Low dimensional behavior of large systems of globally coupled oscillators

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It is shown that, in the infinite size limit, certain systems of globally coupled phase oscillators display low dimensional dynamics. In particular, we derive an explicit finite set of nonlinear ordinary differential equations for the macroscopic evolution of the systems considered. For example, an exact, closed form solution for the nonlinear time evolution of the Kuramoto problem with a Lorentzian oscillator frequency distribution function is obtained. Low dimensional behavior is also demonstrated for several prototypical extensions of the Kuramoto model, and time-delayed coupling is also considered. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2930766]

Because synchronous behavior in large groups consisting of many coupled oscillators has been widely observed in many situations, the behavior of such systems has long been of interest. Since the problem is difficult to solve in general, much work has been done on the simple paradigmatic case of globally coupled phase oscillators. Even in this simple context, however, much remains unclear, particularly when considering situations in which a large oscillator population interacts with external dynamical systems, or when there are communities of interacting oscillators with different community and connection characteristics, etc. In this paper we consider an approach that allows the study of the time evolving dynamical behavior of these types of systems by an exact reduction to a small number of ordinary differential equations. This reduction is achieved by considering a restricted class of states. In spite of this restriction, for at least one significant example (see Ref. 10), consideration of our derived ordinary differential equations appears to yield dynamics in precise agreement with results obtained from considerations not imposing this restriction. Thus we believe that our results may be useful in many other contexts.

I. INTRODUCTION

Understanding the generic behavior of systems consisting of large numbers of coupled oscillators is of great interest because such systems occur in a wide variety of significant applications.[1] Examples are the synchronous flashing of groups of fireflies, coordination of oscillatory neurons governing circadian rhythms in animals,[2] entrainment in coupled oscillatory chemically reacting cells,[3] Josephson junction circuits,[4] neutrino oscillations,[5] bubbly fluids,[6] etc. A key contribution in this area was the introduction of the following model by Kuramoto:[7]

$$d\theta_i(t)/dt = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin[\theta_j(t) - \theta_i(t)], \quad (1)$$

where the state of oscillator i is given by its phase $\theta_i(t)$ ($i=1,2,\ldots,N$), ω_i is the natural frequency of oscillator i, and the coupling constant K specifies the strength of the influence of one oscillator on another. It has been shown[7,8] that in the $N\to\infty$ limit there is a continuous phase transition such that, for K below a critical value ($K<K_c$), no coherent behavior of the system occurs (i.e., there is no global correlation between the oscillator phases), while above the critical coupling strength ($K>K_c$), the system displays global cooperative behavior (i.e., partial or complete synchronization of the phases).

Among other problems related to Eq. (1) that we shall also consider are the case where there is a sinusoidal periodic external drive of strength Λ added to the right-hand side of Eq. (1) (see Refs. 9 and 10),

$$d\theta_i/dt = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \Lambda \sin(\Omega t - \theta_i), \quad (2)$$

and the case where there are several communities of different kinds of oscillators where the evolution of the phases $\theta_i^\sigma(t)$ of oscillators in community σ is given by (see Refs. 11 and 12)

$$d\theta_i^\sigma/dt = \omega_i^\sigma + \sum_{\sigma'=1}^{s} \frac{K_{\sigma\sigma'}}{N_{\sigma'}}\sum_{j=1}^{N_{\sigma'}} \sin(\theta_j^{\sigma'} - \theta_i^\sigma). \quad (3)$$

Here $\sigma=1,2,\ldots,s$, N_σ is the number of oscillators of type σ, and $K_{\sigma\sigma'}$ is the strength of the coupling from oscillators in community σ' to oscillators in community σ. For all three cases [Eqs. (1)–(3)], we are interested in the limit $N\to\infty$. We will also consider such problems with time delayed coupling [e.g., $\theta_j(t)\to\theta_j(t-\tau)$ in Eqs. (1)–(3)].

