$$= \lim_{\substack{y \to x \\ (y-x)^\mu n_\mu = 0}} \{[\partial_\nu \varphi^i(x), \partial_\mu \varphi^j(y)] + [\varphi^i(x), \partial_\nu \partial_\mu \varphi^j(y)]\}$$

$$= [\partial_\nu \varphi^i(x), \partial_\mu \varphi^j(x)] + [\varphi^i(x), \partial_\nu \partial_\mu \varphi^j(x)]. \tag{A·4}$$

Similarly the right-hand side of Eq. (A·3) is reduced to

$$i\hbar \delta_\mu{}^0 \delta(\mathbf{0}) \partial_\nu g^{ij}.$$

Thus Eq. (A·1) is satisfied.

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Note added: Recently Kamo and Kawai (Osaka City Univ. Preprint 1973) proposed a variational method generally applicable. If the Lagrangian is suitably chosen in their formulation so that $\delta^* I=0$, the first Noether theorem can be extended to the case of $R \neq$ constant.

§ 2. Phase shifts at low energy

Since the two types of velocity-dependent potentials, Eqs. (1·1) and (1·2), are essentially the same,[3] we consider only the expression given by Eq. (1·1). Furthermore, in order to accomplish a partial wave analysis, we restrict the function $J(\boldsymbol{r})$ to being spherically symmetric and of finite range

$$\begin{aligned} J(\boldsymbol{r}) &= J(r), \quad r<b\,, \\ &= 0\,, \qquad r>b\,. \end{aligned} \tag{2·1}$$

Under these conditions, the Schrödinger equation for a particle of mass m and energy E, submitted to a potential of the form given by Eq. (1·1), can be separated in spherical polar coordinates.[5] The angular part of the wavefunction is a spherical harmonic function, and the radial part $R_l(r)$ for angular momentum l satisfies the differential equation

$$\begin{aligned} &(1-\lambda J(r))\,(R_l''+(2/r)R_l'-(l(l+1)/r^2)R_l) \\ &\quad +(d(1-\lambda J(r))/dr)R_l'+k^2R_l=0\,, \end{aligned} \tag{2·2}$$

where $k^2=2mE/\hbar^2$ with the conditions[3]

$R_l(r)$	is finite,	(2·3a)
$R_l(r)$	is continuous,	(2·3b)
$(1-\lambda J(r))R_l'(r)$	is continuous	(2·3c)

for all r.

Usual techniques allow to obtain for the l wave phase shift δ_l the expression

$$\tan\delta_l=\frac{L_l(b)j_l(kb)-kj_l'(kb)}{L_l(b)n_l(kb)-kn_l'(kb)}\,, \tag{2·4}$$

where j_l and n_l are the spherical Bessel functions. The primes mean derivatives with respect to the argument, and

$$L_l(b)=(1-\lambda J(b))R_l'(b)/R_l(b) \tag{2·5}$$

is the limit of the product of the logarithmic derivative of the radial wavefunction times $(1-\lambda J(r))$ when r tends to b from the left. Of course, $L_l(b)$ depends on k through $R_l(b)$ and $R_l'(b)$.

The behaviour of $\tan\delta_l(k)$ at low energies can be studied by introducing the series expansion of the Bessel functions and its derivatives for $kb\ll 1$. We obtain

$$\begin{aligned} \tan\delta_l(k) &= -\frac{x^{2l+1}}{(2l+1)!!\,(2l-1)!!} \\ &\times\frac{B+\sum_{n=1}^{\infty}(-1)^n(B-2n)\,(x^2/2)^n/n!\,(2l+3)\,(2l+5)\cdots(2l+2n+1)}{B+2l+1+\sum_{n=1}^{\infty}(B+2l-2n+1)\,(x^2/2)^n/n!\,(2l-1)\,(2l-3)\cdots(2l-2n+1)}\,, \end{aligned} \tag{2·6}$$

where we have denoted

$$x=kb$$

and

$$B=bL_l(b)-l\,. \tag{2·7}$$

The expression (2·6) is very suitable for discussing the limit $k\to 0$. At first sight,

$$\tan\delta_l(k)=0(x^{2l+1}),\quad x=kb\ll 1\,, \tag{2·8}$$

and therefore

$$\delta_l(k)=0(k^{2l+1}),\quad k\to 0\,. \tag{2·9}$$

However, the fact must be taken in consideration that B depends on k through $L_l(b)$.

