Low-dimensional lattices. VI. Voronoi reduction of three-dimensional lattices

By J. H. Conway[1] and N. J. A. Sloane[2]

[1] *Mathematics Department, Princeton University, Princeton, New Jersey 08540, U.S.A.*
[2] *Mathematical Sciences Research Center, AT&T Bell Laboratories, Murray Hill, New Jersey 07974, U.S.A.*

The aim of this paper is to describe how the Voronoi cell of a lattice changes as that lattice is continuously varied. The usual treatment is simplified by the introduction of new parameters called the vonorms and conorms of the lattice. The present paper deals with dimensions $n \leqslant 3$; a sequel will treat four-dimensional lattices. An elegant algorithm is given for the Voronoi reduction of a three-dimensional lattice, leading to a new proof of Voronoi's theorem that every lattice of dimension $n \leqslant 3$ is of the first kind, and of Fedorov's classification of the three-dimensional lattices into five types. There is a very simple formula for the determinant of a three-dimensional lattice in terms of its conorms.

1. Introduction

Our aim in this paper and its sequel is to describe how the Voronoi cell of a lattice changes as that lattice is continuously varied. We simplify the usual treatment by introducing new parameters which we call the vonorms and conorms of the lattice. The present paper studies lattices in one, two and three dimensions, ending with the theorem of Fedorov (1885, 1891) and on the five types of three-dimensional lattices. The sequel will use the same machinery to give a simple proof of the theorem of Delone (1929, 1937, 1938), as corrected by Stogrin (1973), that there are 52 types of four-dimensional lattice.

The main theorem of the present paper is the following.

Theorem 1. *Each three-dimensional lattice is uniquely represented by a projective plane of order 2 labelled with seven numbers, the conorms of the lattice, whose minimum is 0 and whose support is not contained in a proper subspace. Two lattices are isomorphic if and only if the corresponding labellings differ only by an automorphism of the plane.*

As we shall see, in three dimensions our seven 'conorms' are just 0 and the six 'Selling parameters'. However, this apparently trivial replacement of six numbers by seven numbers whose minimum is zero leads to several valuable improvements in the theory.

1. The conorms vary continuously with the lattice. (For the Selling parameters the variation is usually continuous but requires occasional readjustments.)

2. The definition of the conorms makes it apparent that they are invariants of the lattice. (The Selling parameters are almost but not quite invariant.)

3. All symmetries of the lattice arise from symmetries of the conorm function. (Again, this is false for the Selling parameters.)

There are several reasons for studying Voronoi cells of lattices. Besides the

Proc. R. Soc. Lond. A (1992) **436**, 55–68
Printed in Great Britain

applications to packing, covering and quantizing problems (see, for example, Barnes & Sloane 1983; Conway & Sloane 1988*a*; Gruber & Lekkerkerker 1987; Ryskov & Baranovskii 1976, 1979) there are connections with the theory of tilings. Following Gruber & Lekkerkerker (1987, p. 168) we define an n-dimensional *parallelotope* (or *parallelohedron* if $n = 3$) to be a convex body S which admits a lattice tiling (in other words there is a lattice Λ such that the translates $S+u$, $u \in \Lambda$ cover $\mathbb{R}^n$ while their interiors are disjoint). Voronoi (1908*a*, *b*, 1909) conjectured that every parallelohedron is the affine image of the Voronoi cell of some lattice. This was proved for $n \leqslant 4$ by Delone (1929), while for $n \geqslant 5$ the question remains open. In particular there are five three-dimensional parallelohedra, the affine images of the five Voronoi cells described in figure 7 (see Theorem 9).

Voronoi vectors are defined in §2, and vonorms and conorms in §§3–5. All of these quantities are particularly simple in the case of lattices of the 'first kind', defined in §2. Sections 6 and 7 will establish that every lattice of dimension $n \leqslant 3$ is of the first kind. In particular the proof that every three-dimensional lattice is of the first kind is accomplished by means of a new algorithm given in §7 for the 'Voronoi reduction' of the lattice, that is, for finding a specification of the lattice that makes its Voronoi vectors apparent. Figure 5 shows an example. This algorithm is also used to prove Theorem 1. The last section of the paper applies the preceding theory to derive the five types of three-dimensional Voronoi cells (summarized in figure 7, Theorem 9 and table 2). At the beginning of §8 we give especially simple formulae (involving the conorms) for the vertices, edges and faces of the generic Voronoi cell of a three-dimensional lattice, for its edge-lengths and for the determinant of the lattice (see (15) and (17)).

2. Voronoi vectors

Let $\Lambda \subset \mathbb{R}^n$ be a real n-dimensional lattice (as in Conway & Sloane 1988*a*). The *Voronoi cell* $V(u)$ for $u \in \Lambda$ is the set of points of $\mathbb{R}^n$ that are at least as close to u as to any other lattice point:

$$V(u) = \{x \in \mathbb{R}^n : N(x-u) \leqslant N(x-v) \quad \text{all} \quad v \in \Lambda\},$$

where $N(x) = x \cdot x$ denotes the norm of a vector. All the $V(u)$ for $u \in \Lambda$ are congruent convex polytopes. They partition $\mathbb{R}^n$ into the *Voronoi honeycomb* of Λ.

A vector $v \in \Lambda$ is called a *Voronoi vector* if the hyperplane

$$\{x \in \mathbb{R}^n : x \cdot v = \tfrac{1}{2} v \cdot v\}$$

has a non-empty intersection with $V(0)$ (the Voronoi cell containing the origin). A Voronoi vector is *relevant* (or *strict*) if this intersection is an $(n-1)$-dimensional face of $V(0)$, and is otherwise *irrelevant* (or *lax*).

By the *Voronoi reduction* of Λ we mean finding a description of Λ from which its Voronoi vectors are apparent.

The starting point for this investigation is the following theorem.

Theorem 2. *A non-zero vector* $v \in \Lambda$ *is* (i) *a Voronoi vector if and only if* v *is a shortest vector in the class* $v + 2\Lambda$; (ii) *a strict Voronoi vector if and only if* v *and* $-v$ *are the only shortest vectors in* $v + 2\Lambda$.