The problem stated in Eq. (2) was first considered by Sakaguchi.[9] It can, for example, be motivated as a model of circadian rhythm.[2] Circadian rhythm in mammals is governed by the suprachiasmatic nucleus that is located in the brain and consists of a large population of oscillatory neurons. These neurons presumably couple with each other and are also influenced (though the optic nerve) by the daily variation of sunlight [modeled by the term in Eq. (2) involving Λ]. In Ref. 10 we found numerical and analytical evidence that the bifurcations and macroscopic dynamics of Eq. (2) with large N appeared to be similar to what might be expected for the dynamics of a two-dimensional dynamical system. This observation was the motivation for the present paper.

The problem stated in Eq. (3) has been previously considered in Refs. 11 and 12 where the linear stability of the incoherent state was investigated along with numerical solutions for the nonlinear evolution.

II. NATURE OF THE MAIN RESULT

Considering the limit $N\to\infty$, the state of the oscillator system at time t can be described by a continuous distribution function, $f(\omega,\theta,t)$, in frequency ω and phase θ for the problems in Eqs. (1) and (2) or by $f^{\sigma}(\omega,\theta,t)$ with $\sigma=1,2,\ldots,s$ for the problem in Eq. (3), where

$$\int_0^{2\pi} f(\omega,\theta,t)d\theta = g(\omega) \quad \text{or} \quad \int_0^{2\pi} f^{\sigma}(\omega,\theta,t)d\theta = g^{\sigma}(\omega),$$

and $g(\omega)$ and $g^{\sigma}(\omega)$ are time independent oscillator frequency distributions.

Our main result is as follows. For initial distribution functions $f(\omega,\theta,0)$ satisfying a certain set of conditions that we will specify later in this paper, we show that

(i) the evolution of $f(\omega,\theta,t)$ from $f(\omega,\theta,0)$ continues to satisfy the specified conditions;
(ii) for appropriate $g(\omega)$ [or $g^{\sigma}(\omega)$], the macroscopic system state obeys a finite set of nonlinear ordinary differential equations, which we obtain explicitly.

Concerning (i), we define a distribution function $h(\omega,\theta)$ as a function for which $h\geq 0$ and $\int_0^{2\pi}d\theta\int d\omega h=1$, and the distribution functions $h(\omega,\theta)$ satisfying our conditions form a manifold M in the space D of all possible distribution functions. What point (i) says is that initial states in $M\subset D$ evolve to other states in M. Thus M is "invariant" under the dynamics. Concerning point (ii), we use the so-called "order-parameter" description to define the macroscopic system state. We define the order parameter [or parameters in the case of Eq. (3)] subsequently [Eq. (5)] in terms of an integral over the distribution function f [or f^{σ} for Eq. (3)], where this order-parameter integral globally quantifies the degree to which the entire ensemble of oscillators [or ensembles σ for Eq. (3)] behaves in a coherent manner. According to point (ii) the evolution of the order parameters is exactly finite dimensional even though the manifold M and the dynamics of the distribution function f as it evolves in M are infinite dimensional.

The macroscopic dynamics we obtain allows for much simplified investigation of the systems we study. For example, we obtain an exact closed form solution for the nonlinear time evolution of the Kuramoto problem, Eq. (1), for the case of Lorentzian $g(\omega)$. Our formulation will be practically useful if at least some of the macroscopic order-parameter attractors and bifurcations of the full dynamics in the space D are replicated in M. In this regard, we note that numerical solutions of the system (2) for large N have been carried out in Ref. 10, and the resulting macroscopic order-parameter attractors, as well as their bifurcations with variation of system parameters, have been fully mapped out. Comparing these numerical results for the full system [Eq. (2)] with results for the corresponding low dimensional system for the dynamics on M [Eq. (14)], we find that *all* (not just some) of the macroscopic order-parameter attractors and bifurcations of Eq. (2) with Lorentzian $g(\omega)$ are precisely and quantitatively captured by examination of the dynamics on M. These results for the problem given by Eq. (2) suggest that our approach may be useful for other situations such as Eq. (3). Another notable point is that Ref. 10 also reports numerical simulation results for Eq. (2) with large N for the case of a Gaussian oscillator distribution function, $g(\omega)=(2\pi\Delta^2)^{-1/2}\exp[-(\omega-\omega_0)^2/(2\Delta^2)]$, and the macroscopic order-parameter attractors and bifurcations in this case are found to be the same as those in the Lorentzian case (albeit at different parameter values). Thus, at least for problem (2), phenomena for Lorentzian $g(\omega)$ are not special and should give a useful indication of what can be expected for other unimodal distributions $g(\omega)$.

