$$

The final results for σ_2 are as follows:

$$\sigma_2 = HA_{66}\gamma_1\gamma_2 \sum_{j=1}^{2} \frac{1}{(m_j-1)} \left\{ -\tau e^{i\phi} \frac{4l_{2j}(a)}{\rho}[\mathbf{E}(k_j) - \mathbf{F}(k_j)] \right.$$
$$+ \bar{\tau} e^{i3\phi} \left[\frac{8\pi a^2 z_j}{\rho^3} + \frac{4l_{2j}(a)}{3\rho^3}(\rho^2 + 4z_j^2 - 8a^2)\mathbf{E}(k_j) \right.$$
$$\left.\left. + \frac{4l_{2j}(a)}{3\rho^3}\{8a^2 - \rho^2 - 4l_{1j}^2(a) - 4z_j^2\}\mathbf{F}(k_j) - \frac{16a^2 z_j^2}{\rho^3 l_{2j}(a)}\Pi(n_j, k_j) \right] \right\}$$
$$- \frac{1}{2\pi\gamma_3} \left\{ -\tau e^{i\phi} \frac{4l_{23}(a)}{\rho}[\mathbf{E}(k_3) - \mathbf{F}(k_3)] \right.$$
$$- \bar{\tau} e^{i3\phi} \left[\frac{8\pi a^2 z_3}{\rho^3} + \frac{4l_{23}(a)}{3\rho^3}(\rho^2 + 4z_3^2 - 8a^2)\mathbf{E}(k_3) \right.$$
$$\left.\left. + \frac{4l_{23}(a)}{3\rho^3}\{8a^2 - \rho^2 - 4l_{13}^2(a) - 4z_3^2\}\mathbf{F}(k_3) - \frac{16a^2 z_3^2}{\rho^3 l_{23}(a)}\Pi(n_3, k_3) \right] \right\}. \tag{70}$$

The expressions in Eqs. (58, 61) for displacements and Eqs. (64, 65, 67, 70) for stresses comprise a complete solution to the elastic field for uniform shear loading.

7. Results for isotropy. Now attention is focused on the case of an isotropic half space. The solution for normal loading was given previously by Love (1929) (except the displacement component w was not evaluated). The solution for shear loading has not been published previously.

The solution for isotropy is obtained by a limiting form of the transversely isotropic results. For isotropy $\gamma_1, \gamma_2, \gamma_3, m_1, m_2 \to 1$ and each term in the expressions above becomes indeterminate. A limiting procedure is thus required. The appropriate isotropic limits needed for normal loading can be extracted from the paper by Hanson (1992) as

$$\sum_{j=1}^{2} \frac{\gamma_j}{(m_j-1)} f(z_j) = -\frac{(1-2\nu)f(z) + zf'(z)}{2(1-\nu)},$$
$$\sum_{j=1}^{2} \frac{m_j}{(m_j-1)} f(z_j) = f(z) - \frac{z}{2(1-\nu)} f'(z),$$
$$\sum_{j=1}^{2} \frac{\gamma_j^2 - (1+m_j)\gamma_3^2}{\gamma_j(m_j-1)} f(z_j) = \frac{(1+2\nu)f(z) + zf'(z)}{2(1-\nu)}, \tag{71}$$
$$\frac{1}{(\gamma_1-\gamma_2)} \sum_{j=1}^{2} \gamma_j(-1)^j f(z_j) = -f(z) + zf'(z),$$
$$\frac{1}{(\gamma_1-\gamma_2)} \sum_{j=1}^{2} (-1)^{j+1} f(z_j) = -zf'(z),$$
$$H = \frac{1-\nu^2}{\pi E}, \qquad A_{66} = \frac{E}{2(1+\nu)},$$

where $f'(z)$ denotes the derivative of $f(z)$ with respect to z, E is Young's modulus, and ν is Poisson's ratio.

Using the above results, the elastic field for normal loading can be written in terms of the potential functions as

$$\begin{aligned}
u^c &= -\frac{\sigma e^{i\phi}}{4\pi\mu}\left[(1-2\nu)\frac{\partial}{\partial\rho}\psi(\rho,z) + z\frac{\partial^2}{\partial\rho\partial z}\psi(\rho,z)\right],\\
w &= \frac{\sigma(1-\nu)}{2\pi\mu}\left[\frac{\partial}{\partial z}\psi(\rho,z) - \frac{z}{2(1-\nu)}\frac{\partial^2}{\partial z^2}\psi(\rho,z)\right],\\
\sigma_1 &= \frac{\sigma}{2\pi}\left[(1+2\nu)\frac{\partial^2}{\partial z^2}\psi(\rho,z) + z\frac{\partial^3}{\partial z^3}\psi(\rho,z)\right],\\
\sigma_2 &= -\frac{\sigma}{2\pi}\left[(1-2\nu)\Lambda^2\psi(\rho,z) + z\Lambda^2\frac{\partial}{\partial z}\psi(\rho,z)\right],\\
\sigma_{zz} &= \frac{\sigma}{2\pi}\left[\frac{\partial^2}{\partial z^2}\psi(\rho,z) - z\frac{\partial^3}{\partial z^3}\psi(\rho,z)\right],\\
\tau_z &= -\frac{\sigma e^{i\phi}}{2\pi}z\frac{\partial^3}{\partial\rho\partial z^2}\psi(\rho,z).
\end{aligned} \tag{72}$$

It is easily shown that the above representation is in exact agreement with the one used by Love if one identifies $\sigma\psi(\rho,z)$ here with the function χ used by Love. Furthermore, Love's function V is $\sigma\frac{\partial\psi(\rho,z)}{\partial z}$ presently.

