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LOW ENERGY THEOREMS FOR MASSLESS BOSONS : PHOTONS AND GRAVITONS

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A B S T R A C T

The low energy theorems for the scattering of massless bosons are discussed. Photon and graviton scattering are examined in detail, using techniques which make no high-energy assumptions. It is shown that the low energy form for the amplitude is given by the dispersion theoretic Born approximation, and that the energy dependence of the neglected terms is determined by the spin of the scattered boson. It is demonstrated that Schwinger terms and sea-gull terms do not cancel in gravity theory.

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I. - INTRODUCTION

Recently some attention has reverted to the exact low energy theorems for the scattering of massless bosons off massive particles. These theorems were first proved by Low [1] and Gell-Mann and Goldberger [2] for the case of massless spin 1 particles scattering off spin $\frac{1}{2}$ systems, viz., Compton scattering. Pais and Singh [3] have extended Low's considerations to higher energies and Bell [4] has shown that Low's approach does not contain any high-energy assumptions. Abarbanel and Goldberger [5] have given a derivation of the Compton scattering low energy theorem from an S matrix point of view, using the techniques of dispersion theory. Gross and the present author [6] used the method of Abarbanel and Goldberger to give a low energy theorem for massless spin 2 particles scattering off spin 0 systems, viz., graviton scattering.

The purpose of the present paper is to re-establish the graviton scattering low energy theorem by a method which minimizes the assumptions of the derivation. Specifically we show that this theorem follows, in a model independent fashion, from gauge invariance and from assumptions about the analyticity structure of the scattering amplitude at low energies. The present argument differs from the methods previously used to establish the low-energy theorems in that the dispersion theoretic results [5],[6] are established without use of dispersion theory. In order to illustrate it in a simple application, we first use it in Section II to study the case of photon scattering. Then in Section III, we give the low energy theorem for gravitons. In Section IV we discuss the divergence conditions in gravity theory, and show that Schwinger terms do not cancel sea-gull terms in this theory.

II. - SPIN 1 SCATTERING, THE PHOTON CASE

We examine briefly some aspects of the usual derivations of the low energy theorems. The dispersive approach [5),6)] has the attractive feature of using only physically measurable quantities since one works with helicity amplitudes. However, a "no subtraction" hypothesis seems to be required, since one writes unsubtracted dispersion relations for the helicity amplitudes with their kinematical zeros divided out. This state of affairs should be circumvented, as one believes that low energy behaviour is independent of subtractions. Moreover, the dispersive approach, as applied to graviton scattering, suffers from further shortcomings. Firstly the study of kinematic zeros requires a partial wave expansion which fails to exist due to the long range force between matter and gravity, which arises from graviton exchange and which leads to a pole in the forward direction. Secondly the dispersive approach does not yield the optimal estimate for the energy dependence of the neglected terms. Specifically the result obtained is that the neglected terms are quadratic in the graviton energy ; yet an independent argument can be given to show that they are in fact quartic [6)]. A final technical shortcoming of the dispersive method is that a separate argument is given for different spins of the target particle. As the result can be stated in a fashion which makes no reference to the target spin, a more unified treatment is preferable.

The method of Low [1)], in its original form, concentrates on the evaluation of the time-time component of the scattering amplitude tensor. Evidently a specific theoretical framework, such as the L.S.Z. formalism, is required to give a definite expression for this object. In addition to the general assumptions inherent in this formalism, specific assumptions about sea-gull terms and Schwinger terms are made to arrive at the desired fact that the time-time component is given by the time-ordered product of charge densities. Although such assumptions can be justified in definite models of electrodynamics, the situation in gravitation theory seems to be more obscure. Bell's modification [4)] focuses attention on the energy dependence of the single particle contribution ; a program we are unable to carry out for gravitation theory.

The derivation by Gell-Mann and Goldberger, as discussed by Kazes [7], exploits generalized Ward identities for the scattering amplitude continued off the mass shell in all the variables. These Ward identities require specific assumptions about the current commutators, the Schwinger terms in these commutators, and the sea-gull terms in the scattering amplitude. Alternatively one may assume invariance of the underlying theory under gauge transformations of the second kind, which then is sufficient to determine divergence conditions (viz., Ward identities) for the four-point function. Such assumptions can be readily made and justified for electrodynamics. For gravitation theory, however, we are unable to give the complete equal time commutator of the "currents", which are the sources of the gravity field. Thus we prefer a different approach which does not make use of Ward identities, so that no commutators are required.

Our approach then is the following. Consider the scattering of photons, with initial (final) four-momentum and polarization k, ε^{μ} (k', $\varepsilon^{*\nu}$) respectively, off a target of arbitrary spin and initial (final) four-momentum p (p'). The scattering amplitude A is given by

$$A = \varepsilon^{\mu} \varepsilon^{*\nu} T_{\mu\nu} \Big|_{k^2 = k'^2 = 0} , \tag{II.1}$$

where $T_{\mu\nu}$ is the scattering amplitude tensor, with photon momenta continued off their mass shell, but target momenta retaining their mass shell value. The polarization vectors satisfy $k_{\mu} \varepsilon^{\mu} = k'_{\nu} \varepsilon^{*\nu} = 0$. Energy conservation is imposed so that $k + p = k' + p'$. $T_{\mu\nu}$ satisfies a crossing relation

$$T_{\mu\nu}(k, k') = T_{\nu\mu}(-k', -k) , \tag{II.2}$$

which reflects the fact that, to every Feynman diagram contributing to $T_{\mu\nu}$, there corresponds a crossed diagram. Gauge invariance requires that [8]

$$k^{\mu} T_{\mu\nu} = k'^{\nu} T_{\mu\nu} = 0. \tag{II.3}$$

Our first assumption is that the contribution to $T_{\mu\nu}$ which is singular in k and k' (in the precise sense to be given below) can be identified and explicitly separated from $T_{\mu\nu}$. Call this singular part $T^{pole}_{\mu\nu}$. It <u>will not</u> in general satisfy the gauge condition (II.3). However, we may add to $T^{pole}_{\mu\nu}$ a second term, regular in k and k', $T'_{\mu\nu}$, such that $T'_{\mu\nu}$ summed with $T^{pole}_{\mu\nu}$ satisfies (II.3). We call the quantity $T^{pole}_{\mu\nu} + T'_{\mu\nu}$ the gauge invariant pole term, and we have

$$T_{\mu\nu} = T^{Pole}_{\mu\nu} + T'_{\mu\nu} + R_{\mu\nu} \tag{II.4a}$$

$$k^{\mu}\left(T^{Pole}_{\mu\nu} + T'_{\mu\nu}\right) = 0 \tag{II.4b}$$

$$k'^{\nu}\left(T^{Pole}_{\mu\nu} + T'_{\mu\nu}\right) = 0 \tag{II.4c}$$

$$k^{\mu} R_{\mu\nu} = 0 \tag{II.4d}$$

$$k'^{\nu} R_{\mu\nu} = 0. \tag{II.4e}$$

It is assumed that $T^{pole}_{\mu\nu} + T'_{\mu\nu}$ and $R_{\mu\nu}$ separately satisfy the crossing relation (II.2). Thus (II.4e) is derivable from (II.4d) and similarly (II.4c) from (II.4b). (We construct $T^{pole}_{\mu\nu} + T'_{\mu\nu}$ explicitly below.) The precise meaning of this regularity assumption may now be given : we wish to conclude that the relations (II.4d) and (II.4e) require that $R_{\mu\nu}$ be quadratic in the photon momenta, or identically zero.

To see that this is true for regular $R_{\mu\nu}$, we proceed as follows. Considering $R_{\mu\nu}$ as a function of p, p' and k, the last being arbitrary, we differentiate (II.4d) by k^{μ} to get

$$R^{\mu\nu} = -k_{\alpha} \frac{\partial}{\partial k_{\mu}} R^{\alpha\nu}. \tag{II.5}$$

Thus if the limit

$$k \to 0 \quad \frac{\partial}{\partial k_{\mu}} R^{\alpha\nu}$$

exists, viz., if $R^{\mu\nu}$ is differentiable by k at the origin, then (II.5) indicates that $R^{\mu\nu}$ is indeed linear in k or identically zero. The crossing relation or the gauge condition (II.4e) forces $R^{\mu\nu}$ to be linear in k' or identically zero. Thus it is seen that the divergence conditions (II.4d) and (II.4e) trivially enforce $R^{\mu\nu}$ to be of order k and of order k'. We show in Appendix A that in fact $R^{\mu\nu}$ is quadratic in the photon momenta. (Since k and k' are not independent, this is not an obvious consequence of the fact that $R^{\mu\nu}$ is of order k and of order k'.) When k and k' are restricted to their mass shell values, this term is quadratic in the photon frequency. Since no information is available about $R^{\mu\nu}$, we conclude that the terms whose form we can give explicitly, viz., $T^{pole}_{\mu\nu} + T'_{\mu\nu}$, yield a result accurate up to terms quadratic in the photon frequency.

