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LOW ENERGY THEOREM APPROACH TO SINGLE PARTICLE SINGULARITIES IN THE PRESENCE OF MASSLESS BOSONS

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ABSTRACT

By use of low energy theorems, which follow from gauge invariance and analiticity assumptions, we determine the nature of the single particle singularity of a meson propagator in the presence of the known massless bosons : photons and gravitons. In addition to regaining the well-known results for covariant gauges in electrodynamics, we present new results for covariant gauges in gravity theory, and for radiation gauges in both electrodynamics and gravity theory. The gauges in which no infra-red singularities are present are found : for covariant electrodynamics it is of course the Yennie gauge; for covariant gravity theory a similar gauge is given. In radiation gauges it is shown that Schwinger's new gauge has this desirable property for photons, and an analogous gauge is constructed for gravitons. It is established that in these gauges the single particle singularity of the meson propagator becomes a simple pole.

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1. INTRODUCTION

Although for simplicity one-particle singularities in n point functions are usually taken to be simple poles, it is clear that due to the interaction of massless bosons with all particles, the precise singularity may not be of this form. For charged particles, the emission of photons, and for all particles, the emission of gravitons ensures that in general the singularity lies at the beginning of a cut. For a Green's function G, the modification of the single-particle singularity due to electromagnetic interactions has been studied extensively by methods which rely heavily of field theoretic or diagrammatic models describing these interactions [1]. Recently, however, one of us (L.S.) has shown that this limit of G can be determined in a model independent fashion, by use of a dispersion theoretic representation for G and low energy theorems to evaluate the soft photon contribution to $\mathrm{Im}\, G$ [2].

For gravitational interactions, a complete field theoretic description is absent, but model independent low energy theorems have been established [3]. Thus the derivation of the structure of G near its single particle singularity as affected by gravity interactions can be given in a fashion parallel to that for electromagnetic interactions.

The purpose of this paper is to present a unified treatment of this problem taking into account the effects of the known massless bosons : photons and gravitons. We make little reference to field theory, as our results follow from gauge invariance and analyticity assumptions. In Section 2 we set the notation, define the relevant dispersive representation for G, and determine the matrix elements which are to be evaluated in the soft photon and soft graviton limit. We discuss carefully our approach to the problem of infra-red divergences which bedevil any calculation of G.

Section 3 is devoted to the electromagnetic calculation. The treatment parallels that of Ref. 2). In addition to obtaining the well-known previous results about the properties of G in covariant gauges, we discuss the behaviour of G in radiation gauges and show that it is free from infra-red singularities to all orders in the electromagnetic interaction when Schwinger's new radiation gauge is used [4]. In Section 4, an analogous calculation is performed for gravitational and gravitational and electromagnetic interactions combined, in covariant and radiation gauges. The gauges that are free from infra-red divergences are identified.

2. THE GREEN'S FUNCTION

Our object of study is

$$G(p^2) = \int d^4x\, e^{ipx} \langle \Omega | T \varphi(x) \varphi(0) | \Omega \rangle. \tag{2.1}$$

The field φ is renormalized and for simplicity is taken to be a spinless scalar meson with mass m. We assume that $G(p^2)$ satisfies a dispersive representation of the form

$$G(p^2) = \int_{m^2-\eta}^{m^2+\eta} dr^2 \frac{\sigma(r^2)}{p^2 - r^2 + i0} + \tilde{G}(p^2), \tag{2.2}$$

$$\eta \ll m^2,$$

where $\tilde{G}(p^2)$ is regular near $p^2 = m^2$. Equation (2.2) follows from the conventional spectral representation (with arbitrary subtractions) but obviously requires weaker assumptions than the representation over an infinite range. The spectral function $\sigma(p^2)$ is given by

$$\sigma(p^2) = (2\pi)^3 \sum_N \delta^4(p - p_N) \langle \Omega | \varphi | N \rangle \langle N | \varphi^+ | \Omega \rangle. \tag{2.3}$$

The total four momentum of the state $|N\rangle$ is p_N^μ. Since $p^2 = p_N^2 \leq m^2 + \eta < 2m^2$, the only intermediate states N which contribute to $\sigma(p^2)$ are those containing one meson and an arbitrary number of massless particles. Setting

$$|N\rangle = |r; k_1, \ldots, k_n\rangle \tag{2.4}$$

where r is the momentum of the meson $r^2 = m^2$, and k_i is the momentum of the i^{th} soft particle, we have

$$\sigma(p^2) = \int \frac{d^4 r}{2r^0} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int \frac{d^4 k}{(2\pi)^3} \theta(k_0) \delta^4(k^2) \sum_{\varepsilon} \right)_n \qquad (2.5)$$

$$\times\, \delta\left(p - r - \sum_{i=1}^{n} k_i\right) \langle \Omega | \varphi | r; k_1, .., k_n \rangle \langle r; k_1, ... k_n | \varphi^+ | \Omega \rangle .$$

The notation

$$\begin{aligned} (f(k))_0 &= 1 \\ (f(k))_n &= \prod_{i=1}^{n} f(k_i) \end{aligned} \qquad (2.6)$$

has been introduced. The internal sum over ε is a sum over the polarizations of the massless particle.

In offering the expansion (2.5), we sum over intermediate states each containing a <u>definite</u> number of massless particles, a procedure fraught with well-known difficulties. Our point of view is that we presume (2.3) to be a well-defined expression, but that (2.5) is an awkward representation of

$$\sum_{\substack{N \\ p_N^2 < 2m^2}} | N \rangle \langle N | .$$

Since the latter quantity, considered as an over-all sum, is presumed to be well defined (even though each term in the sum is not), we may evaluate it by using the representation (2.5) with an infra-red cut-off, for example by ascribing a mass μ to the massless particles whenever necessary to render an integral finite. This mass is to be put to zero at the end of the calculation. (Thus all quantities which are not identically zero should be kept finite. They can tend to zero after the limit $\mu \to 0$ has been taken.) We shall find, of course, that in general the limit $\mu \to 0$ <u>cannot</u> be taken, even at the end of the calculation, since an infra-red singularity persists in matrix elements with definite number of <u>external</u> massless particles, and in particular in G. (The limit can be taken only in physically observable quantities, like cross-sections describing an infinite number of soft particles.) We interpret, therefore, this remaining infra-red divergence as a consequence of the lack of existence of the external number states. However, the nature of the single particle singularity can be determined and only its strength depends on μ . Moreover, in special gauges, the infra-red singularity disappears; viz., the limit $\mu \to 0$ exists. Then the single particle singularity becomes a pole.

Let us remark that our use of renormalized states with finite number of soft particles implies that the external lines of Feynman diagrams are renormalized in the usual way. This means that we have to define $G(m^2)S^{-1}(m^2) = 1 = S(m^2)G^{-1}(m^2)$, where $S(p^2)$ is the free Green's function with a pole at m^2. This definition is relevant only for external lines and says nothing about the behaviour of $G(p^2)$ in the neighbourhood of $p^2 = m^2$, which is our subject of interest. Indeed, we shall find that in the vicinity of $p^2 = m^2$, $G(p^2)$ is a generalized function, and at the point $p^2 = m^2$ is not defined in the usual functional sense. That is why we need the above-mentioned definition for the external lines.

Returning to (2.5), it is seen that our problem reduces to an evaluation of $\langle \Omega | \varphi | r; k_1, \ldots, k_n \rangle$. Since we must have

$$(r + \sum_i k_i)^2 = m^2 + 2r\cdot\sum_i k_i + (\sum_i k_i)^2 \leq m^2 + \eta$$

$$r\cdot\sum_i k_i + \sum_{i\leq j} k_i\cdot k_j \leq \frac{\eta}{2} + m^2 \qquad (2.7)$$

we may use low energy theorems to evaluate the matrix element. We shall be able to determine, in the electrodynamical treatment, the first two terms in the expansion of the matrix element in the soft photon momenta. This will permit the evaluation of all the singular terms in $G(p^2)$ near $p^2 = m^2$. For gravitational interactions, only the first term in the expansion of the matrix element can be given. However, we shall argue, by analogy with electrodynamics, that the entire singular behaviour of $G(p^2)$ near $p^2 = m^2$ can still be determined. The matrix elements are singular in the soft particle momenta, so that the integrals over k in (2.5) are infra-red divergent. As discussed above, we shall regulate them by a small mass for the soft particle.