To find the isotropic field some additional z derivatives are needed. These are now evaluated. Differentiating Eq. (45) with respect to z and ρ, and using Eqs. (24–26), one may obtain

$$\begin{aligned}
\frac{\partial^3}{\partial\rho\partial z^2}\psi(\rho,z) &= \frac{4z}{\rho l_2(a)(1-k^2)}\left\{-\mathbf{F}(k) + \frac{1+k^2}{1-k^2}\mathbf{E}(k)\right\},\\
\frac{\partial^3}{\partial\rho^2\partial z}\psi(\rho,z) &= \frac{4[l_2^2(a)-\rho^2]}{\rho^2 l_2(a)(1-k^2)}\{\mathbf{F}(k)-\mathbf{E}(k)\} - \frac{4k[a-2\rho k+ak^2]}{\rho l_2(a)(1-k^2)^2}\mathbf{E}(k).
\end{aligned} \tag{73}$$

Since $\psi(\rho,z)$ is harmonic it is easy to see that

$$\begin{aligned}
\frac{\partial^3}{\partial z^3}\psi(\rho,z) &= -\frac{\partial^3}{\partial\rho^2\partial z}\psi(\rho,z) - \frac{1}{\rho}\frac{\partial^2}{\partial\rho\partial z}\psi(\rho,z)\\
&= \frac{4[l_2^2(a)-a^2]}{l_2^3(a)(1-k^2)}\mathbf{F}(k) - \frac{4[\rho^2+z^2-a^2]}{l_2^3(a)(1-k^2)^2}\mathbf{E}(k),
\end{aligned} \tag{74}$$

where Eqs. (45, 73) have been used.

All of the derivatives of the potential function $\psi(\rho,z)$ agree with Love's results except those that involve the complete elliptic integral of the third kind which he did not use. Thus, apart from these terms, the elastic field for isotropy given below is in perfect agreement with Love's solution:

$$\begin{aligned}
u^c = -\frac{\sigma e^{i\phi}}{4\pi\mu\rho}\bigg[&(1-2\nu)\pi a^2 + 2zl_2(a)(3-2\nu)\mathbf{E}(k)\\
&- \frac{2z}{l_2(a)}\{2l_2^2(a) + (1-2\nu)[\rho^2+l_2^2(a)]\}\mathbf{F}(k)\\
&- \frac{2z(1-2\nu)(a^2-\rho^2)}{l_2(a)}\Pi(n,k)\bigg],
\end{aligned} \tag{75}$$

$$w = \frac{\sigma}{2\pi\mu l_2(a)}[4(1-\nu)l_2^2(a)\mathbf{E}(k) - \{4(1-\nu)[\rho^2 - l_1^2(a)] + 2z^2\}\mathbf{F}(k) - 2z^2(1-2\nu)\Pi(n,k)], \tag{76}$$

$$\sigma_1 = \frac{2z\sigma}{\pi}\left[-\frac{[\rho^2+z^2-a^2]}{l_2^3(a)(1-k^2)^2}\mathbf{E}(k) + \left\{\frac{[l_2^2(a)-a^2]}{l_2^3(a)(1-k^2)} + \frac{(1+2\nu)}{l_2(a)}\right\}\mathbf{F}(k) - \frac{(1+2\nu)}{l_2(a)}\Pi(n,k)\right], \tag{77}$$

$$\begin{aligned}\sigma_2 = -\frac{\sigma e^{i2\phi}}{\pi\rho^2}\Bigg\{ &-(1-2\nu)\pi a^2 \\ &+ \frac{2z}{l_2(a)(1-k^2)}[2l_2^2(a) - l_1^2(a) - \rho^2 + (1-2\nu)(1-k^2)l_2^2(a)]\mathbf{F}(k) \\ &- \frac{2z}{l_2(a)(1-k^2)^2}[(1-2\nu)(1-k^2)^2 l_2^2(a) + 2l_2^2(a) \\ &- 2(1-k^2)l_1^2(a) - \rho^2(1+k^2)]\mathbf{E}(k) + \frac{2za^2(1-2\nu)}{l_2(a)}\Pi(n,k)\Bigg\},\end{aligned} \tag{78}$$

$$\sigma_{zz} = \frac{2z\sigma}{\pi}\left[\frac{[\rho^2+z^2-a^2]}{l_2^3(a)(1-k^2)^2}\mathbf{E}(k) + \frac{[a^2-l_1^2(a)]}{l_2^3(a)(1-k^2)}\mathbf{F}(k) - \frac{1}{l_2(a)}\Pi(n,k)\right], \tag{79}$$

$$\tau_z = \frac{2\sigma z^2 e^{i\phi}}{\pi\rho l_2(a)(1-k^2)}\left[\mathbf{F}(k) - \frac{1+k^2}{1-k^2}\mathbf{E}(k)\right]. \tag{80}$$