It is seen that the result that the unknown, neglected term $R^{\mu\nu}$ is quadratic in the photon frequency follows essentially from the fact that there are two gauge conditions (II.4d) and (II.4e) (or alternatively one gauge condition and one crossing condition). When the scattered massless boson carries spin s, there will be 2s gauge condition [8] (or s gauge conditions and one crossing condition). Thus we expect that for boson spins higher than 1, the low energy theorem will be valid up to terms of order boson frequency to the power 2s.

Returning to our discussion of the photon low-energy theorems, we discuss the calculation of the gauge invariant pole contribution $T^{pole}_{\mu\nu} + T'_{\mu\nu}$. One method of doing this is the following. We take for $T^{pole}_{\mu\nu}$ the expression given by the single particle intermediate state Feynman diagrams which arise in the covariant Feynman-Dyson perturbation theory. The vertices and propagators in these diagrams are the complete, physical, renormalized quantities. The appropriate expression is summarized by Fig. 1. Then we explicitly construct $T'_{\mu\nu}$, so that the sum $T^{pole}_{\mu\nu} + T'_{\mu\nu}$ is gauge invariant. This procedure, which is carried out in Appendix C, is rather complicated and uses the Ward identity for the vertex operator, which is a consequence of gauge invariance but does not require current commutators, only current field commutators. However, we assert here that this explicit calculation is irrelevant because our second method, which we now describe, immediately gives $T^{pole}_{\mu\nu} + T'_{\mu\nu}$. In the above, we have argued that once a gauge invariant pole term has been separated from the scattering amplitude, the remainder is necessarily quadratic in the photon frequency. But we can obtain this gauge invariant pole contribution by simply calculating the Born approximation for the scattering of light off a system whose single photon emission amplitude is governed by the appropriate form factors, all evaluated at zero argument, viz., total charge for spin zero targets, charge and magnetic moment for spin $\frac{1}{2}$ targets, etc. That this expression is gauge invariant is assured by the underlying formalism. That this is the entire pole contribution is assured by the fact that the emission of physical photons is governed by form factors at zero argument.

To make the above considerations explicit, we formalize the argument as follows. Start with $T^{pole}_{\mu\nu}$ as given by the single particle Feynman diagrams of Fig. 1. This is certainly the entire pole contribution, but it is not gauge invariant. Thus

$$T^{Pole}_{\mu\nu} = \Gamma^{\mu}(p, p+k) D(p+k) \Gamma^{\nu}(p+k, p') + \Gamma^{\nu}(p, p-k') D(p-k') \Gamma^{\mu}(p-k', p') . \tag{II.6a}$$

Here Γ^{μ} is the complete vertex operator, D is the complete propagator. No commitment about the degrees of freedom of target is made and it is understood that (II.6a) is restricted to the p and p' mass shells. Many form factors contribute to (II.6a) since the vertex operators appearing there always have one leg off the mass shell, e.g., $\Gamma^{\mu}(p,p+k)$ with $(p+k)^2 \neq m^2$. We now note that an expression which is simpler than (II.6a), but has the same poles and residues as (II.6a), hence can serve just as well for $T^{pole}_{\mu\nu}$, is obtained from (II.6a) by replacing the propagator D by D_0, the bare propagator with a pole at the physical mass. Further the form factors with one leg off the mass shell may be replaced by form factors with both legs on the mass shell. Hence we use for $T^{pole}_{\mu\nu}$, instead of (II.6a) the formula

$$T^{Pole}_{\mu\nu} = \Gamma_0^{\mu}(p, p+k) D_0(p+k) \Gamma_0^{\nu}(p+k, p') + \Gamma_0^{\nu}(p, p-k') D_0(p-k') \Gamma_0^{\mu}(p-k', p') . \tag{II.6b}$$

Γ_0^{μ} contains the subscript to remind of the fact that any form factors appearing in a covariant expansion of Γ_0^{μ}, have both legs on the m mass shell. These form factors are functions of the photon momenta k^2, k'^2, which we do not as yet take to be on their mass shell. The expression (II.6b) is not in general gauge invariant ; so we must still construct $T'_{\mu\nu}$. When $k^2 = k'^2 = 0$, (II.6b) is simply the gauge non-invariant pole term of the Feynman-Dyson Born approximation to the scattering amplitude for light off matter, where the electromagnetic interaction is given by $\Gamma^{\mu}_{0}(p,p+k)\big|_{k^2=0}$, (e.g., total charge, magnetic moment, etc.). But we know how to calculate the additional terms which are required to make such pole terms gauge invariant : simply calculate the lowest order scattering amplitude in a gauge invariant theory where the interaction is given by (in momentum space) $\Gamma^{\mu}_{0}(p,p+k)\big|_{k^2=0}$. Thus we can certainly give $T'_{\mu\nu}\big|_{k^2=k'^2=0}$. Now proceeding to the physical amplitude, we have

$$
\begin{aligned}
T^{\mu\nu}\Big|_{k^2=k'^2=0} = {} & \Gamma^{\mu}_{0}(p,p+k)\,D_0(p+k)\,\Gamma^{\nu}_{0}(p+k,p')\Big|_{k^2=k'^2=0} \\
& + \Gamma^{\nu}_{0}(p,p-k')\,D_0(p-k')\,\Gamma^{\mu}_{0}(p-k',p')\Big|_{k^2=k'^2=0} \\
& + T'^{\mu\nu}\Big|_{k^2=k'^2=0} + O\,\omega^2 .
\end{aligned}
\qquad \text{(II.7)}
$$

(Here ω is the initial photon frequency.) This is the desired result. The scattering amplitude, up to terms quadratic in the photon momenta, is given by what is seen to be the Born approximation in the dispersion theoretic sense. The explicit calculation in Appendix B serves to confirm the above result [9]. We point out that the above holds for arbitrary target spins.

As a concrete example of the above, we work out the standard result for spin 0 and spin $\frac{1}{2}$ targets. For spin zero targets, (II.6a) gives the Feynman-Dyson pole terms with

$$
\begin{aligned}
\Gamma^{\mu}(p,p+k) = \Big[& (2p+k)^{\mu} f\big(m^2, (p+k)^2, k^2\big) \\
& + k^{\mu} g\big(m^2, (p+k)^2, k^2\big)\Big] .
\end{aligned}
\qquad \text{(II.8)}
$$

We have decomposed the vertex operator into the invariant form factors f and g. Recall that total physical charge e enters through

$$f(m^2, m^2, 0) = e, \tag{II.9a}$$

and that charge conservation requires that

$$g(m^2, m^2, k^2) = 0. \tag{II.9b}$$

Instead of the pole term (II.6a), we may consider the pole term (II.6b) which differs from (II.6a) by non-pole terms. Therefore we use (II.6b) with

$$\begin{aligned} \Gamma_0^{\mu}(p, p+k) &= (2p+k)^{\mu} f(m^2, m^2, k^2) \\ D_0(p) &= (p^2-m^2)^{-1}. \end{aligned} \tag{II.10}$$

When $k^2 = k'^2 = 0$, (II.6b) has the form

$$T^{\text{Pole}}_{\mu\nu}\Big|_{k^2=k'^2=0} = e^2 \frac{(2p+k)_\mu (2p'+k')_\nu}{(p+k)^2-m^2} + e^2 \frac{(2p-k')_\nu (2p'-k)_\mu}{(p-k')^2-m^2} \tag{II.11a}$$

This is just the pole term of the Born approximation to the scattering amplitude of light off scalar particles of charge e. In a well-known fashion, to make this expression gauge invariant, it is necessary to add a sea-gull term $-2e^2 g_{\mu\nu}$. Hence

$$T'_{\mu\nu}\Big|_{k^2=k'^2=0} = -2e^2 g_{\mu\nu}, \tag{II.11b}$$

and

$$T^{\mu\nu}\Big|_{k^2=k'^2=0} = e^2\frac{(2p+k)^\mu(2p'+k')^\nu}{2p\cdot k} - e^2\frac{(2p-k')^\nu(2p'-k)^\mu}{2p\cdot k'} - 2e^2 g^{\mu\nu} + O\,\omega^2. \tag{II.12}$$

This is the theorem for spin zero target.