For $l\neq 0$, Eq. (2·9) stands, no matter whether $L_l(b)$ goes to zero, remains constant or diverges when $k\to 0$. If it happens that B vanishes identically, the phase shift may become considerably reduced with respect to the values given by Eq. (2·9). On the other hand, if $B=-(2l+1)$, $\delta_l(k)$ would result remarkably increased. But these results would come out from accidents, and do not constitute general features.

For $l=0$, the behaviour of $L_l(b)$ for $k\to 0$ becomes decisive and may invalidate Eq. (2·9). Let us discuss this possibility.

§ 3. S wave phase shift

In the case of a square well velocity-dependent potential

$$J(r)=1,\quad r<b\,,$$
$$=0,\quad r>b\,, \tag{3·1}$$

we obtain for $l=0$

$$L_0(b)=(1-\lambda)Kj_0'(Kb)/j_0(Kb), \tag{3·2}$$

where

$$K=k/(1-\lambda)^{1/2} \tag{3·3}$$

is the wavenumber inside the range of the potential. Now, the behaviour of $L_0(b)$ for $k\to 0$ can be obtained in the following way. Since $K\to 0$ for $k\to 0$, we can use a series expansion for $L_0(b)$ valid when $Kb\ll 1$. In this way, we obtain

$$L_0(b)=(1-\lambda)K\{\cot(Kb)-1/Kb\}$$
$$=-\frac{x^2}{b}\left(\frac{1}{3}+\frac{y^2}{45}+\frac{2y^4}{945}+\cdots+(-1)^{n-1}2^{2n}B_{2n}\frac{y^{2n-2}}{(2n)!}+\cdots\right), \tag{3·4}$$

where B_n are the Bernoulli numbers[9] and where we have denoted

$$x=kb, \quad y=Kb\,. \tag{3·5}$$

By introducing the above expression for $L_0(b)$ in the definition of B, Eq. (2·7), it can be obtained

$$B=-x^2/3-x^4/45(1-\lambda)+0(x^6), \quad l=0, \tag{3·6}$$

and, from Eq. (2·6),

$$\tan\delta_0=\lambda x^5/45(1-\lambda)+0(x^7), \quad x=kb\ll 1\,, \tag{3·7}$$

that is,

$$\delta_0(k)=0(k^5), \quad k\to 0\,, \tag{3·8}$$

a result which has already been reported.[8]

For $l=0$, the Schrödinger equation for the radial wavefunction, Eq. (2·2), reduces to

$$d\{(1-\lambda J(r))R'\}/dr+(2/r)(1-\lambda J(r))R'+k^2R=0\,. \tag{3·9}$$

Let us introduce a new function

$$v(r)=(1-\lambda J(r))R'(r), \tag{3·10}$$

a transformation which has already been used[5] to relate a velocity-dependent potential with a static one in the case $l=0$. Then Eq. (3·9) becomes

$$v'+(2/r)v+k^2R=0\,. \tag{3·11}$$

If we derive this equation with respect to r and use Eq. (3·10), we obtain

$$v''+(2/r)v'+\{k^2-2/r^2+\lambda k^2J(r)/(1-\lambda J(r))\}v=0 \tag{3·12}$$

which resembles a radial Schrödinger equation for $l=1$ with a potential $-E\lambda J(r)/(1-\lambda J(r))$. If we introduce the function

$$w(r)=rv(r), \tag{3·13}$$

the above equation gives

$$w''+\{k^2-2/r^2+\lambda k^2J(r)/(1-\lambda J(r))\}w=0\,. \tag{3·14}$$

For a free particle ($\lambda=0$), transformations similar to Eqs. (3·10) and (3·13) would give

$$w''_{\text{free}}+(k^2-2/r^2)w_{\text{free}}=0\,. \tag{3·15}$$

From Eqs. (3·14) and (3·15) it can be easily obtained for the Wronskian of the two functions $w(r)$ and $w_{\text{free}}(r)$

$$(ww'_{\text{free}}-w'w_{\text{free}})\Big|_0^r=\int_0^r\frac{\lambda k^2J(r)}{1-\lambda J(r)}ww_{\text{free}}dr\,. \tag{3·16}$$

Of course, $w_{\text{free}}(r)$ is essentially $krj_1(kr)$, so both w_{free} and w'_{free} vanish at $r=0$. (In fact, they tend to zero like r^2 and r, respectively, when r tends to zero.)