The usual version of this theorem only gives part (ii) (Voronoi 1908*b*, p. 277; Venkov 1983; Engel 1986, p. 35; Gruber & Lekkerkerker 1987, p. 95; Conway & Sloane 1988*a*, where, owing to an unfortunate printer's error, the statement of the theorem was omitted from the foot of page 474).

Proof. (i) Suppose v is a Voronoi vector yet there is a vector $w \in \Lambda$ with $v-w \in 2\Lambda$, $N(w) < N(v)$. Then $t = \frac{1}{2}(v+w)$ and $u = \frac{1}{2}(v-w)$ belong to Λ. Let P satisfy $P \cdot v = \frac{1}{2}v.v$, $P \cdot t \leqslant \frac{1}{2}t \cdot t$, $P \cdot u \leqslant \frac{1}{2}u \cdot u$. These equations imply $N(v) \leqslant N(w)$, a contradiction. On the other hand, suppose v is a shortest vector in its class $v+2\Lambda$, but is not a Voronoi vector. Then for some $w \in \Lambda$, $\frac{1}{2}v \cdot w > \frac{1}{2}w \cdot w$, so $N(v-2w) < N(w)$, a contradiction. The proof of (ii) (see, for example, Conway & Sloane 1988*a*, p. 475) is similar.

Since there are 2^n-1 non-zero classes of $\Lambda/2\Lambda$, from part (i) of the theorem there are at least $2(2^n-1)$ Voronoi vectors. In a generic (or random) lattice there are no coincidences between the lengths of vectors in distinct classes, and hence there are exactly $2(2^n-1)$ Voronoi vectors, all strict.

A lattice Λ is said to be of *Voronoi's first kind* if it has what we shall call an *obtuse superbase*; that is to say, a set of vectors $v_0, v_1, \ldots, v_n$ such that $v_1, \ldots, v_n$ is an integral basis for Λ and

$$v_0+v_1+\ldots+v_n = 0$$

(this is a *superbase*), and in addition

$$p_{ij} = v_i \cdot v_j \leqslant 0, \quad \text{for} \quad i,j = 0, \ldots, n, \quad i \neq j, \tag{1}$$

(this is the *obtuse* condition). The superbase is *strictly obtuse* if

$$v_i \cdot v_j < 0, \quad \text{for} \quad i,j = 0, \ldots, n, \quad i \neq j. \tag{2}$$

For example, the root lattice A_n $(n \geqslant 1)$ and its dual A_n^* are of the first kind. The $n+1$ cyclic shifts of $(1, -1, 0, \ldots, 0)$ are an obtuse superbase for A_n, and the vectors $(n/(n+1), -1/(n+1), \ldots, -1/(n+1))$ are a strictly obtuse superbase for A_n^*.

The numbers p_{ij} are traditionally called the *Selling parameters* for the superbase or lattice (Selling 1874; Baranovskii 1980), and if we define

$$p_{i|jk\ldots l} = p_{ij}+p_{ik}+\ldots+p_{il} \tag{3}$$

then the inner product matrix for the superbase is

$$\begin{bmatrix} p_{0|12\ldots n} & -p_{01} & -p_{02} & \cdots & -p_{0n} \\ -p_{10} & p_{1|02\ldots n} & -p_{12} & \cdots & -p_{1n} \\ & \cdots & & \cdots & \\ -p_{n0} & -p_{n1} & -p_{n2} & \cdots & p_{n|01\ldots n-1} \end{bmatrix}. \tag{4}$$

Theorem 3. (i) *If Λ is of the first kind with superbase $v_0, \ldots, v_n$ then the $2^{n+1}-2$ subsums*

$$v_S = \sum_{i \in S} v_i, \quad S \subseteq \{0, 1, \ldots, n\},$$

$0 < |S| < n$, *are Voronoi vectors. Also v_S and $v_{\bar{S}} = -v_S$ are congruent modulo 2Λ, but otherwise these vectors are in distinct classes of $\Lambda/2\Lambda$.* (ii) *The vectors v_S are all strict if and only if the superbase is strict.*

Remark. Here $\bar{S}$ is the set complementary to S. We write $v_{ijk\ldots}$ for $v_{\{i,j,k,\ldots\}}$.

Proof. (i) The norm of any vector $v = \sum_{i=0}^{n} m_i v_i \in \Lambda, m_i \in \mathbb{Z}$, is plainly given by Selling's formula

$$N\left(\sum_{i=0}^{n} m_i v_i\right) = \sum_{\substack{i,j=0 \\ i<j}}^{n} p_{ij}(m_i-m_j)^2 \tag{5}$$

(Selling 1874). Now v is unchanged if the m_i are all increased by the same amount, and unchanged modulo 2Λ if the m_i are changed by even integers. So within a given coset of $\Lambda/2\Lambda$ the norm is minimized when all even m_i are replaced by 0 and all odd m_i by 1. (ii) If all $p_{ij} > 0$ and we suppose, as we may, that $\min\{m_0, \ldots, m_n\} = 0$, then the norm is minimized *only* if all m_i are 0 or 1. ∎

3. Vonorms

The *Voronoi norm*, or *vonorm*, vo $(\bar{v})$, of a class $\bar{v} = v + 2\Lambda$ of $\Lambda/2\Lambda$ is the least norm of any vector in that class. Thus the vonorms are the norms of the Voronoi vectors (the *proper* vonorms), together with zero (the *improper* vonorm). By 'the vonorms' we usually mean 'the proper vonorms'. The vonorm map from $\Lambda/2\Lambda$ to $\mathbb{R}$ is obviously an invariant of Λ; we shall see in dimensions $n \leqslant 4$ (and we conjecture in general) that it also characterizes Λ. The quotient $\Lambda/2\Lambda$ can obviously be regarded as a vector space over the field of order 2. Since this important space is the domain of the vonorms we call it *vonorm space*.