Now shear loading is considered. In addition to Eq. (71) the following results from Hanson (1993) are required:

$$\begin{aligned}\sum_{j=1}^{2}\frac{1}{(m_j-1)}f(z_j) &= -f(z) - \frac{zf'(z)}{2(1-\nu)}, \\ \sum_{j=1}^{2}\frac{m_j}{\gamma_j(m_j-1)}f(z_j) &= \frac{(1-2\nu)f(z) - zf'(z)}{2(1-\nu)}, \\ \sum_{j=1}^{2}\frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j^2(m_j-1)}f(z_j) &= \frac{2(1+\nu)f(z)+zf'(z)}{2(1-\nu)}, \\ \frac{1}{(\gamma_1-\gamma_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}}{\gamma_j}f(z_j) &= -f(z) - zf'(z).\end{aligned} \tag{81}$$

Using the above equations, the elastic field in terms of the potential functions is written as

$$u^c = \frac{1}{4\pi\mu}\left[\tau\left\{(2-\nu)\frac{\partial}{\partial z}\psi(\rho,z)+\frac{z}{2}\frac{\partial^2}{\partial z^2}\psi(\rho,z)\right\}+\overline{\tau}\Lambda^2\left\{\nu\chi(\rho,z)-\frac{z}{2}\psi(\rho,z)\right\}\right],$$
$$w = \frac{1}{8\pi\mu}(\tau e^{-i\phi}+\overline{\tau}e^{i\phi})\left[(1-2\nu)\frac{\partial}{\partial\rho}\psi(\rho,z)-z\frac{\partial^2}{\partial\rho\partial z}\psi(\rho,z)\right],$$
$$\sigma_1 = \frac{1}{4\pi}(\tau e^{-i\phi}+\overline{\tau}e^{i\phi})\left[2(1+\nu)\frac{\partial^2}{\partial\rho\partial z}\psi(\rho,z)+z\frac{\partial^3}{\partial\rho\partial z^2}\psi(\rho,z)\right],$$
$$\sigma_2 = \frac{1}{2\pi}\left[\tau\left\{(2-\nu)\Lambda\frac{\partial}{\partial z}\psi(\rho,z)+\frac{z}{2}\Lambda\frac{\partial^2}{\partial z^2}\psi(\rho,z)\right\}+\overline{\tau}\left\{\nu\Lambda^3\chi(\rho,z)-\frac{z}{2}\Lambda^3\psi(\rho,z)\right\}\right],$$
$$\sigma_{zz} = -\frac{1}{4\pi}(\tau e^{-i\phi}+\overline{\tau}e^{i\phi})\left[z\frac{\partial^3}{\partial\rho\partial z^2}\psi(\rho,z)\right],$$
$$\tau_z = \frac{1}{4\pi}\left[\tau\left\{2\frac{\partial^2}{\partial z^2}\psi(\rho,z)+z\frac{\partial^3}{\partial z^3}\psi(\rho,z)\right\}-\overline{\tau}\Lambda^2\left\{z\frac{\partial}{\partial z}\psi(\rho,z)\right\}\right]. \tag{82}$$

Substituting into the above expressions leads to the following elastic field:

$$\begin{aligned}u^c = \frac{1}{4\pi\mu}\Bigg[&\frac{2\tau}{l_2(a)}\{2(2-\nu)l_2^2(a)\mathbf{E}(k)+[z^2-2(2-\nu)(\rho^2-l_1^2(a))]\mathbf{F}(k)\\&\qquad -z^2(5-2\nu)\Pi(n,k)\}\\&+\overline{\tau}\frac{e^{i2\phi}}{3\rho^2}\Bigg\{3\pi a^2 z(1-2\nu)+2l_2(a)[3z^2+2\nu(2a^2-\rho^2-z^2)]\mathbf{E}(k)\\&\qquad +2l_2(a)[-3z^2+2\nu(-2a^2+\rho^2+z^2+l_1^2(a))]\mathbf{F}(k)\\&\qquad -\frac{6a^2z^2(1-2\nu)}{l_2(a)}\Pi(n,k)\Bigg\}\Bigg],\end{aligned}$$

$$\begin{aligned}w = \frac{1}{4\pi\mu}(\cos\phi\tau_x+\sin\phi\tau_y)\frac{1}{\rho l_2(a)}&[\pi a^2 l_2(a)(1-2\nu)-2zl_2^2(a)(1+2\nu)\mathbf{E}(k)\\&+2z[2l_2^2(a)-(1-2\nu)(\rho^2+l_2^2(a))]\mathbf{F}(k)-2z(a^2-\rho^2)(1-2\nu)\Pi(n,k)],\end{aligned}$$

$$\begin{aligned}\sigma_1 = \frac{2}{\pi}(\cos\phi\tau_x+\sin\phi\tau_y)\Bigg[&\frac{1}{\rho l_2(a)(1-k^2)^2}\{z^2(1+k^2)+2l_2^2(a)(1+\nu)(1-k^2)^2\}\mathbf{E}(k)\\&-\frac{1}{\rho l_2(a)(1-k^2)}\{z^2+2l_2^2(a)(1+\nu)(1-k^2)\}\mathbf{F}(k)\Bigg],\end{aligned}$$