III. DERIVATION FOR THE EXAMPLE OF THE KURAMOTO PROBLEM

We now support points (i) and (ii) for the case of the Kuramoto problem, Eq. (1). Following that, we will consider other problems, including those associated with Eqs. (2) and (3). Because of its relative simplicity, in this section we use the Kuramoto problem as an example, but we emphasize that our interest is primarily in developing a method that will be useful in less simple cases, such as the problems stated in Eqs. (2) and (3) (see Sec. IV). Following Kuramoto,[7,8] we note that the summation in Eq. (1) can be written as

$$\frac{1}{N}\sum_j \sin[\theta_j-\theta_i] = \mathrm{Im}\left\{e^{-i\theta_i}\frac{1}{N}\sum_j e^{i\theta_j}\right\} = \mathrm{Im}[re^{-i\theta_i}],$$

where $r=N^{-1}\Sigma\exp(i\theta_j)$. Letting $N\to\infty$ in Eq. (1), $f(\omega,\theta,t)$ satisfies the following initial value problem:

$$\partial f/\partial t + \partial/\partial\theta\{[\omega+(K/2i)(re^{-i\theta}-r^*e^{i\theta})]f\}=0, \tag{4}$$

$$r=\int_0^{2\pi}d\theta\int_{-\infty}^{+\infty}d\omega f e^{i\theta}, \tag{5}$$

where $r(t)$ is the *order parameter*, and Eq. (4) is the continuity equation for the conservation of the number of oscillators. Note that by its definition (5), r satisfies $|r|\leq 1$. Expanding $f(\omega,\theta,t)$ in a Fourier series in θ, we have

$$f=(g(\omega)/2\pi)\left\{1+\left[\sum_{n=1}^{\infty}f_n(\omega,t)\exp(in\theta)+\text{c.c.}\right]\right\},$$

where c.c. stands for complex conjugate. We now consider a restricted class of $f_n(\omega,t)$ such that

$$f_n(\omega,t)=[\alpha(\omega,t)]^n,$$

where $|\alpha(\omega,t)|\leq 1$ to avoid divergence of the series. Substituting this series expansion into Eqs. (4) and (5), we find the remarkable result that this special form of f represents a solution to Eqs. (4) and (5) if

$$\partial\alpha/\partial t+(K/2)(r\alpha^2-r^*)+i\omega\alpha=0, \tag{6}$$

$$r^* = \int_{-\infty}^{+\infty} d\omega \alpha(\omega,t) g(\omega). \tag{7}$$

Thus this special initial condition reduces the θ-dependent system, Eqs. (4) and (5) to a problem (6) and (7), that is θ-independent. However, we emphasize that Eqs. (6) and (7) still constitute an infinite dimensional dynamical system because any initial condition is a *function* of ω, namely $\alpha(\omega,0)$. Performing the summation of the Fourier series using $\Sigma_{n=1}^{\infty} x^n = x/(1-x)$, we obtain