For spin $\frac{1}{2}$ targets, we use (II.6b) with

$$\begin{aligned}\Gamma_0^\mu(p, p+k) &= \gamma^\mu F(m^2, m^2, k^2) \\ &\quad - i k_\omega \sigma^{\mu\omega} G(m^2, m^2, k^2) \\ D_0(p) &= (\not{p}-m)^{-1}.\end{aligned} \tag{II.13}$$

The total charge e and anomalous magnetic moment μ are defined through

$$\begin{aligned}F(m^2, m^2, 0) &= e^2 \\ G(m^2, m^2, 0) &= \mu.\end{aligned} \tag{II.14}$$

When $k^2 = k'^2 = 0$, (II.6b) has the form

$$T_{\mu\nu}^{pole} = \bar{u}(p)\left[\gamma_\mu e - i k^w \sigma_{\mu w}\mu\right]\left[\not{p} + \not{k} - m\right]^{-1}\left[\gamma_\nu e + i k'^w \sigma_{\nu w}\mu\right]u(p')$$
$$+ \bar{u}(p)\left[\gamma_\nu e + i k'^w \sigma_{\nu w}\mu\right]\left[\not{p} - \not{k}' - m\right]^{-1}\left[\gamma_\mu e - i k^w \sigma_{\mu w}\mu\right]u(p').$$

(II.15)

It is seen that this is just the pole terms of the Born approximation to the scattering amplitude of light from a spin $\frac{1}{2}$ particle of total charge e and anomalous magnetic moment μ. No further terms are necessary to make this gauge invariant, since (II.15) already possesses his property, as can be verified directly. Hence (II.15) is the total scattering amplitude up to quadratic photon frequency terms. This is the theorem for spin $\frac{1}{2}$ targets.

III. - SPIN 2 SCATTERING, THE GRAVITON CASE

We now discuss the low energy theorem for spin 2 massless particles, i.e., graviton scattering ; closely following the previous analysis of photon scattering. The kinematics are as before, except that the initial and final polarizations are now described by two component tensors, which may be taken to be of the form

$$\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$$
$$\varepsilon^\mu \varepsilon_\mu = 0, \qquad \text{(III.1)}$$

and similarly for the final polarization tensor $\varepsilon^{*\alpha\beta}$. The scattering amplitude A is given by

$$A = \varepsilon^{\mu\nu} \varepsilon^{*\alpha\beta} T_{\mu\nu,\alpha\beta} \Big|_{k^2 = k'^2 = 0} \qquad \text{(III.2)}$$

where $T_{\mu\nu,\alpha\beta}$ is symmetric in $\mu\nu$ and in $\alpha\beta$, and satisfies the crossing relation

$$T_{\mu\nu,\alpha\beta}(k,k') = T_{\alpha\beta,\mu\nu}(-k',-k). \qquad \text{(III.3)}$$

The polarization tensors are transverse $\varepsilon^\mu k_\mu = \varepsilon^{*\alpha} k'_\alpha = 0$. The gauge condition requires that [8]

$$k^\mu T_{\mu\nu,\alpha\beta} = k'^\alpha T_{\mu\nu,\alpha\beta} = 0. \qquad \text{(III.4)}$$

We now assume $T^{pole}_{\mu\nu,\alpha\beta}$, i.e., that the contribution to $T_{\mu\nu,\alpha\beta}$, which is singular in k and k', can be isolated, and that a regular expression $T'_{\mu\nu,\alpha\beta}$ can be given so that $T^{pole}_{\mu\nu,\alpha\beta}$ + + $T'_{\mu\nu,\alpha\beta}$ is gauge invariant. Thus we have

$$T_{\mu\nu,\alpha\beta} = T^{Pole}_{\mu\nu,\alpha\beta} + T'_{\mu\nu,\alpha\beta} + R_{\mu\nu,\alpha\beta} \tag{III.5a}$$

$$k^{\mu}\left(T^{pole}_{\mu\nu,\alpha\beta} + T'_{\mu\nu,\alpha\beta}\right) = 0 \tag{III.5b}$$

$$k'^{\alpha}\left(T^{pole}_{\mu\nu,\alpha\beta} + T'_{\mu\nu,\alpha\beta}\right) = 0 \tag{III.5c}$$

$$k^{\mu} R_{\mu\nu,\alpha\beta} = 0 \tag{III.5d}$$

$$k'^{\alpha} R_{\mu\nu,\alpha\beta} = 0. \tag{III.5e}$$

It is assumed that $T_{\mu\nu,\alpha\beta}$, $T'_{\mu\nu,\alpha\beta}$ and $R_{\mu\nu,\alpha\beta}$ separately possess the same symmetries as $T_{\mu\nu,\alpha\beta}$, and that $R_{\mu\nu,\alpha\beta}$ is regular in k and k'. Further divergences of $T^{pole}_{\mu\nu,\alpha\beta} + T'_{\mu\nu,\alpha\beta}$, and $R_{\mu\nu,\alpha\beta}$, vanish by virtue of the symmetry in $\mu\nu$ and $\alpha\beta$.

The divergence conditions on $R_{\mu\nu,\alpha\beta}$, the regularity assumption, as well as the symmetries, set restrictions on the k and k' dependence of $R_{\mu\nu,\alpha\beta}$. As in the photon case, part of the restriction is obtained trivially by differentiation. We have from $k^{\mu} R_{\mu\nu,\alpha\beta} = 0$ the result that

$$R^{\mu\nu,\alpha\beta} = -k_{\omega} \frac{\partial}{\partial k_{\mu}} R^{\omega\nu,\alpha\beta} . \tag{III.6a}$$

Now differentiate $k_{\mu} k_{\nu} R^{\mu\nu,\alpha\beta} = 0$ with respect to $k_{\mu'}$ and $k_{\nu'}$, to give

$$2 R^{\mu'\nu',\alpha\beta} + 2 k_{\nu} \frac{\partial}{\partial k_{\nu'}} R^{\nu\mu',\alpha\beta} + 2 k_{\mu} \frac{\partial}{\partial k_{\mu'}} R^{\mu\nu',\alpha\beta}$$

$$+ k_{\mu} k_{\nu} \frac{\partial^2}{\partial k_{\mu'} \partial k_{\nu'}} R^{\mu\nu,\alpha\beta} = 0. \tag{III.6b}$$

The symmetry of $R^{\mu\nu,\alpha\beta}$ has been used in (III.6b). Then (III.6a) gives

$$R^{\mu'\nu',\alpha\beta} = \frac{1}{2} k_{\mu} k_{\nu} \frac{\partial^2}{\partial k_{\mu'} \partial k_{\nu'}} R^{\mu\nu,\alpha\beta} , \tag{III.6c}$$

i.e., $R^{\mu\nu,\alpha\beta}$ is quadratic in k. Crossing or (III.5e) gives the result that $R^{\mu\nu,\alpha\beta}$ is also quadratic in k'. Hence $R^{\mu\nu,\alpha\beta}$ is quadratic in k and k'. In Appendix B, we show how to give a closer estimate for $R^{\mu\nu,\alpha\beta}$. The procedure, analogous to that of the photon case, is very involved due to proliferation of indices. We therefore show only that $R^{\mu\nu,\alpha\beta}$ is cubic in the graviton frequencies, and indicate how the proof proceeds to the end that $R^{\mu\nu,\alpha\beta}$ is quartic in the graviton frequencies. For the simpler problem, where $R^{\mu\nu,\alpha\beta}$ arises from a spin zero target, the result that $R^{\mu\nu,\alpha\beta}$ is quartic in the graviton frequencies has been proven explicitly elsewhere [6].

To complete the discussion of graviton scattering, we need to give the gauge invariant pole contribution $T^{\text{pole}}_{\mu\nu,\alpha\beta} + T'_{\mu\nu,\alpha\beta}$. The pole terms include, in addition to the single particle intermediate states, a term which has a pole in the crossed channel, and which arises from the exchange of a massless boson between matter and gravity. All the pole terms are summarized in Fig. 2. The graviton exchange pole is of the form (for physical gravitons) $\sigma\!\!\lambda/k\cdot k'$. For fixed energy in the forward direction this diverges. However, for fixed angle there is no divergence with vanishing energy of the graviton. The reason for this is that the residue $\sigma\!\!\lambda$ also vanishes as energy decreases to zero, which reflects the fact that gravitons interact with energy. It is seen thus that at fixed angle not in the forward direction, a low energy theorem can be given.

As in the photon case two methods are presented to calculate the gauge invariant pole contribution. One method is to take for $T^{\text{pole}}_{\mu\nu,\alpha\beta}$ the expression summarized by Fig. 2, and to calculate $T'_{\mu\nu,\alpha\beta}$ explicitly. This is done in Appendix D, where the Ward identity for the gravitational vertex is derived. The second method is the assertion that the gauge invariant pole contribution is simply obtained by calculating the Born approximation to the scattering amplitude of gravitons off a system whose emission amplitude for physical gravitons is governed by appropriate form factors at zero argument. The reasoning which establishes this result is the same as in the photon argument. We do not repeat it now, but merely illustrate it in the case of a spin zero target.