Since $w(r)$ can be obtained from $w_{\text{free}}(r)$ by continuous variation of the parameter λ, it is clear that $w(r)$ must be the regular solution of Eq. (3·14). Therefore, the Wronskian vanishes at the origin. If we choose for the upper limit of the integral a value of r in the asymptotic region, where

$$w \sim \sin(kr-\pi/2+\delta_0), \quad w_{\text{free}} \sim \sin(kr-\pi/2), \quad r\to\infty\,, \tag{3·17}$$

Eq. (3·16) becomes

$$k\sin\delta_0 = \int_0^b \{\lambda k^2 J(r)/(1-\lambda J(r))\}\, w(r)\, w_{\text{free}}(r)\, dr\,. \tag{3·18}$$

To obtain an approximate estimate of the magnitude of the phase shift, we can substitute in the above expression $w_{\text{free}}(r)=krj_1(kr)$ instead of $w(r)$; we get

$$k\sin\delta_0 \simeq \lambda k^4 \int_0^b \{J(r)/(1-\lambda J(r))\}\, j_1^2(kr)\, r^2 dr\,. \tag{3·19}$$

When k tends to zero, we can replace $j_1(kr)$ by its approximate value

$$j_1(kr) \simeq kr/3, \quad kr \ll 1 \tag{3·20}$$

and for Eq. (3·19) we obtain finally

$$\sin\delta_0 \simeq (\lambda/9)\, k^5 \int_0^b \{J(r)/(1-\lambda J(r))\}\, r^4 dr, \quad kb \ll 1\,, \tag{3·21}$$

that is,

$$\delta_0(k) \simeq 0(k^5), \quad k\to 0\,. \tag{3·22}$$

We must remark that the particular case of the square well potential as shown in Eq. (3·7) is in agreement with this general result, as it must be.

§ 4. Final remarks

The conclusion of the present study is that, at sufficiently low energies, the S wave phase shifts for a short range velocity-dependent potentials of the type of Eq. (1·1) are much smaller than the P wave phase shifts. In fact, the behaviour of S waves is similar to that presented by D waves.

The transformation given by Eq. (3·10), which has been very useful in our study of the low energy behaviour of the S wave phase shift, provides also an explanation for that behaviour. In view of the asymptotic expression (3·17), the function $w(r)$, obtained from $R(r)$ via the transformations given by (3·10) and (3·13), can be considered a P wave reduced radial wavefunction with the same phase shift as $R(r)$. Now, $w(r)$ satisfies a Schrödinger equation with a potential proportional to k^2, namely

$$-(\hbar^2/2m)\, k^2 \lambda J(r)/(1-\lambda J(r))\,. \tag{4·1}$$

In other words, the S wave phase shift originated by the velocity-dependent potential (1·1) is the same as the P wave phase shift originated by the energy-dependent potential (4·1). At low energies, this phase shift should behave like k^3 for an energy-nondependent potential, since it is a P wave phase shift. However, the fact that the potential Eq. (4·1) is proportional to k^2 introduces a k^2 factor in the phase shift, giving a k^5 behaviour for $\delta_0(k)$ as $k \to 0$.

Hitherto we have restricted our study to velocity-dependent potentials of the form of Eq. (1·1). The addition of a static potential invalidates, in general, the resulting equation (3·22). However, the intensities of the static and velocity-dependent parts of the potential may be such that the S wave phase shift remains smaller than the P wave phase shift for a given range of energies. If velocity-dependent potentials are to be allowed, there is no a priori reason to neglect the P wave contribution at low energies. The phase shift analysis of the scattering data in elementary particle and nuclear low energy physics ought to take this fact into account. Some effort in this direction has been done recently.[10]

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