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi} \frac{(1-|\alpha|)(1+|\alpha|)}{(1-|\alpha|)^2 + 4|\alpha| \sin^2\left[\frac{1}{2}(\theta-\psi)\right]}, \tag{8}$$

where $\alpha \equiv |\alpha| e^{-i\psi}$ and ψ real. For $|\alpha|<1$ we can explicitly verify from Eq. (8) that $f \geq 0$, $\int d\theta f = g(\omega)/2\pi$, and that as $|\alpha| \nearrow 1$ we have $f \to \delta(\theta-\psi) g(\omega)/2\pi$. In order that Eqs. (6) and (7) represent a solution of Eq. (5) for all finite time, we require that, as $\alpha(\omega,t)$ evolves under Eqs. (6) and (7), $|\alpha(\omega,t)| \leq 1$ continues to be satisfied. This can be shown by substituting $\alpha = |\alpha| e^{-i\psi}$ into Eq. (6), multiplying by $e^{i\psi}$, and taking the real part of the result, thus obtaining

$$\partial|\alpha|/\partial t + (K/2)(|\alpha|^2 - 1)\mathrm{Re}[r e^{-i\psi}] = 0. \tag{9}$$

We see from Eq. (9) that $\partial|\alpha|/\partial t = 0$ at $|\alpha|=1$. Hence a trajectory of Eq. (6), starting with an initial condition satisfying $|\alpha(\omega,0)|<1$ cannot cross the unit circle in the complex α-plane, and we have $|\alpha(\omega,t)|<1$ for all finite time, $0 \leq t < +\infty$.

One way to motivate our ansatz, $f_n = \alpha^n$, is to note that the well-known stationary states of the Kuramoto model,[7,8] both the incoherent state ($f = g/2\pi$ corresponding to $\alpha = 0$) and the partially synchronized state with $|r| = \text{const} > 0$, both conform to $f_n = \alpha^n$. Thus one view of the ansatz is that it specifies a family of distribution functions that connect these two states in a natural way.

To proceed further, we now introduce another restriction on our assumed form of f; we require that $\alpha(\omega,t)$ can be analytically continued from real ω into the complex ω-plane, that this continuation has no singularities in the lower half ω-plane, and that $|\alpha(\omega,t)| \to 0$ as $\mathrm{Im}(\omega) \to -\infty$. If these conditions are satisfied for the initial condition, $\alpha(\omega,0)$, then they are also satisfied for $\alpha(\omega,t)$ for $\infty > t > 0$. To see that this is so, we first note that for large negative $\omega_i = \mathrm{Im}(\omega)$, Eq. (6) is approximately $\partial\alpha/\partial t = -|\omega_i|\alpha$, and thus $\alpha(\omega,t) \to 0$ as $\omega_i \to -\infty$ will continue to be satisfied if $\alpha(\omega,0) \to 0$ as $\omega_i \to -\infty$. Next we note from Ref. 13, that $\alpha(\omega,t)$ is analytic in any region of the complex ω-plane for which $\alpha(\omega,0)$ is analytic provided that the solution $\alpha(\omega,t)$ to Eq. (6) exists. To establish existence for $0 \leq t < +\infty$ it suffices to show that the solution to Eq. (6) cannot become infinite at a finite value of t. This can be ruled out by noting that our derivation of Eq. (9) with ω now complex carries through except that there is now an additional term $-\omega_i|\alpha|$ on the left-hand side of the equation. Thus at $|\alpha|=1$ we have $\partial|\alpha|/\partial t = \omega_i|\alpha| < 0$, and we conclude that, if $|\alpha(\omega,0)|<1$ everywhere in the lower half complex ω-plane, then $|\alpha(\omega,t)|<1$ for all finite time $0 \leq t < +\infty$ everywhere in the lower half complex ω-plane.

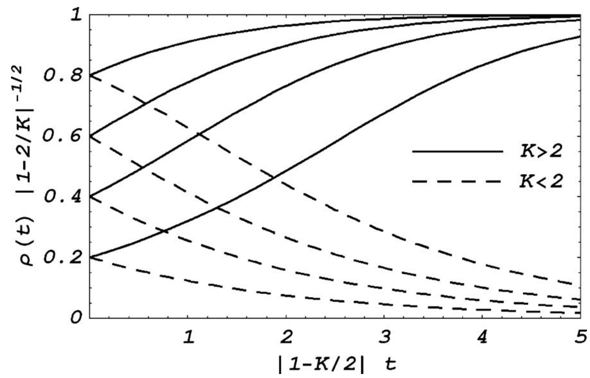

FIG. 1. The order parameter $\rho = |r|$ vs time.