3. ELECTROMAGNETIC INTERACTIONS

A. To evaluate the contribution of the soft photons, we consider at the outset the situation when only the n^{th} photon is soft. We define

$$T_n(r;k_1,\ldots,k_n) = \langle \Omega | \varphi | r;k_1,\ldots,k_n \rangle. \quad (3.1a)$$

We want to consider T_n for $k_n^2 \neq 0$, hence we continue off the photon mass shell by an LSZ formula :

$$T_n = \varepsilon_n^\mu T_{\mu n} \quad (3.1b)$$

$$T_n^\mu = \int d^4x\, e^{ik_n x} \langle \Omega | T \varphi j^\mu(x) | r;k_1,\ldots k_{n-1} \rangle$$

(3.1c)

In offering (3.1c), the assumption is made that the renormalized photon field $A^\mu(x)$, and its time derivative, commute at equal times with φ. This permits the following manipulation, used above

$$\Box T \varphi A^\mu(x) = T \varphi \Box A^\mu(x) = T \varphi j^\mu(x) \quad (3.2)$$

From its definition (3.1c), T_n^μ is seen to satisfy the following Ward identity

$$k_{\mu n} T_n^\mu = e\, T_{n-1}(r;k_1,\ldots,k_{n-1}) \quad (3.3)$$

where e is the physical charge carried by the meson.

T_n^μ contains terms which are singular in k_n^μ. These arise for example from Feynman diagrams in which the soft photon line with momentum k_n^μ and the incoming meson line can be separated from the remainder of the diagram by cutting a single meson line (see Fig. 1). Call this contribution to T_n^μ, given in Fig. 1, $T_n^{\mu(\mathrm{pole})}$. We assume that all contributions to T_n^μ, additional to $T_n^{\mu(\mathrm{pole})}$; viz., $T_n^\mu - T_n^{\mu(\mathrm{pole})}$, are less singular in k_n^μ than $T_n^{\mu(\mathrm{pole})}$. (The precise singularity assumption will be given below.) $T_n^{\mu(\mathrm{pole})}$ of course does not satisfy (3.3). However, we may construct explicitly a term $T_n^{\mu\prime}$, less singular in k_n^μ, so that $T_n^{\mu(\mathrm{pole})} + T_n^{\mu\prime}$ satisfies (3.3). We therefore have

$$T_n^\mu = T_n^{\mu(\mathrm{pole})} + T_n^{\mu\prime} + R_n^\mu \tag{3.4a}$$

$$k_{\mu n} T_n^{\mu\prime} = e\, T_{n-1} - k_{\mu n} T_n^{\mu(\mathrm{pole})} \tag{3.4b}$$

$$k_{\mu n} R_n^\mu = 0 \tag{3.4c}$$

The last equation above is differentiated by $k_{\mu n}$ to yield

$$R_n^\mu = -k_{\nu n} \frac{\partial R_n^\nu}{\partial k_{\mu n}} \tag{3.5}$$

The regularity assumption that we need is that R_n^ν is sufficiently well behaved so that $k_{\nu n}(\partial R_n^\nu / \partial k_{\mu n})$ vanishes as $k_{\nu n}$ goes to zero; viz., that R_n^μ vanishes. The above regularity assumption is weaker than the usual one which requires that $T_n^{\mu(\mathrm{pole})}$ be the <u>total</u> singular contribution to T_n^μ, so that R_n^μ is analytic at $k_n^\mu = 0$ and therefore, according to (3.5), of order k_n. This weaker assumption allows us to establish the low energy theorem to all orders in e, rather than just to lowest order. We return to this point below.

No information about R_n^μ, other than (3.4c) and (3.5), is available from the present treatment. Making the above-mentioned regularity assumption concerning R_n^μ, we conclude that the portion of T_n^μ, which can be given explicitly; viz., $T_n^{\mu(\text{pole})} + T_n^{\mu'}$, determines T_n^μ up to terms which vanish as k_n goes to zero. Therefore to the same degree of accuracy we need to give $T_n^{\mu(\text{pole})} + T_n^{\mu'}$ only up to terms which vanish with k_n. In the following, we shall frequently use the symbol R_i, $i = 1, \dots, n$ to signify a remainder term which vanishes as k_i goes to zero. Repeated use of the same symbol does not imply that the remainders are equal. Occasionally we shall encounter ambiguous quantities of the form $(k_n^2/(2r\cdot k_n+k_n^2))$. We shall lump these in R_n, because they go to zero when k_n goes to zero simultaneously in all its components, which is the only sense relevant for our purposes.

$T_n^{\mu(\text{pole})}$ is given, according to Fig. 1, by

$$T_n^{\mu(\text{pole})} = I^\mu(r;k_n)\,S([r+k_n]^2)\,\Lambda_{n-1}(r+k_n,k_1,\dots,k_{n-1}). \tag{3.6}$$

Here $\Lambda_{n-1}(r;k_1,\dots,k_{n-1})$ does not contain any one-meson line with momentum $r+k_n$ and coincides with $T_{n-1}(r;k_1,\dots,k_{n-1})$ continued off the r^2 mass shell (by an LSZ formula for example)

$$T_{n-1}(r;k_1,\dots,k_{n-1}) = \Lambda_{n-1}(r;k_1,\dots,k_{n-1})\Big|_{r^2=m^2}.$$

$I^\mu(r;k_n)S[(r+k_n)^2]$ is exactly $T_1^\mu(r;k_n)$, the one-photon matrix element (3.1). The function $I^\mu(r;k_n)$ is written in the form

$$\begin{aligned} I^\mu(r;k_n) = {}& (2r+k)^\mu f(r^2=m^2,(r+k)^2,k^2) \qquad (3.7\text{a}) \\ & + k^\mu g(r^2=m^2,(r+k)^2,k^2). \end{aligned}$$

I^μ coincides with the electromagnetic vertex function, when the meson momenta are on the mass shell : $r^2 = m^2$; $(r+k)^2 = m^2$. [This may be verified, for example, by comparing the diagrammatic expansions of T_1 and I^μ.] Therefore it is true that

$$f(m^2, m^2, 0) = e \tag{3.7b}$$

$$g(m^2, m^2, k^2) = 0 \tag{3.7c}$$

From the Ward identity for T_n, (3.4b), we have for the case $n = 1$:

$$S([k+r]^2)\, k_\mu I^\mu(r, k) = e \langle \Omega | \phi | r \rangle = e ; \tag{3.8a}$$

hence (3.7a) and (3.8a) imply

$$f(m^2, (r+k)^2, k^2) = e - \frac{k^2}{2r\cdot k + k^2}\, g(m^2, (r+k)^2, k^2). \tag{3.8b}$$

Therefore

$$S([k+r]^2) I^\mu(r, k) = \frac{2r^\mu + k^\mu}{2r\cdot k + k^2}\, e + \frac{1}{2r\cdot k + k^2}\left[k^\mu - (2r^\mu + k^\mu)\frac{k^2}{2r\cdot k + k^2}\right] \times g(m^2, (r+k)^2, k^2) \tag{3.8c}$$

Since the last term in the above vanishes as $k \to 0$ in all its components, the final result for T_n^μ(pole) is

$$T_n^{\mu\ \text{pole}} = e\,\frac{2r^\mu + k^\mu}{2r\cdot k + k^2}\,\Lambda_{n-1}(r+k_n, k_1, \ldots, k_{n-1}) + R_n \tag{3.9}$$

Next we determine $T_n^{\mu\,\prime}$. According to (3.4b), (3.6) and (3.8a), it is sufficient to take any $T_n^{\mu\,\prime}$ which satisfies

$$k_{\mu n} T_n^{\mu\,\prime}(r; k_1, \ldots, k_n) = e\left[T_{n-1}(r; k_1, \ldots, k_{n-1}) - \Lambda_{n-1}(r+k_n; k_1, \ldots, k_{n-1})\right] \tag{3.10}$$

By use of the intermediate value theorem, $\Lambda_{n-1}(r+k_n; k_1, \ldots, k_{n-1})$, is represented in the form

$$\Lambda_{n-1}(r+k_n; k_1, \ldots, k_{n-1}) = T_{n-1}(r; k_1, \ldots, k_{n-1}) + k_n^\mu \left[\frac{\partial}{\partial k_n^\mu}\Lambda_{n-1}(r+k_n; k_1, \ldots, k_{n-1})\right]_{k_n = k_n^*} \tag{3.11}$$

Here k^* is an intermediate point for k. From (3.10) we see that it is sufficient to take

$$T_n^{\mu\,\prime} = -e\left[\frac{\partial}{\partial k_{\mu n}}\Lambda_{n-1}(r+k_n; k_1, \ldots, k_{n-1})\right]_{k_n = k_n^*} \tag{3.12a}$$

We recognize that

$$\frac{\partial}{\partial k_{\mu n}} \Lambda_{n-1}(r+k_n; k_1, \ldots, k_{n-1})\Big|_{k_n = k_n^*}$$

$$= \frac{\partial}{\partial r'_\mu} \Lambda_{n-1}(r'; k_1, \ldots, k_{n-1})\Big|_{r' = r + k_n^*} . \tag{3.12b}$$