The pole contribution of Fig. 2 has the form

$$T^{\text{Pole}}_{\mu\nu,\alpha\beta} = \Gamma_{\mu\nu}(p, p+k)\, D(p+k)\, \Gamma_{\alpha\beta}(p+k, p') + \Gamma_{\alpha\beta}(p, p-k')\, D(p-k')\, \Gamma_{\mu\nu}(p-k', p') + U_{\mu\nu,\alpha\beta,\gamma\delta}(k, k')\, \frac{d^{\gamma\delta,\varepsilon\varphi}(k-k')}{(k-k')^2}\, \Gamma_{\varepsilon\varphi}(p, p'). \tag{III.7}$$

$k^2 g_{\mu\nu} - k_\mu k_\nu$ or $k'^2 g_{\alpha\beta} - k'_\alpha k'_\beta$. Such terms do not contribute to the final scattering amplitude. Thus the form for $T^{pole}_{\mu\nu,\alpha\beta}$ which we use is

$$T^{Pole}_{\mu\nu,\alpha\beta} = \frac{G^2}{16} F_1(m^2, m^2, k^2)\, F_1(m^2, m^2, k'^2) \times$$

$$\Big\{ D_0(p+k)(2p_\mu + k_\mu)(2p_\nu + k_\nu)(2p'_\alpha + k'_\alpha)(2p'_\beta + k'_\beta)$$

$$+ D_0(p-k')(2p_\alpha - k'_\alpha)(2p_\beta - k'_\beta)(2p'_\mu - k_\mu)(2p'_\nu - k_\nu)\Big\}$$

$$+ U_{\mu\nu,\alpha\beta,\gamma\delta}(k,k') \frac{d^{\gamma\delta,\varepsilon\varphi}(k-k')}{(k-k')^2} \frac{G}{4}(p+p')_\varepsilon (p+p')_\varphi . \qquad \text{(III.10)}$$

To calculate $T'_{\mu\nu,\alpha\beta}\big|_{k^2=k'^2=0}$, we note that $T^{pole}_{\mu\nu,\alpha\beta}\big|_{k^2=k'^2=0}$ is exactly the pole contribution to the Born approximation for the scattering of gravitons off spinless matter of mass m. Such a pole contribution is not gauge invariant. The additional term needed is a sea-gull, which is given in Fig. 3. The explicit form of this is calculable from the Feynman rules. Therefore $T'_{\mu\nu,\alpha\beta}\big|_{k^2=k'^2=0}$ can be evaluated. Thus the scattering amplitude up to terms quartic in the graviton frequency is given by the Born approximation [10]. Note that only one form factor (at zero momentum transfer) contributes, which is a surprising result ; i.e., F_2 plays no role in graviton scattering to this order.

The explicit calculation from the Feynman rules of the sea-gull term and of the entire Born term has been presented elsewhere [6]. We record here the final result for the total cross-section for unpolarized gravitons, in the matter rest frame and zero graviton energy.

$$\frac{d\sigma}{d\Omega} = \frac{G^2 m^2}{2} \left[\frac{1 + 6\cos^2\theta + \cos^4\theta}{(1 - \cos\theta)^2} \right], \qquad \text{(III.11)}$$

where θ is the scattering angle. The calculation in Appendix D serves to confirm this result.

This completes our discussion of the low energy theorem of graviton scattering. We repeat that it has been unnecessary to appeal to anything beyond gauge invariance of the scattering amplitude and the identification of the gauge invariant pole contribution.

IV. - DIVERGENCE CONDITION IN GRAVITY THEORY

It has been possible to give the low energy theorem for photons and gravitons, without reference to current commutators or divergence conditions. For gravitons these commutators are not known. In the present section, we shall show that the derivation of divergence conditions in gravitation theory cannot proceed in the same heuristic fashion as in electrodynamics ; that is Schwinger terms (terms proportional to divergences of delta functions) do not cancel sea-gull terms.

Let us first recall the situation in electrodynamics. Consider the scattering amplitude tensor $M^{\mu\nu}$, off the mass shell in all the variables.[7] Then the divergence condition is

$$k_\mu M^{\mu\nu} = D^{-1}(p')\,D(p'-k)\,\Gamma^\nu(p'-k, p) - \Gamma^\nu(p', p+k)\,D(p+k)\,D^{-1}(p). \quad \text{(IV.1)}$$

On the mass shell this vanishes as it must by gauge invariance. This divergence condition is obtained either by exploiting gauge invariance of the second kind of the underlying theory, or by explicit calculation using an expression for the scattering amplitude, given by the L.S.Z. formalism. The second method requires the knowledge of current commutators. Experience in electrodynamics shows that for purposes of calculating this divergence, one can pretend that $M^{\mu\nu}$ is given by a time ordered product of currents and that the commutator of the currents does not contain Schwinger terms. (The true state of affairs of course is that $M^{\mu\nu}$ contains, in addition to the time ordered product of currents, sea-gull terms whose divergence just cancels the Schwinger term in the commutator.)

One might hope that a similar state of affairs obtains in gravity theory. However, we shall show that this is not the case.

In order to begin the derivation of the analogue of (IV.1) for gravity theory, we consider a four-point function consisting of the time ordered product of two spin zero matter fields and two currents - sources of the gravity field. The complete source is the energy-momentum tensor of matter and of the gravity field. However, since we have no information here about the gravity field energy-momentum tensor, we shall assume that we may take the source to be given just by the energy-momentum tensor of matter. The effective content of this assumption in the present context is that the commutation relations of the complete currents with themselves are the same as those of the matter energy-momentum tensor. Having made this assumption, it is straightforward to calculate the divergence condition. The result is

$$k_\mu M^{\mu\nu,\alpha\beta} = (p'-k)^\nu D^{-1}(p') D(p'-k) \Gamma^{\alpha\beta}(p-k',p) - (p+k)^\nu \Gamma^{\alpha\beta}(p',p+k) D(p+k) D^{-1}(p) + (p-p')^\nu \Gamma^{\alpha\beta}(p',p) \qquad \text{(IV.2a)}$$

$$k'_\alpha M^{\mu\nu,\alpha\beta} = -(p'+k')^\beta D^{-1}(p') D(p'+k') \Gamma^{\mu\nu}(p+k,p) + (p-k')^\beta \Gamma^{\mu\nu}(p',p-k') D(p-k') D^{-1}(p) + (p'-p)^\beta \Gamma^{\mu\nu}(p',p). \qquad \text{(IV.2b)}$$

Equation (IV.2b) follows from (IV.2a) by crossing symmetry.

$$M^{\mu\nu,\alpha\beta}(p,p';k,k') = M^{\alpha\beta,\mu\nu}(p,p';-k',-k). \qquad \text{(IV.3)}$$

Schwinger terms have been ignored in the derivation of (IV.2). The first two terms in (IV.2a) and (IV.2b) represent the commutator of the source with the field, while the last term arises from the source-source commutator.

There are two things wrong with (IV.2) as a candidate for the divergence condition for the off mass shell scattering amplitude. First, the gauge invariance condition is not regained on the mass shell, since the term $(p-p')^{\nu}\Gamma^{\alpha\beta}(p',p)$ does not vanish. Second (IV.2) is not crossing symmetric. By this we mean that if we subtract k'_{α} times the first equation, (IV.2a), from k_{μ} times the second equation, (IV.2b), the result is not zero. Thus the point of view familiar from electrodynamics, where one ignores Schwinger terms and sea-gull terms, is not effective in gravity theory.

We now enquire if we might modify the divergence condition (IV.2) in a simple fashion to overcome its defects. In deriving (IV.2), we have ignored the Schwinger terms in the equal time commutator of $H^{0\nu}$ with $H^{\alpha\beta}$ ($H^{\mu\nu}$ is the matter energy-momentum tensor.) However, it is well known [11] that this commutator does in fact contain terms proportional to derivatives of the delta function. If we concentrate on the first derivative of the delta function, then it can be verified by model calculations that the proportionality factors are linear combinations of energy-momentum tensor components [11),12)]. Thus if single derivative Schwinger terms are included, (IV.2a) is replaced by

$$\begin{aligned} k_{\mu} M^{\mu\nu,\alpha\beta} = {} & (p'-k)^{\nu} D^{-1}(p') D(p'-k) \Gamma^{\alpha\beta}(p-k',p) \\ & - (p+k)^{\nu} \Gamma^{\alpha\beta}(p',p+k) D(p+k) D^{-1}(p) \\ & + (p-p')^{\nu} \Gamma^{\alpha\beta}(p',p) \\ & + k^{i} S^{i\nu|\alpha\beta}_{\omega\varphi} \Gamma^{\omega\varphi}(p',p) \,. \end{aligned} \tag{IV.4}$$

The last, non-covariant term in (IV.4) is the effect of Schwinger terms. $S^{i\nu|\alpha\beta}_{\omega\varphi}$ is a numerical tensor, generating the linear combinations of the vertex function $\Gamma^{\omega\varphi}$. It is seen therefore that even if the relation (IV.4) is extended into some invariant form, for example by replacing the last term by $k_{\mu} S^{\mu\nu,\alpha\beta}_{\omega\varphi} \Gamma^{\omega\varphi}_{(p',p)}$, so that the entire equation also is crossing symmetric, such an expression cannot be the divergence condition, since the first of the above defects persists, viz. it does not vanish on the mass shell. Therefore the divergence condition on the amplitude cannot be obtained by this cavalier elimination of sea-gull and Schwinger terms, which is valid in electrodynamics. To obtain the divergence condition, one must carefully account for these singular objects. Alternatively, one might study the gauge invariance of the underlying Lagrangian. We do not pursue these considerations any further.