$$\sigma_{zz} = \frac{2}{\pi}(\cos\phi\tau_x+\sin\phi\tau_y)\frac{z^2}{\rho l_2(a)(1-k^2)}\left[\mathbf{F}(k)-\frac{1+k^2}{1-k^2}\mathbf{E}(k)\right], \tag{83}$$

$$\sigma_2 = \frac{1}{2\pi}\left[\tau e^{i\phi}\left\{\frac{1}{\rho l_2(a)(1-k^2)^2}[4(2-\nu)l_2^2(a)(1-k^2)^2+2z^2(1+k^2)]\mathbf{E}(k) - \frac{2}{\rho l_2(a)(1-k^2)}[2(2-\nu)l_2^2(a)(1-k^2)+z^2]\mathbf{F}(k)\right\}\right.$$
$$+\bar{\tau}e^{i3\phi}\left\{-\frac{4\pi a^2 z(1-2\nu)}{\rho^3} + \frac{2}{3\rho^3 l_2(a)(1-k^2)}[2\nu l_2^2(a)(1-k^2)\{8a^2-\rho^2-4z^2-4l_1^2(a)\} - 3z^2\{\rho^2+4l_1^2(a)-4l_2^2(a)\}]\mathbf{F}(k)\right.$$
$$+\left[\frac{4l_2(a)}{3\rho^3}\{\nu(\rho^2+4z^2-8a^2)-6z^2\}+\frac{2z^2(1+k^2)}{\rho l_2(a)(1-k^2)^2}\right]\mathbf{E}(k)$$
$$\left.\left.+\frac{8(1-2\nu)a^2z^2}{\rho^3 l_2(a)}\Pi(n,k)\right\}\right],$$

$$\tau_z = \frac{1}{\pi}\left[\tau\left\{-\frac{z(\rho^2+z^2-a^2)}{l_2^3(a)(1-k^2)^2}\mathbf{E}(k)+\frac{z(3l_2^2(a)-2l_1^2(a)-a^2)}{l_2^3(a)(1-k^2)}\mathbf{F}(k)-\frac{2z}{l_2(a)}\Pi(n,k)\right\}\right.$$
$$-\bar{\tau}e^{i2\phi}\left\{-\frac{z}{\rho^2 l_2^3(a)(1-k^2)^2}[l_1^2(a)(3l_1^2(a)-2l_2^2(a)-\rho^2)+l_2^2(a)(2l_2^2(a)-\rho^2)]\mathbf{E}(k)\right.$$
$$\left.\left.+\frac{z(2l_2^2(a)-l_1^2(a)-\rho^2)}{\rho^2 l_2(a)(1-k^2)}\mathbf{F}(k)\right\}\right].$$

8. Elastic field on the z-axis. In the previous sections the elastic field is given for a general point within the half space. It is apparent that some of these expressions contain powers of the polar radius ρ in their denominators. However, the numerators also tend to zero, giving a finite limit on the z-axis. To aid in a numerical analysis of these expressions, the elastic field along the z-axis is now evaluated analytically. To this end the following expansions are used:

$$\begin{aligned}
l_1(a) &= \frac{a}{(a^2+z^2)^{1/2}}\rho - \frac{az^2}{2(a^2+z^2)^{5/2}}\rho^3 + O(\rho^5),\\
\frac{1}{l_2(a)} &= \frac{1}{(a^2+z^2)^{1/2}} - \frac{z^2}{2(a^2+z^2)^{5/2}}\rho^2 + O(\rho^4),\\
k &= \frac{a}{a^2+z^2}\rho - \frac{az^2}{(a^2+z^2)^3}\rho^3 + O(\rho^5),\\
\mathbf{E}(k) &= \frac{\pi}{2} - \frac{\pi a^2}{8(a^2+z^2)^2}\rho^2 + O(\rho^4),\\
\mathbf{F}(k) &= \frac{\pi}{2} + \frac{\pi a^2}{8(a^2+z^2)^2}\rho^2 + O(\rho^4),\\
\Pi(n,k) &= \frac{\pi(a^2+z^2)^{1/2}}{2z} + \frac{\pi[z-(a^2+z^2)^{1/2}]}{4(a^2+z^2)^{3/2}}\rho^2 + O(\rho^4), \quad z>0.
\end{aligned} \tag{84}$$

Using the above results, the elastic field for normal loading, along the z-axis becomes

$$
\begin{aligned}
u^c &= \sigma_2 = \tau_z = 0, \\
w &= 2\pi H\sigma \sum_{j=1}^{2} \frac{m_j}{(m_j-1)}\{(a^2+z_j^2)^{1/2} - z_j\}, \\
\sigma_1 &= 4\pi H\sigma A_{66} \sum_{j=1}^{2} \frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j(m_j-1)} \left\{ \frac{z_j}{(a^2+z_j^2)^{1/2}} - 1 \right\}, \\
\sigma_{zz} &= \frac{\sigma}{(\gamma_1-\gamma_2)} \sum_{j=1}^{2} \gamma_j(-1)^{j+1} \left\{ \frac{z_j}{(a^2+z_j^2)^{1/2}} - 1 \right\}.
\end{aligned} \tag{85}
$$