Theorem 4. *The vonorms of a lattice of the first kind are the numbers*

$$N(v_S) = \sum_{i\in S,\, j\in \bar{S}} p_{ij} = p_{ab\ldots c|de\ldots f}\,(say), \tag{6}$$

where $S = \{a, b, \ldots, c\}$, $\bar{S} = \{d, e, \ldots, f\}$.

Proof. This follows immediately from Theorem 3. ∎

4. Characters

As usual we define a *real character* on an n-dimensional lattice Λ to be a function χ from Λ to $\{\pm 1\}$ with the property that $\chi(u+v) = \chi(u)\chi(v)$ for all $u, v \in \Lambda$. The 2^n real characters form a group which is abstractly isomorphic to the vector space dual of the vonorm space $\Lambda/2\Lambda$. We call this dual space *conorm space*. From now on, 'character' will always mean 'real character'.

Theorem 5. *The characters of a lattice of the first kind correspond to the subsets S of $\{0, \ldots, n\}$ for which $|S|$ is even. They are defined by*

$$\chi_S(v_i) = \begin{cases} -1, & i \in S, \\ +1, & i \notin S. \end{cases} \tag{7}$$

Proof. It is easy to check that these are characters, and since there are $\frac{1}{2}\cdot 2^{n+1} = 2^n$ of them, there is no other. ∎

5. Conorms

The *conjugate norm*, or *conorm*, co (χ) corresponding to a character χ is defined by

$$\mathrm{co}\,(\chi) = -\frac{1}{2^{n-1}} \sum_{\bar{v}\in \Lambda/2\Lambda} \chi(\bar{v})\,\mathrm{vo}\,(\bar{v}). \tag{8}$$

The conorm co (1) corresponding to the trivial character $\chi_0 = 1$ is called the *improper* conorm. By 'the conorms' we usually mean 'the proper conorms'. The conorms are, apart from a scale factor, the Fourier transforms of the vonorms, and so carry exactly the same information as the vonorms. In fact

$$\mathrm{vo}\,(\bar{v}) = \Sigma\,\mathrm{co}\,(\chi), \tag{9}$$

where the sum is taken over all χ with $\chi(\bar{v}) = -1$. Since $\mathrm{vo}\,(\bar{v}) > 0$ if $\bar{v}$ is not the zero class, (9) implies that the support of the conorm function cannot be contained in a proper subspace of conorm space.

Theorem 6. *For a lattice of the first kind the proper conorms are given by*

$$\mathrm{co}\,(\chi_S) = \begin{cases} p_{ij}, & \text{if } S = \{i,j\}, \\ 0, & \text{otherwise}. \end{cases}$$

So for a lattice of the first kind the conorms are the Selling parameters supplemented by zeros.

The proof is an easy calculation.

6. One-dimensional and two-dimensional lattices

An n-dimensional lattice is represented by a point in a vector space of dimension $N = \frac{1}{2}n(n+1)$, the space of Gram matrices. On the other hand there are 2^n-1 proper vonorms (or conorms), a number which is always greater than or equal to N, as we see in table 1.

Table 1

n	$N = \frac{1}{2}n(n+1)$	2^n-1	difference
1	1	1	0
2	3	3	0
3	6	7	1
4	10	15	5
5	15	31	16

In this section we briefly discuss the one-dimensional and two-dimensional cases, and show that there the vonorms are exactly enough to parametrize the space of lattices and that every lattice is of the first kind.

Dimension 1. Let Λ have Gram matrix (a), and generator v_1, with $N(v_1) = a$. Then $v_0 = -v_1$ and v_1 form an obtuse superbase, the proper vonorm is a, and the proper conorm is $-v_0 \cdot v_1 = p_{01} = a$.

Dimension 2. Suppose Λ is generated by vectors v_1, v_2 having Minkowski-reduced Gram matrix

$$\begin{bmatrix} a & -h \\ -h & b \end{bmatrix},$$

with $0 \leqslant 2h \leqslant a \leqslant b$. Then $v_0 = -(v_1+v_2)$, v_1, v_2 form an obtuse superbase, with Selling parameters p_{ij} determined by

$$a = p_{01}+p_{12}, \quad h = p_{12}, \quad b = p_{02}+p_{12}.$$

The Voronoi vectors are $\pm v_0$, $\pm v_1$, $\pm v_2$, and if $h = p_{12} \neq 0$ there is no other, while if

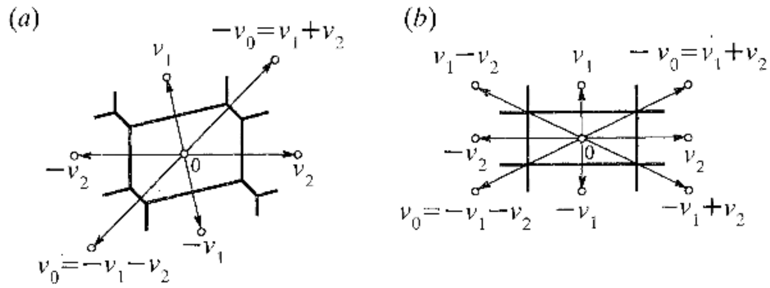

Figure 1. Voronoi cells (heavy lines) of two-dimensional lattices. (*a*) If $p_{12} \neq 0$ there are three pairs $\pm v_0$, $\pm v_1$, $\pm v_2$ of Voronoi vectors, all strict, and the cell is a hexagon. (*b*) If $p_{12} = 0$ there are four pairs $\pm v_1$, $\pm v_2$, $\pm(v_1+v_2)$, $\pm(v_1-v_2)$ of Voronoi vectors, but only $\pm v_1$, $\pm v_2$ are strict, and the cell is a rectangle.

$p_{12} = 0$ there is an additional pair of Voronoi vectors $\pm(v_1-v_2)$. The corresponding Voronoi cells are shown in figure 1.

The proper vonorms are $a = p_{01}+p_{12}$, $b = p_{02}+p_{12}$, $c = p_{01}+p_{02} = a+b-2h$. These may be any three positive numbers satisfying the triangle inequalities

$$b+c \geqslant a, \quad c+a \geqslant b, \quad a+b \geqslant c. \tag{10}$$

The proper conorms p_{01}, p_{02}, p_{12} may be any three non-negative numbers, although at least two must be strictly positive for Λ to be a proper lattice. Note that

$$\det \Lambda = ab-h^2 = p_{01}p_{02}+p_{01}p_{12}+p_{02}p_{12}$$

(compare with (15) below).