Therefore inserting (3.11) into (3.9), and adding this to (3.12a), gives

$$T_n^{\mu\,(pole)} + T_n^{\mu\,\prime} = e\left[\frac{2r^\mu + k_n^\mu}{2r\cdot k + k_n^2} T_{n-1} + \left(\frac{r^\mu k_n^\nu}{r\cdot k_n} - g^{\mu\nu}\right) \frac{\partial}{\partial r'^\nu} \Lambda_{n-1}\Big|_{\substack{r' = r + k^* \\ r'^2 = m^2 + \varepsilon}}\right] . \tag{3.12c}$$

The function Λ_{n-1} is a function of the invariants

$$r'^2,\ k_i\cdot r',\ \varepsilon_i \cdot r',\ k_i\cdot k_j,\ \varepsilon_i\cdot\varepsilon_j;\ i, j = 1, \ldots, n-1.$$

The derivative with respect to r'_ν can be written as a derivative with respect to the invariants

$$r'^2,\ k_i\cdot r'_j,\ \varepsilon_i\cdot r'_j$$

$$\frac{\partial}{\partial r'^\nu}\Lambda_{n-1}\Big|_{r' = r + k^*} = \sum_c c_\nu \frac{\partial}{\partial r'\cdot c}\Lambda_{n-1}\Big|_{r' = r + k^*} + 2r'_\nu \frac{\partial}{\partial r'^2}\Lambda_{n-1}\Big|_{r' = r + k^*} \tag{3.13a}$$

We now let k^* go to zero, except in the last term in (3.13a), we keep $r'^2 = r^2 + \mathcal{E} = m^2 + \mathcal{E}$, so as to avoid an infra-red divergence. Thus we replace (3.13a) by

$$\sum_c c_\nu \frac{\partial}{\partial r \cdot c} \Lambda_{n-1} \Big|_{r^2 = m^2} + 2 r_\nu \frac{\partial}{\partial r^2} \Lambda_{n-1} \Big|_{r^2 = m^2 + \mathcal{E}}$$

$$= \sum_c c_\nu \frac{\partial}{\partial r \cdot c} T_{n-1} + 2 r_\nu \frac{\partial}{\partial r^2} \Lambda_{n-1} \Big|_{r^2 = m^2 + \mathcal{E}} . \tag{3.13b}$$

The second term does not contribute to (3.12c). Thus the final result for T_n is

$$T_n = \Big(A_n + \sum_c B_n^c\Big) T_{n-1} + R_n$$

$$A_n = e \frac{(2r + k_n)\cdot \varepsilon_n}{2r \cdot k_n + k_n^2} \tag{3.14}$$

$$B_n^c = e\left(\frac{r\cdot \varepsilon_n}{r \cdot k_n} c \cdot k_n - c \cdot \varepsilon_n\right)\frac{\partial}{\partial r \cdot c} .$$

We have derived this result under the assumption that $T_n^\mu - T_n^{\mu(\mathrm{pole})}$ is sufficiently regular so that R_n goes to zero when k_n does. This assumption is justified by considering the diagrams that contribute to T_n^μ. Moreover it is known that to all orders in e [5]:

$$T_n \Big|_{k_n^2 = 0} = e \frac{r\cdot \varepsilon_n}{r \cdot k_n} T_{n-1} + O(1) \tag{3.15}$$
$$+ O\, k_n \log k_n .$$

Comparing (3.15) with (3.14) and recalling that R_n is assumed not to contain O (1), we find

$$R_n = O\, k_n \log k_n . \tag{3.16a}$$

If we were working to lowest order in e we would expect R_n to be of order k_n. Hence we may rewrite (3.16a)

$$R_n = O\, e\, k_n + O\, e^2 k_n \log k_n . \tag{3.16b}$$

In the above we have kept terms of the form $\mathcal{E}_n \cdot k_n$, that is for the sake of generality, we do not impose the transversality condition $\mathcal{E}_n \cdot k_n = 0$. The reason for this is that we shall want to discuss covariant gauges which necessitate the presence of fictitious timelike and longitudinal photons.

Formula (3.15) was derived by use of the condition $k_n \ll k_i, r$; $i = 1, \ldots, n-1$. Let us now expand T_{n-1} under the restriction that $k_n \ll k_{n-1} \ll k_i, r$; $i = 1, \ldots, n-2$. We use (3.14) again and after some manipulation we get an expression in k_n and k_{n-1} which is symmetric in these two variables except for a term of the form

$$\left(B_n^{k_{n-1}} + B_n^{\varepsilon_{n-1}} \right) A_{n-1} . \tag{3.17}$$

However, this term can be easily symmetrized by replacing the quantity $r \cdot k_{n-1}$ occurring in (3.17) by $r \cdot (k_n + k_{n-1})$ which is legitimate since $k_n \ll k_{n-1}$. The resulting expansion for T_n is now symmetric in k_n and k_{n-1} and is therefore valid for k_n, $k_{n-1} \ll k_i, r$; $i = 1, \ldots, n-2$; irrespective of the relative magnitude of k_n and k_{n-1}. Repeating the argument for all the photons yields finally

$$T_n = (A)_n \left[1 + \sum_{i<j} \frac{B_{ij}}{A_i A_j} + O \sum_i k_i^2 \log k_i \right] \tag{3.18a}$$

$$B_{ij} = \frac{-e^2}{r\cdot(k_i+k_j)}\left(k_i^{\mu}\frac{r\cdot\varepsilon_i}{r\cdot k_i}-\varepsilon_i^{\mu}\right)\left(k_{j\mu}\frac{r\cdot\varepsilon_j}{r\cdot k_j}-\varepsilon_{j\mu}\right). \tag{3.18b}$$

The first term in (3.18a), $(A)_n$, is the well-known factorized contribution of soft photons in any process [6]. This expression arises from diagrams where the soft photons are attached to external lines. The second term is analogous to the $O(1)$ term given by Low [7] in his Bremsstrahlung low energy theorem. It shall be seen that this term is irrelevant for our result, and that the modification of the single particle singularity is determined only by the first term (which contains a portion which is of the same order of magnitude in the k_i's as the second term.)

B. We are now in a position to evaluate (2.5). For this purpose we shall need $|T_n|^2$ which is of the form

$$|T_n|^2 = (A^2)_n\left[1+2\sum_{i<j}\frac{B_{ij}}{A_iA_j}+O\sum_i k_i^2\log k_i\right] \tag{3.19a}$$

$$(A^2)_n = (a^2)_n\left[1+\sum_i\frac{d_i}{a_i}+\sum_{i<j}O(k_ik_j)\right] \tag{3.19b}$$

$$a = e\,\frac{2r\cdot\varepsilon}{2r\cdot k+k^2},\quad d = e\,\frac{2k\cdot\varepsilon}{2r\cdot k+k^2} \tag{3.19c}$$

$$|T_n|^2 = (a^2)_n\left[1+\sum_i\frac{d_i}{a_i}+2\sum_{i<j}\frac{B_{ij}}{a_ia_j}+\sum_i O\,k_i^2\log k_i\right] \tag{3.19d}$$

Therefore, from (2.5), we have

$$\sigma(p^2) = \sigma_1(p^2) + \sigma_2(p^2) \tag{3.20a}$$

$$\sigma_1(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \int \frac{d^3r}{2r^0} e^{irx} \sum_n \frac{1}{n!}$$
$$\times \left(\int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\,\theta(k^0) \sum_\varepsilon \right)_n \tag{3.20b}$$
$$\times \; (a^2)_n \left[1 + \sum_i \frac{d_i}{a_i} \right]$$

$$\sigma_2(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \int \frac{d^3r}{2r^0} e^{irx} \sum_n \frac{1}{n!} \tag{3.20c}$$
$$\times \left(\int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\,\theta(k^0) \sum_\varepsilon \right)_n$$
$$\times \; (a^2)_n \left[2 \sum_{i<j} \frac{B_{ij}}{a_i a_j} + \sum_i k_i^2 \log k_i \right].$$

We have used the integral representation for the delta function appearing in (2.5). Next the polarization sums are performed by defining the polarization tensor $\Pi^{\mu\nu}$

$$\Pi^{\mu\nu}(k) = \sum_{\text{polarizations}} \varepsilon^\mu \varepsilon^\nu. \tag{3.21}$$

$$\sigma_1(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \int \frac{d^3r}{2r^0} e^{irx} \sum_n \frac{1}{n!} \times$$
$$\left[\left(\frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\theta(k^0) g(k) \right)_n + n \left(\int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\,\theta(k^0)\, g(k) \right)_{n-1} \right. \tag{3.22a}$$
$$\times \int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\,\theta(k^0)\, h(k).$$

$$\sigma_2(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ip x} \int \frac{d^3r}{2r^0} \sum_n \frac{1}{n!}$$

$$\times \Big[n(n-1)\Big(\int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\,\theta(k^0)\, g(k) \Big)^{n-2} H_1$$

$$+ n \Big(\int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\,\theta(k^0)\, g(k) \Big)_{n-1} H_2 \Big]$$