For shear loading, Eqs. (84) lead to

$$
\begin{aligned}
w &= \sigma_1 = \sigma_{zz} = \sigma_2 = 0, \\
u^c &= -\pi H\gamma_1\gamma_2\tau \sum_{j=1}^{2} \frac{1}{(m_j-1)}\{(a^2+z_j^2)^{1/2} - z_j\} + \frac{\gamma_3\tau}{2A_{44}}\{(a^2+z_3^2)^{1/2} - z_3\}, \\
\tau_z &= \frac{\gamma_1\gamma_2\tau}{2(\gamma_1-\gamma_2)} \sum_{j=1}^{2} \frac{(-1)^{j+1}}{\gamma_j} \left\{ 1 - \frac{z_j}{(a^2+z_j^2)^{1/2}} \right\} - \frac{\tau}{2}\left\{ 1 - \frac{z_3}{(a^2+z_3^2)^{1/2}} \right\}.
\end{aligned} \tag{86}
$$

9. Elastic field at the surface. Now the elastic field at the surface is evaluated. As $z \to 0$, it is easy to verify

$$
\lim_{z\to 0} l_1(a) = \min(a,\rho), \qquad \lim_{z\to 0} l_2(a) = \max(a,\rho), \tag{87}
$$

where min is the minimum of the two values and max is the maximum. Thus k and n become

$$
\begin{aligned}
k &= \frac{\rho}{a}, \quad \rho < a; \qquad k = \frac{a}{\rho}, \quad \rho > a, \\
n &= 1, \quad \rho < a; \qquad n = \frac{a^2}{\rho^2}, \quad \rho > a.
\end{aligned} \tag{88}
$$

The only difficulty in evaluating the displacements and stresses at the surface results from the complete elliptic integral of the third kind. From Eq. (15), it is seen that when $z \to 0$ with $\rho < a$ ($n \to 1$), this elliptic integral tends to infinity. However, in the expressions for the elastic field this elliptic integral is multiplied by z which tends to zero. To evaluate this limit, Eq. (23) can be used. From this equation it is readily apparent that

$$
\lim_{z\to 0} z\Pi(n,k) = \frac{\pi a}{2}, \quad \rho < a; \qquad \lim_{z\to 0} z\Pi(n,k) = 0, \quad \rho > a. \tag{89}
$$

An additional consideration is the form of the expressions for the elastic field. For transverse isotropy, the expressions are given as the sum of two terms for normal loading, and a third term is included in some of the expressions for shear loading. When $z \to 0$,

$z_j \to 0$ as well and the two-term summations are only on the elastic parameters. These can be simplified as follows:

$$\sum_{j=1}^{2} \frac{\gamma_j}{(m_j - 1)} = -\gamma_1\gamma_2 \sum_{j=1}^{2} \frac{m_j}{\gamma_j(m_j - 1)} = -\alpha,$$
$$\sum_{j=1}^{2} \frac{m_j}{(m_j - 1)} = -\sum_{j=1}^{2} \frac{1}{(m_j - 1)} = 1, \tag{90}$$
$$\sum_{j=1}^{2} \frac{(m_j + 1)}{\gamma_j(m_j - 1)} = \frac{-1}{2\pi H A_{44}\gamma_1\gamma_2},$$

where the relation $m_1 m_2 = 1$ was used in the first two equations above, and Eq. (51) was used in the last one. In addition to α, the elastic parameters G_1, G_2, and β defined by Fabrikant (1989) are used. These are given as

$$\alpha = \frac{(A_{11}A_{33})^{1/2} - A_{13}}{A_{11}(\gamma_1 + \gamma_2)}, \quad \beta = \frac{\gamma_3}{2\pi A_{44}}, \quad G_1 = \beta + \gamma_1\gamma_2 H, \quad G_2 = \beta - \gamma_1\gamma_2 H. \tag{91}$$

Note that all of the above elastic parameters are real quantities.

Using the above analysis, the elastic field on the surface is easily found. For normal loading the results are

$$u^c = -H\alpha\pi\sigma(x + iy), \quad \rho < a; \qquad u^c = -H\alpha\pi\sigma(x + iy)\frac{a^2}{\rho^2}, \quad \rho > a,$$
$$w = 4H\sigma a\mathbf{E}\left(\frac{\rho}{a}\right), \quad \rho < a; \qquad w = \frac{4H\sigma}{\rho}\left[\rho^2\mathbf{E}\left(\frac{a}{\rho}\right) - (\rho^2 - a^2)\mathbf{F}\left(\frac{a}{\rho}\right)\right], \quad \rho > a,$$
$$\sigma_1 = \frac{2\sigma}{\gamma_1\gamma_2}[-1 + (G_1 - G_2)\pi\alpha A_{66}], \quad \rho < a; \qquad \sigma_1 = 0, \quad \rho > a, \tag{92}$$
$$\sigma_{zz} = -\sigma, \quad \rho < a; \qquad \sigma_{zz} = 0, \quad \rho > a,$$
$$\sigma_2 = 0, \quad \rho < a; \qquad \sigma_2 = 4H\alpha\pi\sigma A_{66}e^{i2\phi}\frac{a^2}{\rho^2}, \quad \rho > a,$$
$$\tau_z = 0, \quad \rho < a; \qquad \tau_z = 0, \quad \rho > a.$$