V. - CONCLUSION

We have given a general derivation of low energy theorems for the scattering of massless bosons of spin 1 and 2, using a technique which does not use any high-energy assumptions, nor does it commit one to any model for the exact scattering amplitude. We have shown that the dispersion theoretic Born approximation gives the scattering amplitude up to terms in the boson frequency whose order is determined by the spin of the boson. We do not discuss spins greater than 2, since it can be shown that such massless bosons do not couple at low energy [8]. We do not have a theorem for spin zero bosons. The reason for this is that the main ingredient of low energy theorems is the limitation that masslessness, viz., gauge invariance, places on the amplitude. These limitations are consequences of the fact that a massless particle can exist in, at most, two spin states. But a spin zero object has only one spin state available, so that masslessness does not place any further restrictions. The physically uninteresting, but conceptually intriguing case of graviton scattering - with its attendant peculiarities - is thus seen to be completely tractable within conventional techniques. However, it is seen that divergence condition in gravity theory cannot be given in the same simple fashion as in electrodynamics, since the Schwinger terms and sea-gull terms do not cancel.

VI. - ACKNOWLEDGEMENT

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APPENDIX A

We show that $R^{\mu\nu}(p,p',k,k')$ which satisfies

$$R^{\mu\nu}(p,p',k,k') = R^{\nu\mu}(p,p',-k',-k) \tag{A.1}$$

$$k_\mu R^{\mu\nu} = 0, \tag{A.2}$$

must be quadratic in the photon momenta (under the usual regularity assumptions). According to (II.5) we have the fact that $R^{\mu\nu}$ is linear in k

$$R^{\mu\nu} = k_\alpha R^{\mu\nu\alpha}(p,p',k), \tag{A.3}$$

where we have eliminated k' by the energy conservation relation. Crossing, viz., (A.1), now gives

$$R^{\mu\nu} = -k'_\alpha R^{\nu\mu\alpha}(p,p',-k'). \tag{A.4}$$

We expand $R^{\nu\mu\alpha}$ in powers of k :

$$R^{\nu\mu\alpha}(p,p',-k') = R_0^{\nu\mu\alpha} - k_\beta R_1^{\mu\nu|\alpha\beta}(p,p',k). \tag{A.5}$$

(A.5) is exact by definition, when $R_0^{\nu\mu\alpha} \equiv R^{\nu\mu\alpha}(p,p',p'-p)$. Therefore from (A.4)

$$R^{\mu\nu} = -k'_\alpha R_0^{\nu\mu\alpha} + k'_\alpha k_\beta R_1^{\mu\nu|\alpha\beta}(p,p',k), \tag{A.6}$$

and from (A.1)

$$R^{\mu\nu} = k_\alpha R_0^{\mu\nu\alpha} + k_\alpha k'_\beta R_1^{\nu\mu|\alpha\beta}(p, p', -k'). \qquad \text{(A.7)}$$

It is to be remembered that k' in the above is not an independent variable, but only short for $p + k - p'$.

We almost have the desired result. It remains to show that the first term on the right-hand side of (A.6) and (A.7) is in fact quadratic in the photon momenta, viz., that $R_0^{\mu\nu\alpha}$ is linear in $(p'-p) = (k-k')$. To do this, we combine (A.6) and (A.7) :

$$k_\alpha R_0^{\mu\nu\alpha} + k_\alpha k'_\beta R_1^{\nu\mu|\alpha\beta}(p, p', -k')$$
$$= -k'_\alpha R_0^{\nu\mu\alpha} + k'_\alpha k_\beta R_1^{\mu\nu|\alpha\beta}(p, p', k). \qquad \text{(A.8)}$$

We equate terms independent of k and linear in k, to get

$$(p-p')_\alpha R_0^{\nu\mu\alpha} = 0 \qquad \text{(A.9)}$$

$$R_0^{\mu\nu\alpha} + R_0^{\nu\mu\alpha} = (p-p')_\beta R_1^{\mu\nu|\beta\alpha}(p, p', 0)$$
$$- (p-p')_\beta R_1^{\nu\mu|\alpha\beta}(p, p', p'-p). \qquad \text{(A.10)}$$

Next we impose (A.2) on (A.7) and equate to zero terms quadratic in k

$$k_\mu k_\alpha \left[R_0^{\mu\nu\alpha} + (p-p')_\beta R_1^{\nu\mu|\alpha\beta}(p, p', p'-p)\right] = 0.$$
(A.11)

To show that $R_0^{\mu\nu\alpha}$, is proportional to $p'-p$, we assume the contrary and prove it to be zero. Assuming $R_0^{\mu\nu\alpha}$ depends on $P \equiv p+p'$, but not on $Q \equiv p'-p$, we have from (A.9), (A.10) and (A.11) the fact that $R_0^{\mu\nu\alpha}$ is antisymmetric in all its indices and transverse to Q. Since $R_0^{\mu\nu\alpha}$ is independent of Q, the only way $Q_\mu R_0^{\mu\nu\alpha}$ can vanish is if $R_0^{\mu\nu\alpha}$ is proportional to P^μ, since $P \cdot Q = 0$. But $R_0^{\mu\nu\alpha}$ is transverse to Q when contracted with <u>any</u> index ; thus $R_0^{\mu\nu\alpha}$ is proportional to $P^\mu P^\nu P^\alpha$, which violates the antisymmetric nature of $R_0^{\mu\nu\alpha}$, and proves the desired result.

APPENDIX B

We give a calculation for gravitons analogous to that given above for photons. The problem is almost intractable due to the proliferation of indices. We therefore simplify it by showing that $R^{\mu\nu,\alpha\beta}$ is cubic rather than quartic in the graviton momenta. We consider $R^{\mu\nu,\alpha\beta}$ to be a function of p, p' and k. We then have

$$k_\mu R^{\mu\nu,\alpha\beta} = 0 \tag{B.1}$$

$$R^{\mu\nu,\alpha\beta}(p,p',k) = R^{\alpha\beta,\mu\nu}(p,p',-k'). \tag{B.2}$$

$R^{\mu\nu,\alpha\beta}$ is symmetric in $\mu\nu$ and in $\alpha\beta$. We have from (III.6) the fact that

$$\begin{aligned} R^{\mu\nu,\alpha\beta} &= k_\omega k_\varphi R^{\mu\nu,\alpha\beta,\omega\varphi}(p,p',k) \\ &= k'_\omega k'_\varphi R^{\alpha\beta,\mu\nu,\omega\varphi}(p,p',-k'). \end{aligned} \tag{B.3}$$

$R^{\alpha\beta,\mu\nu,\omega\varphi}$ may be chosen symmetric in $\omega\varphi$. Expanding

$$R^{\alpha\beta,\mu\nu,\omega\varphi}(p,p',-k') = R_0^{\alpha\beta,\mu\nu,\omega\varphi} - k_\gamma R_1^{\alpha\beta,\mu\nu,\omega\varphi,\gamma}(p,p',k)$$

$$R_0^{\alpha\beta,\mu\nu,\omega\varphi} \equiv R^{\alpha\beta,\mu\nu,\omega\varphi}(p,p',p'-p), \tag{B.4}$$

we have

$$R^{\mu\nu,\alpha\beta} = k'_\omega k'_\varphi R_0^{\alpha\beta,\mu\nu,\omega\varphi} - k_\gamma k'_\omega k'_\varphi R_1^{\alpha\beta,\mu\nu,\omega\varphi,\gamma}(p,p';k)$$
$$= k_\omega k_\varphi R_0^{\mu\nu,\alpha\beta,\omega\varphi} + k'_\gamma k_\omega k_\varphi R_1^{\mu\nu,\alpha\beta,\omega\varphi,\gamma}(p,p';-k'). \quad (B.5)$$

Equating terms independent of k, and linear in k, in the above yields

$$(p-p')_\omega (p-p')_\varphi R_0^{\alpha\beta,\mu\nu,\omega\varphi} = 0$$
$$2(p-p')_\omega R_0^{\alpha\beta,\mu\nu,\omega\gamma} = (p-p')_\omega (p-p')_\varphi R_1^{\alpha\beta,\mu\nu,\omega\varphi,\gamma}(p,p';0) \quad (B.6)$$

Finally, imposing (B.1) on the second equation in (B.5) and equating terms cubic in k gives

$$k_\mu k_\omega k_\varphi \left[R_0^{\mu\nu,\alpha\beta,\omega\varphi} + (p-p')_\gamma R_1^{\mu\nu,\alpha\beta,\omega\varphi,\gamma}(p,p';p'-p)\right] = 0. \quad (B.7)$$

To prove that $R^{\mu\nu,\alpha\beta}$ is cubic in the photon momenta, it is sufficient to show that $R_0^{\mu\nu,\alpha\beta,\omega\varphi}$ is proportional to $(p.p') = (k'-k)$. We assume the contrary, and obtain from (B.6) and (B.7)

$$Q_\omega Q_\varphi R_0^{\alpha\beta,\mu\nu,\omega\varphi} = 0$$
$$Q_\omega R_0^{\alpha\beta,\mu\nu,\omega\varphi} = 0$$
$$k_\mu k_\omega k_\varphi R_0^{\mu\nu,\alpha\beta,\omega\varphi} = 0$$
$$Q \equiv p' - p. \quad (B.8)$$

According to the first two equations above and the symmetry in $\omega\varphi$, we have that

$$R_0^{\alpha\beta,\mu\nu,\omega\varphi} = P^\omega P^\varphi R_0^{\alpha\beta,\mu\nu} \quad (B.9)$$
$$P \equiv p' + p.$$

The last equation in (B.8) shows that $k_{\mu} R_0^{\alpha\beta,\mu\nu} = 0$, which forces $R_0^{\alpha\beta,\mu\nu}$ to vanish, and establishes the theorem.