Regarding the initial condition $\alpha(\omega,0)$, we note that, if $|\alpha(\omega,0)| \leq 1$ for ω real, if the continuation $\alpha(\omega,0)$ is analytic everywhere in the lower half ω-plane, and if the continuation satisfies $|\alpha(\omega,0)| \to 0$ as $\omega_i \to -\infty$, then the continuation satisfies $|\alpha(\omega,0)|<1$ everywhere in the lower half complex ω-plane.[14] Examples of possible initial conditions are $k \exp(-i\omega c)$ with $\mathrm{Re}(c)>0$ and $|k| \leq 1$, $k/(\omega-d)$ with $|k| \leq \mathrm{Im}(d)$, and $\int_0^\infty k(c)\exp(-i\omega c)dc$ with $\int_0^\infty |k(c)| dc \leq 1$.

We can now specify the invariant manifold M on which our dynamics takes place. It is the space of functions of the real variables (ω,θ) of the form given by Eq. (8) where $|\alpha(\omega,t)| \leq 1$ for real ω; $\alpha(\omega,t)$ can be analytically continued from the real ω-axis into the lower half ω-plane; and, when continued into the lower half ω-plane, $\alpha(\omega,t)$ has no singularities there and approaches zero as $\omega_i \to -\infty$.

Now taking $g(\omega)$ to be Lorentzian

$$g(\omega) = g_L(\omega) \equiv (\Delta/\pi)[(\omega-\omega_0)^2 + \Delta^2]^{-1},$$

we can do the ω integral in Eq. (7) by closing the contour by a large semicircle in the lower half ω-plane. Writing $g_L(\omega) = (2\pi i)^{-1}[(\omega-\omega_0-i\Delta)^{-1} - (\omega-\omega_0+i\Delta)^{-1}]$, we see that the integral is given by the residue of the pole at $\omega = \omega_0 - i\Delta$. By a change of variables $(\theta,\omega) \to [\theta-\omega_0 t, (\omega-\omega_0)/\Delta]$, we can, without loss of generality set $\omega_0=0$, $\Delta=1$. Thus we obtain $r(t) = \alpha^*(-i,t)$. Putting this result into Eq. (6) and setting $\omega = -i$, we obtain the nonlinear evolution of the order parameter $r = \rho e^{-i\phi}$ ($\rho \geq 0$ and ϕ real),

$$d\rho/dt + \left(1 - \tfrac{1}{2}K\right)\rho + \tfrac{1}{2}K\rho^3 = 0, \tag{10}$$

and $d\phi/dt = 0$. Thus the dynamics is described by the single real nonlinear, first order, ordinary differential equation, Eq. (10). The solution of Eq. (10) is

$$\frac{\rho(t)}{R} = \left| 1 + \left\{ \left[\frac{R}{\rho(0)} \right]^2 - 1 \right\} e^{[1-(1/2)K]t} \right|^{-1/2}, \tag{11}$$

where $R = |1-(2/K)|^{1/2}$. We see that for $K < K_c = 2$, the order parameter goes to zero exponentially with increasing time, while for $K>2$ it exponentially asymptotes to the finite value $[1-(2/K)]^{1/2}$, in agreement with the known time-asymptotic results for the case $g = g_L$ (e.g., see Ref. 8). Plots of the nonlinear evolution of $\rho(t)$ are shown in Fig. 1. Linearization of Eq. (10) yields an exponential damping rate of $[1-(K/2)]$ for perturbations around $\rho=0$ for $K<2$, which becomes unstable for $K > K_c = 2$, at which point the stable nonlinear equilibrium at $\rho=\sqrt{1-(2/K)}$ comes into existence. For $K>K_c$ linearization of Eq. (10) around the equilibrium $\rho=\sqrt{1-(2/K)}$ yields a corresponding perturbation damping rate $[(K/2)-1]$. For $g=g_L$ the latter damping rate can also be obtained from the recent stability analyses of solutions of Eqs. (4) and (5).[10,15] We emphasize that our solution for $r(t)$ obeys two uncoupled first order real ordinary differential equations [Eq. (10) and $d\phi/dt=0$], while the problem for $\alpha(\omega,t)$ [Eqs. (6) and (7)] is an infinite dimensional dynamical system [i.e., to obtain $\alpha(\omega,t)$ we need to specify an initial *function* of ω, $\alpha(\omega,0)$]. This is further reflected by the fact that linearization of Eqs. (6) and (7) about their equilibria yields a problem with a continuous spectrum of neutral modes.[15,16] Thus the microscopic dynamics in M of the distribution function is infinite dimensional, while the macroscopic dynamics of the order parameter is low dimensional.