The vonorms have also a geometric interpretation.

Theorem 7. *The vonorms $p_{i|jk}$ of a two-dimensional lattice Λ are the three smallest norms of primitive vectors (ignoring the distinction between vectors and their negatives).*

Proof. If the conorms are p_{ij} then the norm of $v = \sum_i m_i v_i$ is $\sum_{i<j} p_{ij}(m_i-m_j)^2$. If two of the m_i are equal, v is a multiple of one of the v_i. If not, its norm is at least $p_{01}+p_{02}+p_{12}$, which exceeds all the vonorms $p_{ij}+p_{jk}$. ∎

7. Three-dimensional lattices

The main result of this section is the following theorem of Voronoi (1908*a*, *b*, 1909).

Theorem 8. *Any three-dimensional lattice Λ is of the first kind.*

Proof. We establish this by giving an algorithm for the Voronoi reduction of Λ, which computes the Selling parameters p_{ij} for the desired obtuse superbase.

Given any base v_1, v_2, v_3 for Λ, we call the seven subsums $v_1, v_2, \ldots, v_{123} = v_1+v_2+v_3$ the *putative Voronoi vectors*, their norms $N(v_1), N(v_2), \ldots, N(v_{123})$ the *putative* (proper) *vonorms*, and the six numbers $p_{ij} = -v_i \cdot v_j$ $(0 \leqslant i < j \leqslant 3)$ the *putative* (proper) *conorms*, for the superbase $v_0 = -v_1-v_2-v_3, v_1, v_2, v_3$. (These putative quantities will be correct if this superbase is obtuse.)

The non-zero cosets of $\Lambda/2\Lambda$ naturally form a projective plane of order 2, which we draw as in figure 2*a*. The lines of this plane correspond to the points of the dual plane as in figure 2*b*.

The putative vonorms $p_{i|jkl} = N(v_i)$, $p_{ij|kl} = N(v_{ij}) = N(v_{kl})$ are marked at the nodes

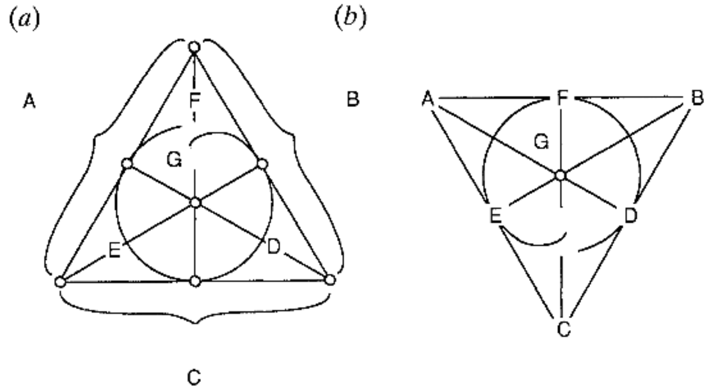

Figure 2. Two dual projective planes. The lines A, B, ..., G of vonorm space (*a*) correspond to the points of conorm space (*b*).

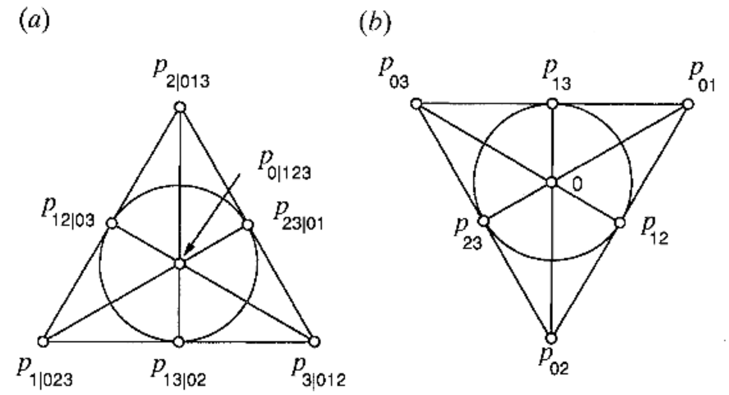

Figure 3. (*a*) Projective plane labelled with putative vonorms; (*b*) dual plane labelled with putative conorms.

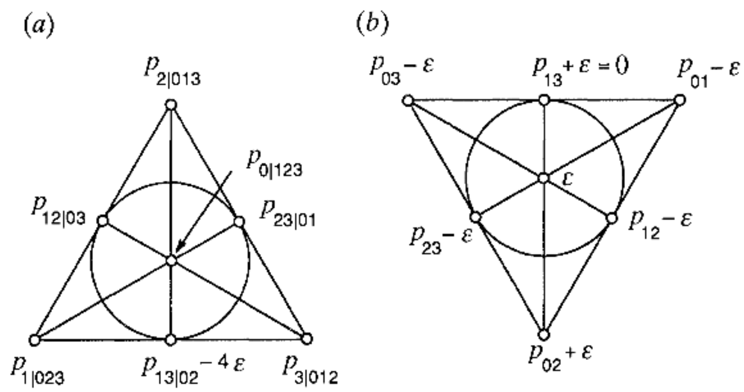

Figure 4. Putative vonorms (*a*) and conorms (*b*) for superbase adjacent to that described by figure 3.

of figure 3*a* and the corresponding putative conorms 0 and p_{ij} at the nodes of the dual plane in figure 3*b*.

Any non-trivial character takes the value $+1$ at the three points of a line and -1 at the remaining four points, so four times the typical putative conorm in figure 3*b* is equal to the sum of the four numbers off a line minus the sum of the three numbers on that line in figure 3*a*.