(3.22b)

$$g(k) = e^2 \frac{4 r_\mu r_\nu \Pi^{\mu\nu}(k)}{[2r\cdot k + k^2]^2} \qquad (3.22c)$$

$$h(k) = e^2 \frac{4 r_\mu k_\nu \Pi^{\mu\nu}(k)}{[2r\cdot k + k^2]^2} \qquad (3.22d)$$

$$H_1 = -e^4 \int \frac{d^4k_1\, d^4k_2}{(2\pi)^6} \delta(k_1^2)\,\delta(k_2^2)\,\theta(k_1^0)\,\theta(k_2^0)$$

$$\times\; e^{ix\cdot(k_1+k_2)} \frac{r_\alpha r_\beta}{r\cdot k_1\, r\cdot k_2\, r\cdot(k_1+k_2)}$$

$$\times \left(\frac{k_1^w r^w}{k_1\cdot r} - g^{w\mu} \right) \left(\frac{k_{2w} r^\nu}{k_2\cdot r} - g_w{}^\nu \right) \Pi^{\alpha\mu}(k_1)\, \Pi^{\beta\nu}(k_2)$$

(3.22e)

$$H_2 = \int \frac{d^3k}{(2\pi)^3} e^{ikx} g(k)\, O(k^2 \log k). \qquad (3.22f)$$

The sum over n can be performed by noting that

$$\left(\int \frac{d^4k}{(2\pi)^3} h(k) e^{ikx} \delta(k^2)\theta(k^0) \right)_n = (F_1)^n$$

$$F_1 = \int \frac{d^4k}{(2\pi)^3} h(k)\, \delta(k^2)\, \theta(k^0) e^{ikx}. \qquad (3.23)$$

We also define

$$F_2 = \int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\theta(k^0) e^2 \frac{4 r_\mu k_\nu \Pi^{\mu\nu}(k)}{[2r\cdot k + k^2]^2}. \qquad (3.24)$$

Then (3.22) becomes

$$\sigma_1(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \int \frac{d^3r}{2r^0} e^{irx} e^{F_1} [1 + F_2] \qquad (3.25a)$$

$$\sigma_2(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \int \frac{d^3r}{2r^0} e^{irx} e^{F_1} [H_1 + H_2]. \qquad (3.25b)$$

To the same degree of accuracy, we can exponentiate F_2. We rewrite F_1+F_2 in the following form, which differs from the previous expressions by irrelevant terms

$$F = F_1 + F_2 = \int \frac{d^4k}{(2\pi)^3} e^{ikx} \delta(k^2)\, \theta(k^0) \times \left[\frac{e^2 [2r+k]^\mu [2r+k]^\nu \Pi_{\mu\nu}}{[2r\cdot k + k^2]^2} \right] \tag{3.26a}$$

Accordingly, σ_1 becomes

$$\sigma_1 = \int \frac{d^4x}{(2\pi)^4} e^{ipx} \int \frac{d^3r}{2r^0} e^{irx} e^F \tag{3.26b}$$

It is recognized that the quantity in parentheses in (3.26a) is just $\sum_\varepsilon A^2$. That σ_2 is unimportant will emerge below. The significance of F is clear : it is

$$\sum_{\text{one photon}} \langle \Omega | \varphi | r; k \rangle \langle r; k | \varphi^+ | \Omega \rangle$$

where the matrix element has been evaluated in the soft photon limit. Equation (3.26b) is yet another example of the exponentiation of soft photons.

C. To proceed, we must give a form to $\Pi^{\mu\nu}$. We discuss first explicitly covariant gauges, and we denote these with the subscript c. When the quantization is carried out in an explicitly covariant fashion, as in Gupta-Bleuler formalism, one must introduce fictitious timelike longitudinal photons to define the complete set of photon states, with the special Gupta-Bleuler metric. Therefore the polarization sum (3.21) has the explicit form

$$\Pi_c^{\mu\nu} = \sum_{\lambda=1}^{4} \eta_\lambda \, \varepsilon_\lambda^\mu \, \varepsilon_\lambda^\nu \tag{3.27}$$

($\eta_\lambda = -1$ when $\varepsilon_\lambda^{\ 2} = 1$, $\eta_\lambda = 1$ when $\varepsilon_\lambda^{\ 2} = -1$).

As we are unwilling to introduce any external vectors, we must have

$$\Pi_c^{\mu\nu} = -\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) - d(k^2)\frac{k^\mu k^\nu}{k^2} \tag{3.28}$$

[In the above formula we ignore the fact that $k^2 = 0$, we shall give meaning to terms involving k^{-2} presently.] The function $d(k^2)$ reflects the q number gauge invariance of the theory under the transformation

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x) \tag{3.29a}$$

It can be shown that d is given by

$$\frac{d(k^2)}{k^2} = -i\int e^{ikx} d^4x \langle \Omega | T\Lambda(x)\Lambda(0) | \Omega \rangle \tag{3.29b}$$

where Λ is the q valued gauge function with transforms A^μ from the Landau gauge (defined by $\partial_\mu A^\mu = 0$) [8]. It is also true that d must be constant, independent of k^2, if it is required that $\Box A^\mu = j^\mu$, as in (3.2). The photon polarizations transform according to

$$\varepsilon^\mu \to \varepsilon^\mu + (\sqrt{d} - 1)\frac{\varepsilon \cdot k \, k^\mu}{k^2} \tag{3.29c}$$

and the free photon propagator has the form

$$D_0^{\mu\nu} = \frac{1}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \frac{d(k^2)k^\mu k^\nu}{k^2}\right]. \tag{3.29d}$$

It is seen that (3.28) is the most general, properly normalized, second rank tensor consistent with Lorentz invariance, depending only on k. Therefore we may consider (3.28) as the definition of d, avoiding explicit reference to gauge transformations of the photon field.

Collecting (3.26b) and (3.28), we get

$$F_c = -e^2\int \frac{d^4k}{(2\pi)^3}\, e^{ikx}\,\theta(k^0)\, \times \left[\frac{(2r+k)^2\,\delta(k^2)}{(2r.k+k^2)^2} + (d(k^2)-1)\,\frac{\delta(k^2)}{k^2}\right] \tag{3.30a}$$

We must now give a definition of the terms involving k^{-2}. It is seen that they always appear in the combination $\delta(k^2)/k^2$, therefore it is natural to set this quantity equal to $-\delta'(k^2)$. The correctness of this prescription can be verified in the following fashion. Calculate the lowest order meson self-energy diagram, where the photon propagator is gauged as in (3.28), and take the imaginary part. This expression is the same as the one given above with the interpretation $\delta(k^2)/k^2 = -\delta'(k^2)$. Finally we must introduce a small photon mass in (3.30a) to avoid infra-red divergences. Therefore (3.30a) becomes

$$F_c = -e^2 \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx} \Big[\delta(k^2-\mu^2) \Big(\frac{m^2}{(r\cdot k)^2} + \frac{1}{(r\cdot k)} \Big) - \big(d(k^2)-1\big) \frac{\partial}{\partial k^2} \delta(k^2-\mu^2) \Big]$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2) \Big[\frac{m^2}{(r\cdot k)^2} + \frac{1}{r\cdot k} \Big] - \frac{\partial}{\partial \mu^2} e^2 \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2) \big[d(k^2) - 1 \big]$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2) \Big[\frac{m^2}{(r\cdot k)^2} + \frac{1}{r\cdot k} \Big] - \frac{\partial}{\partial \mu^2} e^2 \big(d(\mu^2)-1\big) \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2) \qquad (3.30b)$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2) \Big[\frac{m^2}{(r\cdot k)^2} + \frac{1}{r\cdot k} + d'(0) \Big] - e^2 \big(d(0)-1\big) \frac{\partial}{\partial \mu^2} \int \frac{d^4k}{(2\pi)^3}\, \Theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2)$$

It is clear that $d'(0)$ is not important compared to the other terms in the brackets of the last equation in (3.30b). Hence the final expression for F_c that we need consider is

$$F_c = -e^2 \int \frac{d^4k}{(2\pi)^3}\, \theta(k^0)\, e^{ikx}\, \delta(k^2-\mu^2) \left[\frac{m^2}{(\gamma\cdot k)^2} + \frac{1}{\gamma\cdot k}\right] \tag{3.31}$$
$$- e^2 \left(d(0)-1\right) \frac{\partial}{\partial\mu^2} \int \frac{d^4k}{(2\pi)^3}\, \theta(k_0)\, e^{ikx}\, \delta(k^2-\mu^2).$$

It is understood that all terms which vanish with $\mu^2 \to 0$ are ignored.