IV. GENERALIZATIONS

A. Other distributions $g(\omega)$

So far we have restricted our discussion to the case of the Lorentzian $g_L(\omega)$. We now consider

$$g(\omega)=g_4(\omega)\equiv(\sqrt{2}/\pi)(\omega^4+1)^{-1},$$

which decreases with increasing ω as ω^{-4}, in contrast to $g_L(\omega)$ which decreases as ω^{-2}. The distribution $g_4(\omega)$ has four poles at $\omega=(\pm 1\pm i)/\sqrt{2}$. Proceeding as before, we apply the residue method to the integral equation (7) to obtain

$$r(t)=\tfrac{1}{2}[(1+i)r_1(t)+(1-i)r_2(t)],$$

where

$$r_{1,2}=\alpha^*[(\mp 1-i)/\sqrt{2},t]$$

and $r_{1,2}(t)$ obey the two coupled nonlinear ordinary differential equations,

$$dr_{1,2}/dt+(K/2)[r^*r_{1,2}^2-r]+[(1\mp i)/\sqrt{2}]r_{1,2}=0. \tag{12}$$

Thus we obtain a system of two first order complex nonlinear differential equations. Indeed, the above considerations can be applied to any $g(\omega)$ that is a rational function of ω [i.e., $g(\omega)=P_1(\omega)/P_2(\omega)$ where $P_1(\omega)$ and $P_2(\omega)$ are polynomials in ω]. The requirement that $g(\omega)$ be normalized [$\int g(\omega)d\omega=1$] and real puts restrictions on the possible $P_{1,2}(\omega)$, e.g., $P_2(\omega)$ must have an even degree, $2m$, and all its roots must come in complex conjugate pairs (it cannot have a root on the real ω axis). Such a $g(\omega)$ has m poles in the lower half ω-plane, and application of our method yields m complex, first order ordinary differential equations for m complex order parameters. For instance, for the example, $g(\omega)=g_4(\omega)$, above, there are two poles in $\mathrm{Im}(\omega)<0$, namely, $\omega=(\pm 1-i)/\sqrt{2}$, and these two poles result in the two order parameters r_1 and r_2.

B. External driving

We now consider the Kuramoto problem with an external drive, Eq. (2). Again taking the $N\to\infty$ limit for the number of oscillators, we obtain

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial\theta}\left\{f\left[(\omega-\Omega)+\frac{1}{2i}(Kr+\Lambda)e^{-i\theta}-\frac{1}{2i}(Kr+\Lambda)^*e^{i\theta}\right]\right\}=0, \tag{13}$$

with $r(t)$ given by Eq. (5). In writing Eq. (13), we have utilized a change of variables $\theta\to\theta+\Omega t$ to remove the $e^{i\Omega t}$ time dependence that would otherwise appear multiplying the Λ terms. Again assuming that $g(\omega)$ is a Lorentzian with unit width $\Delta=1$ peaked at $\omega=\omega_0$, and proceeding as before, we obtain the following equation for $r(t)$:

$$dr/dt+\tfrac{1}{2}[(Kr+\Lambda)^*r^2-(Kr+\Lambda)]+[1+i(\Omega-\omega_0)]r=0. \tag{14}$$