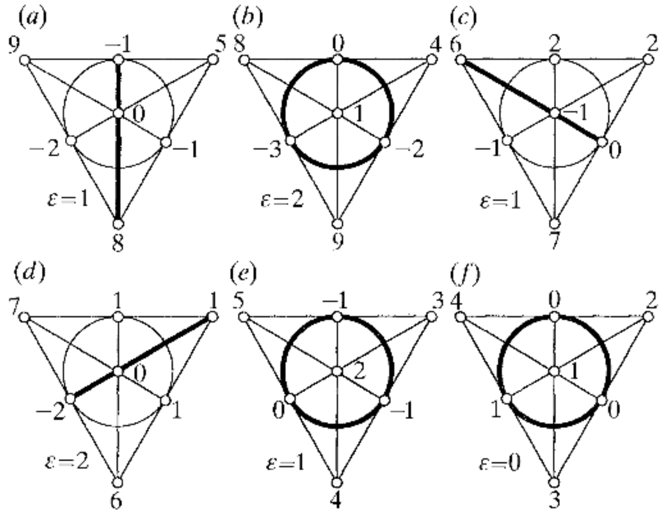

Figure 5. Illustration of algorithm for Voronoi reduction of three-dimensional lattice. (*f*), which gives the conorms corresponding to an obtuse superbase, is the final answer.

Now suppose that one conorm is negative, say $p_{13} = -\epsilon$. We study what happens when we change to the 'adjacent' superbase v'_0, v'_1, v'_2, v'_3 defined by

$$v'_0 = v_{01},\ v'_1 = -v_1,\ v'_2 = v_{12},\ v'_3 = v_3, \tag{11}$$

so that

$$v'_{12} = v_2,\ v'_{23} = -v_0,\ v'_{13} = v_1 - v_3. \tag{12}$$

Note that six of these seven vectors agree (up to sign) with six of our seven original putative Voronoi vectors. Their norms are shown in figure 4*a* and their conorms in figure 4*b*.

We see in figure 4*a* that just one putative vonorm has changed, being decreased by 4ϵ (since $N(v'_{31}) = N(v_3 - v_1) = N(v_{31}) - 4\epsilon$), and in figure 4*b* that three putative conorms on a line have increased by ϵ while those off that line have decreased by ϵ. The line of conorms which increase by ϵ is that joining the positions of the conorms that are 0 in figures 3*b* and 4*b* respectively, and indeed ϵ is determined by the condition that one of the new conorms shall be 0.

This leads to our algorithm for Voronoi reduction, which we illustrate by an example. Let Λ have Gram matrix

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix},$$

which we border with a leading row and column chosen so that the sum of the four entries in any row or column is 0:

$$\begin{bmatrix} 22 & -5 & -8 & -9 \\ -5 & 3 & 1 & 1 \\ -8 & 1 & 5 & 2 \\ -9 & 1 & 2 & 6 \end{bmatrix}.$$

This corresponds to the putative conorms shown in figure 5*a*.

We choose a line joining the 0 entry to a negative number $-\epsilon$, increase the three numbers on this line by ϵ and decrease the numbers off it by ϵ. The node originally labelled $-\epsilon$ is now labelled 0. This corresponds to a change from the original superbase to an adjacent one. Figure 5*b* shows the result of applying this process (with $\epsilon = 1$) to the vertical line in figure 5*a*. In each of figures 5*a*–*f* the heavily drawn lines indicates the next modification. Note that figure 5*f* has two zeros. In such a case it does not matter which of them we regard as 'the' 0 entry, because we can change from one to the other by a move with $\epsilon = 0$. Since figure 5*f* contains no negative label its labels are the conorms corresponding to an obtuse superbase, and we stop. (The vonorms are then easily obtained from (9).)

This algorithm is correct, because each step simply replaces the putative conorms for one superbase by those for an adjacent superbase. To show that it terminates we consider the putative vonorms and compare figures 3*a* and 4*a*. Since just one putative vonorm is decreased, the putative Voronoi vectors at all times have norms below some fixed bound, and so the process of reducing their norms must eventually terminate.

Given any three-dimensional lattice, the algorithm produces conorms corresponding to an obtuse superbase, and establishes theorem 8.

On the other hand the conorms are (by definition) unique, and so we have now also proved Theorem 1.

There is an analogous algorithm for the Voronoi reduction of a two-dimensional lattice. The algorithm replaces putative conorms $\alpha, \beta, \gamma = -\epsilon$, where $\epsilon > 0$, by $\alpha - 2\epsilon$, $\beta - 2\epsilon$, $\gamma + 2\epsilon = \epsilon$, adding 2ϵ to one conorm and subtracting 2ϵ from the other two. (Note that because $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$ are positive, at least one conorm is negative.) After a finite number of such steps all the putative conorms are non-negative and the algorithm terminates.

Remark. The algorithms described here are theoretical rather than practical. In practice a reduction algorithm such as that of Lenstra *et al.* (1982) (see also Lagarias 1991) would be used before applying our algorithm.

8. The five parallelohedra

In this section we study the Voronoi cells of three-dimensional lattices and prove Fedorov's theorem.

Let Λ be an arbitrary three-dimensional lattice, with obtuse superbase v_0, v_1, v_2, v_3 and conorms p_{ij}. A vector $t \in \mathbb{R}^3$ will be specified by its inner products

$$(t \cdot v_0, t \cdot v_1, t \cdot v_2, t \cdot v_3) = (y_0, y_1, y_2, y_3) = y, \tag{13}$$

say, where $y_0 + y_1 + y_2 + y_3 = 0$.

We first show that the Voronoi cell for a generic three-dimensional lattice Λ is a truncated octahedron, or permutohedron, as in figure 6. Corresponding to each Voronoi vector v_S, $S \subseteq \{0,1,2,3\}$, $1 \leqslant |S| \leqslant 3$, there is a face $F_{S|\bar{S}}$ (say), where $\bar{S} = \{0,1,2,3\} \backslash S$, of the Voronoi cell. The face $F_{i|jkl}$ (where $\{i,j,k,l\}$ is any permutation of $\{0,1,2,3\}$) contains the points with $y_i = \frac{1}{2}p_{i|jkl}$, the face $F_{ij|kl}$ contains the points with $y_i + y_j = -y_k - y_l = \frac{1}{2}p_{ij|kl}$, and the face $F_{ijk|l}$ contains the points with $y_i + y_j + y_k = -y_l = \frac{1}{2}p_{l|ijk}$. Then $F_{\bar{S}|S} = -F_{S|\bar{S}}$ and $F_{S|\bar{S}}$ are opposite faces.