The evaluation of the integrals to determine σ_i is described in Appendix A. The result is that σ_2 is negligible and

$$\sigma_1(p^2) = \frac{1}{m^2}\left(\frac{m}{\mu}\right)^{\beta} e^{\alpha/\pi - C\beta}\, \frac{x_+^{\beta-1}}{\Gamma(\beta)} \tag{3.32}$$
$$\times \left[1 - \left(\frac{1}{2}+\frac{\beta}{4}\right)x + O\, x^2 \log x\right]$$

Here $\alpha = e^2/4\pi$, $\beta = \alpha(d(0)-3)/2\pi$, C is Euler's constant, Γ is the gamma function, $x = p^2 m^{-2} - 1$, and x_+^{γ} is the generalized function defined for example by

$$e^{i\gamma\pi/2}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx\, e^{-ixy}}{(x+i0)^{\gamma}} = \frac{y_+^{\gamma-1}}{\Gamma(\gamma)}. \tag{3.33}$$

Inserting (3.32) in (2.2) yields

$$G(p^2) = \frac{Z}{(p^2-m^2)^{1-\beta}} \left[1 - \left(\frac{1}{2}+\frac{\beta}{4}\right)\frac{p^2-m^2}{m^2}\right] + O \text{ constant}$$
$$Z = \mu^{-\beta} e^{\frac{\alpha}{\pi} - C\beta}\, \Gamma(1-\beta). \tag{3.34}$$

In this formula we must have $\beta < 1$, otherwise the term proportional to $(p^2-m^2)^{\beta-1}$ becomes of order constant and such terms are not significant. The term proportional to $(p^2-m^2)^{\beta}$ may be kept only when $\beta < 0$.

In connection with the above formula for $G(p^2)$, we wish to make the following observations. The first term in $G(p^2)$, i.e., the most singular, arises from the most singular term in T_n: viz., the term that goes as $1/k_1,\ldots,k_n$. The next term in $G(p^2)$ involves contributions from the next term in T_n: viz., the term that goes as

$$\frac{1}{k_2\ldots k_n}+\frac{1}{k_1 k_3\ldots k_n}+\ldots+\frac{1}{k_1\ldots k_{n-1}}$$

However, a portion of this less singular part of T_n does not contribute to G in our limit. Specifically it is seen that that part which leads to H_1, (3.22e), is irrelevant. Tracing this back, it is seen that the terms B_{ij} in (3.19a) are not important. The entire singular portion of G, i.e., the portion given above in (3.34), is determined just by A_n, viz., by the approximation

$$T_n = \left(\frac{e(2r+k)\cdot\varepsilon}{2r\cdot k+k^2}\right)_n \tag{3.35}$$

This is the contribution of photons emitted from external lines.

For $d(0)=3$, we regain the result that the propagator has no infra-red divergences. In that case it is seen also that the single particle singularity becomes a simple pole. If we take $d=3$, we have the Yennie gauge in which the free photon propagator has the form

$$D_0^{\mu\nu}\Big|_{d=3}=\frac{1}{k^2+i\varepsilon}\left[g^{\mu\nu}+2\frac{k^\mu k^\nu}{k^2}\right]. \tag{3.36a}$$

This rather arbitrary value for d takes on a special significance in position space. The Fourier transform of (3.36a) is

$$\int \frac{d^4k}{(2\pi)^4} e^{-ikx} D_0^{\mu\nu}\Big|_{d=3} = \frac{-i}{4\pi^2} \frac{1}{x-i0} \left[g^{\mu\nu} - \frac{x^\mu x^\nu}{x^2} \right] \tag{3.36b}$$

Therefore the photon propagator is transverse in position space, in the Yennie gauge.

In conclusion, we wish to point out that covariant gauges require unphysical intermediate photons. It is well known of course that the Feynman gauge requires timelike and longitudinal photons which, however, remain on their mass shell. The present investigation shows that other covariant gauges $(d(0) \neq 1)$ require timelike and longitudinal photons which are slightly off their mass shell, so that a differentiation with respect to their mass can be performed.

D. We now repeat the calculation in the special radiative gauge, recently used by Schwinger [4). The covariant calculation forced us to introduce fictitious timelike and longitudinal photons. Let us now insist that the intermediate state photons are physical. Therefore they are described by two polarization vectors ε_λ^μ, $\lambda = 1,2$ which are constrained by

$$k_\mu \varepsilon_\lambda^\mu = 0, \quad k^2 = 0$$
$$g_{\mu\nu} \varepsilon_\lambda^\mu \varepsilon_{\lambda'}^\nu = -\delta^{\lambda\lambda'} \tag{3.37}$$

The polarization tensor, which we now write with the subscript R, satisfies, by virtue of (3.37)

$$\Pi_R^{\mu\nu} = \sum_{\lambda=1}^{2} \varepsilon_\lambda^\mu \varepsilon_\lambda^\nu \qquad (3.38)$$

$$\Pi_R^{\mu\nu} = \Pi_R^{\nu\mu}$$

(3.39a)

$$k_\mu \Pi_R^{\mu\nu} = 0$$

(3.39b)

$$\Pi_R^{\mu\omega} \, \Pi_{R\omega}{}^{\nu} = - \Pi_R^{\mu\nu}$$

$$\Pi_{R\mu}{}^{\mu} = -2 \qquad (3.39c)$$

(3.39d)

It is easy to see that $\Pi_R^{\mu\nu}$ cannot be a function only of k. We therefore must abandon explicit Lorentz covariance by introducing another vector η^μ, on which $\Pi_R^{\mu\nu}$ may depend. One can verify that $\eta \cdot k \neq 0$ if (3.39) is to be satisfied. The most general form consistent with (3.39) is

$$\Pi_R^{\mu\nu} = -g^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{\eta \cdot k} - \frac{\eta^2}{(\eta \cdot k)^2} k^\mu k^\nu \qquad (3.40)$$

So far η^μ is arbitrary as long as $\eta \cdot k \neq 0$. However, if we impose explicit rotational invariance, η^μ must be timelike, and without loss of generality, we may take $\eta^2 = 1$.

Combining (3.40) with (3.26a), we get for F_R

$$F_R = \frac{e^2}{(2\pi)^3} \int d^4k \, e^{ikx} \, \theta(k^0) \, \delta(k^2) \frac{r_\mu r_\nu}{(r \cdot k)^2} \Pi^{\mu\nu}_R . \quad (3.41)$$

With the general form (3.40) for $\Pi^{\mu\nu}_R$, F_R and hence G are infrared divergent. We do not examine the general case here, but further restrict η^μ by an argument which leads to Schwinger's form for $\Pi^{\mu\nu}_R$.

Recall that $\Pi^{\mu\nu}_R$ arises from a summation over the intermediate states which contain one boson with momentum r, $r^2 = m^2$, and one photon with momentum k, $k^2 = 0$. It is natural to require that $\Pi^{\mu\nu}_R$ depend only on the momenta already present in the problem. Therefore we must have

$$\eta^\mu = \frac{a r^\mu + b k^\mu}{\sqrt{(ar + bk)^2}} \quad (3.42)$$

where a and b are arbitrary constants.

Schwinger's [4] form for $\Pi^{\mu\nu}$ is

$$\Pi^{\mu\nu}_{Schwinger} = -g^{\mu\nu} + \frac{k^\mu \bar{k}^\nu + k^\nu \bar{k}^\mu}{k \cdot \bar{k}} \quad (3.43)$$

where

$$\bar{k}^\mu \equiv (-k^0, \underline{k}) . \quad (3.44)$$

This is seen to be just (3.40), when use is made of the formula

$$\bar{k}^{\mu} = k^{\mu} - 2\eta^{\mu}(k\cdot\eta) \tag{3.45}$$

in the η^{μ} rest frame. If we work in the r^{μ} rest frame, we may set $\eta^{\mu} = r^{\mu}/m$. This makes (3.43) equivalent to (3.40) if b is chosen to be zero in (3.42).

With the choice (3.42) for η^{μ}, $\eta_{\mu} r_{\nu} \Pi^{\mu\nu}_{R} = 0$, hence $F_R = 0$. Therefore from (3.26a)

$$\sigma(p^2) = \delta(p^2 - m^2) \tag{3.46a}$$

and

$$G(p^2) = \frac{1}{p^2 - m^2 + i0} \, . \tag{3.46b}$$

Therefore we have established that in the radiation gauge, with η^{μ} chosen as above, all infra-red divergences vanish. Also the single particle singularity of G_{-} becomes an isolated pole. This state of affairs has been remarked upon by Schwinger [4], however, only to lowest order in e^2.

4. GRAVITATIONAL INTERACTIONS

A. In the present Section we discuss gravitational interactions and gravitational and electromagnetic interactions combined. The treatment follows closely that of Section 3. Hence we only outline it here, emphasizing the differences.