Equilibria are obtained by setting $dr/dt=0$ in Eq. (14). Depending on parameters (K,Ω,Λ), there are either one or three such equilibria.[10] Also, depending on the parameters, there may be an attracting limit cycle. Whether the equilibria are attractors for Eq. (14) depends on their stability which can be assessed by linearization around the equilibria. The equilibria obtained for Eq. (14) and their stability are the same as obtained by the analysis of the full system (13) as performed in Ref. 10. Furthermore, the bifurcations and stability of the limit cycle are the same as numerically found in Ref. 10. Thus, for this problem, it appears that the important observable macroscopic dynamics is contained entirely within the invariant manifold M.

C. Communities of oscillators

Turning now to the problem of coupled communities of Kuramoto systems given by Eq. (3), we introduce different Lorentzians for each community,

$$g^{\sigma}(\omega)=\pi^{-1}[(\omega-\omega_\sigma)^2+\Delta_\sigma^2]^{-1},$$

and proceed as before. We obtain a coupled system of equations for the order parameter associated with each community σ,

$$dr_\sigma/dt+(-i\omega_\sigma+\Delta_\sigma)r_\sigma+\frac{1}{2}\sum_{\sigma'=1}^{s}K_{\sigma\sigma'}[r_{\sigma'}^*r_\sigma^2-r_{\sigma'}]=0, \tag{15}$$

where $\sigma=1,2,\ldots,s$. Thus we obtain s complex coupled differential equations where s is the number of communities. We conjecture that, for s large enough (e.g., $s\geq 2$ or 3) and appropriate parameter values, there may be chaotic attracting solutions for Eq. (15). It would be particularly interesting to see whether such solutions in M are also attractors for the macroscopic order-parameter behavior of the full system (3), e.g., by comparing numerical solutions of Eqs. (3) and (15).

D. Time-delayed coupling

In applications time delay in the coupling between dynamical units in a network is often present. For example, the propagation speed of signals between units is finite (e.g., along axons in a neural network), and there may also be an inherent response time of a unit to information that it receives. Thus time delay has been extensively studied in the context of networks of coupled systems, and in particular for the case of coupled phase oscillators.[17–19] It has been found for such systems that time delay can substantially modify the dynamics, leading to a much richer variety of behaviors. In the context of Eqs. (1)–(3), for example, the response of oscillator i at time t to input from oscillator j is now related to the state θ_j of oscillator j at time $(t-\tau_{ji})$, where τ_{ji} is the time delay for this interaction. Assuming that all the delay times are the same, $\tau_{ji}=\tau$, independent of i and j, the quantities $\theta_j(t)$ appearing in the summations in Eqs. (1)–(3) must now be replaced by $\theta_j(t-\tau)$. Again such a generalization can be straightforwardly incorporated into our method. For example, for the external drive problem [Eq. (2) and Sec. IV B] we have in place of Eq. (2),

$$\frac{d\theta_i(t)}{dt} = \omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin[\theta_j(t-\tau) - \theta_i(t)] + \Lambda \sin[\Omega t - \theta_i(t)]. \tag{16}$$

Going to a rotating frame, $\theta_i'(t)=\theta_i(t)-\Omega t$, $\omega'=\omega-\Omega$, Eq. (16) becomes

$$\frac{d\theta_i'(t)}{dt} = \omega_i' + \frac{K}{N}\sum_{i=1}^{N} \sin[\theta_j'(t-\tau) - \theta_i'(t) - \Omega\tau] - \Lambda \sin \theta_i'(t). \tag{17}$$

The summation in Eq. (17) is

$$\frac{K}{N}\mathrm{Im}\left\{ e^{-i[\theta_i'(t)+\Omega\tau]}\sum_{j=1}^{N} e^{i\theta_j'(t-\tau)}\right\} = K\,\mathrm{Im}\left\{e^{-i[\theta_i'(t)+\Omega\tau]}r(t-\tau)\right\}. \tag{18}$$