The proof to terms quartic in the graviton momenta makes use of further constraints analogous to (B.6) and (B.7), which may be derived from (B.5). It is also necessary to remember that $R^{\mu\nu,\alpha\beta}$ may be chosen to be traceless in $\mu\nu$ and $\alpha\beta$. We do not pursue this proof any further here, but note that the simpler problem where $R^{\mu\nu,\alpha\beta}$ arises from a spin zero target (or is spin averaged for higher spin targets) has been solved explicitly elsewhere [6], where it is shown that $R^{\mu\nu,\alpha\beta}$ is quartic in the graviton momenta.

APPENDIX C

We give an explicit calculation of the gauge invariant pole term $T^{pole}_{\mu\nu} + T'_{\mu\nu}$, starting from an expression for $T^{pole}_{\mu\nu}$, given by Fig. 1. We then explicitly construct $T'_{\mu\nu}$. This approach is tedious, and has the additional disadvantage that it requires the Ward identity for the vertex function. The analytic form of $T^{pole}_{\mu\nu}$ is, according to Fig. 1,

$$T^{Pole}_{\mu\nu} = \tilde{\Gamma}_\nu(p', p+k)\,\tilde{D}(p+k)\,\tilde{\Gamma}_\mu(p+k, p) + \tilde{\Gamma}_\mu(p', p-k')\,\tilde{D}(p'-k)\,\tilde{\Gamma}_\nu(p-k', p). \tag{C.1}$$

No specific commitment about the spin or the other degrees of freedom of the target particle is made. It is understood that appropriate wave-functions sandwich (C.1) if necessary, and that $p^2=p'^2=m^2$. The tilda in (C.1) serves to remind of this fact.

Next we exploit the Ward identity for the vertex operator, which, it is remembered, is also a consequence of gauge invariance and requires no current commutators for its derivation.

$$k_\mu \Gamma^\mu(p, p+k) = D^{-1}(p+k) - D^{-1}(p). \tag{C.2}$$

Therefore

$$\begin{aligned} k^\mu T^{Pole}_{\mu\nu} &= \tilde{\Gamma}_\nu(p', p+k) - \tilde{\Gamma}_\nu(p-k', p) \\ &\quad - \Gamma_\nu(p', p+k)\,\tilde{D}(p+k)\,\tilde{D}^{-1}(p) \\ &\quad + \tilde{D}^{-1}(p')\,\tilde{D}(p'-k)\,\tilde{\Gamma}_\nu(p-k', p) \\ &= \tilde{\Gamma}_\nu(p', p+k) - \tilde{\Gamma}_\nu(p-k', p). \end{aligned} \tag{C.3}$$

The second equation in (C.3) differs from the first by terms which vanish on the mass shell. According to (C.3) and (II.4b) we must have

$$k^\mu T'_{\mu\nu} = -\tilde{\Gamma}_\nu(p', p+k) + \tilde{\Gamma}_\nu(p'-k, p). \tag{C.4a}$$

Since the only condition that $T'_{\mu\nu}$ needs to fulfil is given by (C.4a), we may impose a further requirement, consistent with (C.4a), and that is that (C.4a) is valid also off the mass shell, viz.

$$\begin{aligned} k^\mu T''_{\mu\nu} &= -\Gamma_\nu(p', p+k) + \Gamma_\nu(p'-k, p) \\ T'_{\mu\nu} &= \tilde{T}''_{\mu\nu} . \end{aligned} \tag{C.4b}$$

$T''_{\mu\nu}$ is taken to satisfy the same symmetry as $T_{\mu\nu}$.

Once we have arrived at (C.4b), we have made contact with the method of Gell-Mann and Goldberger [2] and Kazes [7]. Accordingly, we may take over their result, which is that (C.4b) and (II.2) are sufficient to determine $T''_{\mu\nu}$ up to terms quadratic in the photon momenta. The form for $T''_{\mu\nu}$ is

$$\begin{aligned} T''_{\mu\nu} = &- \frac{d}{dp^\mu} \Gamma_\nu(p,p) - k^\alpha \frac{d}{dp^\nu} \Gamma_{\mu|\alpha}(p,p) \\ &+ k'^\alpha \frac{d}{dp^\mu} \Gamma_{\nu|\alpha}(p,p) \end{aligned} \tag{C.5 a}$$

$$\Gamma_{\mu|\alpha}(p,p) \equiv \frac{\partial}{\partial p'^\alpha} \Gamma_\mu(p',p) \Big|_{p'=p} ; \tag{C.5b}$$

The expression for the scattering amplitude tensor may be taken to be

$$S^{\mu\nu} \equiv \Gamma^{\nu}(p', p+k)\, D(p+k)\, \Gamma^{\mu}(p+k, p)$$
$$+ \Gamma^{\mu}(p', p-k')\, D(p-k')\, \Gamma^{\nu}(p-k', p)$$
$$- \frac{d}{dp_{\mu}} \Gamma^{\nu}(p,p) - k^{\alpha} \frac{d}{dp_{\nu}} \Gamma^{\mu}{}_{|\alpha}(p,p) + k'^{\alpha} \frac{d}{dp_{\mu}} \Gamma^{\nu}{}_{|\alpha}(p,p) \qquad (C.6)$$

$$T^{\mu\nu} = \tilde{S}^{\mu\nu} + O(\omega^2). \qquad (C.7)$$

The term in $S^{\mu\nu}$, independent of the photon variables, can be simplified, using the differential form of the Ward identity :

$$S_0^{\mu\nu} = \Gamma^{\nu}(p,p)\, D(p)\, \Gamma^{\mu}(p,p) + \Gamma^{\mu}(p,p)\, D(p)\, \Gamma^{\nu}(p,p)$$
$$- \frac{d}{dp_{\mu}} \Gamma^{\nu}(p,p)$$
$$= D^{-1}(p) \left(\frac{d^2}{dp_{\mu}\, dp_{\nu}} D(p) \right) D^{-1}(p). \qquad (C.8)$$

Finally we note that any contribution to $S^{\mu\nu}$, proportional to p^{μ} or p^{ν}, may be ignored, since in the end p is taken in its rest frame, and the polarization vectors ε^{μ} and $\varepsilon^{*\nu}$ may be taken to have zero time component.

A P P E N D I X   D

We give an explicit calculation of the gauge invariant pole term $T^{\text{pole}}_{\mu\nu,\alpha\beta} + T'_{\mu\nu,\alpha\beta}$ in graviton scattering, analogous to the above derivation in photon scattering. For simplicity, we restrict ourselves to spin zero targets. The pole terms are given, in addition to terms analogous to those of Fig. 1, by a term which arises from graviton exchange. Thus we take $T^{\text{pole}}_{\mu\nu,\alpha\beta}$ to be given by terms whose diagramatic representation is as in Fig. 2, and whose analytic form is

$$\begin{aligned} T^{\text{Pole}}_{\mu\nu,\alpha\beta} = {} & \tilde{\Gamma}_{\alpha\beta}(p',p+k)\,\tilde{D}(p+k)\,\tilde{\Gamma}_{\mu\nu}(p+k,p) \\ & + \tilde{\Gamma}_{\mu\nu}(p',p-k')\,\tilde{D}(p-k')\,\tilde{\Gamma}_{\alpha\beta}(p-k',p) \\ & + \tilde{\Gamma}_{\gamma\delta}(p',p)\,\frac{d^{\gamma\delta,\varepsilon\varphi}(k-k')}{(k-k')^2}\,U_{\varepsilon\varphi,\mu\nu,\alpha\beta}(k,k'). \end{aligned} \tag{D.1}$$