We assert that the vertices of the Voronoi cell are the 24 points p_{ijkl} (where again $\{i,j,k,l\}$ is any permutation of $\{0,1,2,3\}$) with coordinates

$$\left.\begin{aligned} y_i &= \tfrac{1}{2}(+p_{ij}+p_{ik}+p_{il}), \quad & y_j &= \tfrac{1}{2}(-p_{ji}+p_{jk}+p_{jl}),\\ y_k &= \tfrac{1}{2}(-p_{ki}-p_{kj}+p_{kl}), \quad & y_l &= \tfrac{1}{2}(-p_{li}-p_{lj}-p_{lk}).\end{aligned}\right\} \tag{14}$$

(Note that $p_{klji} = -p_{ijkl}$ and p_{ijkl} are opposite vertices.) In fact, each such point p_{ijkl} belongs to the three faces $F_{i|jkl}$, $F_{ij|kl}$ and $F_{ijk|l}$. (In figure 6 p_{ijkl} is simply labelled $ijkl$.) Also $F_{i|jkl}$ is a hexagonal face containing the six vertices $p_{i\alpha\beta\gamma}$, where $\{\alpha, \beta, \gamma\} = \{j, k, l\}$, and $F_{ij|kl}$ is a rhombic face containing the four vertices p_{ijkl}, p_{ijlk}, p_{jikl}, p_{jilk}.

Similar (although less symmetrical) coordinates for the Voronoi cell were given by Barnes (1956) and Barnes & Sloane (1983).

The determinant of Λ, the squared volume of the Voronoi cell, is the determinant of the Gram matrix

$$\begin{bmatrix} p_{1|023} & -p_{12} & -p_{13} \\ -p_{12} & p_{2|013} & -p_{23} \\ -p_{13} & -p_{23} & p_{3|012} \end{bmatrix},$$

which is equal to

$$\begin{aligned} &p_{01}p_{02}p_{03}+p_{01}p_{02}p_{13}+p_{01}p_{02}p_{23}+p_{01}p_{03}p_{12}+p_{01}p_{03}p_{23} \\ &\quad +p_{01}p_{12}p_{13}+p_{01}p_{12}p_{23}+p_{01}p_{13}p_{23}+p_{02}p_{03}p_{12}+p_{02}p_{03}p_{13} \\ &\quad +p_{02}p_{12}p_{13}+p_{02}p_{12}p_{23}+p_{02}p_{13}p_{23}+p_{03}p_{12}p_{13}+p_{03}p_{12}p_{23}+p_{03}p_{13}p_{23}. \end{aligned}$$

Upon examination of figure 3*b* we see that this can be written as

$$\det \Lambda = \sum_{\{P, Q, R\}=\Delta} \mathrm{co}\,(P)\,\mathrm{co}\,(Q)\,\mathrm{co}\,(R), \tag{15}$$

where $\{P, Q, R\}$ runs through all 28 *triangles* in figure 3*b* that is to say, through all bases for the conorm space. (For 12 of these triangles the product is 0.)

If a vector t is specified by $y = (y_0, y_1, y_2, y_3)$, as in (13), then its norm is given by

$$N(t) = \frac{1}{\det \Lambda} y^{\mathrm{T}} D y, \tag{16}$$

where $D = (d_{ij})$, $d_{ii} = p_{jk}p_{kl} + p_{kl}p_{lj} + p_{lj}p_{jk}$, $d_{ij} = -\frac{1}{2}(p_{ik}p_{jl} + p_{il}p_{jk})$ $(i \neq j)$, $0 \leqslant i, j \leqslant 3$.

The Voronoi cell has six families of parallel edges. If we denote the vector along a typical edge by e_{ij} $(0 \leqslant i < j \leqslant 3)$, as in figure 6, then e_{ij} has coordinates $y_i = p_{ij} = -y_j$, $y_k = y_l = 0$. From (16) we find that

$$N(e_{ij}) = \frac{1}{\det \Lambda} \sum_{\{P, Q, R\}=\Delta} \mathrm{co}^2\,(P)\,\mathrm{co}\,(Q)\,\mathrm{co}\,(R), \tag{17}$$

where the sum is over all triangles in figure 3*b* containing that node P for which $\mathrm{co}\,(P) = p_{ij}$. Note that (17) vanishes only when $\mathrm{co}\,(P)$ does, since otherwise the support of the conorm function includes at least one base containing P.

It follows that two lattices in which the same conorms are zero have combinatorially equivalent Voronoi cells (since one can be continuously deformed into the other without any edges being lost). There are only five choices for the locations of the zeros (see top of figure 7):

one zero,

two zeros,

three collinear zeros,

three non-collinear zeros,

four zeros,

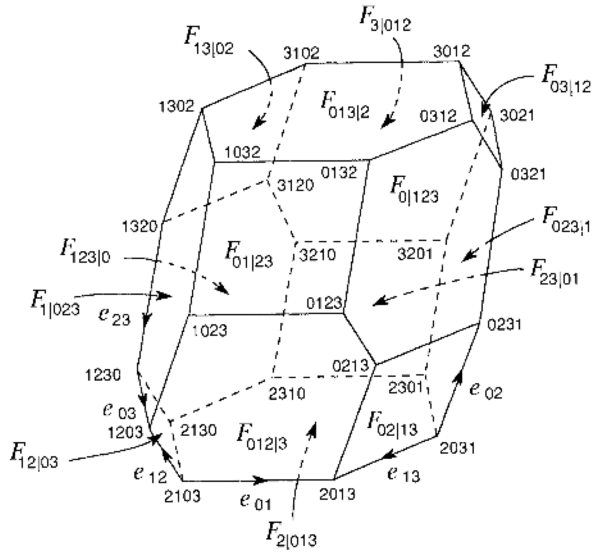

Figure 6. Voronoi cell for generic three-dimensional lattice.