The expressions for shear loading lead to

$$u^c = 2G_1\tau a\mathbf{E}\left(\frac{\rho}{a}\right) + G_2\bar{\tau}e^{i2\phi}\frac{2a}{3\rho^2}\left[(2a^2 - \rho^2)\mathbf{E}\left(\frac{\rho}{a}\right) + 2(\rho^2 - a^2)\mathbf{F}\left(\frac{\rho}{a}\right)\right], \quad \rho < a,$$
$$u^c = 2G_1\tau\frac{1}{\rho}\left[\rho^2\mathbf{E}\left(\frac{a}{\rho}\right) - (\rho^2 - a^2)\mathbf{F}\left(\frac{a}{\rho}\right)\right]$$
$$+ \frac{2}{3\rho}G_2\bar{\tau}e^{i2\phi}\left[(2a^2 - \rho^2)\mathbf{E}\left(\frac{a}{\rho}\right) + (\rho^2 - a^2)\mathbf{F}\left(\frac{a}{\rho}\right)\right], \quad \rho > a,$$

$$w = H\pi\alpha(x\tau_x + y\tau_y), \quad \rho < a; \qquad w = H\pi\alpha(x\tau_x + y\tau_y)\frac{a^2}{\rho^2}, \quad \rho > a,$$

$$\sigma_1 = -8A_{66}\left[H\gamma_1\gamma_2 - \frac{\beta\gamma_3(\gamma_1+\gamma_2)}{\gamma_1\gamma_2}\right](\cos\phi\tau_x + \sin\phi\tau_y)\frac{a}{\rho}\left[\mathbf{E}\left(\frac{\rho}{a}\right) - \mathbf{F}\left(\frac{\rho}{a}\right)\right], \quad \rho < a,$$
$$\sigma_1 = -8A_{66}\left[H\gamma_1\gamma_2 - \frac{\beta\gamma_3(\gamma_1+\gamma_2)}{\gamma_1\gamma_2}\right](\cos\phi\tau_x + \sin\phi\tau_y)\left[\mathbf{E}\left(\frac{a}{\rho}\right) - \mathbf{F}\left(\frac{a}{\rho}\right)\right], \quad \rho > a, \tag{93}$$
$$\sigma_{zz} = 0, \quad \rho < a; \qquad \sigma_{zz} = 0, \quad \rho > a,$$

$$\sigma_2 = A_{66}G_1\tau e^{i\phi}\frac{4a}{\rho}\left[\mathbf{E}\left(\frac{\rho}{a}\right) - \mathbf{F}\left(\frac{\rho}{a}\right)\right]$$
$$+ \frac{4A_{66}G_2 a}{3\rho^3}\bar{\tau}e^{i3\phi}\left[(\rho^2 - 8a^2)\mathbf{E}\left(\frac{\rho}{a}\right) + (8a^2 - 5\rho^2)\mathbf{F}\left(\frac{\rho}{a}\right)\right], \quad \rho < a,$$
$$\sigma_2 = 4A_{66}G_1\tau e^{i\phi}\left[\mathbf{E}\left(\frac{a}{\rho}\right) - \mathbf{F}\left(\frac{a}{\rho}\right)\right]$$
$$+ \frac{4A_{66}G_2}{3\rho^2}\bar{\tau}e^{i3\phi}\left[(\rho^2 - 8a^2)\mathbf{E}\left(\frac{a}{\rho}\right) + (4a^2 - \rho^2)\mathbf{F}\left(\frac{a}{\rho}\right)\right], \quad \rho > a,$$
$$\tau_z = -\tau, \quad \rho < a; \qquad \tau_z = 0, \quad \rho > a.$$

The above results can be specialized for isotropic materials using the values in Eq. (71) for H and A_{66} while the additional results needed are

$$\gamma_1 = \gamma_2 = \gamma_3 = 1, \quad \alpha = \frac{1-2\nu}{2(1-\nu)}, \quad \beta = \frac{1+\nu}{\pi E},$$
$$G_1 = \frac{(2-\nu)(1+\nu)}{\pi E}, \quad G_2 = \frac{\nu(1+\nu)}{\pi E}. \tag{94}$$

10. Summary and discussion. This paper has evaluated displacement and stress fields in a transversely isotropic half space when a uniform traction is prescribed over a circular area on the surface. The solution is evaluated in terms of closed-form expressions containing complete elliptic integrals of the first, second, and third kinds. The solution for an isotropic half space was evaluated as a special case. The solution for an isotropic half space was previously solved using direct integration methods by Love (1929). He evaluated all derivatives of the potential function necessary to obtain the stress field but his solution does not provide the complete displacement field. A further difference in the solutions results from the integral I_3 in Eq. (31). Love evaluated it in terms of complete and incomplete elliptic integrals of the first and second kinds whereas here it is evaluated in a simpler form involving the complete elliptic integral of the third kind. The present solution analytically evaluates the elastic field along the z-axis and at the surface $z = 0$. A new relation is also established between different forms of the complete elliptic integral of the third kind. Thus, using Love's method, the radial derivatives of the potential function were evaluated and this new relation allowed the final expression to be written in a consistent, numerically friendly form.