Here $\tilde{\Gamma}_{\alpha\beta}$ is the gravity vertex operator, D is the matter propagator, $d^{\gamma\delta,\varepsilon\varphi}(k)/k^2$ is the graviton propagator and $U_{\varepsilon\varphi,\mu\nu,\alpha\beta}$ is the three-graviton vertex. The tilde reminds that the target is on the mass shell. Since we are performing the calculation to lowest order in G^2, we may take the lowest order perturbation expression for $d^{\gamma\delta,\varepsilon\varphi}(k-k')\,U_{\varepsilon\varphi,\mu\nu,\alpha\beta}(k,k')$. The explicit, complicated, expression for this has been given elsewhere [6]. We do not offer the details here, as the only property we shall need is that, on the p and p' mass shell.

$$\begin{aligned} & k^{\mu}\,\tilde{\Gamma}_{\gamma\delta}(p',p)\,\frac{d^{\gamma\delta,\varepsilon\varphi}(k-k')}{(k-k')^2}\,U_{\varepsilon\varphi,\mu\nu,\alpha\beta}(k,k') = \\ & (k'-k)_{\nu}\,\tilde{\Gamma}_{\alpha\beta}(p',p) - k^{\mu}\,\tilde{\Gamma}_{\mu\alpha}(p',p)\,g_{\nu\beta} - k^{\mu}\tilde{\Gamma}_{\mu\beta}(p',p)\,g_{\nu\alpha}. \end{aligned} \tag{D.2}$$

This follows from the explicit form of the lowest order graviton propagator and the three-graviton vertex. (It may also be true more generally.)

Next we shall need a Ward identity for $\Gamma_{\alpha\beta}(p',p)$. To derive such an identity in the usual fashion, it is necessary to know the commutator of the matter field, with the time component of the current, viz., with the time component of the source of the gravity field. The source of the gravity field is the total, symmetric energy-momentum tensor density, which is the sum of the matter energy-momentum tensor density and the gravitational field energy momentum tensor density. However, the gravity tensor, as it does not involve any matter variables, commutes at equal times with the matter field operator. The zero component of the matter tensor is just the four-momentum density. When the usual assumption, that the density of the generator of an invariance group does not contain Schwinger terms in its equal-time commutation relation with the fields, is made, then the relevant commutator for the present problem is determined to be proportional to the derivative of the field operator. Finally since the field operators themselves commute at equal times, we have for the Ward identity of gravitation theory for scalar particles the following expression

$$k_\mu \Gamma^{\mu\nu}(p,p+k) = p^\nu D^{-1}(p+k) - (p+k)^\nu D^{-1}(p). \tag{D.3}$$

This Ward identity plays a role analogous to that of the Ward identity of electrodynamics. Thus one derives in the usual fashion the (weak) equivalence principle from (D.3). Using (D.1), (D.2) and (D.3) we can give the divergence condition which determines $T'_{\mu\nu,\alpha\beta}$

$$\begin{aligned} -k^\mu T'_{\mu\nu,\alpha\beta} = k^\mu T^{Pole}_{\mu\nu,\alpha\beta} &= p_\nu \tilde{\Gamma}_{\alpha\beta}(p',p+k) \\ &- p'_\nu \tilde{\Gamma}_{\alpha\beta}(p-k',p) - (k'-k)_\nu \tilde{\Gamma}_{\alpha\beta}(p',p) \\ &+ k^\mu \tilde{\Gamma}_{\mu\alpha}(p',p) g_{\nu\beta} + k^\mu \tilde{\Gamma}_{\mu\beta}(p',p) g_{\nu\alpha}. \end{aligned} \tag{D.4}$$

Finally we set the ansatz that $T'_{\mu\nu,\alpha\beta} = \tilde{T}''_{\mu\nu,\alpha\beta}$, where

$$k_\mu T''_{\mu\nu,\alpha\beta} = -p_\nu \Gamma_{\alpha\beta}(p';p+k) + p'_\nu \Gamma_{\alpha\beta}(p-k';p) + (k'-k)_\nu \Gamma_{\alpha\beta}(p';p) - k^\mu \Gamma_{\mu\alpha}(p';p)\, g_{\nu\beta} - k^\mu \Gamma_{\mu\beta}(p';p)\, g_{\nu\alpha}, \tag{D.5}$$

and $T''_{\mu\nu,\alpha\beta}$ satisfies the same symmetries as $T_{\mu\nu,\alpha\beta}$. $T''_{\mu\nu,\alpha\beta}$ may be determined from this equation. The complete calculation through terms cubic in the graviton momenta is very involved. We do not give it here, but merely verify that the present considerations give the correct zero energy form. $T''_{\mu\nu,\alpha\beta}$, to zero order in graviton frequency, follows from (D.5), and is given by

$$T''_{\mu\nu,\alpha\beta} = -\tfrac{1}{2} p_\nu \frac{d}{dp^\mu} \Gamma_{\alpha\beta}(p,p) - \tfrac{1}{2} p_\mu \frac{d}{dp^\nu} \Gamma_{\alpha\beta}(p,p) - \tfrac{1}{2} g_{\nu\beta} \Gamma_{\mu\alpha}(p,p) - \tfrac{1}{2} g_{\nu\alpha} \Gamma_{\mu\beta}(p,p) - \tfrac{1}{2} g_{\mu\beta} \Gamma_{\nu\alpha}(p,p) - \tfrac{1}{2} g_{\mu\alpha} \Gamma_{\nu\beta}(p,p). \tag{D.6}$$

The total, zeroth order, scattering amplitude is

$$S_{\mu\nu,\alpha\beta} = \Gamma_{\alpha\beta}(p,p) D(p) \Gamma_{\mu\nu}(p,p) + \Gamma_{\mu\nu}(p,p) D(p) \Gamma_{\alpha\beta}(p,p) + \Gamma_{\gamma\delta}(p,p) F^{\gamma\delta,}{}_{\mu\nu,\alpha\beta} - \tfrac{1}{2} p_\nu \frac{d}{dp^\mu} \Gamma_{\alpha\beta}(p,p) - \tfrac{1}{2} p_\mu \frac{d}{dp^\nu} \Gamma_{\alpha\beta}(p,p) - \tfrac{1}{2} g_{\nu\beta} \Gamma_{\mu\alpha}(p,p) - \tfrac{1}{2} g_{\nu\alpha} \Gamma_{\mu\beta}(p,p) - \tfrac{1}{2} g_{\mu\beta} \Gamma_{\nu\alpha}(p,p) - \tfrac{1}{2} g_{\mu\alpha} \Gamma_{\nu\beta}(p,p) \tag{D.7}$$

$$T_{\mu\nu,\alpha\beta} = S_{\mu\nu,\alpha\beta} + O\omega.$$

(In offering expressions (D.6) and (D.7), we have suppressed terms which do not contribute to the final result ; viz., expressions proportional to $g^{\mu\nu}$ or $g^{\alpha\beta}$, and terms antisymmetric in μ and ν, or α and β, or $\mu\nu$ and $\alpha\beta$.)

Examining (D.7) we see that $S_{\mu\nu,\alpha\beta}$ is given by three parts. The first part, comprising the first two terms in (D.7) is the contribution from the single particle intermediate states. The second part, consisting of the third term in (D.7) is the graviton exchange contribution. The expression $F^{\gamma\delta,}{}_{\mu\nu,\alpha\beta}$ is the limit of $\left\{ d^{\gamma\delta,\varepsilon\varphi}(k-k')/(k-k')^2 \right\}$ $U_{\varepsilon\varphi,\mu\nu,\alpha\beta}(k,k')$, as k and k' go simultaneously to zero in all their components. The third part consisting of the remainder of (D.7) is the additional term which must be added to maintain gauge invariance. In perturbation theory, it corresponds to the "sea-gull" term pictured in Fig. 3.