The low energy theorem evaluation of T_n, when gravitational interactions are taken into account, follows that of Section 3. However, we are now unable to give T_n to the same degree of accuracy as in electrodynamics. Recall that in that instance T_n involves terms of order k_n^{-1}, constant and $k_n \log k_n$, and that it is possible to give exactly the k_n^{-1} and the constant terms. In order to do this, it is necessary to separate the singular contribution to T_n. In the electrodynamical case this singular contribution is easily determined as it involves for all n only the one term summarized by Fig. 1. In the gravitational instance, however, at every $n \geqslant 2$, there are additional graphs arising from graviton exchange which contribute a $\mathcal{O}$ constant term (see Fig. 2). These graphs are necessary to preserve gauge invariance; they reflect the non-Abelian nature of the gravitational gauge group. We are unable to give a closed expression for these graphs for arbitrary n.

Therefore the only term in T_n which we can give precisely is the most singular. The determination of this can be accomplished in the same fashion as in the electrodynamical case, exploiting gravitational Ward identities [9]. An alternate and simpler way to give the most singular terms in T_n is to recall that this contribution arises from the diagrams in which the gravitons are emitted from external lines. This expression has been evaluated by Weinberg [3] and is given by

$$T_n = (A)_n + \text{less singular terms} \tag{4.1a}$$

$$A = \frac{(8\pi G)^{1/2}}{2} \frac{(2r^\mu + k^\mu)(2r^\nu + k^\nu)\,\varepsilon_{\mu\nu}}{2r\cdot k + k^2} \tag{4.1b}$$

Here G is Newton's constant and $\varepsilon_{\mu\nu}$ is the graviton polarization tensor. Since we have given the most singular term exactly, we can again give the most singular contribution to $G(p^2)\big|_{p^2 \approx m^2}$. Only a part of the next, less singular contribution to T_n has been given in (4.1); the part involving

$$\frac{2r^\mu k^\nu + 2r^\nu k^\mu}{2r \cdot k + k^2}$$

However, we recall from the electrodynamical investigation that the second term in the expansion of $G(p^2)$ near the position of its one-particle singularity is determined just by this contribution from the T_n. We therefore make the assumption that this state of affairs persists in gravity theory, i.e., we assume that A is sufficient to determine the second term in the expansion of $G(p^2)$.

The argument now proceeds as before to the conclusion that the contribution to the spectral function from soft graviton is just the exponentiation of the lowest order result.

$$\sigma_1(p^2) = \int \frac{d^4x}{(2\pi)^4}\, e^{-ipx} \int \frac{d^3r}{2r^0}\, e^{irx}\, e^F \tag{4.2a}$$

$$F = \int \frac{d^4k}{(2\pi)^3}\, \theta(k^0)\, \delta(k^2)\, e^{ikx} \times \frac{2\pi G\, (2r^\mu + k^\mu)(2r^\nu + k^\nu)(2r^\alpha + k^\alpha)(2r^\beta + k^\beta)}{(2r \cdot k + k^2)^2}\, \Pi_{\mu\nu,\alpha\beta} \tag{4.2b}$$

$$\Pi_{\mu\nu,\alpha\beta} = \sum_{\text{polarizations}} \varepsilon_{\mu\nu}\, \varepsilon_{\alpha\beta} \tag{4.2c}$$

B. If we perform covariant quantization, the most general form for $\Pi_{\mu\nu,\alpha\beta}$ is

$$\begin{aligned}\Pi_c^{\mu\nu,\alpha\beta} = & -\frac{1}{2}\left[g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta}\right] \\ & + \frac{d_1(k^2)}{4}\left[g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta}\right] \\ & + \frac{d_2(k^2)}{4k^2}\left[k^\mu k^\nu g^{\alpha\beta} + k^\alpha k^\beta g^{\mu\nu}\right] \\ & + \frac{d_3(k^2)}{8k^2}\left[k^\mu k^\alpha g^{\nu\beta} + k^\nu k^\alpha g^{\mu\beta} + k^\mu k^\beta g^{\nu\alpha} + k^\nu k^\beta g^{\mu\alpha}\right] \\ & + \frac{d_4(k^2)}{(k^2)^2} k^\mu k^\nu k^\alpha k^\beta . \end{aligned} \tag{4.3}$$

In offering (4.3) we have imposed only symmetry under interchange $\mu \leftrightarrow \nu$, $\alpha \leftrightarrow \beta$; $\mu\nu \leftrightarrow \alpha\beta$. The d_i's are related to the q number gauge transformations of the theory, however, we do not pursue this connection but merely use (4.3) to define the d_i's. With this value for $\Pi^{\mu\nu,\alpha\beta}$ F becomes

$$F_c = \pi G \int \frac{d^4k}{(2\pi)^3}\theta(k^0)\delta(k^2)e^{ikx}$$

$$\left[\frac{([2r+k]^2)^2}{(2r\cdot k+k^2)^2}\left(1+d_1(k^2)\right) + (2r+k)^2\left(d_2(k^2)+d_3(k^2)\right) \right. \tag{4.4a}$$

$$\left. + \frac{2d_4(k^2)}{(k^2)^2}(2r\cdot k+k^2)^2\right].$$

The singular expressions $\delta(k^2)/k^2$ and $\delta(k^2)/(k^2)^2$ are interpreted as before by

$$\frac{\delta(k^2)}{k^2} = -\delta'(k^2)$$
$$\frac{\delta(k^2)}{(k^2)^4} = \frac{1}{2}\delta''(k^2) \tag{4.4b}$$

A graviton mass λ is introduced to remove infra-red divergences in (4.4a). As before, it is found that derivatives with respect to the argument of the d_i's do not contribute. Hence the expression to be evaluated is

$$\begin{aligned} F_c = {} & 4\pi G m^2 \left(1+d_1(0)\right) \int \frac{d^4k}{(2\pi)^3}\, \theta(k^0)\, \delta(k^2-\lambda^2)\, e^{ikx} \left[\frac{m^2}{(r\cdot k)^2} + \frac{2}{r\cdot k}\right] \\ & + 4\pi G m^2 \left(d_2(0)+d_3(0)\right) \frac{\partial}{\partial\lambda^2} \int \frac{d^4k}{(2\pi)^3}\, \theta(k^0)\, \delta(k^2-\lambda^2)\, e^{ikx} \left(1+\frac{r\cdot k}{m^2}\right) \\ & + 4\pi G m^2 d_4(0) \frac{\partial^2}{\partial\lambda^4} \int \frac{d^4k}{(2\pi)^3}\, \theta(k^0)\, \delta(k^2-\lambda^2)\, e^{ikx} \frac{(r\cdot k)^2}{m^2} \\ & + 8\pi G m^2 d_4(0) \frac{\partial}{\partial\lambda^2} \int \frac{d^4k}{(2\pi)^3}\, \theta(k^0)\, \delta(k^2-\lambda^2)\, e^{ikx} \frac{r\cdot k}{m^2}\,. \end{aligned} \tag{4.5}$$

The evaluation of integrals proceeds as in the electromagnetic calculation, and is described in Appendix B. The result for Γ_1 is

$$\sigma_1(p^2) = \frac{1}{m^2}\left(\frac{m}{\lambda}\right)^{\beta'} e^{\frac{\alpha'}{\pi} - C\beta'} \frac{X_+^{\beta'-1}}{\Gamma(\beta')}\left[1 - \frac{\beta'}{4}X\right] \tag{4.6a}$$

$$\alpha' = -Gm^2\left[1 + d_1 - \frac{1}{4}d_4\right] \tag{4.6b}$$

$$\beta' = \frac{Gm^2}{\pi}\left[1 + d_1 - \frac{d_2}{2} - \frac{d_3}{2} - \frac{d_4}{4}\right] \tag{4.6c}$$

The Green's function $G(p^2)$ becomes

$$G(p^2) = \frac{Z'}{(p^2-m^2+i0)^{1-\beta'}}\left[1 - \frac{\beta'}{4}\left(\frac{p^2-m^2}{m^2}\right)\right] + O\ \text{constant} \tag{4.7a}$$

$$Z' = \lambda^{-\beta'} e^{\frac{\alpha'}{\pi} - C\beta'}\,\Gamma(1-\beta') \tag{4.7b}$$

We recall that the term of order $(p^2-m^2)^{-1+\beta'}$ is obtained without any assumptions, while the term of order $(p^2-m^2)^{\beta'}$ requires the assumption that the situation here is as in electrodynamics, viz., that no contributions to T_n other than $(A)_n$ are important.

For the $(p^2-m^2)^{-1+\beta'}$ to be significant, we must have $\beta' < 1$; while for the $(p^2-m^2)^{\beta'}$ term to be significant we must have $\beta' < 0$.