It has not been forgotten that this problem could have been solved by application of Hankel transform methods. This solution method was used by Terazawa (1916) for normal pressure on an isotropic half space, and the details are reviewed by Sneddon (1951). From Sneddon's comments it seems apparent that Terazawa evaluated the inverse transform integrals in terms of elliptic integrals (as they must be). However, the authors have not been able (at present) to obtain this paper and possibly compare expressions. The isotropic solution for uniform shear loading over a circular area was solved by Muki (1960) using Hankel transform methods. The evaluation of the elastic field into elliptic integrals was not published and no comparisons can be made.

In any event, the inverse Hankel transform integrals (which are available in the literature) arising in the Hankel transform method could be evaluated using results derived in a paper by Eason et al. (1955). They evaluated various integrals of Lipschitz-Hankel type containing products of Bessel functions in terms of complete elliptic integrals of all three kinds. The parameters in these complete elliptic integrals are analogous to those used by Love (1929), although arrived at in a different way. In particular, their integral evaluations are given by slightly different expressions for different ranges in the parameters. Using the new results in the present paper, particularly Eq. (23) and the parameters $l_1(a)$ and $l_2(a)$, it is anticipated that the integral evaluations in the paper by Eason et al. (1955) can be put in a much more convenient form. This possibility is presently being explored by the authors.

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Appendix A. The z derivative of the potential function ψ is evaluated here. From Eq. (28) it is easy to see that

$$\frac{\partial}{\partial z}\psi(\rho, z) = \int_0^{2\pi}\int_0^a \frac{\rho_0\, d\rho_0\, d\phi_0}{R}. \tag{A1}$$

The reciprocal of the distance R can be written as (see Fabrikant, 1988)

$$\frac{1}{R} = \frac{2}{\pi}\int_0^{l_1(\rho_0)} \frac{1}{[\rho^2 - x^2]^{1/2}[\rho_0^2 - g^2(x)]^{1/2}}\lambda\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right) dx, \tag{A2}$$

where

$$\lambda(k, \omega) = \frac{1 - k^2}{1 + k^2 - 2k\cos\omega}, \qquad g^2(x) = x^2 + \frac{x^2 z^2}{\rho^2 - x^2}. \tag{A3}$$

If Eq. (A2) is substituted into (A1), the integral on ϕ_0 can easily be evaluated, giving

$$\frac{\partial}{\partial z}\psi(\rho, z) = 4\int_0^a\int_0^{l_1(\rho_0)} \frac{1}{[\rho^2 - x^2]^{1/2}[\rho_0^2 - g^2(x)]^{1/2}}\, dx\, \rho_0\, d\rho_0. \tag{A4}$$

Interchanging the order of integration leads to

$$\begin{aligned}\frac{\partial}{\partial z}\psi(\rho, z) &= 4\int_0^{l_1(a)} \frac{1}{[\rho^2 - x^2]^{1/2}}\int_{g(x)}^a \frac{\rho_0\, dp_0}{[\rho_0^2 - g^2(x)]^{1/2}}\, dx \\ &= 4\int_0^{l_1(a)} \frac{[a^2 - g^2(x)]^{1/2}}{[\rho^2 - x^2]^{1/2}}\, dx.\end{aligned} \tag{A5}$$

Substituting for $g(x)$ and algebraic manipulations result in

$$\frac{\partial}{\partial z}\psi(\rho,z) = 4\int_0^{l_1(a)} \frac{[l_1^2(a)-x^2]^{1/2}[l_2^2(a)-x^2]^{1/2}}{\rho^2-x^2}\,dx. \tag{A6}$$

This integral can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial z}\psi(\rho,z) &= 4[l_1^2(a)-\rho^2][l_2^2(a)-\rho^2]\int_0^{l_1(a)} \frac{dx}{(\rho^2-x^2)[l_1^2(a)-x^2]^{1/2}[l_2^2(a)-x^2]^{1/2}}\\ &\quad + 4[l_1^2(a)-\rho^2]\int_0^{l_1(a)} \frac{dx}{[l_1^2(a)-x^2]^{1/2}[l_2^2(a)-x^2]^{1/2}}\\ &\quad + 4\int_0^{l_1(a)} \frac{[l_2^2(a)-x^2]^{1/2}}{[l_1^2(a)-x^2]^{1/2}}\,dx.\end{aligned} \tag{A7}$$