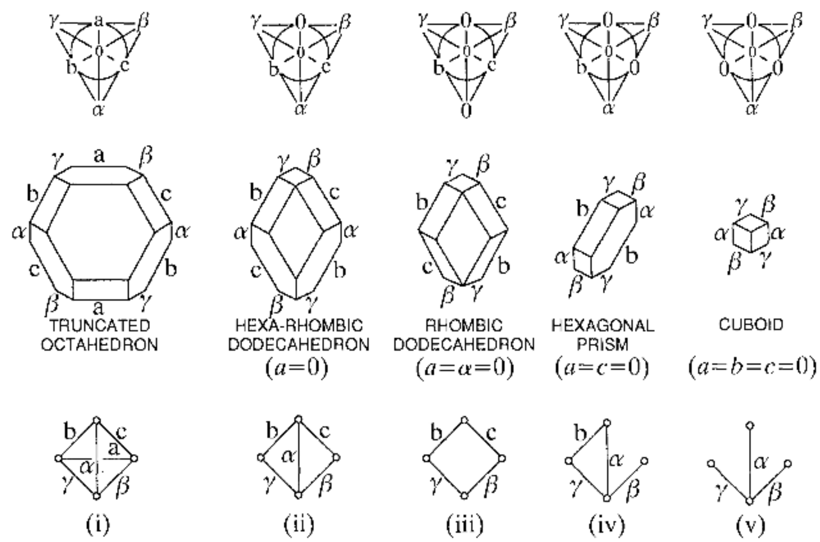

Figure 7. The five parallelohedra (middle), together with conorms (top) and Delone graphs (bottom) for the corresponding lattices.

since the non-zero conorms may not be collinear (by the remark preceding Theorem 6). Thus we have established the following theorem of Fedorov (1885, 1891).

Theorem 9. *There are just five combinatorially distinct possibilities for the shape of the Voronoi cell of a three-dimensional lattice.*

The five cases are obviously distinct, for we see from (17) that if the conorms are varied in such a way that $\mathrm{co}(P)$ approaches 0 then the corresponding family of parallel edges all shrink to points and the polyhedron simplifies. Figure 7 shows the effects of successive simplifications. (During these simplifications the remaining edges may change their lengths and directions, but we ignore this in the figures.) The labels on the edges are the corresponding conorms, taken from the top row of diagrams.

If we shrink its edges labelled a, the truncated octahedron of figure 7 (i) becomes the hexarhombic dodecahedron of figure 7(ii). This has one family of four parallel edges labelled α, whose shrinkage leads to a rhombic dodecahedron (figure 7(iii)), and four families of six parallel edges such as those labelled c, whose shrinkage leads to a hexagonal prism (figure 7(iv)). (Our term 'hexarhombic dodecahedron' is an abbreviation for 'hexagonal-rhombic dodecahedron'. We prefer this to the less specific name 'elongated dodecahedron' of Fejes Tóth (1964) and Coxeter (1973).)

Shrinking any of the families of parallel edges in figure 7(iii) or (iv) leads to a cuboid (figure 7(v)) or to a polyhedron of zero volume.

The Delone diagram. An n-dimensional lattice of the first kind may also be described by its *Delone diagram* (after Delone 1937, 1938; see also Ryskov 1972; Ryskov & Baranovskii 1979). This has $n+1$ nodes labelled $0, 1, \ldots, n$, with an edge labelled with the Selling parameter p_{ij} joining nodes i and j whenever $p_{ij} \neq 0$. The Delone diagrams for the five types of three-dimensional lattices are given in the bottom row of figure 7 (cf. Delone 1938, p. 138, fig. 37). These diagrams are very useful, but have the disadvantage of not displaying all the symmetries of the situation. For example, in figure 7(ii) it is possible to interchange b and γ independently of c and β: this is obvious from the conorm picture but not from the Delone diagram. Similarly in figure 7(iii) it is possible to permute all four parameters b, c, β, γ freely, and in figure 7(iv) to permute b, α, γ. There is, however, a simple mnemonic: a permutation of the parameters that does not affect the circuits of the Delone diagram to which they belong leads to another specification of the same lattice.

The conorm representation has no such defect. Two projective planes labelled with conorms represent the same lattice precisely when there is a conorm-preserving collineation between them.

We call a three-dimensional lattice *primitive* (or *primary*), *secondary*, *tertiary*, etc., according to whether it has $1, 2, 3, \ldots,$ conorms equal to 0. Table 2 lists the five types of three-dimensional lattices, together with the corresponding section of figure 7, the name of the Voronoi cell, and in the final column the (classically) integral lattice of smallest determinant of each type. The latter are obtained by setting all non-zero conorms in figure 7 equal to 1; the notation is that of Conway & Sloane (1988*b*).

The dual lattice. If the conorms of a three-dimensional lattice Λ are as shown in figure 3*b*, then those of its dual Λ^* are, after multiplication by $\det \Lambda$, as shown in figure 8, where we set

$$pp = \min\{p_{01}\,p_{23}, p_{02}\,p_{13}, p_{03}\,p_{12}\}, \tag{18}$$

$$pp_{ijk} = p_{ij}\,p_{jk} + p_{jk}\,p_{ki} + p_{ki}\,p_{ij}. \tag{19}$$

From this it is immediate that the dual of a

fully decomposable lattice is fully decomposable,

simply decomposable lattice is simply decomposable,

secondary or indecomposable tertiary lattice is primary,

whereas the dual of a primary lattice is primary, secondary or indecomposable tertiary according to whether the minimum in (18) is attained once, twice or thrice.

There is an analogous classification to that of table 2 in every dimension. In two dimensions the primary or indecomposable lattices are those with hexagonal Voronoi cells (figure 1*a*), and the others are secondary or decomposable with rectangular Voronoi cells (figure 1*b*).