Thus, to include delay, it suffices to replace the term $[Kr(t)+\Lambda]$ in Eqs. (13) and (14) by $[Ke^{-i\Omega\tau}r(t-\tau)+\Lambda]$. For example, making this substitution in Eq. (14) and setting $\Lambda=0$, $\Omega=\omega_0$ yields the following first order delay-differential equation for the order-parameter of the standard Kuramoto model with coupling delay,

$$\frac{dr(t)}{dt} - \frac{K}{2}\{e^{-i\omega_0\tau}r(t-\tau) - e^{i\omega_0\tau}r^*(t-\tau)[r(t)]^2\} + r(t) = 0, \tag{19}$$

which returns Eq. (10) for $\tau\to 0$. We note that our reduced descriptions with delay [e.g., Eq. (19)] are [in contrast to Eqs. (10), (14), and (15)] now infinite dimensional dynamical systems. For small $|r|$, linearizing Eq. (19) about the incoherent state ($r=0$), and setting $r\sim e^{st}$ yields a dispersion relation for s,

$$s+1=(K/2)\exp[-(s+i\omega_0)\tau], \tag{20}$$

in agreement with Ref. 18. In addition, steady synchronized states can be found (as in Ref. 19) by setting $r(t)=r_0e^{i\eta t}$ in Eq. (19) and solving the result,

$$i\eta - \frac{K}{2}[e^{-i(\omega_0+\eta)\tau} - r_0^2 e^{i(\omega_0+\eta)\tau}] + 1 = 0, \tag{21}$$

for the real constants η and r_0. Furthermore, through linearization of Eq. (19) about $r=r_0e^{i\eta t}$, our formulation can be used to study the previously unaddressed problem of assessing the stability of the steady synchronized states, Eq. (21).

E. The millennium bridge problem, Ref. 20

Another example is that of the observed oscillation of London's Millennium Bridge induced by the pacing phase entrainment of pedestrians walking across the bridge as modeled by Eqs. (52) and (53) of Eckhardt *et al.*[20] In that case, assuming a Lorentzian distribution of natural pacing frequencies for the pedestrians, one can use the method given in our paper to obtain an ordinary differential equation for the mechanical response of the bridge coupled to another ordinary differential equation for the order parameter describing the collective state of the pedestrians.

V. DISCUSSION AND CONCLUSION

Low dimensional descriptions of the classical Kuramoto problem [Eq. (1)] have been previously attempted. An early such attempt was made by Kuramoto and Nishikawa[21] who used a heuristic approach resulting in an integral equation for $r(t)$. On the basis of their work they predict that for small $|r(0)|$ the order-parameter $r(t)$ initially grows (decays) exponentially in time for $K>K_c$ ($K<K_c$) (later shown rigorously and quantitatively in Ref. 16). Crawford,[22] using center manifold theory, obtains [Eq. (108) of Ref. 22] an equation of the form $d\rho/dt=a(K-K_c)\rho+b\rho^3+O(\rho^5)$ for K near K_c. Another work of interest is that of Watanabe and Strogatz[23] who consider the case where all oscillators have the same frequency for both finite and infinite N. By use of a nonlinear transformation of the phase variables $\theta_i(t)$, these authors show that the dynamics reduces to a solution of three coupled first order ordinary differential equations. Thus, while macroscopic behavior of order-parameter dynamics has been previously addressed for the standard Kuramoto problem, it has, until now, never been demonstrated fully (e.g., without the restriction of Ref. 22 to small amplitude, or the restriction of Ref. 23 to identical frequencies). Our paper does this and also demonstrates that our technique can be usefully applied to a host of other important related problems.

Our work also suggests other future lines of study. For example, can any rigorous results be obtained relevant to whether our macroscopic order-parameter attractors obtained by considering f in the manifold M have general validity?[24] Are there interesting qualitative differences between the behavior for Lorentzian $g(\omega)$ as compared to other monotonic symmetric oscillator distribution functions $g(\omega)$? What other systems, in addition to those discussed in Sec. IV, can our method be applied to?

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