Introducing the new integration variable t by the relation $x = l_1(a)t$, the above integrals become

$$\begin{aligned}&\int_0^{l_1(a)} \frac{dx}{[l_1^2(a)-x^2]^{1/2}[l_2^2(a)-x^2]^{1/2}}\\ &\qquad = \frac{1}{l_2(a)}\int_0^1 \frac{dt}{(1-t^2)^{1/2}(1-k^2t^2)^{1/2}} = \frac{1}{l_2(a)}\mathbf{F}(k),\end{aligned}$$

$$\int_0^{l_1(a)} \frac{[l_2^2(a)-x^2]^{1/2}}{[l_1^2(a)-x^2]^{1/2}}\,dx = l_2(a)\int_0^1 \frac{(1-k^2t^2)^{1/2}}{(1-t^2)^{1/2}}\,dt = l_2(a)\mathbf{E}(k), \tag{A8}$$

$$\begin{aligned}&\int_0^{l_1(a)} \frac{dx}{(\rho^2-x^2)[l_1^2(a)-x^2]^{1/2}[l_2^2(a)-x^2]^{1/2}}\\ &\qquad = \frac{1}{\rho^2 l_2(a)}\int_0^1 \frac{dt}{(1-nt^2)(1-t^2)^{1/2}(1-k^2t^2)^{1/2}} = \frac{1}{\rho^2 l_2(a)}\Pi(n,k).\end{aligned}$$

The final form of this function is

$$\frac{\partial}{\partial z}\psi(\rho,z) = \frac{4}{l_2(a)}\{l_2^2(a)\mathbf{E}(k) - [\rho^2 - l_1^2(a)]\mathbf{F}(k) - z^2\Pi(n,k)\}. \tag{A9}$$

Appendix B. Here the radial derivatives of the function $\Phi(\rho,z)$ are evaluated. Using Love's method it can be shown that

$$\frac{\partial}{\partial \rho}\Phi(\rho,z) = a^2\rho\int_0^{2\pi} \frac{\sin^2\theta}{\overline{R}}\,d\theta, \qquad \overline{R} = [\rho^2 + a^2 - 2\rho a\cos\theta + z^2]^{1/2}. \tag{B1}$$

Using the relation $\sin^2\theta = 1 - \cos^2\theta$ and some algebra provides the result

$$\int_0^{2\pi} \frac{\sin^2\theta}{\overline{R}}\,d\theta = \int_0^{2\pi} \frac{d\theta}{\overline{R}} - \frac{\rho^2+a^2+z^2}{2a\rho}\int_0^{2\pi} \frac{\cos\theta\,d\theta}{\overline{R}} + \frac{1}{2a\rho}\int_0^{2\pi} \overline{R}\cos\theta\,d\theta. \tag{B2}$$

Performing additional algebraic manipulations on the middle term and using $\cos\theta\, d\theta = d(\sin\theta)$ to integrate the last integral by parts leads to

$$\int_0^{2\pi} \frac{\sin^2\theta}{\overline{R}} d\theta = \int_0^{2\pi} \frac{d\theta}{\overline{\overline{R}}} + \frac{\rho^2 + a^2 + z^2}{4a^2\rho^2} \int_0^{2\pi} \overline{R}\, d\theta - \frac{[\rho^2 + a^2 + z^2]^2}{4a^2\rho^2} \int_0^{2\pi} \frac{d\theta}{\overline{\overline{R}}} - \frac{1}{2} \int_0^{2\pi} \frac{\sin^2\theta}{\overline{\overline{R}}} d\theta. \tag{B3}$$

Collecting terms and introducing the parameters $l_1(a)$ and $l_2(a)$ allows one to obtain

$$\int_0^{2\pi} \frac{\sin^2\theta}{\overline{R}} d\theta = \frac{[l_1^2(a) + l_2^2(a)]}{6a^2\rho^2} \int_0^{2\pi} \overline{R}\, d\theta - \frac{[l_2^2(a) - l_1^2(a)]^2}{6a^2\rho^2} \int_0^{2\pi} \frac{d\theta}{\overline{\overline{R}}}. \tag{B4}$$

The second integral above is I_2, which was evaluated in Eq. (32), whereas the first integral is part of I_1 and was evaluated in Eq. (33). Substituting these evaluations into the above equation and then substituting Eq. (B4) into Eq. (B1) leads to Eq. (56).

To evaluate the second radial derivative, Love's method is again used to write it as

$$\frac{\partial^2}{\partial\rho^2}\Phi(\rho, z) = \int_0^{2\pi} \frac{[a^2\cos^2\theta - a\rho\cos\theta]}{\overline{R}} d\theta. \tag{B5}$$

Now substituting $\cos^2\theta = 1 - \sin^2\theta$ allows this integral to be written as

$$\frac{\partial^2}{\partial\rho^2}\Phi(\rho, z) = \frac{1}{2} \int_0^{2\pi} \overline{R}\, d\theta + \frac{a^2 - \rho^2 - z^2}{2} \int_0^{2\pi} \frac{d\theta}{\overline{\overline{R}}} - a^2 \int_0^{2\pi} \frac{\sin^2\theta}{\overline{R}} d\theta. \tag{B6}$$

If Eq. (B4) is substituted for the last term above and the terms are collected, one obtains

$$\frac{\partial^2}{\partial\rho^2}\Phi(\rho, z) = \frac{2\rho^2 - a^2 - z^2}{6\rho^2} \int_0^{2\pi} \overline{R} d\theta + \frac{[l_2^2(a) - l_1^2(a)]^2 + 3\rho^2(a^2 - \rho^2 - z^2)}{6\rho^2} \int_0^{2\pi} \frac{d\theta}{\overline{\overline{R}}}. \tag{B7}$$

Again, relating the above integrals to I_1 and I_2 allows the final result to be written as in Eq. (57).

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