It is seen that the analogue of the Yennie gauge here is $\beta' = 0$ or

$$d_4 = 4 + 4d_1 - 2d_2 - 2d_3 \tag{4.8a}$$

$$\begin{aligned}\Pi_C^{\mu\nu,\alpha\beta} = {} & -\frac{1}{2}\Big[g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta} \\ & \qquad - 8\,k^\mu k^\nu k^\alpha k^\beta/(k^2)^2\Big] \\ & + \frac{d_1}{4}\Big[g^{\mu\nu}g^{\alpha\beta} + g^{\mu\alpha}g^{\nu\beta} + 16\,k^\mu k^\nu k^\alpha k^\beta/(k^2)^2\Big] \\ & + \frac{d_2}{4k^2}\Big[k^\mu k^\nu\left(g^{\alpha\beta} - 4k^\alpha k^\beta/k^2\right) \\ & \qquad + k^\alpha k^\beta\left(g^{\mu\nu} - 4k^\mu k^\nu/k^2\right)\Big] \\ & + \frac{d_3}{8k^2}\Big[k^\mu k^\alpha\left(g^{\nu\beta} - 4k^\nu k^\beta/k^2\right) \\ & \qquad + k^\nu k^\alpha\left(g^{\mu\beta} - 4k^\mu k^\beta/k^2\right) \\ & \qquad + k^\mu k^\beta\left(g^{\nu\alpha} - 4k^\nu k^\alpha/k^2\right) \\ & \qquad + k^\nu k^\beta\left(g^{\mu\alpha} - 4k^\mu k^\alpha/k^2\right).\end{aligned} \tag{4.8b}$$

In this gauge, infra-red divergences vanish, and the single-particle singularity is a simple pole.

C. Next, we perform the calculation in the radiation gauge. We require the gravitons to be physical. This means that we describe them by complex polarization tensors of the form

$$\varepsilon^{\mu\nu} = \varepsilon^{\mu}\varepsilon^{\nu}$$
$$\varepsilon^{\alpha\beta} = \varepsilon^{*\alpha}\varepsilon^{*\beta} \tag{4.9a}$$

$$k_{\mu}\varepsilon^{\mu} = 0$$
$$\varepsilon_{\mu}\varepsilon^{\mu} = 0 \tag{4.9b}$$

(4.9c)

$$\Pi_R^{\mu\nu,\alpha\beta} = \sum_{\lambda=1}^{2} \varepsilon_{\lambda}^{\mu}\,\varepsilon_{\lambda}^{\nu}\,\varepsilon_{\lambda}^{\alpha *}\,\varepsilon_{\lambda}^{\beta *} \tag{4.10}$$

The conditions (4.9) imply that $\Pi_R^{\mu\nu,\alpha\beta}$ satisfies

$$\Pi_R^{\mu\nu,\alpha\beta} = \Pi_R^{\nu\mu,\alpha\beta} = \Pi_R^{\mu\nu,\beta\alpha} = \Pi_R^{\alpha\beta,\mu\nu *} \tag{4.11a}$$

$$k_{\mu}\Pi_R^{\mu\nu,\alpha\beta} = 0 \tag{4.11b}$$

$$\Pi_{R\ \mu}^{\mu\ ,\alpha\beta} = 0 \tag{4.11c}$$

$$\Pi_R^{\mu\nu,}{}_{\nu}{}^{\alpha} = -\Pi_R^{\mu\alpha} \tag{4.11d}$$

$$\Pi_R^{\mu\nu,\alpha\beta}\,\Pi_{R\beta}{}^{\gamma} = -\Pi_R^{\mu\nu,\alpha\gamma} \tag{4.11e}$$

$$\Pi_R^{\mu\nu,}{}_{\alpha\beta}\,\Pi_R^{\alpha\beta,\gamma\delta} = \Pi_R^{\mu\nu,\gamma\delta} \tag{4.11f}$$

In the above, the two index object $\Pi_R^{\mu\nu}$ is defined in (3.38) and (3.40). Conditions (4.11) imply the following most general form for the polarization tensor

$$\Pi_R^{\mu\nu,\alpha\beta} = -\frac{1}{2}\left[\Pi_R^{\mu\nu}\Pi_R^{\alpha\beta} - \Pi_R^{\mu\alpha}\Pi_R^{\nu\beta} - \Pi_R^{\nu\alpha}\Pi_R^{\mu\beta}\right] \qquad (4.12)$$

Inserting (4.12) in (4.2b) gives

$$F_R = \int \frac{d^4k}{(2\pi)^3}\,\Theta(k^0)\,\delta(k^2)\,e^{ikx}\,4\pi G\,\frac{\left(r_\mu r_\nu \Pi_R^{\mu\nu}\right)^2}{(r\cdot k)^2} \qquad (4.13)$$

Hence with the choice (3.42) for η^μ, we get $F_R = 0$ and

$$\Gamma_1(p^2) = \delta(p^2 - m^2) \qquad (4.14a)$$

$$G(p^2) = \frac{1}{p^2 - m^2 + i0} \ . \qquad (4.14b)$$

This establishes the result that in Schwinger's radiation gauge (generalized to spin two massless bosons) the infra-red divergence is absent and the single-particle singularity becomes an isolated pole.

D. In conclusion, we consider electromagnetic and gravitational interactions combined. The most singular contribution to $G(p^2)$ can be readily evaluated. To obtain the next term, it is again necessary to make the assumption that the situation is the same as in electrodynamics, viz., that the entire contribution comes from the photons and gravitons attached only to external lines. Making this assumption, we obtain for covariant gauges

$$\sigma_1(p^2) = \frac{1}{m^2}\left(\frac{m}{\lambda}\right)^{\beta''} e^{\frac{\alpha''}{\pi} - C\beta''} \frac{x_+^{\beta''-1}}{\Gamma(\beta'')}\left[1 - \left(\frac{1}{2} + \frac{\beta''}{4}\right)x\right] \tag{4.15a}$$

$$\alpha'' = \alpha + \alpha'$$

$$\beta'' = \beta + \beta'$$

and

$$G(p^2) = \frac{Z''}{(p^2 - m^2 + i0)^{1-\beta''}}\left[1 - \left(\frac{1}{2} + \frac{\beta''}{4}\right)\left(\frac{p^2 - m^2}{m^2}\right)\right] + \text{O constant}. \tag{4.15b}$$

$$Z'' = Z'Z$$

It is clear that in Schwinger's radiation gauge we have as before

$$\sigma_1(p^2) = \delta(p^2 - m^2)$$
$$G(p^2) = \frac{1}{p^2 - m^2 + i0} \tag{4.16}$$

CONCLUSION

We have shown that gauge invariance and analyticity assumptions, inasmuch as they give low energy theorems, are sufficient to determine single-particle singularity of a propagator on the presence of massless particles. In select gauges, this singularity becomes a simple pole.

This treatment can be easily generalized to the case of non-zero spin particle propagators, as well as to the case of a single particle singularity of scattering matrix elements.

The latter objects are of course gauge independent, hence an infra-red singularity is always present.

APPENDIX A

We evaluate the integrals (3.31), (3.25a) and (3.25b). Although much of the integration can be performed exactly, we shall frequently approximate, keeping only terms which are relevant for the final result. We seek

$$\sigma_i(p^2) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx} \int \frac{d^3r}{2r^0} e^{irx} \mathcal{F}_i \qquad \text{(A.1)}$$

where

$$\mathcal{F}_1 = e^{F_c}$$

$$\mathcal{F}_2 = e^{F_1}(H_1 + H_2)$$

We evaluate the above in the p rest frame, by writing the integral in the form

$$\sigma_i(p^2) = \int_{0}^{\infty} dx^0 \frac{e^{-ip^0x^0}}{(2\pi)} \int d^3x \int \frac{d^3r}{(2\pi)^3} e^{-i\underline{r}\cdot\underline{x}} \frac{e^{ir^0x^0}}{2r^0} \mathcal{F}_i \qquad \text{(A.2)}$$

$\mathcal{F}_i$ is a function of $\underset{\sim}{x}$, $\underset{\sim}{r}$ and x^0. We expand

$$\frac{e^{ir^0x^0}}{2r^0} \mathcal{F}_i$$

in powers of $\underset{\sim}{r}$, keeping only the first three terms in σ_1 and only the first term in σ_2; the remainder being irrelevant for our purposes.

$$\frac{e^{ir^0 x^0}}{2r^0}\mathcal{F}_i = \frac{e^{imx^0}}{2m}\left[\mathcal{F}_i + r_j\frac{\partial}{\partial r_j}\mathcal{F}_i + \frac{r_j r_k}{2}\frac{\partial^2}{\partial r_j \partial r_k}\mathcal{F}_i\right. \tag{A.3}$$

$$+ \frac{r^2}{2m}\left(ix^0 - \frac{1}{m}\right)\mathcal{F}_i \,.$$

All quantities are taken at $\underset{\sim}{r} = 0$ after differentiation. It turns out that the third term in the brackets and the second term in the parentheses are unimportant. Hence, performing the $\underset{\sim}{r}$ integrals gives

$$\sigma_1(p^2) \approx \int_{-\infty}^{\infty}\frac{dx^0}{(2\pi)}\,\frac{e^{i(m-p^0)x^0}}{2m}\int d^3x \tag{A.4a}$$

$$\times\left[\mathcal{F}_1\,\delta^3(\underline{x}) + i\frac{\partial}{\partial x^j}\delta^3(\underline{x})\frac{\partial}{\partial r^i}\mathcal{F}_1 - \frac{ix^0}{2m}\nabla^2\delta^3(\underline{x})\,\mathcal{F}_1\right]$$

$$\sigma_2(p^2) \approx \int_{-\infty}^{\infty}\frac{dx^0}{(2\pi)}\,\frac{e^{i(m-p^0)x^0}}{2m}\int d^3x\;\mathcal{F}_2\,\delta^3(\underline{x}) \,. \tag{A.4b}$$