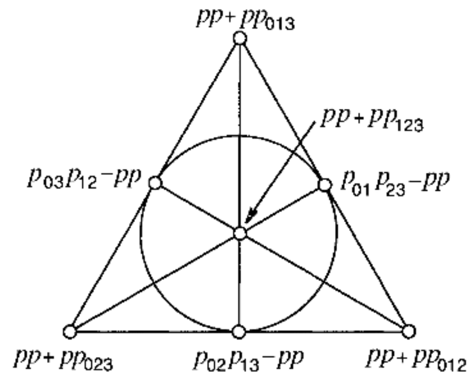

Figure 8. Conorms (multiplied by det Λ) for Λ^*, where Λ is defined by the conorms in figure 3*b*.

Table 2. The five types of three-dimensional lattice

type	figure 7	Voronoi cell	canonical example
primary	(*a*)	truncated octahedron	$2A_3^*$, det = 16
secondary	(*b*)	hexarhombic dodecahedron	$(A_1^2 8_1)^{+2}$, det = 8
indecomposable tertiary	(*c*)	rhombic dodecahedron	A_3, det = 4
decomposable tertiary or simply decomposable	(*d*)	hexagonal prism	$A_2 I_1$, det = 3
quaternary or fully decomposable	(*e*)	cuboid	I_3, det = 1

Delone (1929, 1937, 1938), as corrected by Stogrin (1973), enumerated the Voronoi cells of four-dimensional lattices. There are three primary ones and 52 in all. In a sequel to the present paper we shall show that the conorm method enables us to enumerate these very simply, and we shall also give a detailed account of their properties.

References

Baranovskii, E. P 1980 The Selling reduction domain of positive quadratic forms in five variables. *Trudy̆ mat. Inst. V. A. Steklova* **152**, 5–33. (English translation in *Proc. Steklov Inst. Math.*, Issue 3, pp. 5–35 (1982).)

Barnes, E. S. 1956 The covering of space by spheres. *Can. J. Math.* **8**, 293–304.

Barnes, E. S. & Sloane, N. J. A. 1983 The optimal lattice quantizer in three dimensions. *SIAM J. Algebraic Discrete Methods* **4**, 30–41.

Conway, J. H. & Sloane, N. J. A. 1988*a* *Sphere packings, lattices and groups*. New York: Springer-Verlag.

Conway, J. H. & Sloane, N. J. A. 1988*b* Low-dimensional lattices. I. Quadratic forms of small determinant. *Proc. R. Soc. Lond.* A **418**, 17–41.

Coxeter, H. S. M. 1973 *Regular polytopes*. New York: Dover.

Delone, B. N. 1929 Sur la partition regulière de l'espace à 4-dimensions. *Izv. Akad. Nauk SSSR Otdel. Fiz.-Mat. Nauk*, pp. 79–110 and 145–164.

Delone, B. N. 1937 Geometry of positive quadratic forms. *Usp. mat. Nauk* **3**, 16–62. Delone, B. N. 1938 *Usp. mat. Nauk*. **4**, 102–164. (In Russian.)

Engel, P. 1986 *Geometric crystallography*. Dordrecht, Holland: Reidel.

Fedorov, E. S. 1885 Elements of the study of figures. *Zap. miner. Obshch.* **21** (2), 1–279. (Reprinted in *Izv. Akad. Nauk SSSR* (1953).)

Fedorov, E. S. 1891 The symmetry of regular systems of figures. *Zap. Miner. Obshch.* **28** (2), 1–146. (In Russian.) (English translation in *Symmetry of crystals*, ACA Monograph no. 7, pp. 50–131. New York: ACA (1971).)

Fejes Tóth, L. 1964 *Regular figures*. Oxford: Pergamon.

Gruber, P. M. & Lekkerkerker, C. G. 1987 *Geometry of numbers*, 2nd edn. Amsterdam: North-Holland.

Lagarias, J. C. 1991 Point lattices. In *Handbook of combinatorics*. Amsterdam: North-Holland. (In the press.)

Lenstra, A. K., Lenstra, H. W. Jr & Lovász, L. 1982 Factoring polynomials with rational coefficients. *Math. Ann.* **261**, 515–534.

Ryskov, S. S. 1972 Maximal finite groups of integral $n \times n$ matrices and full groups of integral automorphisms of positive quadratic forms (Bravais models). *Trudy̆ mat. Inst. V. A. Steklova* **128**, 183–211. (In Russian.) (English translation in *Proc. Steklov Inst. Math.* **128**, 217–250 (1972).)

Ryskov, S. S. & Baranovskii, E. P. 1976 *C*-types of *n*-dimensional lattices and 5-dimensional primitive parallelohedra (with application to the theory of covering). *Trudy̆ mat. Inst. V. A. Steklova* **137**. (In Russian.) (English translation in *Proc. Steklov Inst. Math.*, Issue 4 (1978).)

Ryskov, S. S. & Baranovskii, E. P. 1979 Classical methods in the theory of lattice packings. *Usp. mat. Nauk* **34** (4), 3–63. (In Russian.) (English translation in *Russian Math. Surveys* **34** (4), 1–68 (1979).)

Selling, E. 1874 Ueber die binären und ternären quadratischen Formen. *J. reine angew. Math.* **77**, 143–229.

Stogrin, M. I. 1973 Regular Dirichlet-Voronoi partitions for the second triclinic group. *Trudy̆ mat. Inst. V. A. Steklova*, no. 123. (In Russian.) (English translation in *Proc. Steklov Inst. Math.*, no. 123 (1973).)

Venkov, B. B. 1983 Voronoi parallelohedra for certain unimodular lattices *Zap. Nauch. Sem. Leningrad Otdel. Mat. Inst. V. A. Steklova* **132**. 57–61. (In Russian.) (English translation in *J. Soviet Math.* **30**, 1833–1836 (1985).)

Voronoi, G. F. 1908*a* Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *J. reine angew. Math.* **133**, 97–178.

Voronoi, G. F. 1908*b* *J. reine angew. Math.* **134**, 198–287.

Voronoi, G. F. 1909 *J. reine angew. Math.* **136**, 67–181.

Received 7 March 1991; accepted 17 April 1991