The expression (D.7) may be simplified. For this we shall need the differential form of (D.3)

$$\Gamma^{\varepsilon\varphi}(p,p) = p^{\varepsilon}\frac{\partial}{\partial p_{\varphi}} D^{-1}(p) - g^{\varepsilon\varphi} D^{-1}(p). \tag{D.8}$$

(This is not manifestly symmetric in ε and φ. Symmetry is established by recalling that $D^{-1}(p)$, for spin zero particles, is a function only of p^2, so that $p^{\varepsilon}(\partial/\partial p_{\varphi})$ $D^{-1}(p)$ is proportional to $p^{\varepsilon}p^{\varphi}$.) Next we recall that the scattering amplitude is given by contracting the above with $\varepsilon^{\mu}\varepsilon^{\nu}$ and $\varepsilon^{*\alpha}\varepsilon^{*\beta}$, with $\varepsilon^{\mu}\varepsilon_{\mu} = \varepsilon^{*\alpha}\varepsilon^{*}_{\alpha} = 0$. Also p may be taken to be in its rest frame and the polarization tensors may be chosen without a time component. Hence terms in (D.7) proportional to p^{μ}, p^{ν}, p^{α}, p^{β}, $g^{\mu\nu}$ and $g^{\alpha\beta}$ do not contribute. It is seen that the first two terms in (D.7) do not contribute. The third term does contribute. In the remainder, the only terms that are not proportional to a momentum, are those that involve, e.g., $g^{\nu\beta}\Gamma^{\mu\alpha}$. However, from (D.8) it is seen that $\Gamma^{\mu\alpha}$ is proportional either to a momentum, $p^{\mu}p^{\alpha}$, or to D^{-1} which vanishes on the mass shell. Thus in the zero energy limit we may take for $S_{\mu\nu,\alpha\beta}$

$$S_{\mu\nu,\alpha\beta} = \Gamma_{\gamma\delta}(p,p)\, F^{\gamma\delta,}{}_{\mu\nu,\alpha\beta} \tag{D.9}$$

i.e., only the graviton exchange term contributes. We insert (D.8) ; the portion of $\Gamma_{\gamma\delta}$ proportional to $g^{\gamma\delta} D^{-1}$ does not contribute, since D^{-1} vanishes on the mass shell. Thus

$$S_{\mu\nu,\alpha\beta} = p_\gamma \frac{\partial D^{-1}(p)}{\partial p^\delta} F^{\gamma\delta,}{}_{\mu\nu,\alpha\beta}\,. \tag{D.10}$$

D^{-1} has the form $(p^2-m^2)(1+\sum(p^2))$, where $\sum(p^2)$ vanishes on the mass shell. Therefore, the final result for the zero energy scattering amplitude for gravitons off spin zero particles is

$$S_{\mu\nu,\alpha\beta} = 2 p_\gamma p_\delta F^{\gamma\delta,}{}_{\mu\nu,\alpha\beta} = 2m^2 F^{00,}{}_{\mu\nu,\alpha\beta} \tag{D.11}$$

The cross-section which follows from (D.11) is then given by (III.11).

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9) A subtle point arises in connection with this solution for $T^{pole}_{\mu\nu}$ + $+ T'_{\mu\nu}$. We have succeeded in giving $T'_{\mu\nu}\big|_{k^2=k'^2=0}$, which is all that is required for physical applications. However, it is not obvious that this solution is unique, viz., that if we could obtain the form of $T'_{\mu\nu}$ when $k^2 \neq 0 \neq k'^2$, that the limit of this $T'_{\mu\nu}$ as k^2 and $k'^2 \to 0$ is the above given $T'_{\mu\nu}\big|_{k^2=k'^2=0}$. Explicit calculation for spin 0 and spin $\frac{1}{2}$ targets does in fact show that the solution is unique. Furthermore, the calculation in Appendix C also shows that the solution we give here is the correct, unique solution. I am indebted to Dr. J.S. Bell for calling my attention to this point.

10) Comments similar for those of note 9) apply here.

11) D. Boulware and S. Deser - J.Math.Phys. 8, 1468 (1967).

12) The first derivative Schwinger term in the $[H^{o\ell}, H^{mn}]$ (ℓ, m, n = 1, 2, 3) equal time commutator is not of this form.

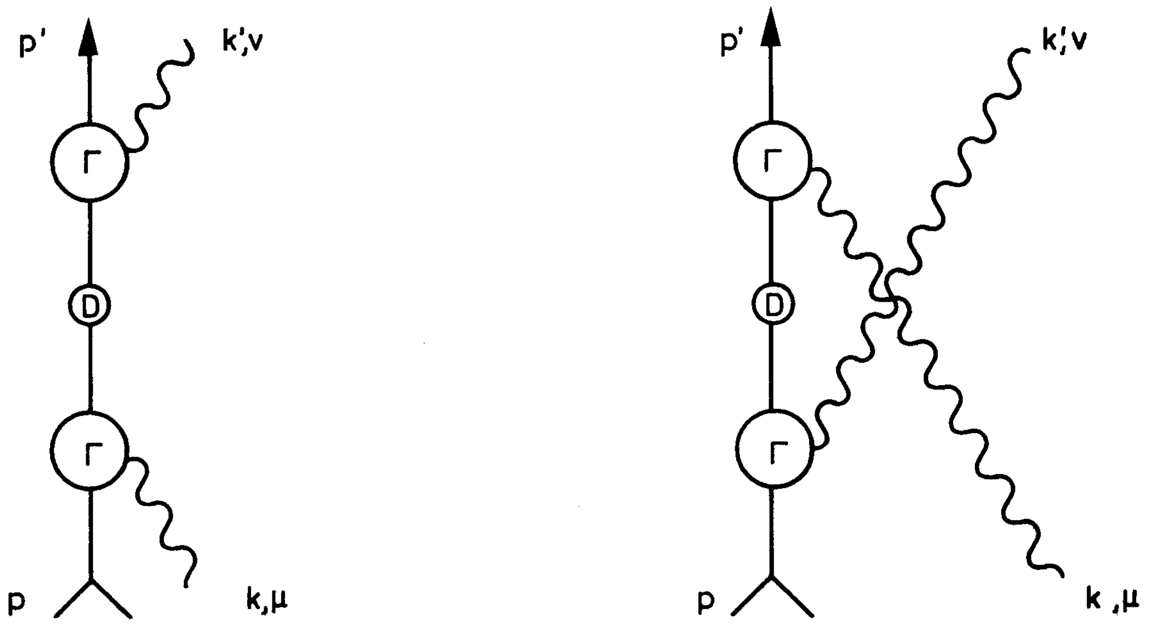
FIGURE CAPTIONS

Figure 1 Pole terms in scattering of photons.

Figure 2 Pole terms in scattering of gravitons

Figure 3 Sea-gull term in graviton scattering.

1)



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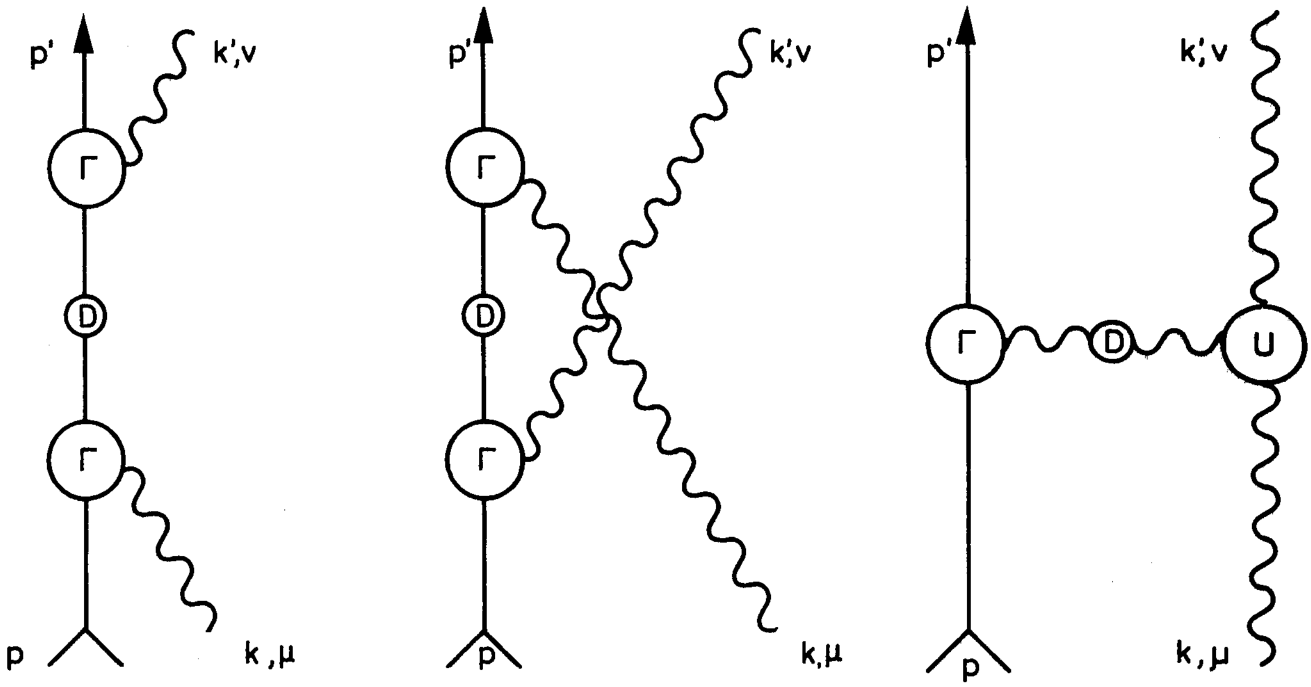
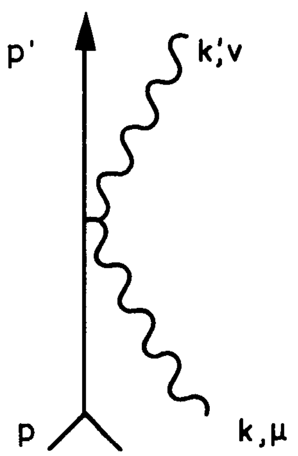


3)



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