The $\underline{x}$ integral now yields

$$\sigma_1(p^2) = \int_{-\infty}^{\infty}\frac{dx^0}{(2\pi)}\,\frac{e^{i(m-p^0)x^0}}{2m}\left[\mathcal{F}_1 - i\frac{\partial}{\partial x^i}\frac{\partial}{\partial r^i}\mathcal{F}_1 - \frac{ix^0}{2m}\nabla_x^2\,\mathcal{F}_1\right]_{\underline{x}=\underline{0}=\underline{r}} \tag{A.5}$$

$$\sigma_2(p^2) = \int_{-\infty}^{\infty}\frac{dx^0}{(2\pi)}\,\frac{e^{i(m-p^0)x^0}}{2m}\,\mathcal{F}_2\Big|_{\underline{x}=\underline{0}=\underline{r}} \,.$$

To evaluate (A.5), we need the formula for F_c. This is given by the observation that from its definition F_c is

$$
\begin{aligned}
F_c = {} & i e^2 m^2 \int_0^\infty \alpha\, d\alpha\, D^+(x+\alpha\tau, \mu^2) \\
& - e^2 \int_0^\infty d\alpha\, D^+(x+\alpha\tau, \mu^2) \\
& - i e^2 \left[d(0) - 1 \right] \frac{\partial}{\partial \mu^2} D^+(x+\alpha\tau, \mu^2)
\end{aligned}
\tag{A.6}
$$

where $D^+(x, \mu^2)$ is defined by

$$
\begin{aligned}
D^+(x,\mu^2) &= \int \frac{d^4k}{i(2\pi)^3}\, \theta(k^0)\, \delta(k^2-\mu^2)\, e^{ikx} \\
&= \frac{1}{4\pi}\, \varepsilon(x^0)\, \delta(x^2) - \frac{\mu i}{8\pi\sqrt{x^2}}\, \theta(x^2) \left[N_1(\mu\sqrt{x^2}) \right. \\
&\quad \left. - i\, \varepsilon(x^0)\, J_1(\mu\sqrt{x^2}) \right] - \frac{\mu i}{4\pi^2\sqrt{-x^2}}\, \theta(-x^2)\, K_1(\mu\sqrt{-x^2}).
\end{aligned}
\tag{A.7}
$$

In the above, $\varepsilon(x^0) \equiv x^0/|x|$ and J_1, N_1, K_1 are Bessel functions. In (A.6) only terms which do not vanish with μ^2 are kept. The evaluation of (A.6) proceeds by exploiting the explicit form for D^+. We find

$$F_c = \frac{e^2}{16\pi^2}\left[3-d_0\right]\left[\log\frac{\mu^2|x^2|}{4} + 2C\right]$$
$$- \frac{e^2}{8\pi} i \left[\left(\frac{y}{\sqrt{y^2-x^2}} - 1\right)\Theta\left(\sqrt{y^2-x^2} - y\right)\right.$$
$$- \left(\frac{y}{\sqrt{y^2-x^2}} + 1\right)\Theta\left(-\sqrt{y^2-x^2} - y\right)$$
$$\left. + 1 + \frac{(1-d_0)}{2}\Theta(x^2)\,\varepsilon(x^0)\right]$$
$$- \frac{e^2}{8\pi^2}\left(1 - \frac{i}{my}\right)\frac{1}{\sqrt{1 - x^2/y^2}}\log\left|\frac{1-\sqrt{1-x^2/y^2}}{1+\sqrt{1-x^2/y^2}}\right| \tag{A.8}$$
$$- \frac{e^2}{8\pi m}\frac{1}{\sqrt{y^2-x^2}}\left[\Theta\left(\sqrt{y^2-x^2} - y\right) - \Theta\left(-\sqrt{y^2-x^2} - y\right)\right].$$

With this value for F_c, we determine $\mathcal{F}_1$ and $\mathcal{F}_2$. The contribution to σ_2 from H_1 vanishes by parity while the contribution from H_2 is of order $|p_0-m|^{1+\beta}\log|p_0-m|$, $(\beta \equiv (e^2/8\pi^2)(d-3))$, hence it is negligible. Inserting (A.8) into (A.4b) yields a one-dimensional integral which can be evaluated with the help of the integral given in (3.33). The result then is (3.32).

APPENDIX B

We need to perform a calculation, analogous to that of Appendix A, in order to evaluate (4.2a) and (4.4a). The approximate evaluation of σ_1 proceeds by the same device of expanding in a power series, as in (A.2) to (A.5) above. The evaluation of F_c involves, in addition to the three integrals discussed in (A.6), integrals of the form

$$I_1 = \frac{\partial}{\partial \lambda^2} \int \frac{d^4k}{(2\pi)^3} \theta(k^0)\, \delta(k^2 - \lambda^2)\, e^{ik\cdot x} (r.k) \tag{B.1}$$

$$I_2 = \frac{\partial^2}{\partial \lambda^2} \int \frac{d^4k}{(2\pi)^3} \theta(k^0)\, \delta(k^2 - \lambda^2)\, e^{ik\cdot x} (r.k)^2 \tag{B.2}$$

These may be recast in the form

$$I_1 = \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \lambda^2} D^+(x + \alpha r, \lambda^2) \tag{B.3}$$

$$I_2 = -\frac{\partial^2}{\partial\alpha\partial\alpha} \frac{\partial^2}{\partial\lambda\partial\lambda}\, i\, D^+(x + \alpha r, \lambda^2) \tag{B.4}$$

where only terms which do not vanish when α and λ go to zero are kept.

We find

$$I_1 = -x\cdot r\left[\frac{i}{8\pi^2(x^2-i\varepsilon)} + \frac{\delta(x^2)\,\varepsilon(x^0)}{8\pi}\right]$$

(B.5)

$$I_2 = \frac{-i}{32\pi}\,\varepsilon(x^0)\left[\delta(x^2)\,2(x\cdot r)^2 + \theta(x^2)m^2\right] + \frac{1}{32\pi^2}\left[\frac{2(x\cdot r)^2}{x^2-i\varepsilon} + m^2\left(\log\frac{\lambda^2|x^2|}{4}+2C\right)\right]$$

(B.6)

with the results F_c and σ_1 can be evaluated as before.

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FIGURE CAPTIONS

Figure 1 : $T_n^{(pole)}$ in electrodynamics.

Figure 2 : Contribution of graviton exchange to T_2.

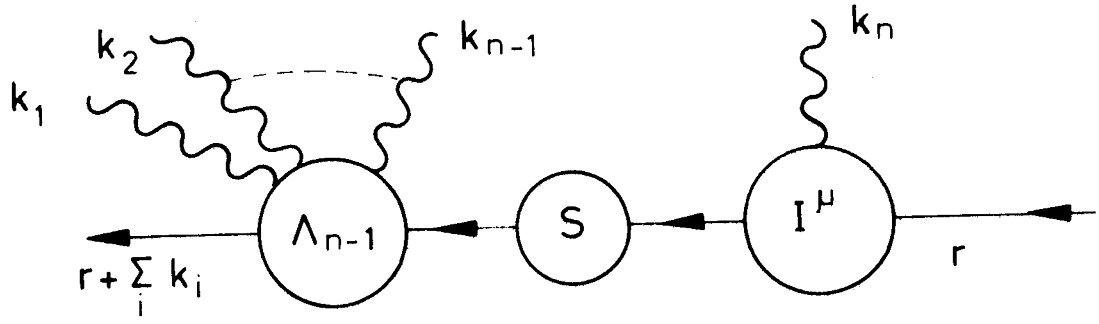

FIG.1

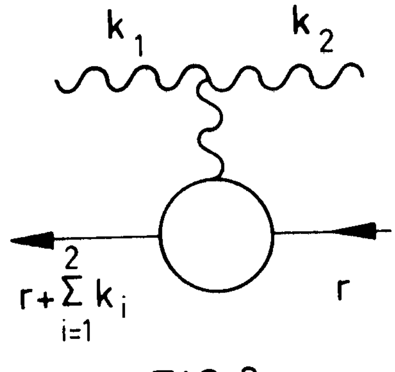

FIG.2

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